

Approximation of Second Order Boundary Value Problems by Galerkin Method with Chebyshev Polynomial Basis

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Abstract

In this paper, the technique of Galerkin weighted residual method is used to obtain the numerical solution of second order linear boundary value problems. By the use of Linear transformation, the problem was first converted from its original form and interval to its equivalent form in the interval of the chebyshev polynomial used as basis of the trial solution. Then the converted form of the problem is solved using the Galerkin weighted residual method. Three numerical examples with Neumann boundary conditions were considered. The approximate solutions of the examples compared favourably with the exact solutions on using a very few Chebyshev polynomials. The method is very effective and accurate.

Keywords: Neumann boundary conditions, Galerkin weighted residual method, Chebyshev polynomials.

1. Introduction

Many second order linear and nonlinear boundary value problems arising in science and engineering can be solved either analytically or numerically. Thus, many authors have attempted solving second order Boundary Value Problem (BVP) to obtain a more accurate results by using a numerous methods, such as least square method, finite difference method, Sinc-Galerkin method, and also some other methods involving polynomial and non-polynomial spline functions.

Since the piecewise polynomials can be differentiated and integrated easily. They can be used to approximate any function to any accuracy desired. So Bernstein polynomials have been studied by many authors, spline functions have been studied extensively for solving only linear BVP. Arshad Khan, (2004) obtained a parametric cubic spline solution of two-point boundary value problems, Feng and Li, (2010) solved a second-order Neumann boundary value problem with singular non-linearity for exact three positive solutions, Also Lima and Carpentier, (2000) obtained a numerical solution of a singular boundary-value problem in non-Newtonian fluid mechanics, Rashidinia and Jalilian, (2010) introduced a spline solution of two point boundary value problems, Viswanadham et al. (2010) obtained a numerical solution of a fourth order boundary value problems by Galerkin method with Quintic B-splines basis, Das et al. (2010) produced a method for solution of nonlinear second order multi-point boundary value problems, Bhatti and Bracken, (2007) solved linear and non-linear differential equation numerically by Galerkin method in a Bernstein polynomials basis, recently Islam and Shirin, (2011) solved linear and non-linear differential equation numerically by Galerkin method in a Bernoulli polynomials basis and Rahman et al, (2012) solved linear differential equation numerically by Galerkin method in a Hermite polynomials basis. But none has attempted, to the knowledge of the present authors, to solve second order Boundary Value Problem using Chebyshev polynomials. Therefore, the purpose of this paper is to use Chebyshev polynomials as trial function for solving linear second order BVP exploiting Galerkin weighted residual method.

In this paper we give a short description of Chebyshev polynomials first in section 2. Then the formulation is derived in section 3 to solve second order boundary value problem with Neumann boundary conditions, in details, in section 3. Finally, three examples of Neumann boundary value problems are solved to verify the proposed formulation.

2. Chebyshev Polynomials

The Chebyshev Polynomials of the first kind is denoted as $T_n(x)$, and defined as one of the solutions of the Chebyshev differential equation:

$$(1 - x^2)y'' - xy' + n^2y = 0, \quad x \in [-1,1] \quad , \quad (1)$$

where $n \geq 0$ and $n \in \mathbb{N}$,

$$T_n(x) = \cos \{n \cos^{-1} x\}, \quad n > 0 \quad (2)$$

Putting $x = \cos \theta$ in (2) we have

$$T_n(x) = \cos n\theta \in [-1, 1] \quad (3)$$

The recurrence relation is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (4)$$

2.1 Chebyshev Polynomials in the interval $[a, b]$

The Chebyshev polynomial on $[a, b]$ is defined by

$$T_n(x) = \cos \left[n \cos^{-1} \left(\frac{2x - a - b}{b - a} \right) \right]; \quad a \leq x \leq b \quad (5)$$

And the recurrence relation on $[a, b]$ is given as

$$T_{n+1}(x) = 2 \left[\frac{2x - a - b}{b - a} \right] T_n(x) - T_{n-1}(x) \quad (6)$$

Thus, the first few terms of Chebyshev polynomial on the interval $[0, 1]$ is given below.

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 2x - 1 \\ T_2(x) &= 8x^2 - 8x + 1 \\ T_3(x) &= 32x^3 - 48x^2 + 8x - 1 \\ T_4(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\ T_5(x) &= 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \end{aligned} \quad (7)$$

3. Galerkin Weighted Residual Method

3.1 Conversion of Second Order Boundary Value Problems into Matrix Form

We consider a general second order linear boundary value problem given by

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = r(x) ; \quad a \leq x \leq b \quad (8)$$

Subject to the boundary conditions

$$\alpha_0 u(a) + \alpha_1 u'(a) = \gamma_1, \quad \beta_0 u(b) + \beta_1 u'(b) = \gamma_2 \quad (9)$$

where $p(x)$, $q(x)$ and $r(x)$ are given functions and γ_1, γ_2 are constants and $u(x)$ is the unknown function to be determined.

To use Chebyshev polynomials over $[0, 1]$, we convert the boundary value problem to an equivalent boundary value problem on $[0, 1]$.

3.2 Conversion of boundary value problem over arbitrary finite domain $[a, b]$ into its equivalent form over the interval $[0, 1]$.

Using the linear transformation

$$X = mx + n \quad (10)$$

and solving for x , we get

$$x = (b - a)X + a \quad (11)$$

Now differentiating (11) with respect to x , we have

$$1 = (b - a) \frac{dX}{dx} \quad (12)$$

That is

$$dx = (b - a)dX \quad (13)$$

Substituting (11) and (13) into (8) and (9) and after simplification we get,

$$\frac{d}{dX} \left(\frac{1}{(b-a)^2} p((b-a)X + a) \frac{du}{dX} \right) + q((b-a)X + a)u = r((b-a)X + a) \quad (14)$$

Subject to the boundary conditions

$$\alpha_0 u(0) + \frac{\alpha_1}{(b-a)} u'(0) = \gamma_1, \quad \beta_0 u(1) + \frac{\beta_1}{(b-a)} u'(1) = \gamma_2 \quad (15)$$

(14) and (15) are rewritten as

$$\frac{d}{dX} \left(p(x) \frac{du}{dX} \right) + q(x)u = r(x); \quad 0 \leq x \leq 1 \quad (16)$$

Subject to the boundary conditions

$$\alpha_0 u(0) + \frac{\alpha_1}{(b-a)} u'(0) = \gamma_1, \quad \beta_0 u(1) + \frac{\beta_1}{(b-a)} u'(1) = \gamma_2 \quad (17)$$

where

$$p(x) = \frac{1}{(b-a)^2} p((b-a)X + a), \quad q(x) = q((b-a)X + a),$$

$$\text{and } r(x) = r((b-a)X + a).$$

Now, consider

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u - r(x) = 0; \quad 0 \leq x \leq 1 \quad (18)$$

Subject to the boundary conditions

$$\alpha_0 u(0) + \frac{\alpha_1}{(b-a)} u'(0) = \gamma_1, \quad \beta_0 u(1) + \frac{\beta_1}{(b-a)} u'(1) = \gamma_2 \quad (19)$$

We want to solve the BVP of the form (20) by Galerkin weighted residual method using the polynomials, described in Section 2, as basis functions. For this, we assume the approximate solution of the differential equation (20) as

$$\tilde{u}(x) = \sum_{i=0}^n c_i T_i(x) \quad (20)$$

where $T_i(x)$ are Chebyshev polynomials of degree i and c_i are unknown constants, to be determined.

Using (20) in (18), the Galerkin weighted residual equations corresponding to the differential equation (20) is given by

$$\int_0^1 \left[\frac{d}{dx} \left(p(x) \frac{d u}{dx} \right) + q(x) u \right] T_j(x) dx = \int_0^1 r(x) T_j(x) dx \quad (21)$$

Subject to the boundary conditions

$$\alpha_0 u(0) + \frac{\alpha_1}{b-a} u'(0) = \gamma_1, \quad \beta_0 u(1) + \frac{\beta_1}{b-a} u'(1) = \gamma_2 \quad (22)$$

Equation (21) is splitted into:

$$\begin{aligned} \int_0^1 \left[\frac{d}{dx} \left(p(x) \frac{d u}{dx} \right) \right] T_j(x) dx + \int_0^1 q(x) u T_j(x) dx \\ = \int_0^1 r(x) T_j(x) dx \end{aligned} \quad (23)$$

The first term of (23) is integrated by parts. For convenience, let

$$\int_0^1 \left[\frac{d}{dx} \left(p(x) \frac{d u}{dx} \right) \right] T_j(x) dx = I(x) \quad (24)$$

so that (23) becomes

$$I(x) + \int_0^1 q(x) u T_j(x) dx = \int_0^1 r(x) T_j(x) dx \quad (25)$$

On integrating, we have

$$I(x) = p(x) T_j(x) \frac{d u}{dx} \Big|_0^1 - \int_0^1 p(x) \frac{d u}{dx} \frac{dT_j}{dx} \quad (26)$$

$$I(x) = p(1) T_j(1) \tilde{u}'(1) - p(0) T_j(0) \tilde{u}'(0) - \int_0^1 p(x) \frac{d u}{dx} \frac{dT_j}{dx} \quad (27)$$

Consider (22)

$$\alpha_0 u(0) + \frac{\alpha_1}{b-a} u'(0) = \gamma_1, \quad \beta_0 u(1) + \frac{\beta_1}{b-a} u'(1) = \gamma_2$$

Making $u'(1)$ the subject of formula, we have;

$$u'(1) = \frac{\gamma_2 (b-a) - \beta_0 (b-a) u(1)}{\beta_1} \quad (28)$$

Then

$$\tilde{u}'(1) = \frac{\gamma_2 (b-a)}{\beta_1} - \frac{\beta_0 (b-a) \tilde{u}(1)}{\beta_1} \quad (29)$$

Similarly;

$$\tilde{u}'(0) = \frac{\gamma_1(b-a)}{\alpha_1} - \frac{\alpha_0(b-a)\tilde{u}(0)}{\alpha_1} \quad (30)$$

Therefore (27) becomes,

$$\begin{aligned} I(x) &= p(1) T_j(1) \left[\frac{\gamma_2(b-a)}{\beta_1} - \frac{\beta_0(b-a)\tilde{u}(1)}{\beta_1} \right] \\ &\quad - p(0) T_j(0) \left[\frac{\gamma_1(b-a)}{\alpha_1} - \frac{\alpha_0(b-a)\tilde{u}(0)}{\alpha_1} \right] \quad (31) \\ &\quad - \int_0^1 p(x) \frac{d u}{d x} \frac{d T_j(x)}{d x} \end{aligned}$$

On expansion we have

$$\begin{aligned} I(x) &= p(1) T_j(1) \frac{\gamma_2(b-a)}{\beta_1} - p(1) T_j(1) \frac{\beta_0(b-a)\tilde{u}(1)}{\beta_1} \\ &\quad - p(0) T_j(0) \frac{\gamma_1(b-a)}{\alpha_1} + p(0) T_j(0) \frac{\alpha_0(b-a)\tilde{u}(0)}{\alpha_1} \quad (32) \\ &\quad - \int_0^1 p(x) \frac{d u}{d x} \frac{d T_j(x)}{d x} \end{aligned}$$

substituting (32) into (26) we obtain

$$\begin{aligned} &p(1) T_j(1) \frac{\gamma_2(b-a)}{\beta_1} - p(1) T_j(1) \frac{\beta_0(b-a)\tilde{u}(1)}{\beta_1} - p(0) T_j(0) \frac{\gamma_1(b-a)}{\alpha_1} \\ &\quad + p(0) T_j(0) \frac{\alpha_0(b-a)\tilde{u}(0)}{\alpha_1} - \int_0^1 p(x) \frac{d u}{d x} \frac{d T_j(x)}{d x} \\ &\quad + \int_0^1 q(x) u T_j(x) dx = \int_0^1 r(x) T_j(x) dx \quad (33) \end{aligned}$$

From (20) we have that

$$\frac{d^2 u}{dx^2} = \sum_{i=0}^n c_i T_i(x) \quad (34)$$

Also substituting $x = 0$ and $x = 1$ in (20) we obtain

$$\tilde{u}(0) = \sum_{i=0}^n c_i T_i(0) \quad \text{and} \quad \tilde{u}(1) = \sum_{i=0}^n c_i T_i(1) \quad (35)$$

On putting (20), (34) and (35) into (33), we get

$$\begin{aligned} & p(1) T_j(1) \frac{\gamma_2(b-a)}{\beta_1} - p(1) T_j(1) \frac{\beta_0(b-a) \sum_{i=0}^n c_i T_i(1)}{\beta_1} - p(0) T_j(0) \frac{\gamma_1(b-a)}{\alpha_1} \\ & + p(0) T_j(0) \frac{\alpha_0(b-a) \sum_{i=0}^n c_i T_i(0)}{\alpha_1} + \int_0^1 \left[- \sum_{i=0}^n c_i p(x) \frac{dT_i}{dx} \frac{dT_j}{dx} \right. \\ & \left. + q(x) \sum_{i=0}^n c_i T_i(x) T_j(x) \right] dx = \int_0^1 r(x) T_j(x) dx \end{aligned} \quad (36)$$

By simplification we get,

$$\begin{aligned} & \sum_{i=0}^n c_i \left\{ \int_0^1 \left[- p(x) \frac{dT_i}{dx} \frac{dT_j}{dx} + q(x) T_i(x) T_j(x) \right] dx - \frac{p(1) T_j(1) \beta_0(b-a) T_i(1)}{\beta_1} \right. \\ & \left. + \frac{p(0) T_j(0) \alpha_0(b-a) T_i(0)}{\alpha_1} \right\} = \int_0^1 r(x) T_j(x) dx - \frac{p(1) T_j(1) \gamma_2(b-a)}{\beta_1} \\ & + \frac{p(0) T_j(0) \gamma_1(b-a)}{\alpha_1} \end{aligned} \quad (37)$$

or equivalently in matrix form

$$\sum_{i=0}^n c_i K_{i,j} = F_j; \quad j = 0, 1, 2, \dots, n \quad (38)$$

where

$$K_{i,j} = \int_0^1 \left[-p(x) \frac{dT_i}{dx} \frac{dT_j}{dx} + q(x) T_i(x) T_j(x) \right] dx - \frac{p(1) T_j(1) \beta_0 (b-a) T_i(1)}{\beta_1} + \frac{p(0) T_j(0) \alpha_0 (b-a) T_i(0)}{\alpha_1} \quad (39)$$

and

$$F_j = \int_0^1 r(x) T_j(x) dx - \frac{p(1) T_j(1) \gamma_2 (b-a)}{\beta_1} + \frac{p(0) T_j(0) \gamma_1 (b-a)}{\alpha_1} \quad (40)$$

Solving the system (38), we find the values of the constants $c_i (i = 0, 1, 2, \dots, n)$, and then substituting these parameters in (20), we get the approximate solution of the BVP (18).

The absolute error, E of this formulation is defined by

$$E = |u(x) - \tilde{u}(x)|$$

0.0.4 Discussion of the Boundary Condition used

0.0.5 Neumann Boundary Conditions

The boundary condition given by

$$\alpha_0 u(a) + \alpha_1 u'(a) = \gamma_1, \quad \beta_0 u(b) + \beta_1 u'(b) = \gamma_2$$

is said to be Neumann boundary conditions if

$$\alpha_0 = 0, \alpha_1 \neq 0, \beta_0 = 0, \beta_1 \neq 0$$

The approximate solution of the differential equation (18) consisting of Neumann boundary conditions can be obtained by substituting

$\alpha_0 = 0$, and $\beta_0 = 0$ in equation (39) and (40) as

$$\sum_{i=0}^n c_i K_{i,j} = F_j ; \quad j = 0,1,2 \dots n \quad (41)$$

Where

$$K_{i,j} = \int_0^1 \left[-p(x) \frac{dT_i}{dx} \frac{dT_j}{dx} + q(x) T_i(x) T_j(x) \right] dx \quad (42)$$

and

$$F_j = \int_0^1 r(x) T_j(x) dx - \frac{p(1) T_j(1) \gamma_2 (b-a)}{\beta_1} + \frac{p(0) T_j(0) \gamma_1 (b-a)}{\alpha_1} \quad (43)$$

4. Numerical Examples

tocsubsection Numerical Example Example 1

Consider the boundary value problem

$$\frac{d}{dx} \left(\frac{du}{dx} \right) + u = x^2 e^{-x} ; \quad 0 \leq x \leq 10 \quad (44)$$

Subject to the boundary conditions

$$u'(0) = 0, \quad u'(10) = 0 \quad (45)$$

The exact solution is

$$u(x) = -\frac{\cos 10 + 99 e^{-10}}{2 \sin 10} \cos x - \frac{1}{2} \sin x + \frac{1}{2} e^{-x} (1+x)^2 \quad (46)$$

0.0.6 Solution

An equivalent boundary value problem (44) over [0,1] is given as

$$\frac{1}{100} \frac{d^2 u}{dx^2} + u = 100 x^2 e^{-10x}, \quad 0 \leq x \leq 1 \quad (47)$$

Subject to the boundary conditions

$$u'(0) = 0, \quad u'(1) = 0 \quad (48)$$

And the exact solution is

$$u(x) = -\frac{\cos 10 + 99 e^{-10}}{2 \sin 10} \cos 10 x - \frac{1}{2} \sin 10 x + \frac{1}{2} e^{-10 x} (1 + 10 x)^2 \quad (49)$$

By the procedure of the method, we have:

$$\sum_{i=0}^n c_i K_{i,j} = F_j; \quad j = 0, 1, 2 \dots n \quad (50)$$

Where

$$K_{i,j} = \int_0^1 \left[-\frac{1}{100} \frac{dT_i}{dx} \frac{dT_j}{dx} + T_i(x) T_j(x) \right] dx \quad (51)$$

and

$$F_j = \int_0^1 100 x^2 e^{-10 x} T_j(x) dx \quad (52)$$

solving the system (50) for $n = 12$, we obtain the approximate solution

$$\begin{aligned} \tilde{u}(x) = & \frac{1043252849}{1374389534} \frac{40353}{72} x^{12} - \frac{1618316846}{4398046511} \frac{9250217}{104} x^{11} + \frac{1233926347}{1759218604} \frac{24902891}{4416} x^{10} \\ & - \frac{5196704812}{92309885} x^9 + \frac{5176698080}{86855225} x^8 - \frac{2630640853}{652094511} x^7 \\ & + \frac{7036874417}{3814795717} \frac{7664}{9731660087} x^6 - \frac{7036874417}{6791710317} \frac{7664}{1843228687} x^5 + \frac{2814749767}{1251458691} \frac{10656}{620333121} x^4 \\ & + \frac{4503599627}{3752532456} \frac{370496}{407501161} x^3 + \frac{1801439850}{1482092425} \frac{9481984}{3324395527} x^2 + \frac{2251799813}{2878021799} \frac{685248}{7922187} x \\ & - \frac{2882303761}{1231526856} \frac{51711744}{219145437} x - \frac{1152921504}{606846976} x + \frac{4611686018}{427387904} \end{aligned} \quad (53)$$

to subsection Numerical Example Example 2

Consider the linear boundary value problem

$$\frac{d}{dx} \left(\frac{du}{dx} \right) + u = \cos x, \quad 0 \leq x \leq 5 \quad (54)$$

Subject to the Neumann conditions

$$u'(0) = 0, \quad u'(5) = 0 \quad (55)$$

The exact solution is

$$\begin{aligned} \frac{1}{4} (-6 \cos^2 5 + 2 + 10 \cot 5 + \cot 5 \sin 10 + 2 \cos 10) \cos x \\ + \frac{1}{4} (2 \cos^3 x + 2x \sin x + \sin x \sin 2x) \end{aligned} \quad (56)$$

The boundary value problem (54) over $[0,1]$ is equivalently to the boundary value problem

$$\frac{1}{25} \frac{d^2 u}{dx^2} + u = \cos 5x, \quad 0 \leq x \leq 1 \quad (57)$$

Subject to the Neumann boundary conditions

$$u'(0) = 0, \quad u'(1) = 0 \quad (58)$$

The exact solution is

$$\begin{aligned} \frac{1}{4} (-6 \cos^2 5 + 2 + 10 \cot 5 + \cot 5 \sin 10 + 2 \cos 10) \cos 5x \\ + \frac{1}{4} (2 \cos^3 5x + 10x \sin 5x + \sin 5x \sin 10x) \end{aligned} \quad (59)$$

By the procedure of the method we have:

$$\sum_{i=0}^n c_i K_{i,j} = F_j ; \quad j = 0,1,2 \dots n \quad (60)$$

where

$$K_{i,j} = \int_0^1 \left[-\frac{1}{25} \frac{dT_i}{dx} \frac{dT_j}{dx} + T_i(x) T_j(x) \right] dx \quad (61)$$

and

$$F_j = \int_0^1 \cos 5x T_j(x) dx \quad (62)$$

solving the system (60) for $n = 12$, we obtain the approximate solution

$$\begin{aligned} \tilde{u}(x) = & \frac{1428537820}{5629499534} x^{12} - \frac{072213}{21312} x^{12} - \frac{1996674976}{1125899906} x^{11} + \frac{4365953}{842624} x^{11} + \frac{4784493111}{1125899906} x^{10} - \frac{4979425}{842624} x^{10} \\ & - \frac{5167948587}{58403053} x^9 - \frac{1555364510}{432048671} x^9 - \frac{6540868682}{55745695} x^7 \\ & + \frac{1801439850}{2636861630} x^6 - \frac{9481984}{191266279} x^6 - \frac{7205759403}{2813420005} x^5 + \frac{7927936}{114082735} x^5 - \frac{7205759403}{2148660982} x^4 + \frac{7927936}{6391780028} x^4 - \frac{71}{71} x^4 \\ & + \frac{3602879701}{4653733638} x^3 - \frac{8963968}{90600223} x^3 + \frac{4611686018}{1143284318} x^2 - \frac{427387904}{0031340572} x^2 - \frac{3689348814}{7419103232} x^2 \\ & - \frac{7378697629}{3501146970} x - \frac{4838206464}{858437} x + \frac{7378697629}{1131159252} - \frac{4838206464}{3446880722} - \frac{89}{89} \\ & - \frac{1180591620}{7174113034} - \frac{24}{24} - \frac{4722366482}{8696452136} - \frac{96}{96} \end{aligned} \quad (63)$$

tocsubsection Numerical Example Example 3

Consider the linear boundary value problem

$$\frac{d}{dx} \left(\frac{du}{dx} \right) + u = -1, \quad 0 \leq x \leq 1 \quad (64)$$

subject to the boundary conditions

$$u'(0) = \frac{1 - \cos 1}{\sin 1}, \quad u'(1) = -\frac{1 - \cos 1}{\sin 1} \quad (65)$$

The exact solution is

$$u(x) = \cos x + \frac{1 - \cos 1}{\sin 1} \sin x - 1 \quad (66)$$

By the procedure of the method we have

$$\sum_{i=0}^{m-1} c_i K_{i,j} = P_j, \quad j = 0, 1, 2, \dots, m \quad (67)$$

where

$$K_{i,j} = \int_0^1 \left[\frac{dT_i}{dx} \frac{dT_j}{dx} + q(x) T_i(x) T_j(x) \right] dx \quad (68)$$

and

$$P_j = \int_0^1 -T_j''(x) dx + \frac{1 - \cos 1}{\sin 1} T_j'(1) + \frac{1 - \cos 1}{\sin 1} T_j'(0) \quad (69)$$

solving the system (64) for $m=10$, we obtain the approximate solution

$$\begin{aligned} \tilde{u}(x) = & \frac{5886344557}{188894659} x^{10} + \frac{167779}{1568655536} x^9 + \frac{4946212338}{5705735037} x^8 + \frac{7640388984}{4175801344} x^7 \\ & + \frac{6269}{8883209538} x^6 + \frac{1827012858}{66349} x^5 + \frac{663323001}{792281631} x^4 + \frac{1441470004}{4264337593} x^3 \\ & + \frac{8355602888}{923666400} x^2 + \frac{543950336}{4179699184} x + \frac{620340767}{620340767} \\ & + \frac{3369126560}{200824960} x^{10} + \frac{5305735077}{3651670423} x^9 + \frac{4175801344}{9472512860} x^8 \\ & + \frac{169020077}{880158553} x^7 + \frac{418832088}{699} x^6 + \frac{360944361}{9835449047} x^5 \\ & + \frac{8410711880}{53} x^4 + \frac{4066481920}{7303340847} x^3 + \frac{8945025720}{6517539555} x^2 \\ & + \frac{1643496047}{5497} x + \frac{811263341}{103849971} \\ & + \frac{4606681655}{7069655257} x^{10} + \frac{7880051440}{0609926584} x^9 \\ & + \frac{641}{40192} x^8 + \frac{138630020}{915534249} x^7 \\ & + \frac{2076998733}{4139310514} x^6 + \frac{1219853168}{80884} x^5 \end{aligned} \quad (70)$$

5. Table of Results

Table 1: Table of Results for Example 1

x	Exact	Approximate	Absolute Error	Absolute Error
		Chebyshev Polynomial, 12		Hermite, 12
.00	0.267044623200002	0.267044818597408	1.95397406 0886488 $\times 10^{-7}$	
.10	0.099412588679783	0.099373341901186	3.92467785 9690195 $\times 10^{-5}$	1.5213348 $\times 10^{-1}$
.20	0.473563254587092	0.473510891411736	5.23631753 5561863 $\times 10^{-5}$	8.9251515 $\times 10^{-2}$
.30	1.087104964438696	1.087159547950565	5.45835118 6882204 $\times 10^{-5}$	2.8328132 $\times 10^{-1}$
.40	1.108720558635556	1.108682114267377	3.84443681 7868058 $\times 10^{-5}$	3.8052691 $\times 10^{-1}$
.50	0.383163629150378	0.383165593911502	1.96476112 3796066 $\times 10^{-5}$	3.0792232 $\times 10^{-1}$
.60	0.536056278303779	0.877641840349425	5.20932753 8302500 $\times 10^{-5}$	6.3392590 $\times 10^{-1}$
.70	0.877589747074042	0.877641840349425	5.20932753 8302500 $\times 10^{-5}$	5.6018947 $\times 10^{-1}$
.80	0.369487868273324	0.369437743521433	5.01247518 9103416 $\times 10^{-5}$	5.5869004 $\times 10^{-1}$
.90	0.498988815996907	0.498952747240357	3.60687565 5037811 $\times 10^{-5}$	5.2580667 $\times 10^{-1}$
.00	0.918362556053612	0.918362752364901	1.96311288 6693708 $\times 10^{-7}$	5.3035772 $\times 10^{-1}$

Table 2: Table of Results for Example 2

x	Exact	Approximate	Absolute Error	Absolute Error
		Chebyshev Polynomial, 12		Hermite, 12
.00	- 0.239532288831 864	- 0.23953228883 1873	9.15933995 3157542 $\times 10^{-15}$	
.10	- 0.090352975037 481	- 0.09035299318 1597	1.81441156 2661711 $\times 10^{-8}$	6.7277990 $\times 10^{-1}$
.20	0.291315644418 219	0.29131566725 5802	2.28375826 0279711 $\times 10^{-8}$	2.1316277 $\times 10^{-1}$
.30	0.731177396132 015	0.73117737296 9259	2.31627558 2447958 $\times 10^{-8}$	1.3334725 $\times 10^{-1}$
.40	1.008978031073 958	1.00897804856 5136	1.74911776 1740138 $\times 10^{-8}$	4.8501662 $\times 10^{-1}$
.50	0.939989944044 937	0.93998993954 4206	4.50073134 3171083 $\times 10^{-9}$	2.5722287 $\times 10^{-1}$
.60	0.448815180726 876	0.44881517131 3601	9.41327465 9753057 $\times 10^{-9}$	3.5905097 $\times 10^{-1}$
.70	- 0.389559034758 166	- 0.38955901788 4770	1.68733957 9170555 $\times 10^{-8}$	1.9719354 $\times 10^{-1}$
.80	- 1.357036238030 049	- 1.35703625515 0527	1.71204783 6832653 $\times 10^{-8}$	5.1685119 $\times 10^{-1}$
.90	- 2.148950364432 671	- 2.14895035304 9155	1.13835159 0768139 $\times 10^{-8}$	5.3573739 $\times 10^{-1}$
.00	- 2.465256939196 901	- 2.46525693919 6895	6.21724893 7900877 $\times 10^{-15}$	5.3107857 $\times 10^{-1}$

Table 3: Table of Results for Example 3

x	Exact	Approximate	Absolute Error	Absolute Error
		Chebyshev Polynomial, 10		Bernoulli, 10
.00	0.0000000000000000 00	0.0000000000000000	0.0000000000000000	0.000000000000
.10	0.0495434093618 00	0.049543409361801	4.51028103 7539698 $\times 10^{-16}$	1.932482 $\times 10^{-14}$
.20	0.0886001279101 83	0.088600127910183	2.91433543 9641036 $\times 10^{-16}$	2.672862 $\times 10^{-14}$
.30	0.1167799138238 47	0.116779913823847	5.27355936 6969494 $\times 10^{-16}$	1.992850 $\times 10^{-14}$
.40	0.1338012039969 42	0.133801203996942	1.11022302 4625157 $\times 10^{-16}$	1.013079 $\times 10^{-14}$
.50	0.1394939273245 49	0.139493927324548	9.99200722 1626409 $\times 10^{-16}$	2.942091 $\times 10^{-14}$
.60	0.1338012039969 42	0.133801203996942	1.11022302 4625157 $\times 10^{-16}$	1.010303 $\times 10^{-14}$
.70	0.1167799138238 47	0.116779913823847	6.10622663 5438361 $\times 10^{-16}$	1.981748 $\times 10^{-14}$
.80	0.0886001279101 83	0.088600127910183	3.33066907 3875470 $\times 10^{-16}$	2.670086 $\times 10^{-14}$
.90	0.0495434093618 00	0.049543409361801	4.37150315 9461554 $\times 10^{-16}$	1.942890 $\times 10^{-14}$
.00	0.0000000000000000 00	0.0000000000000000	0.0000000000000000	0.000000000000

Discussion of Result

Tables 4.1 and 4.2 display the results of example 1 and 2 respectively using the degree of the chebyshev polynomial, $n = 12$, and it is clearly seen that the method with chebyshev polynomial perform better than that of Hermite polynomial.

Tables 4.3 display the result of example 3 for $n = 10$ and it is clearly seen that the method with chebyshev polynomial perform better than that of Bernoulli basis

6. Summary and Conclusion

We have discussed in this paper, in details, the formulation of a linear second order boundary value problems into matrix form by Galerkin weighted residual method to obtain an approximate solution to the second order Neumann boundary value problems using Chebyshev polynomials as our trial functions. Three numerical examples were tested. The approximate solutions compared favourably with the exact solutions.

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