

COEFFICIENT BOUNDS FOR CERTAIN BAZILEVIČ MAPS

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Abstract. Following Babalola [3], we obtain the best possible upper bound for the coefficients of functions in the class $B_n^\lambda(\gamma)$, using a technique due to Nehari and Netanyahu [9] and an application of certain integral iteration of Carathéodory-type functions. The sharp bound on the Fekete-Szegő functional in $B_n^\lambda(\gamma)$ is also obtained.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

In a recent paper, Babalola [4] provided a new approach to the study of the well-known Bazilevič functions, given as

$$f(z) = \left\{ \frac{\alpha}{1 + \beta^2} \int_0^z [p(t) - i\beta] t^{-\left(1 + \frac{i\alpha\beta}{1 + \beta^2}\right)} g(t)^{\left(\frac{\alpha}{1 + \beta^2}\right)} dt \right\}^{\frac{1 + i\beta}{\alpha}},$$

where the parameter β is no longer assumed to be zero, as in many previous works (see e.g. [1, 3, 8, 10, 13, 14]). The new method involved a modification of the class of Carathéodory functions. The modified class is denoted here by P_ξ and consists of analytic functions

$$h(z) = \xi + p_1 z + \dots$$

on E , with positive real part, where $\operatorname{Re} \xi = 1$.

The class P_ξ is of Carathéodory-type. We see that $h \in P_\xi$ if and only if $h(0) = \xi$ and $\operatorname{Re} h(z) > 0$. The well-known class P of Carathéodory maps coincides with P_ξ for $\xi = 1$ and it is easy to see that $p \in P$ if and only if $h(z) = p(z) + \xi - 1 \in P_\xi$. The function given by $H_0(z) = (\xi + (2 - \xi)z)/(1 - z) = \xi + 2z + 2z^2 + \dots$ plays a central role in the study of the class P_ξ , especially regarding extremal problems.

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Using the new definition, Babalola inspired new investigations of the class of Bazilevič functions [4, 5]. In particular, using the Sălăgean derivative operator, D^n , $n = 0, 1, 2, \dots$, defined by $D^n f(z) = D(D^{n-1}f(z)) = z(D^{n-1}f(z))'$, with $D^0 f(z) = f(z)$ (see [12]), he gave the following definition.

DEFINITION 1.1. (see [5]) Let $\eta > 0$, $\lambda = \eta + i\mu$ and $\xi = \lambda/\eta$ be some constants. A function $f \in A$ belongs to the class $B_n(\lambda)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\xi, \quad z \in E.$$

We note that we obtain the class of Bazilevič functions in the case $\lambda = \alpha/(1 + i\beta)$, $\alpha > 0$.

Now, denote by $P_\xi(\gamma)$ the subclass of functions $h \in P_\xi$ with $\operatorname{Re} h(z) > \gamma$, where $0 \leq \gamma < 1$ and $z \in E$.

DEFINITION 1.2. With all parameters defined above, a function $f \in A$ belongs to the class $B_n^\lambda(\gamma)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\xi(\gamma), \quad z \in E.$$

If $\xi = 1$ (that is $\lambda = \eta$) in Definition 1.2, we get the class $T_n^\eta(\gamma)$ introduced in [10] (see also [2]).

Following Babalola [2], we define an integral iteration of $h \in P_\xi(\gamma)$ as follows.

DEFINITION 1.3. Let $h \in P_\xi(\gamma)$. The n th complex-parameter integral iteration of $h(z)$, $z \in E$, is defined by

$$h_n(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} h_{n-1}(t) dt, \quad n = 1, 2, \dots,$$

with $h_0(z) = h(z) = \xi + (1 - \gamma)p_1 z + \dots$

In series form, the above iteration gives $h_n(z) = \xi + (1 - \gamma) \sum_{k=1}^\infty p_{n,k} z^k$, where $p_{n,k} = (1 - \gamma) \lambda^n p_k / (\lambda + k)^n$ is such that

$$|p_{n,k}| \leq \frac{2(1 - \gamma)|\lambda|^n}{|(\lambda + k)|^n}, \quad k = 1, 2, \dots$$

The function $H_n(z)$, defined by

$$H_n(\gamma, z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} H_{n-1}(\gamma, t) dt, \quad n = 1, 2, \dots,$$

where $H_0(\gamma, z) = \gamma + (1 - \gamma)(1 + z)/(1 - z) + \xi - 1 = [\xi + (2(1 - \gamma) - \xi)z]/(1 - z)$, also plays a central role for extremal problems with respect to the iteration $h_n(z)$.

In the present paper, we follow the work of Babalola [3], using a technique due to Nehari and Netanyahu [9] and an application of the integral iteration $h_n(z)$, to obtain the best possible bounds for the coefficients of the functions

in the class $B_n^\lambda(\gamma)$ and their Fekete-Szego functional. The two coefficient problems dealt with in this paper are well-known in the theory of geometric functions (see [7, 11]). In the next section, we state (and prove, where necessary) the relevant lemmas which we then apply, in Section 3, to prove our results.

2. PRELIMINARY LEMMAS

In [4], Babalola noted that most of the inequalities for P remain unperturbed by the new normalization. The proofs of the first two lemmas are similar to those of given by Nehari and Netanyahu for [9, Lemmas I and II].

LEMMA 2.1. *If $p(z) = \xi + b_1z + b_2z^2 + \dots$ and $q(z) = \xi + c_1z + c_2z^2 + \dots$ belongs to P_ξ , then $r(z) = \xi + \frac{1}{2} \sum_{k=1}^{\infty} b_k c_k z^k$ also belongs to P_ξ .*

LEMMA 2.2. *Let $h(z) = \xi + \sum_{k=1}^{\infty} d_k z^k$ and $\xi + G(z) = \xi + \sum_{k=1}^{\infty} b'_k z^k$ be functions in P_ξ . Set*

$$\beta_m = \frac{1}{2^m} \left[\xi + \frac{1}{2} \sum_{\epsilon=1}^m \binom{m}{\epsilon} d_\epsilon \right], \quad \beta_0 = \xi.$$

If B_ν is defined by

$$\sum_{m=1}^{\infty} (-1)^{m+1} \beta_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} B_\nu z^\nu,$$

then $|B_\nu| \leq 2$, $\nu = 1, 2, \dots$

COROLLARY 2.3. *Let $h_n(z)$ be the n th integral iteration of $h_0(z) = \xi + \sum_{k=0}^{\infty} p_k z^k$ with $\operatorname{Re} h_n(z) > \gamma$ and let $\xi + G(z) = \xi + \sum_{k=0}^{\infty} b'_k z^k$ be a function in P_ξ . Define β_m as in the previous lemma and ϕ_m as*

$$(2) \quad \phi_m = \frac{(1-\gamma)\lambda^n}{(\lambda+m)^n} \beta_m, \quad \phi_0 = (1-\gamma)\xi.$$

If A_ν is defined by

$$(3) \quad \sum_{m=1}^{\infty} (-1)^{m+1} \phi_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} A_\nu z^\nu,$$

then

$$(4) \quad |A_\nu| \leq \frac{2(1-\gamma)|\lambda|^n}{|\lambda+\nu|^n}, \quad \nu = 1, 2, \dots$$

Proof. The proof follows as in [3], in view of (2). □

LEMMA 2.4 ([3]). *Let $J(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series. Then the m^{th} integer product of $J(z)$ is*

$$J^m(z) = \left(\sum_{k=0}^{\infty} c_k z^k \right)^m = \sum_{k=0}^{\infty} c_k^{(m)} z^k,$$

where $c_k^{(1)} = c_k$ and

$$c_k^{(m)} = \sum_{j=0}^k c_j c_{k-j}^{(m-1)}, \quad m \geq 2.$$

LEMMA 2.5 ([3, p. 145]). Let $m = 1, 2, \dots$, $n = 0, 1, 2, \dots$ and ρ_l , $l = 1, 2, \dots$, take values in the set $M = \{0, 1, 2, \dots, m\}$ such that $\rho_1 + \rho_2 + \dots + \rho_m = m$. If $\alpha > 0$ is a real number, then we have the inequality

$$\prod_{l=1}^m \frac{\alpha^{\rho_l}}{(\alpha + l)^{\rho_l}} \leq \frac{\alpha}{\alpha + m - 1}.$$

LEMMA 2.6 ([5]). Let $h = \xi + p_1 z + p_2 z^2 + \dots \in P_\xi$. Then, for any real number τ , we have the sharp inequality

$$\left| p_2 - \tau \frac{p_1^2}{2} \right| \leq 2 \max \{1, |1 - \tau|\}.$$

Before we state and prove our main result, we compute the leading coefficients A_ν , in the expression (3), as follows: From (3) we have

$$(5) \quad \sum_{m=1}^{\infty} (-1)^{m+1} \phi_{m-1} G^m(z) = \phi_0 G(z) - \phi_1 G^2(z) + \dots = \sum_{\nu=1}^{\infty} A_\nu z^\nu,$$

with $G(z) = \sum_{\nu=1}^{\infty} b'_\nu z^\nu$, and, applying Lemma 2.4, we have

$$(6) \quad G^m(z) = \left(\sum_{\nu=1}^{\infty} b'_\nu z^\nu \right)^m = \sum_{\nu=m}^{\infty} G_\nu^{(m)} z^\nu, \quad m = 1, 2, \dots$$

$G_\nu^{(m)}$ has the general form

$$(7) \quad G_\nu^{(m)} = \sum_{\rho \in J_{\nu,m}} G_{\nu,\rho} \prod_{l=1}^m (b'_l)^{\rho_l}, \quad \text{where } G_{\nu,\rho} = \frac{m!}{\rho_1! \rho_2! \dots \rho_l!},$$

for some multi-index $\rho = (\rho_1, \rho_2, \dots, \rho_m)$ and the set $J_{\nu,m} = \{\rho \mid \sum_{l=1}^m \rho_l = m, \sum_{l=1}^m l \rho_l = \nu\}$. Using (6) and (7) in (5), we obtain

$$\sum_{m=1}^{\infty} (-1)^{m+1} \phi_{m-1} G^m(z) = \sum_{\nu=1}^{\infty} \left(\sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)} \right) z^\nu,$$

which implies that

$$\sum_{\nu=1}^{\infty} \left[\sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)} \right] z^\nu = \sum_{\nu=1}^{\infty} A_\nu z^\nu,$$

with

$$A_\nu = \sum_{m=1}^{\nu} (-1)^{m+1} \phi_{m-1} G_\nu^{(m)}.$$

By Corollary 2.3, the coefficients A_ν satisfy inequality (4), if $\xi + G(z) = \xi + b'_1 z + b'_2 z^2 + \dots$ is a function in the class P_ξ , and, by Lemma 2.1, we may set $b'_l = \frac{1}{2} b_l c_l$, where $\xi + b_1 z + b_2 z^2 + \dots$ is in P_ξ and $H(z) = \xi + c_1 z + c_2 z^2 + \dots$ is an arbitrary function in P_ξ . Then, taking into account also (7), we have

$$(8) \quad |A_\nu| = \left| \sum_{m=1}^{\nu} (-1)^{m+1} \frac{\phi_{m-1}}{2^m} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\gamma)|\lambda|^n}{|\lambda + \nu|^n}.$$

Using (2) in (8), yields

$$|A_\nu| = \left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1} (1-\gamma) \lambda^n}{2^m (\lambda + m - 1)^n \lambda} \beta_{m-1} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\lambda + \nu|^n}.$$

Using Lemma 2.5, we get, for $\nu = 1, 2, \dots$,

$$(9) \quad \begin{aligned} & \sum_{m=1}^{\nu} (-1)^{m+1} \frac{\beta_{m-1}}{2^m |\lambda|} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^m \frac{(1-\gamma)^{\rho_l} \eta^{\rho_l+1} |\lambda|^{n\rho_l - \rho_l - 1}}{|\lambda + l|^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \\ & \leq \sum_{m=1}^{\nu} \frac{(-1)^{m+1} (1-\gamma) |\lambda|^n}{|\lambda + m - 1|^n |\lambda|} \frac{\beta_{m-1}}{2^m} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right). \end{aligned}$$

Using (8) in (9), we get

$$(10) \quad \begin{aligned} & \left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1} \beta_{m-1}}{2^m \lambda} \left(\sum_{j=1}^{\nu} G_j \prod_{l=1}^m \frac{(1-\gamma)^{\rho_l} \eta^{\rho_l+1} \lambda^{n\rho_l - \rho_l - 1}}{(\lambda + l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \\ & \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda + \nu|^n}, \end{aligned}$$

which implies that

$$(11) \quad \left| \sum_{m=1}^{\nu} \frac{(-1)^{m+1} (1-\gamma)^m \eta^{m+1} \lambda^{mn-m-2} \beta_{m-1}}{2^m} \Phi_\nu \right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda + \nu|^n},$$

where

$$\Phi_\nu = \sum_{j=1}^{\nu} G_j \prod_{l=1}^m \frac{b_l^{\rho_l} c_l^{\rho_l}}{(\lambda + l)^{n\rho_l}}.$$

3. MAIN RESULT

THEOREM 3.1. *Let $\eta > 0$, μ be a real number, $\lambda = \eta + i\mu$, $\xi = \lambda/\eta$ and $0 \leq \gamma < 1$. If $f \in B_n^\lambda(\gamma)$, then*

$$|a_k| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda + k - 1|^n}, \quad k = 2, 3, \dots$$

The inequalities are sharp. The equalities are obtained for $f(z)$ satisfying

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} = \frac{\xi + [2(1-\gamma) - \xi] z^{k-1}}{1 - z^{k-1}}, \quad k = 2, 3, \dots$$

Proof. Let $f \in B_n^\lambda(\gamma)$. Then there exists an analytic function $h \in P_\xi(\gamma)$ such that

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} = h(z) = \gamma + (1-\gamma)p(z) + \xi - 1,$$

for some $p(z) = 1 + p_1 z + p_2 z^2 + \dots \in P$. Hence

$$\frac{f(z)}{z} = \left(1 + (1-\gamma)\eta \lambda^{n-1} \sum_{k=1}^{\infty} \frac{p_k z^k}{(\lambda+k)^n} \right)^{\frac{1}{\lambda}}.$$

Expanding binomially and employing Lemma 2.4, we have

$$(12) \quad f(z) = z + \sum_{k=2}^{\infty} \tilde{B}_1 C_{k-1}^{(1)} z^2 + \sum_{k=2}^{\infty} \tilde{B}_2 C_{k-1}^{(2)} z^3 + \dots + \sum_{k=2}^{\infty} \tilde{B}_m C_{k-1}^{(m)} z^k + \dots,$$

where

$$\tilde{B}_m = \frac{(1-\gamma)^m \eta^m \lambda^{m(n-2)} \prod_{j=0}^{m-1} (1-j\lambda)}{m!}$$

and $C_k^{(m)}$, $m = 1, 2, \dots$; $k = m, m+1, \dots$ is defined by

$$(13) \quad \sum_{k=1}^{\infty} C_k^{(m)} z^k = \left(\sum_{k=1}^{\infty} q_k z^k \right)^m,$$

where $q_k = \frac{p_k}{(\lambda+k)^n}$ and $C_k^{(m)}$ has a similar description as $G_\nu^{(m)}$ in (7)

$$C_k^{(m)} = \sum_{j=1}^k C_{k,\rho} \prod_{l=1}^m q_l^{\rho_l}.$$

Comparing the coefficients in (1) and (12), we have

$$(14) \quad a_k = \sum_{m=1}^{k-1} \frac{(1-\gamma)^m \eta^m \lambda^{m(n-2)} \prod_{j=0}^{m-1} (1-j\lambda)}{m!} \left(\sum_{j=1}^{k-1} C_j \prod_{l=1}^m q_l^{\rho_l} \right).$$

Now, comparing (14) and the term in absolute value in (11) with $\nu = k-1$ and noting that C_j in (14) and G_j in (11) have similar descriptions as C_j mentioned earlier, we conclude that the inequalities in (4) hold if we are able to find two members $h(z) = \xi + d_1 z + d_2 z^2 + \dots$ and $H(z) = \xi + c_1 z + c_2 z^2 + \dots$ of P_ξ which give rise to the constants β_m (as required by Lemma 2.2) and c_l . For $H \in P_\xi$, a natural choice is the function $H(z) = \frac{\xi + (2-\xi)z}{1-z} = \xi + 2z + 2z^2 + \dots$,

which turns out to be suitable. Thus, we have $c_l = 2$, $l = 1, 2, \dots$. Then (11) yields

$$(15) \quad \left| \sum_{m=1}^{k-1} (-1)^{m+1} (1-\gamma)^m \eta^{m+1} \lambda^{mn-m-2} \beta_{m-1} \Phi_{k-1} \right| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi| |(\lambda+k-1)|^n}.$$

Also, comparing (14) and the terms in the absolute value in (15), we have

$$(-1)^{m+1} \frac{\beta_{m-1}}{\lambda} = \frac{\prod_{j=0}^{m-1} (1-j\lambda)}{m! \eta \lambda^{m-1}},$$

that is

$$(16) \quad \beta_{m-1} = \frac{\prod_{j=1}^{m-1} (j\lambda - 1)}{m! \eta \lambda^{m-2}}, \quad \beta_0 = \xi.$$

Now, we define

$$(17) \quad \frac{1}{2^{m-1}} \left[\xi + \frac{1}{2} \sum_{\epsilon=1}^{m-1} \binom{m-1}{\epsilon} d_{\epsilon} \right] = \frac{\prod_{j=1}^{m-1} (j\lambda - 1)}{m! \eta \lambda^{m-2}},$$

for some d_{ϵ} , $\epsilon = 1, 2, \dots, m-1$, and we need to find $h(z)_k$ corresponding to each a_k , $k = 2, 3, 4, \dots$, such that the coefficients d_{ϵ} of each $h(z)_k$ satisfy (17). In view of (15), we consider the following cases for $m = 1, 2, \dots, k-1$, $k = 2, 3, 4, \dots$

- (i) For $k = 2$, $m = 1$, using (16), we have $\beta_0 = \xi$ and, by (17), we have $d_{\epsilon} = 0$, for all ϵ . Hence we obtain $h(z)_2 = \xi$.
- (ii) For $k = 3$, $m = 1, 2$, using (17), we have $d_1 = -2/\eta$. Hence, we obtain

$$h(z)_3 = \xi - \frac{1}{\eta} + \frac{1}{\eta} \left(\frac{1-z}{1+z} \right) = \xi - \frac{2}{\eta} z + \dots$$

- (iii) For $k = 4$, $m = 1, 2, 3$, using (17), we have

$$\frac{1}{4} \left[\xi + \frac{1}{2} (2d_1 + d_2) \right] = \frac{(\lambda-1)(2\lambda-1)}{6\eta\lambda}$$

and, taking $d_1 = 0$, we obtain $\frac{d_2}{2} = \frac{\lambda^2 - 6\lambda + 2}{3\eta\lambda}$, where

$$|d_2| = \frac{2}{3} \left| \frac{\lambda^2 - 6\lambda + 2}{\eta\lambda} \right| \leq 2.$$

We define

$$h(z)_4 = \frac{2(\lambda-1)(2\lambda-1)}{3\eta\lambda} - \left(\frac{\lambda^2 - 6\lambda + 2}{3\eta\lambda} \right) \left(\frac{1-z^2}{1+z^2} \right),$$

where

$$|\lambda^2 + 2| \leq (3\eta + 6)|\lambda|.$$

Then

$$h(z)_4 = \xi + \frac{2(\lambda^2 - 6\lambda + 2)}{3\eta\lambda} z^2 + \dots$$

(iv) For $k = 5$, we have $m = 1, 2, 3, 4$. So, using (17), we get

$$\frac{1}{8} \left[\xi + \frac{1}{2}(3d_1 + 3d_2 + d_3) \right] = \frac{(\lambda - 1)(2\lambda - 1)(3\lambda - 1)}{24\eta\lambda^2},$$

where, taking $d_1 = d_2 = 0$, we obtain

$$\frac{d_3}{2} = \frac{6\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\eta\lambda^2},$$

with

$$|d_3| = \frac{2}{3} \left| \frac{3\lambda^3 - 11\lambda^2 + 6\lambda - 1}{\eta\lambda^2} \right| \leq 2,$$

and we define

$$h(z)_5 = \frac{6\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\eta\lambda^2} - \left(\frac{3\lambda^3 - 11\lambda^2 + 6\lambda - 1}{3\eta\lambda^2} \right) \left(\frac{1 - z^3}{1 + z^3} \right),$$

where

$$|3\lambda^3 - 11\lambda^2 - 1| \leq (3\eta|\lambda| - 6)|\lambda|.$$

Hence

$$h(z)_5 = \xi + \frac{2(3\lambda^3 - 11\lambda^2 + 6\lambda - 1)}{3\eta\lambda^2} z^3 + \dots$$

(v) For $k \geq 6$, we have $m = 1, 2, 3, \dots, k - 1$, and we set $d_1 = \frac{-2\xi}{(m-1)}$, $d_2 = d_4 = \dots = d_\tau = \sigma$, where τ equals $m - 1$, if $m - 1$ is even, and $m - 2$, otherwise. Also, $d_3 = d_5 = \dots = d_\omega = 0$, where ω equals $m - 1$, if $m - 1$ is odd, and $m - 2$, otherwise. Thus, we have

$$\frac{\sigma_m}{2} = \frac{2^{m-1}\xi \prod_{j=1}^{m-1} \left(\frac{j\lambda-1}{j\lambda} \right)}{m \left(\binom{m-1}{2} + \binom{m-1}{4} + \dots + \binom{m-1}{\tau} \right)},$$

for all $\epsilon = 1, 2, \dots, m - 1$ such that $|d_\epsilon| \leq 2$. Thus, setting $m = k - 1$, we find that $h(z)_k$, $k \geq 6$, is given by

$$h(z)_k = \xi - \frac{2\xi}{k-2}z + \frac{\sigma_{k-1}}{2}z^2 + \frac{\sigma_{k-1}}{2}z^4 + \dots$$

That $h(z)_k$ belongs to P_ξ follows from the fact that P_ξ is, like P , a convex family. The proof is complete. \square

THEOREM 3.2. Let $\eta > 0$, μ be a real number, $\lambda = \eta + i\mu$, $\xi = \lambda/\eta$ and $0 \leq \gamma < 1$. If $f \in B_n^\lambda(\gamma)$, then

$$|a_3 - \rho a_2^2| \leq \frac{2(1-\gamma)|\lambda|^{n-1}}{|\xi||\lambda+2|^n} \max\{1, |1-M|\},$$

where

$$M = \frac{(2\rho + \lambda - 1)(1 - \gamma)(\lambda + 2)^2\eta\lambda^{n-2}}{(\lambda + 1)^{2n}}.$$

The inequalities are sharp. For each ρ , equalities are obtained by the same extremal function defined in Theorem 3.1.

Proof. Careful computations for (6) yield

$$a_2 = \frac{(1-\gamma)\eta\lambda^{n-2}p_1}{(\lambda+1)^n}$$

$$a_3 = \frac{(1-\gamma)\eta\lambda^{n-2}p_2}{(\lambda+2)^n} + \frac{(1-\lambda)(1-\gamma)^2\eta^2\lambda^{2n-4}p_1^2}{2(\lambda+1)^{2n}}.$$

Hence,

$$|a_3 - \rho a_2^2| = \frac{(1-\gamma)\eta\lambda^{n-2}}{(\lambda+2)^n} \left| p_2 - \frac{(2\rho + \lambda - 1)(1-\gamma)(\lambda+2)^n\eta\lambda^{n-2}}{(\lambda+1)^{2n}} \frac{p_1^2}{2} \right|.$$

By choosing

$$\tau = \frac{(2\rho + \lambda - 1)(1-\gamma)(\lambda+2)^n\eta\lambda^{n-2}}{(\lambda+1)^{2n}}$$

and using Lemma 2.6, the result then follows. \square

For $\lambda = \alpha/(1+i\beta)$, we have the following corollaries for generalized Bazilevič maps (with $g(z) = z$), whose family is denoted here by $B_n^{\alpha,\beta}(\gamma)$.

COROLLARY 3.3. *Let $f \in B_n^{\alpha,\beta}(\gamma)$. Then*

$$|a_k| \leq \frac{2(1-\gamma)\alpha^n}{\sqrt{(1+\beta^2)(\alpha^2+\beta^2)[\alpha^2+2\alpha(k-1)+(1+\beta^2)(k-1)^2]^n}}, \quad k = 2, 3, \dots$$

The inequalities are sharp.

COROLLARY 3.4. *Let $f \in B_n^{\alpha,\beta}(\gamma)$. Then*

$$|a_3 - \rho a_2^2| \leq \frac{2(1-\gamma)\alpha^n \sqrt{(1+\beta^2)^n}}{\sqrt{(\alpha^2+\beta^2)[(\alpha+2)^2+4\beta^2]^n}} \max\{1, |1-T|\},$$

where

$$T = \frac{(1-\gamma)(\alpha+2+i\beta)^2\alpha^{n-1}[2\rho+\alpha-1+i\beta(2\rho-1)]}{(1+\beta^2)(1+i\beta)^{n+1}}.$$

The inequalities are sharp.

Finally, we remark that, with appropriate choices of the defining parameters, our results agree with the existing results.

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