



## CLASS OF RATIO ESTIMATORS WITH KNOWN FUNCTIONS OF AUXILIARY VARIABLE FOR ESTIMATING FINITE POPULATION VARIANCE

AUDU AHMED<sup>1\*</sup>, ADEDAYO AMOS ADEWARA<sup>2</sup>  
AND RAN VIJAY KUMAR SINGH<sup>3</sup>

<sup>1</sup>Department of Mathematics, Usmanu Danfodiyo University, Sokoto, Nigeria.

<sup>2</sup>Department of Statistics, University of Ilorin, Kwara State, Nigeria.

<sup>3</sup>Department of Mathematics, Kebbi State University of Science and Technology Aliero, Nigeria.

### AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration among the authors. Author AA formulates the estimators and established their properties. Authors RVKS and AAA help in checking the solutions gathered the initial data and help to interpret the results. All the authors read and approved the final manuscript.

**Received:** 8<sup>th</sup> January 2016

**Accepted:** 25<sup>th</sup> March 2016

**Published:** 28<sup>th</sup> April 2016

**Original Research Article**

### ABSTRACT

In this paper, we have suggested a class of improved ratio estimators for finite population variance. The proposed class of estimators is obtained by using unknown weight on some existing estimators. The MSE of the proposed estimators have been obtained and the conditions for their efficiency over some existing variance estimators have been established. The present proposed family of finite variance estimators, having obtaining the optimal values of the constants, exhibit significant improvement over the existing estimators. The empirical study is also conducted to corroborate the theoretical results and the results show that the proposed class of estimators is more efficient.

**Keywords:** Efficiency; mean square error; ratio estimator; finite population; variance.

## 1 Introduction

The variation of produce or yields in the manufacturing industries and pharmaceutical laboratories are sometime a matter of concern to researchers (Ahmed et al. [1]). The use of parameters of auxiliary variable, being constant with unit (e.g. population mean, population standard deviation, e.t.c.) or unit free constant (e.g. Coefficient of variation, Kurtosis, e.t.c.), can enhance the efficiency at the estimation stage. In recent past, this concept has been utilized by several authors to improve the efficiency of ratio and product type estimators for estimating population mean as well population variance of study variable.

In this paper, an improved class of ratio estimators for estimating finite population variance has been proposed with objective to produce efficient estimators and their properties have been established.

Let  $\Omega = (1, 2, 3, \dots, N)$  be a population of size  $N$  and  $Y, X$  be two real valued functions having values  $(Y_i, X_i) \in \mathbb{R}^+ > 0$  on the  $i^{th}$  unit of  $U(1 \leq i \leq N)$ . We assume positive correlation  $\rho > 0$  between

\*Corresponding author: Email: [ahmed.audu@udusok.edu.ng](mailto:ahmed.audu@udusok.edu.ng);

the study variable  $Y$  and auxiliary variable  $X$ . Let  $S_y^2$  and  $S_x^2$  be the finite population mean squares of  $Y$  and  $X$  respectively and  $s_y^2$  and  $s_x^2$  be respective sample mean squares based on the random sample of size  $n$  drawn without replacement.

Singh et al. [2] defined the general family of estimators for estimating finite population variance  $S_y^2$  of the study variable  $Y$  as

$$\eta = s_y^2 \left[ \frac{aS_x^2 + b}{\alpha(as_x^2 - b) + (1-\alpha)(aS_x^2 - b)} \right] \quad (1.1)$$

where  $a$  and  $b$  are constants based on auxiliary variable  $X$  like coefficient of skewness, kurtosis and correlation coefficient etc.  $\alpha$  is the constant that minimizes the mean square error (MSE) of the estimator. Table 1 shows some members of  $\eta$ -family for different values of  $a$ ,  $b$  and  $\alpha$ .

The MSEs/Variance of the estimators in Table 1 are given below:

$$Var(\eta_0) = \gamma S_y^2 (\psi_{40} - 1) \quad (1.2)$$

$$MSE(\eta_i) = \begin{cases} S_y^4 \gamma [(\psi_{40} - 1) + (\psi_{04} - 1) - 2(\psi_{22} - 1)] & i = 1 \\ S_y^4 \gamma [(\psi_{40} - 1) + h_i^2 (\psi_{04} - 1) - 2h_i (\psi_{22} - 1)] & i = 2, 3, 4, 5, 6 \end{cases} \quad (1.3)$$

$$h_2 = \frac{S_x^2}{S_x^2 - C_x}, h_3 = \frac{S_x^2}{S_x^2 - \beta_2(x)}, h_4 = \frac{S_x^2 \beta_2(x)}{S_x^2 \beta_2(x) - C_x}, h_5 = \frac{S_x^2 C_x}{S_x^2 C_x - \beta_2(x)}, h_6 = \frac{S_x^2}{S_x^2 + \beta_2(x)}$$

where

$$s_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2, s_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2, S_y^2 = \frac{1}{N-1} \sum_{i=1}^n (Y_i - \bar{Y})^2, S_x^2 = \frac{1}{N-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N X_i, \bar{Y} = \frac{1}{N} \sum_{i=1}^N Y_i, \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

$$\gamma = \frac{1}{n}, \psi_{rs} = \frac{\lambda_{rs}}{\lambda_{20}^{r/2} \lambda_{02}^{s/2}} \text{ and } \lambda_{rs} = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^r (X_i - \bar{X})^s$$

The MSE of  $\eta$  to second order approximation is given below:

$$MSE(\eta) = \gamma S_y^2 [(\psi_{40} - 1) + \alpha^2 \theta^2 (\psi_{04} - 1) - 2\alpha \theta (\psi_{22} - 1)] \quad (1.4)$$

The  $MSE(\eta)$  expression is minimized for the optimum values of  $\alpha$  given by equation (1.5). This is obtained by partial differentiation of equation (1.4) with respect to  $\alpha$ .

$$MSE(\eta)_{\min} = \gamma S_y^4 \left[ (\psi_{40} - 1) - \frac{(\psi_{22} - 1)^2}{(\psi_{04} - 1)} \right] \quad (1.5)$$

**Table 1. Some member of  $\eta$ -family for different values of  $a$ ,  $b$  and  $\alpha$**

| Estimator                                                                             | $a$          | $b$           | $\alpha$ |
|---------------------------------------------------------------------------------------|--------------|---------------|----------|
| $\eta_0 = s_y^2$                                                                      | 1            | 0             | 0        |
| <b>Sample variance</b>                                                                |              |               |          |
| $\eta_1 = s_y^2 \left( \frac{S_x^2}{S_x^2} \right)$                                   | 1            | 0             | 1        |
| <b>Isaki [3]</b>                                                                      |              |               |          |
| $\eta_2 = s_y^2 \left( \frac{S_x^2 - C_x}{S_x^2 - C_x} \right)$                       | 1            | $C_x$         | 1        |
| <b>Kadilar and Cingi [4]</b>                                                          |              |               |          |
| $\eta_3 = s_y^2 \left( \frac{S_x^2 - \beta_2(x)}{S_x^2 - \beta_2(x)} \right)$         | 1            | $\beta_2(x)$  | 1        |
| $\eta_4 = s_y^2 \left( \frac{S_x^2 \beta_2(x) - C_x}{S_x^2 \beta_2(x) - C_x} \right)$ | $\beta_2(x)$ | $C_x$         | 1        |
| $\eta_5 = s_y^2 \left( \frac{S_x^2 C_x - \beta_2(x)}{S_x^2 C_x - \beta_2(x)} \right)$ | $C_x$        | $\beta_2(x)$  | 1        |
| $\eta_6 = s_y^2 \left( \frac{S_x^2 + \beta_2(x)}{S_x^2 + \beta_2(x)} \right)$         | 1            | $-\beta_2(x)$ | 1        |
| <b>Upadhyaya and Singh [5]</b>                                                        |              |               |          |

Many other researchers including Kadilar and Cingi [6], Yadav and Kadilar [7], Singh and Solanki [8], Gupta and Shabir [9], Subramani and Kumarapandian [10], Singh and Vishwakarma [11], Singh et al. [2], Sanaullah et al. [12] and Solanki and Singh [13] have significantly contributed to the improvement of both ratio and product mean & variance estimators in sampling survey.

## 2 Proposed Estimator

After studying the related finite population variance estimators stated in section 1, we proposed the following class of ratio-type estimators which utilize the known values of population parameters related to auxiliary variable,

$$\tau_1^* = k_1 s_y^2 \left( \frac{S_x^2}{S_x^2} \right), \tau_2^* = k_2 s_y^2 \left( \frac{S_x^2 - C_x}{S_x^2 - C_x} \right), \tau_3^* = k_3 s_y^2 \left( \frac{S_x^2 - \beta_2(x)}{S_x^2 - \beta_2(x)} \right), \tau_4^* = k_4 s_y^2 \left( \frac{S_x^2 \beta_2(x) - C_x}{S_x^2 \beta_2(x) - C_x} \right),$$

$$\tau_5^* = k_5 s_y^2 \left( \frac{S_x^2 C_x - \beta_2(x)}{S_x^2 C_x - \beta_2(x)} \right), \tau_6^* = k_6 s_y^2 \left( \frac{S_x^2 + \beta_2(x)}{S_x^2 + \beta_2(x)} \right)$$

In order to obtain the MSE, we defined  $e_0 = \frac{S_y^2 - S_y^2}{S_y^2}$  and  $e_1 = \frac{S_x^2 - S_x^2}{S_x^2}$ , thus

$$\left. \begin{aligned} E(e_0) = E(e_1) = 0, E(e_0^2) &= \gamma(\psi_{40} - 1) \\ E(e_1^2) &= \gamma(\psi_{04} - 1), E(e_0 e_1) = \gamma(\psi_{22} - 1) \end{aligned} \right\} \quad (2.1)$$

Expressing  $\tau_i^*$ ,  $i = 1, 2, 3, 4, 5, 6$ , in terms of  $e_0$  and  $e_1$ , we have

$$\tau_i^* = \left. \begin{aligned} k_i S_y^2 (1 + e_0)(1 + e_1)^{-1} & \quad i = 1 \\ k_i S_y^2 (1 + e_0)(1 + h_i e_1)^{-1} & \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \quad (2.2)$$

We now assume that  $|e_1| < 1$  and  $|h_i e_1| < 1$  so that  $(1 + e_1)^{-1}$  and  $(1 + h_i e_1)^{-1}$  are expandable. Expanding the right hand side of (2.2) up to second degree approximation, subtract  $S_y^2$  from its both sides, square the both sides and taking expectation using the results in equation (2.1), we obtain the MSEs of the proposed estimators as:

$$MSE(\tau_i^*) = \left\{ \begin{aligned} S_y^4 \left\{ \gamma \left[ \begin{aligned} k_i^2 (\psi_{40} - 1) + (3k_i^2 - 2k_i)(\psi_{04} - 1) \\ -2(2k_i^2 - k_i)(\psi_{22} - 1) \end{aligned} \right] + (k_i - 1)^2 \right\} & \quad i = 1 \\ S_y^4 \left\{ \gamma^2 \left[ \begin{aligned} k_i^2 (\psi_{40} - 1) + (3k_i^2 - 2k_i)h_i^2 (\psi_{04} - 1) \\ -2(2k_i^2 - k_i)h_i (\psi_{22} - 1) \end{aligned} \right] + (k_i - 1)^2 \right\} & \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \quad (2.3)$$

The  $MSE(\eta_i)$ ,  $i = 1, 2, 3, 4, 5, 6$  expressions are minimized for the optimum values of k given by

$$k_i^* = \frac{\gamma[(\psi_{04} - 1) - (\psi_{22} - 1)] + 1}{\gamma[3(\psi_{04} - 1) - 4(\psi_{22} - 1) + (\psi_{40} - 1)] + 1} = \frac{A_1}{B_1}, \quad i = 1 \quad (2.4)$$

$$k_i^* = \frac{\gamma[h_i^2 (\psi_{04} - 1) - h_i (\psi_{22} - 1)] + 1}{\gamma[3h_i^2 (\psi_{04} - 1) - 4h_i (\psi_{22} - 1) + (\psi_{40} - 1)] + 1} = \frac{A_2}{B_2}, \quad i = 2, 3, 4, 5, 6 \quad (2.5)$$

Replacing  $k_i$  by  $k_i^*$ ,  $i = 1, 2, 3, 4, 5, 6$  in equation (2.3), we obtain the minimum MSE as

$$MSE_{\min}(\tau_i^*) = \left\{ \begin{aligned} S_y^4 \left( 1 - \frac{A_1^2}{B_1} \right), & \quad i = 1 \\ S_y^4 \left( 1 - \frac{A_2^2}{B_2} \right), & \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \quad (2.6)$$

### 3 Efficiency Comparisons

In this section efficiencies of the proposed estimators are compared with efficiencies of some estimators in the literature

The  $\tau_i^*$  – family of estimators of the population variance is more efficient than  $\eta_0$  if,

$$\begin{aligned} &MSE_{\min}(\tau_i^*) < Var(\eta_0) \quad i = 1, 2, 3, 4, 5, 6 \\ &\left. \begin{aligned} &\left(1 - \frac{A_1^2}{B_1}\right) < \gamma[(\psi_{40} - 1)] \quad i = 1 \\ &\left(1 - \frac{A_2^2}{B_2}\right) < \gamma[(\psi_{40} - 1)] \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \end{aligned} \quad (3.1)$$

The  $\tau_i^*$  – family of estimators of the population variance is more efficient than  $\eta_i$  – family if,

$$\begin{aligned} &MSE_{\min}(\tau_i^*) < MSE(\eta_i) \quad i = 1, 2, 3, 4, 5, 6 \\ &\left. \begin{aligned} &\left(1 - \frac{A_1^2}{B_1}\right) < \gamma[(\psi_{40} - 1) + (\psi_{04} - 1) - 2(\psi_{22} - 1)] \quad i = 1 \\ &\left(1 - \frac{A_2^2}{B_2}\right) < \gamma[(\psi_{40} - 1) + h_i^2(\psi_{04} - 1) - 2h_i(\psi_{22} - 1)] \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \end{aligned} \quad (3.2)$$

The  $\tau_i^*$  – family of estimators of the population variance is more efficient than  $\eta$  if,

$$\begin{aligned} &MSE_{\min}(\tau_i^*) < MSE_{\min}(\eta) \quad i = 1, 2, 3, 4, 5, 6 \\ &\left. \begin{aligned} &\left(1 - \frac{A_1^2}{B_1}\right) < \gamma[(\psi_{40} - 1) - (\psi_{04} - 1)] \quad i = 1 \\ &\left(1 - \frac{A_2^2}{B_2}\right) < \gamma[(\psi_{40} - 1) - (\psi_{04} - 1)] \quad i = 2, 3, 4, 5, 6 \end{aligned} \right\} \end{aligned} \quad (3.3)$$

When conditions (3.1), (3.2) and (3.3) are satisfied, we can conclude that the family of proposed class of estimators is more efficient than the  $\eta$  – family estimator.

### 4 Empirical Study

In order to investigate the merits of the proposed estimators, we have considered the following two real populations given as:

Data 1: Subramani and Kumarapandiyan [10]

$$N = 49, n = 20, \bar{Y} = 116.1633, \bar{X} = 98.6765, \rho = 0.6904, S_y = 98.8286$$

$$S_x = 102.9709, C_y = 0.8508, C_x = 1.0435, \psi_{40} = 4.9245, \psi_{04} = 5.9878, \psi_{22} = 4.6977$$

Data 2: Subramani and Kumarapandiyan [10]

$$N = 80, n = 20, \bar{Y} = 51.8264, \bar{X} = 11.2646, \rho = 0.9413, S_y = 18.3569$$

$$S_x = 8.4563, C_y = 0.3542, C_x = 0.7507, \psi_{40} = 2.2667, \psi_{04} = 2.8664, \psi_{22} = 2.2209$$

**Table 2. MSE of different estimators**

| Estimator                                         | Data 1   | Data 2   | Estimator                                        | Data 1  | Data 2   |
|---------------------------------------------------|----------|----------|--------------------------------------------------|---------|----------|
| $\eta_0$                                          | 18719098 | 7191.859 | $\eta_{opt}$                                     | 5683700 | 3657.427 |
| $\eta - family$<br>(Singh et al. [13] estimators) |          |          | $\tau^* - family$<br>(Newly proposed estimators) |         |          |
| $\eta_1$                                          | 7235316  | 3924.948 | $\tau_1^*$                                       | 5675690 | 3463.389 |
| $\eta_2$                                          | 7236528  | 4003.906 | $\tau_2^*$                                       | 5676029 | 3515.270 |
| $\eta_3$                                          | 7242278  | 4249.508 | $\tau_3^*$                                       | 5677637 | 3673.896 |
| $\eta_4$                                          | 7235519  | 3952.035 | $\tau_4^*$                                       | 5675746 | 3481.236 |
| $\eta_5$                                          | 7241987  | 4372.127 | $\tau_5^*$                                       | 5677555 | 3751.556 |
| $\eta_6$                                          | 7228378  | 3658.199 | $\tau_6^*$                                       | 5673748 | 3284.891 |

**Table 3. PRE of different estimators with respect to  $S_y^2$**

| Estimator                                         | Data 1   | Data 2   | Estimator                                        | Data 1   | Data 2   |
|---------------------------------------------------|----------|----------|--------------------------------------------------|----------|----------|
| $\eta_0$                                          | 100.00   | 100.00   | $\eta_{opt}$                                     | 311.6813 | 179.6324 |
| $\eta - family$<br>(Singh et al. [13] estimators) |          |          | $\tau^* - family$<br>(Newly proposed estimators) |          |          |
| $\eta_1$                                          | 258.7184 | 183.2345 | $\tau_1^*$                                       | 329.8118 | 207.6538 |
| $\eta_2$                                          | 258.6751 | 179.6211 | $\tau_2^*$                                       | 329.7922 | 204.5891 |
| $\eta_3$                                          | 258.4697 | 169.2398 | $\tau_3^*$                                       | 329.6988 | 195.7556 |
| $\eta_4$                                          | 258.7112 | 181.9786 | $\tau_4^*$                                       | 329.8086 | 206.5892 |
| $\eta_5$                                          | 258.4801 | 164.4934 | $\tau_5^*$                                       | 329.7035 | 191.7033 |
| $\eta_6$                                          | 258.9668 | 196.5956 | $\tau_6^*$                                       | 329.9247 | 218.9375 |

The numerical demonstration to justify the appropriateness of the suggested class of variance estimator has been conducted using the two data sets. Tables 2 and 3 show the mean square errors of proposed estimators

and that of some existing estimators and their percentage relative efficiencies of different estimators with respect to  $S_y^2$  respectively.

## 5 Conclusions

From section 3, the theoretical conditions obtained for the efficiencies of the proposed estimators supported by the numerical illustration in section 4 given in the Tables 2 and 3. From the result, we infer that the suggested variance estimators produce a better estimate of finite population variance than the existing estimators in the sense of having higher percentage relative efficiency which implies lesser mean square error.

## Competing Interests

Authors have declared that no competing interests exist.

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