Stability Analysis of Tuberculosis Model

By

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Abstract

In this paper we study MSEIR and MSIR Tuberculosis models. Existence, Uniqueness and stability of the problem are proved using the concept of Bellman and Cooke's theorem. The equilibrium solutions with stability results are analyzed. The dynamics of the compartments were described by a system of ordinary differential equations and all the equations were solved and analysed using the stability concept. It was found out that the disease free equilibrium states for each model are stable when the number of recovery individuals is greater than the death rate.

Keywords: Bacillus Calmette-Guerin, Exposed TB infection, Active TB infection, Disease-free equilibrium state; Endemic equilibrium state; Stability analysis and immunity.

1 Introduction

Tuberculosis (TB) is caused by bacteria (Mycobacterium tuberculosis) that most often a ect the lungs. TB is spread from person to person through the air. When people with lung TB cough, sneeze or spit, they propel the TB germs into air. A person needs to inhale only a few of these germs to become infected (WHO, 2014).

Moreover, the host immune system arrests infection by forming a structure called granuloma and stamp down the imminent threat of active infection. However, inside the granuloma, M. tuberculosis switches to a domant or non-replicative state under stressful condition for its survival. This infection persists in humans due to its stability to shift to a clinically latent state without revealing any overt disease symp- toms and it is termed as latent tuberculosis infection (LTBI)(George S. and Melissa C.S., 2007). About one-third of the world population is infected with LTBI and individual harboring LTBI carry a lifetime risk of reactivation to active disease. If these persons are co-infected with

HIV,then the risk elevates to 8-10% per year (Patel K., 2011; Blower et al, 1996; Zhao et al. 2010; JAIDS, 2009).

The World Health Organization (WHO) (WHO, 2014) in her 18th global report publication on tuber- culosis (TB) in a series that started in 1997, provides a comprehensive and up-to-date assessment of the TB epidemic and of the progress made in TB prevention, care and control at global, regional and country level, using data reported by 197 countries and territories. The available research and the estimates are frightening on the transmission of Mother-Child tuberculosis:

- Over half a million women and at least 64,000 children die of TB each year
- TB is common among pregnant women, and has been reported in up to 11% of pregnant women living with HIV in high burden settings
- TB causes 6% - 10% of all maternal mortality, up to 15% in high prevalence settings and 15%-34% of indirect obstetric maternal mortality.
- There is an increased risk of transmitting TB & HIV to infants born to mothers with TBHIV co-infection.
- TB has resulted in nearly 10 million cumulative orphans due to parental deaths.

TB remains a major global health problem although TB treatment has saved the lives of more than 22 million people, according to report, which also reveals that the number of people who fell ill with TB in 2012 was 8.6 million, with 1.3 million TB death globally (including 320,000 deaths among HIV-positive people). The report underlines two major challenges: the drug-resistant TB crisis and the fact that around 3 million people (one in three people falling ill with TB) are currently being missed by health systems. A further challenge identi ed relates to the TB and HIV co-epidemic. (Eurosurveillance article 2013).

The intersection of HIV and TB possess a serious threat to HIV.-infected women and their children. The majority of patients with TB in sub-Saharan Africa are infected with HIV, and together the overlapping epidemics synergistic ally accelerates illness and death. The HIV epidemic requires new strategies to control TB in high-burden areas especially as women of reproductive age are disproportionately a ected by the epidemic. Intensi ed case nding for TB has the potential to reduce morbidity and mortality for people living with HIV, especially pregnant women, their families, and infants.(Patel K., 2011; JAIDS, 2009; Farkas M. 2001)

When the incidence of an infection starts to increase resulting to infant death, people start to look at how best to combat the outbreak. Launching nationwide vaccination campaigns (or even vaccinating a small group of a population) can be a costly and time consuming endeavour, so any tool that will enable the campaign to become directed or to predict the outcome is highly valuable. Therefore, with the employment of mathematical model, hopefully the problem will be solved. (Keeling, et al 2008)

1.1 Mathematical Formulation

For model building, we have subdivided the total population $P(t)$ into four and five compartments with the notation MSIR (Passive Immune Infant, Susceptible, Infectious and Recovery) and MSEIR (Passive Immune Infant, Susceptible, Expose, Infectious and Recovery) respectively. We describe the transmission of dynamics of the tuberculosis models by the following system of Ordinary Differential Equations and Diagram of flow:

Model Equations

MSIR (Passive Immune Infant, Susceptible, Infectious and Recovery)

$$\frac{dM}{dt}\theta\rho(\alpha\mu)M \tag{1}$$

$$\frac{dS}{dt} = \alpha M + (1-\theta)\rho - \beta SI - \mu S \tag{2}$$

$$\frac{dI}{dt} = \beta SI - (\gamma + \delta + \mu)I \tag{3}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{3}$$

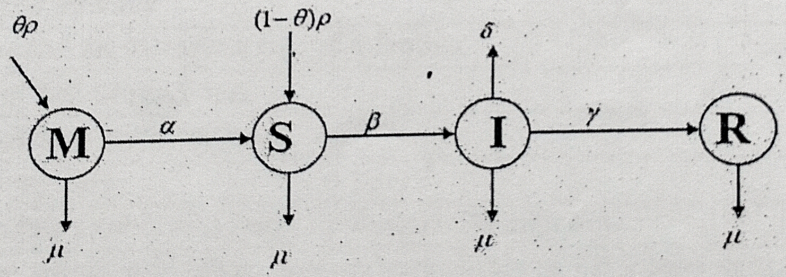


Fig. 1: MSIR model

And MSEIR denote respectively, the classes of Passive Immune Infant, Susceptible, Exposed, Infectious and Recovery.

$$\frac{dM}{dt} = \theta\rho - (\alpha + \mu)M \tag{4}$$

$$\frac{dS}{dt} = \alpha M + (1-\theta)\rho - \beta SI - \mu S + \kappa R \tag{5}$$

$$\frac{dE}{dt} = \beta SI - (\gamma + \eta + \mu)E \tag{6}$$

$$\frac{dI}{dt} = (\gamma + \eta)E - (\delta + \sigma + \mu)I \tag{7}$$

$$\frac{dR}{dt} = \sigma I - (\kappa + \mu)R \tag{8}$$

where $P = M + S + E + I + R$ is the total size of the population. The table below shows detail descriptions of the model parameters

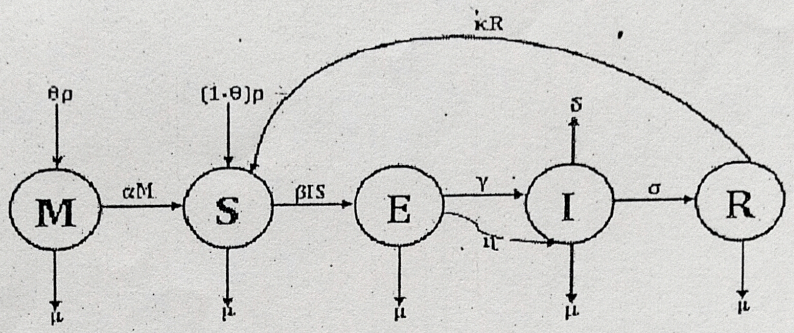


Fig. 2: MSEIR model

Table 1: Description of Parameters for the model

Parameter	Description
$\theta\rho$	The proportion of incoming individual
μ	Natural Death
α	Vaccine efficacy
β	Instantaneous incidence rate of infection
γ	Rate of fast progression
η	Rate of slow progression
σ	Recovery rate
δ	Death rate caused as a result of chronic TB infection
κ	Rate at which individuals become susceptible to TB infection again

2 Equilibrium State of MSIR Model

At equilibrium state,

$$\frac{dM}{dt} = \frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0 \tag{9}$$

The Disease Free Equilibrium (DFE) state was obtained as:

$$M = \frac{\theta\rho}{\alpha + \mu} \tag{10}$$

$$S = \frac{\alpha\rho + (1-\theta)\mu\rho}{\mu(\alpha + \mu)} \tag{11}$$

$$I = R = 0 \tag{12}$$

That is

$$P_0 = \left\{\frac{\theta\rho}{\alpha+\mu}, \quad \frac{\alpha\rho + (1-\theta)\mu\rho}{\mu(\alpha+\mu)}, \quad 0, \quad 0\right\} \tag{13}$$

At endemic equilibrium (EE), the following results were also obtained:

$$I = \frac{(\alpha+\mu)[\alpha\beta - \mu(\gamma+\mu+\delta)] - \beta\theta\mu\rho}{\beta(\alpha+\mu)(\gamma+\mu+\delta)} \tag{14}$$

$$R = \frac{\gamma\{(\alpha+\mu)[\alpha\beta - \mu(\gamma+\mu+\delta)] - \beta\theta\mu\rho\}}{\beta\mu(\alpha+\mu)(\gamma+\mu+\delta)} \tag{15}$$

$$S = \frac{(\gamma+\mu+\delta)}{\beta}. \tag{16}$$

This can as well be written as

$$P^* = \left\{\frac{\theta\rho}{\alpha+\mu}, \quad \frac{(\gamma+\mu+\delta)}{\beta}, \quad \frac{(\alpha+\mu)[\alpha\beta - \mu(\gamma+\mu+\delta)] - \beta\theta\mu\rho}{\beta(\alpha+\mu)(\gamma+\mu+\delta)}, \right. \tag{17}$$
$$\left. \frac{\gamma\{(\alpha+\mu)[\alpha\beta - \mu(\gamma+\mu+\delta)] - \beta\theta\mu\rho\}}{\beta\mu(\alpha+\mu)(\gamma+\mu+\delta)}\right\}$$

2.1 Equilibrium State of the MSEIR Model

$$\frac{dM}{dt} = \frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

Similarly to MSIR, we solve the model equations to obtain the equilibrium states for each compartment.

The Disease Free Equilibrium (DFE) state was obtained as:

$$M = \frac{\theta\rho}{\alpha+\mu} \tag{18}$$

$$S = \frac{\{\alpha + (1-\theta)\mu\}\rho}{\mu(\alpha+\mu)} \tag{19}$$

$$E = I = R = 0 \tag{20}$$

That is

$$P_0 = \left\{\frac{\theta\rho}{\alpha+\mu}, \quad \frac{\{\alpha+(1-\theta)\mu\}\rho}{\mu(\alpha+\mu)}, \quad 0, \quad 0, \quad 0\right\} \tag{21}$$

At endemic equilibrium (EE) of MSEIR, the following results were also obtained:

$$I = \frac{(\alpha+\mu)(\gamma+\eta)\rho\beta - (\gamma+\eta)(\kappa+\mu)\beta\rho\theta\mu - (\alpha+\mu)(\kappa+\mu)(\gamma+\eta+\mu)(\sigma+\delta+\mu)\mu}{\beta(\alpha+\mu)(\kappa+\mu)(\gamma+\mu+\eta)(\delta+\sigma+\mu) - (\alpha+\mu)(\gamma+\eta)\beta\sigma\kappa} \tag{22}$$

$$E = \frac{(\delta+\sigma+\mu)\{(\alpha+\mu)(\gamma+\eta)\rho\beta - (\gamma+\eta)(\kappa+\mu)\beta\rho\theta\mu - (\alpha+\mu)(\kappa+\mu)(\gamma+\eta+\mu)(\sigma+\delta+\mu)\mu\}}{(\gamma+\eta)\{\beta(\alpha+\mu)(\kappa+\mu)(\gamma+\mu+\eta)(\delta+\sigma+\mu) - (\alpha+\mu)(\gamma+\eta)\beta\sigma\kappa\}} \tag{23}$$

$$R = \frac{\sigma\{(\alpha+\mu)(\gamma+\eta)\rho\beta - (\gamma+\eta)(\kappa+\mu)\beta\rho\theta\mu - (\alpha+\mu)(\kappa+\mu)(\gamma+\eta+\mu)(\sigma+\delta+\mu)\mu\}}{(\kappa+\mu)\{\beta(\alpha+\mu)(\kappa+\mu)(\gamma+\mu+\eta)(\delta+\sigma+\mu) - (\alpha+\mu)(\gamma+\eta)\beta\sigma\kappa\}} \tag{24}$$

$$S = \frac{(\gamma+\mu+\eta)(\delta+\sigma+\mu)}{\beta(\gamma+\eta)} \tag{25}$$

3 Stability Analysis of the Model Equations

Having established the equilibrium states, we then investigate the stability of the equilibrium states. To obtain this, we examine the behavior of the models near the equilibrium states.

$$0 = \theta\rho - (\alpha+\mu)M \tag{26}$$

$$0 = \alpha M + (1-\theta)\rho - \beta SI - \mu S \tag{27}$$

$$0 = \beta SI - (\gamma+\delta+\mu)I \tag{28}$$

$$0 = \gamma I - \mu R \tag{29}$$

$$J=\begin{pmatrix} -(\alpha+\mu) & 0 & 0 & 0 \\ \alpha & -(\beta I+\mu) & 0 & 0 \\ 0 & \beta I & -(\gamma+\mu+\delta-\beta S) & 0 \\ 0 & 0 & \gamma & -\gamma \end{pmatrix}$$

The characteristic equation for the Jacobi's Matrix

$$|J-\lambda \mathrm{I}|=0=\begin{vmatrix} -(\alpha+\mu+\lambda) & 0 & 0 & 0 \\ \alpha & -(\beta I+\mu+\lambda) & 0 & 0 \\ 0 & \beta I & -(\gamma+\mu+\delta-\beta S+\lambda) & 0 \\ 0 & 0 & \gamma & -(\mu+\lambda) \end{vmatrix}$$

$$|J-\lambda \mathrm{I}|=(\alpha+\mu+\lambda)\{(\beta I+\mu+\lambda)[-(\mu+\lambda)\ (\gamma-\mu-\delta+\beta S-\lambda)]\}=0$$

3.1 Stability of the Zero Equilibrium State

At DFE state;

$\lambda_1=-\alpha-\mu, \quad \lambda_2=-\mu, \quad \lambda_3=-\mu, \quad \lambda_4=\frac{\alpha\beta\rho+(1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)}-(\gamma+\mu+\delta)$

It can be seen that $\lambda_1, \lambda_2, \lambda_3$ are all negative, meaning, Tuberculosis will be completely eradicate at this state and for the λ_4, need to be negative i.e. $\frac{\alpha\beta\rho+(1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)}<(\gamma+\mu+\delta)$ where $\frac{\alpha\beta\rho+(1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)}$ is said to be number of latent infections produced and $(\gamma+\mu+\delta)$ is the total removal rate from the infections class.

3.2 Stability of the Non-zero Equilibrium using Bellman & Cook's Theorem

Let $H(z)=P(z, e^{x})$, where $P(z, w)$ is a polynomial in with principal term. Suppose, $H(iy)$, $y\in\Re$, is separated into its real and imaginary parts:

$$H(iy)=F(y)+iG(y)$$

If the zeros of $H(y)$ have negative real parts, then, the zeros of $F(y)$ and $G(y)$ are real, simple and alternate and

$$J=F(0)G'(0)-F'(0)G(0)>0,\ \forall y\in\Re \tag{30}$$

Conversely, all zeros of $H(z)$ will be in the left half plane provided that either of the following condition is satisfied:

1. All the zeros of $F(y)$ and $G(y)$ are real, simple and alternate and the inequality (30) is satisfied at least for one y
2. All zeros of $F(y)$ are real and for each zero, the relation (30) is satisfied
3. All zeros of $G(y)$ are real and for each zero, the relation (30) is satisfied

The above fundamental theorem establishes the analysis of the stability of characteristic equation as stated by (Momoh, et al. 2011).

Applying Bellman and Cooke's Theorem for endemic equilibrium state

$$|J-\lambda I| = 0 = \lambda^4 + (4\mu - \beta\beta S + \gamma + \delta + \beta I + \alpha)\lambda^3 + (-3\beta\mu S + 3\gamma\mu + 6\mu^2 + 3\delta\mu + 3\mu\beta I + 3\alpha\mu - \beta^2 SI - \alpha\beta S + \gamma\beta I + \gamma\alpha + \beta\delta I + \delta\alpha + \alpha\beta I)\lambda^2 + (3\gamma\mu^2 + 3\delta\mu^2 + 3\beta\mu^2 I + 3\alpha\mu^2 - 2\mu\beta^2 SI - 2\alpha\mu\beta S + 2\gamma\mu\beta I + 2\mu\alpha\gamma + 2\mu\delta\alpha + 2\mu\alpha\beta I - \beta^2\alpha SI + \alpha\beta\gamma I + \alpha\beta\delta I - 3\beta\mu^2 S + 4\mu^3)\lambda + \mu(-\beta S + \gamma + \mu + \delta)(\beta I + \mu)(\alpha + \mu)$$

$$H(iq) = 0 = (iq)^4 + (4\mu - \beta\beta S + \gamma + \delta + \beta I + \alpha)(iq)^3 + (-3\beta\mu S + 3\gamma\mu + 6\mu^2 + 3\delta\mu + 3\mu\beta I + 3\alpha\mu - \beta^2 SI - \alpha\beta S + \gamma\beta I + \gamma\alpha + \beta\delta I + \delta\alpha + \alpha\beta I)(iq)^2 + (3\gamma\mu^2 + 3\delta\mu^2 + 3\beta\mu^2 I + 3\alpha\mu^2 - 2\mu\beta^2 SI - 2\alpha\mu\beta S + 2\gamma\mu\beta I + 2\mu\alpha\gamma + 2\mu\delta\alpha + 2\mu\alpha\beta I - \beta^2\alpha SI + \alpha\beta\gamma I + \alpha\beta\delta I - 3\beta\mu^2 S + 4\mu^3)(iq) + \mu(-\beta S + \gamma + \mu + \delta)(\beta I + \mu)(\alpha + \mu)$$

By comparison with the statement of Bellman and Cooke

$$H(iy) = F(y) + iG(y)$$

Where

$$F(q) = q^4 + (-3\beta\mu S + 3\gamma\mu + 6\mu^2 + 3\delta\mu + 3\mu\beta I + 3\alpha\mu - \beta^2 SI - \alpha\beta S + \gamma\beta I + \gamma\alpha + \beta\delta I + \delta\alpha + \alpha\beta I)q^2 + \mu(-\beta S + \gamma + \mu + \delta)(\beta I + \mu)(\alpha + \mu)$$

$$F(0) = \mu(-\beta S + \gamma + \mu + \delta)(\beta I + \mu)(\alpha + \mu)$$

$$F'(q) = 4q^3 + 2(-3\beta\mu S + 3\gamma\mu + 6\mu^2 + 3\delta\mu + 3\mu\beta I + 3\alpha\mu - \beta^2 SI - \alpha\beta S + \gamma\beta I + \gamma\alpha + \beta\delta I + \delta\alpha + \alpha\beta I)q$$

$$F'(0) = 0$$

$$G(q) = (4\mu - \beta\beta S + \gamma + \delta + \beta I + \alpha)q^3 + \begin{pmatrix} 3\gamma\mu^2 + 3\delta\mu^2 + 3\beta\mu^2 I + 3\alpha\mu^2 - 2\mu\beta^2 SI - 2\alpha\mu\beta S + 2\gamma\mu\beta I + 2\mu\alpha\gamma \\ +2\mu\delta\alpha + 2\mu\alpha\beta I - \beta^2\alpha SI + \alpha\beta\gamma I + \alpha\beta\delta I - 3\beta\mu^2 S + 4\mu^3 \end{pmatrix} q$$

$$G(0) = 0$$

$$G'(0) = \begin{pmatrix} 3\gamma\mu^2 + 3\delta\mu^2 + 3\beta\mu^2 I + 3\alpha\mu^2 - 2\mu\beta^2 SI - 2\alpha\mu\beta S + 2\gamma\mu\beta I + 2\mu\alpha\gamma \\ +2\mu\delta\alpha + 2\mu\alpha\beta I - \beta^2\alpha SI + \alpha\beta\gamma I + \alpha\beta\delta I - 3\beta\mu^2 S + 4\mu^3 \end{pmatrix}$$

Therefore, using equation (30)

$$J = F(0)G'(0) > 0$$

Substitute S, I, and R in the above expression to obtain value for J.

3.4 The Stability Analysis for MSEIR Model

To investigate the stability of the model equation (5) – (9), we write the equations in Jacobi matrix as we did for MSIR as follow:

$$J = \begin{pmatrix} -(\alpha+\mu) & 0 & 0 & 0 & 0 \\ \alpha & -(\beta I+\mu) & 0 & -\beta S & \kappa \\ 0 & \beta I & -(\gamma+\eta+\mu) & \beta S & 0 \\ 0 & 0 & (\eta+\gamma) & -(\sigma+\delta+\mu) & 0 \\ 0 & 0 & 0 & \sigma & -(\kappa+\mu) \end{pmatrix}$$

The characteristic equation for the Jacobi Matrix is

$$|J-\lambda I| = \begin{vmatrix} -(\alpha+\mu+\lambda) & 0 & 0 & 0 & 0 \\ \alpha & -(\beta I+\mu+\lambda) & 0 & -\beta S & \kappa \\ 0 & \beta I & -(\gamma+\eta+\mu+\lambda) & \beta S & 0 \\ 0 & 0 & (\eta+\gamma) & -(\sigma+\delta+\mu+\lambda) & 0 \\ 0 & 0 & 0 & \sigma & -(\kappa+\mu+\lambda) \end{vmatrix} = 0$$

$$= (\alpha+\mu+\lambda)\{\beta I(\gamma+\eta)(-S\beta(\kappa+\mu+\lambda)+\kappa\sigma) + S\beta(\gamma+\eta)(\kappa+\mu+\lambda)(\beta I+\mu+\lambda) - (\kappa+\mu+\lambda)(\sigma+\delta+\mu+\lambda)(\gamma+\sigma+\mu+\lambda)(\beta I+\mu+\lambda)\}$$

3.5 Stability of the Zero Equilibrium State

At zero equilibrium state, $(M,S,E,I,R) = \left\{\frac{\theta\rho}{\alpha+\mu}, \frac{\alpha\rho+(1-\theta)\mu\rho}{\mu(\alpha+\mu)}, 0,0,0\right\}$

$\lambda_1 = -(\alpha+\mu)$, $\lambda_2 = -(\kappa+\mu)$ $\lambda_3 = -(\mu)$

From the characteristic equation, we see that the first three eigenvalues all have negative real parts. We now establish the necessary and sufficient conditions for the remaining two eigenvalues to have negative real part. The remaining two eigenvalues of the equation will have negative real part if and only if Trace (A) <0 and det. A >0 i.e. The Routh-Hurwitz theorem states that the equilibrium state will be asymptotically stable if and only if the the eigenvalues of the characteristics equation of Det. A have negative real part (Kalu and Inyama, 2012).

This implies

$(\gamma+\eta+\mu+\lambda)(\sigma+\delta+\mu+\lambda) - \beta S(\gamma+\eta) > 0$

$$(\gamma+\eta+\mu+\lambda)(\sigma+\delta+\mu+\lambda) > \beta S(\gamma+\eta)$$

This means that the necessary and sufficient condition for DFE of this model to be asymptotically stable, is that the product of total contraction and total breakdown of latent infection class given by $\beta S(\gamma+\eta)$ must be less than the total removal rate from both infectious and latent class given by $(\gamma+\eta+\mu)(\sigma+\delta+\mu)$.

3.6 Stability of the Non-Zero Equilibrium State

In similar approach to MSIR by applying Bellman and Cooke's Theorem for the endemic equilibrium state i.e. expanding the characteristic equations of Jacobi matrix of MSEIR model and apply the theorem, we also have

$$J = F(0)G'(0) > 0$$

where

$$\begin{aligned}F(0) = {} & I\alpha\beta\gamma\sigma\mu + I\alpha\beta\gamma\delta\kappa + I\alpha\beta\gamma\delta\mu + I\alpha\beta\gamma\kappa\mu + I\alpha\beta\gamma\mu^2 + I\alpha\beta\sigma\delta\kappa + I\alpha\beta\sigma\delta\mu \\ & - I\alpha\beta\sigma\eta\kappa + 2I\alpha\beta\sigma\kappa\mu + 2I\alpha\beta\sigma\mu^2 + I\alpha\beta\delta\kappa\mu + I\alpha\beta\delta\mu^2 + I\alpha\beta\kappa\sigma^2 \\ & + I\alpha\beta\kappa\mu^2 + I\alpha\beta\mu^3 + I\alpha\beta\mu\sigma^2 + I\beta\gamma\sigma\mu^2 + I\beta\gamma\delta\kappa\mu + I\beta\gamma\delta\mu^2 + I\beta\gamma\kappa\mu^2 \\ & + I\beta\gamma\mu^3 + I\beta\sigma^2\mu^2 + I\beta\sigma\delta\kappa\mu + I\beta\delta\sigma\mu^2 + I\beta\sigma\eta\kappa\mu + 2I\sigma\kappa\mu^2 + 2I\beta\sigma\mu^3 \\ & + I\beta\kappa\delta\mu^2 + I\beta\delta\mu^3 + I\beta\kappa\mu^3 + I\beta\kappa\sigma^2 + I\beta\mu^4 - S\alpha\beta\gamma\kappa\mu - S\alpha\beta\gamma\mu^2 \\ & - S\alpha\beta\eta\kappa\mu - S\alpha\beta\eta\mu^2 + \alpha\sigma\delta\mu^2 - S\beta\gamma\kappa\mu^2 - S\beta\gamma\mu^3 - S\beta\eta\kappa\mu^2 - S\beta\eta\mu^3 \\ & + \alpha\gamma\kappa\sigma\mu + \alpha\gamma\sigma\mu^2 + \alpha\gamma\kappa\delta\mu + \alpha\gamma\delta\mu^2 + \alpha\gamma\kappa\mu^2 + \alpha\gamma\delta\mu^3 + \alpha\sigma^2\mu^2 \\ & + \alpha\sigma\delta\kappa\mu + 2\alpha\sigma\kappa\mu^2 + 2\alpha\sigma\mu^3 + \alpha\delta\kappa\mu^2 + \alpha\delta\mu^3 + \alpha\kappa\mu^3 + \alpha\sigma^2\kappa\mu + \alpha\mu^4 \\ & + \gamma\sigma\kappa\mu^2 + \gamma\sigma\mu^2 + \kappa\mu^4 + \gamma\delta\kappa\mu^2 + \gamma\delta\mu^2 + \gamma\delta\mu^3 + \gamma\kappa\mu^3 + \gamma\mu^4 + \sigma^2\mu^3 \\ & + \sigma\delta\kappa\mu^2 + \sigma\delta\mu^3 + 2\sigma\kappa\mu^3 + 2\sigma\mu^4 + \delta\kappa\mu^3 + \delta\mu^4 + \kappa\sigma^2 + \mu^5\end{aligned}$$

and

$$G'^{(0)} = I\alpha\beta\gamma\sigma + I\alpha\beta\gamma\delta + I\alpha\beta\gamma\kappa + 2I\alpha\beta\gamma\mu + I\alpha\beta\sigma\delta + 2I\alpha\beta\sigma + 4I\alpha\beta\sigma\mu + I\alpha\beta\delta\kappa + 2I\alpha\beta\delta\mu + 2I\alpha\beta\kappa\mu + I\alpha\beta\sigma^2 + 3I\alpha\beta\mu^2 + 2I\beta\gamma\sigma\mu + I\beta\gamma\delta\kappa + 2I\beta\gamma\kappa\mu + 3I\beta\gamma\mu^2 + I\beta\sigma\delta\kappa + 2I\beta\sigma\delta\mu + I\beta\sigma\eta\kappa + 4I\beta\sigma\kappa\mu + 6I\beta\sigma\mu^2 + 2I\beta\delta\kappa\mu + 3I\beta\delta\mu^2 + I\beta\sigma^2\kappa + 3I\beta\kappa\mu^2 + 4I\beta\mu^3 + 2I\beta\sigma^2 - S\alpha\beta\gamma\kappa - 2S\alpha\beta\gamma\mu - S\alpha\beta\eta\kappa - 2S\alpha\beta\eta\mu + 2S\gamma\kappa\mu - 3S\beta\gamma\mu^2 - 2S\beta\eta\kappa\mu - 3S\beta\eta\mu^2 + \alpha\gamma\kappa\sigma + 2\alpha\gamma\sigma\mu + \alpha\gamma\kappa\delta + 2\alpha\gamma\delta\mu + 2\alpha\gamma\kappa\mu + 3\alpha\gamma\mu^2 + \alpha\sigma\delta\kappa + 2\alpha\sigma\delta\mu + 4\alpha\sigma\kappa\mu + 6\alpha\sigma\mu^2 + 2\alpha\delta\kappa\mu + 3\alpha\delta\mu^2 + \alpha\kappa\sigma^2 + 3\alpha\kappa\mu^2 + 4\alpha\mu^3 + 2\alpha\mu\sigma^2 + 2\gamma\sigma\kappa\mu + 3\gamma\sigma\mu^2 + 2\gamma\delta\kappa\mu + 3\gamma\delta\mu^2 + 3\gamma\kappa\mu^2 + 4\gamma\mu^3 + 2\sigma\delta\kappa\mu + 3\sigma\delta\mu^2 + 6\sigma\kappa\mu^2 + 8\sigma\mu^3 + 5\mu^4 + 3\delta\kappa\mu^2 + 4\delta\mu^3 + 4\kappa\mu^3 + 2\kappa\sigma^2\mu + 3\sigma^2\mu^2$$

4 Discussion of the Results and Conclusion

4.1 Results and Discussion

In this paper, stability analysis of two tuberculosis models have been investigated and found out that at DFE, the following were observed:

i. MSIR Model

$$\lambda_1 = -\alpha - \mu, \quad \lambda_2 = -\mu, \quad \lambda_3 = -\mu, \quad \lambda_4 = \frac{\alpha\beta\rho + (1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)} - (\gamma + \mu + \delta)$$

Which is obvious that $\lambda_1, \lambda_2, \lambda_3$ are all negative, meaning, Tuberculosis will be completely eradicate at this state and for the λ_4, need to be negative

This implies $\frac{\alpha\beta\rho + (1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)} < (\gamma + \mu + \delta)$

where $\frac{\alpha\beta\rho + (1-\theta)\beta\mu\rho}{\mu(\alpha+\mu)}$ is said to be number of latent infections produced and $(\gamma + \mu + \delta)$ is the total removal rate from the infections class.

For stability at endemic equilibrium of MSIR, table 2 shows the results of simulated values.

TABLE 2: SIMULATION OF STABILITY RESULT OF MSIR MODEL

α	B	Γ	δ	ρ	Θ	μ	J	REMARK
0.2	0.1	0.1	0.1	0.9	0.5	0.0	0.0000	Unstable
0.2	0.2	0.1	0.1	0.9	0.5	0.1	0.0000	Unstable
0.2	0.3	0.1	0.2	0.9	0.5	0.1	-1.557796869E-13	Unstable
0.2	0.4	0.3	0.2	0.9	0.5	0.1	0.0000	Unstable
0.2	0.5	0.4	0.2	0.9	0.5	0.1	0.0000	Unstable
0.2	0.6	0.5	0.2	0.9	0.5	0.1	-6.218437552E-14	Unstable
0.2	0.7	0.5	0.2	0.9	0.5	0.1	2.993554720E-14	Stable
0.2	0.8	0.6	0.2	0.9	0.5	0.1	0.0000	Unstable
0.2	0.9	0.7	0.0	0.9	0.5	0.1	0.0000	Unstable
0.2	1.0	0.8	0.0	0.9	0.5	0.1	0.0000	Unstable

ii. MSEIR model

iii. $\lambda_1 = -(\alpha + \mu),\ \lambda_2 = -(\kappa + \mu),\quad \lambda_3 = -(\mu)$

It can as well be seen that the first three eigenvalues are also negative and for the last two to be stable require to be negative. This will be achieved by having the following inequality

$$(\gamma + \eta + \mu + \lambda)(\sigma + \delta + \mu + \lambda) > \beta S(\gamma + \eta)$$

where $(\gamma + \eta + \mu)(\sigma + \delta + \mu)$ is the total removal rate from both infectious and latent exposed class and must be greater than the production of total contraction and total breakdown of latent class $\beta S(\gamma + \eta)$. At endemic equilibrium state, table 3 below gives results of the simulated values.

TABLE 3: SIMULATION OF STABILITY RESULT OF MSEIR MODEL

A	β	γ	α	δ	η	κ	ρ	μ	θ	J	REMARK
0.2	0.1	0.1	0.1	0.1	0.05	0.1	0.9	0.0	0.5	0.000002187000000	Stable
0.2	0.2	0.2	0.1	0.1	0.10	0.0	0.9	0.1	0.5	3.097500000E-7	Stable
0.2	0.3	0.3	0.1	0.2	0.15	0.0	0.9	0.1	0.5	-3.863700922E-7	Unstable
0.2	0.4	0.4	0.3	0.2	0.20	0.2	0.9	0.1	0.5	0.0001103040000	Stable
0.2	0.5	0.5	0.4	0.2	0.25	0.3	0.9	0.1	0.5	0.0006216124297	Stable
0.2	0.6	0.6	0.5	0.2	0.30	0.4	0.9	0.1	0.5	0.002427660300	Stable
0.2	0.7	0.7	0.5	0.2	0.35	0.4	0.9	0.1	0.5	0.00361368150	Stable
0.2	0.8	0.8	0.6	0.2	0.40	0.5	0.9	0.1	0.5	0.01027592198	Stable
0.2	0.9	0.9	0.7	0.0	0.45	0.6	0.9	0.1	0.5	0.04130270245	Stable
0.2	1.0	1.0	0.8	0.0	0.50	0.7	0.9	0.1	0.5	0.09568518464	Stable

4.2 Conclusion

This paper proposed two deterministic models: standard MSIR and MSEIR model of tuberculosis. We have successfully proved that the DFE were stable for the eigenvalues being negative in both MSIR and MSEIR. And however, it is also important that at endemic equilibrium, for the population to sustained, the recovery rate from infectious class γ and σin MSIR and MSEIR respectively must be greater or equal to the natural death and the death rate due to infection else the population will tend towards extinction i.e. the total contraction and total breakdown of latent infection class must be less than the total removal rate from both infectious and latent exposed class.

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