

COUSINS OF THE BERNOULLI INEQUALITY

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ABSTRACT

We present elementary inequalities of Bernoulli type, which have applications in the theory of inequalities, especially the geometric functions theory.

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INTRODUCTION

An infinite sequence $c_0, c_1, \dots, c_k, \dots$ of nonnegative numbers is said to be convex null if $c_k \rightarrow 0$ as $k \rightarrow \infty$ and $c_0 - c_1 \geq c_1 - c_2 \geq \dots \geq c_k - c_{k+1} \geq \dots \geq 0$ (see [3]).

Convex null sequences have most intriguing functional relationship with analytic functions, especially functions which map the unit disk $E = \{z \in \mathbb{D} : |z| < 1\}$ onto the right half plane. The pioneer result in this direction is due to Fejer [2] who proved that if $\{c_k\}_{k=0}^{\infty}$ is a convex null sequence, then the function $p(z) = c_0/2 + c_1z + c_2z^2 + \dots$, $z \in E$, is analytic in E and $\operatorname{Re} p(z) > 0$. Further researches have shown that convex null sequences preserve many known geometric structures of the image of the unit disk, namely, subordination, convexity, starlikeness, quasi-convexity, close-to-convexity and so on.

In this short note we present some variants of an elementary inequality presented in [1]. These inequalities are quite similar to the well-known Bernoulli inequality - one of the most elementary inequalities, which have wide applications in science and engineering - which states as follows:

If $x > -1$ is a real number. Then for all real numbers σ excluding the open interval $(0,1)$,

$$(1+x)^{\sigma} \geq 1 + \sigma x.$$

In the original form of the inequality σ was a natural number. It is well known that for $0 < \sigma < 1$, the inequality is false. In fact the converse of the inequality holds in that case. Various extensions of this famous inequality have been obtained and applied to solve many optimal problems [4]. The inequalities presented in this paper have proved very useful in the investigation of convex nullity of certain sequences associated with some problems in geometric function theory. They also add to the body of applicable inequalities in theory and applications.

2. The INEQUALITIES

Theorem 1. Let a be a positive real number and x such that $-a > x > a$. Then for all real numbers σ excluding the open interval $(-1, 0)$, we have

$$(1) \quad \left(\frac{x}{x-a}\right)^{\sigma} + \left(\frac{x}{x+a}\right)^{\sigma} \geq 2.$$

The inequality is also true for $x \in \{-a, a\}$ if $\sigma \leq -1$.

Proof. First suppose $\sigma \geq 0$. Denote the left hand side of (1) by $h(\sigma)$. Differentiating with respect to σ , we have

$$h'(\sigma) = \left(\frac{x}{x-a}\right)^{\sigma} \log\left(\frac{x}{x-a}\right) + \left(\frac{x}{x+a}\right)^{\sigma} \log\left(\frac{x}{x+a}\right)$$

Then if $x > a$ we have

$$\begin{aligned} h'(\sigma) &> \left(\frac{x}{x-a}\right)^{\sigma} \left[\log\left(\frac{x}{x-a}\right) + \log\left(\frac{x}{x+a}\right) \right] \\ &= \left(\frac{x}{x+a}\right)^{\sigma} \log\left(\frac{x^2}{x^2 - a^2}\right) \\ &> 0. \end{aligned}$$

Otherwise if $x < -a$, then we have

$$h'(\sigma) > \left(\frac{x}{x-a}\right)^{\sigma} \left[\log\left(\frac{x}{x-a}\right) + \log\left(\frac{x}{x+a}\right) \right]$$

$$= \left(\frac{x}{x-a} \right)^{\sigma} \log \left(\frac{x^2}{x^2 - a^2} \right)$$

> 0

Hence $h(\sigma)$ is an increasing function of σ on $[0, \infty)$. Thus $h(\sigma) \geq h(0) = 2$ for $\sigma \geq 0$. Next suppose $\sigma \leq -1$. Take $\sigma = -\mu, \mu \geq 1$, then we can write (1) as

$$(2) \quad \left(1 - \frac{a}{x} \right)^{\mu} + \left(1 + \frac{a}{x} \right)^{\mu} \geq 2$$

which follows on applying the Bernoulli inequality to each term on the left hand side of (2) if $x > a$. Finally if $x < -a$, replace x by $-x$ in (2). The cases $x \in \{-a, a\}$ is obvious from (2). Thus (2) is valid for $\mu \in [1, \infty)$, that is $(\sigma \in (-\infty, -1])$. This proves the inequality.

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For $\sigma \geq 1$, the inequalities can be improved as follows:

Theorem 2. Let a, b be positive real numbers with $a < b$ and x such that $x \in [-b, -a) \cup (a, b]$. Then for $\sigma \geq 1$, we have

$$\alpha \left(\frac{x}{x-a} \right)^{\sigma} + \beta \left(\frac{x}{x+a} \right)^{\sigma} \geq \begin{cases} \frac{(\alpha+\beta)b^2 + (\alpha-\beta)ab}{b^2 - a^2} & \text{if } \alpha \geq \beta, \\ \frac{(\alpha+\beta)b^2 + (\beta-\alpha)ab}{b^2 - a^2} & \text{if } \alpha \leq \beta. \end{cases}$$

Proof. Let
$$h(\sigma, x) = \alpha \left(\frac{x}{x-a} \right)^{\sigma} + \beta \left(\frac{x}{x+a} \right)^{\sigma}.$$

Suppose $\alpha \geq \beta$. Then following as in the proof of Theorem 1, $h(\sigma, x)$ is increasing on $(a, b]$ so that $h(\sigma, x) \geq h(1, x)$. By elementary calculus it can also be proved, without loss of generality, that $h(1, x)$ is a decreasing function of x on $(a, b]$. Hence we have $h(\sigma, x) \geq h(1, b)$ as required. For $x \in [-b, -a)$ set $x = -y$ with $y \in (a, b]$ so that

$$h(\sigma, y) = \alpha \left(\frac{y}{y+a} \right)^\sigma + \beta \left(\frac{y}{y-a} \right)^\sigma$$

and the second inequality follows with $\alpha \leq \beta$.

Now suppose $\alpha \leq \beta$, the above argument is reversed starting with $x \in [-b, -a)$. This completes the proof.

In applying the above inequalities, the positive real number b is determined according to the requirements of the application. This is demonstrated in Example 2 in the next section. That the

above result improves the inequalities in Theorem 1 for $\sigma \geq 1$, will also become clear in the same example.

Now observe that the limiting case $\sigma \rightarrow \infty$ gives $-a > x > a$ and the above theorem recasts as

Corollary 1. Let a be a positive real number and x such that $-a > x > a$. Then for $\sigma \geq 1$, we have

$$\alpha \left(\frac{x}{x-a} \right)^\sigma + \beta \left(\frac{x}{x+a} \right)^\sigma \geq \alpha + \beta.$$

The case $\alpha = \beta = 1$ agrees with Theorem 1 for $\sigma \geq 1$.

3. APPLICATIONS

Applications of the inequalities have been found in the determination of convex-nullity of some infinite sequences applicable in the theory of analytic and univalent functions.

Example 1. The sequence $\{\alpha_k\}_{k=0}^{\infty} = \left(\frac{\mu}{\mu+k}\right)^{\sigma}$ with $\mu, \sigma > 0$ is convex null.

Proof. It is obvious that $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. We need to prove that $\alpha_k - 2\alpha_{k+1} + \alpha_{k+2} \geq 0$, $k = 0, 1, \dots$. That is

$$\left(\frac{\mu}{\mu+k}\right)^{\sigma} - 2\left(\frac{\mu}{\mu+k+1}\right)^{\sigma} + \left(\frac{\mu}{\mu+k+2}\right)^{\sigma} \geq 0$$

In other words we have to prove that

$$\left(\frac{\mu+k+1}{\mu+k}\right)^{\sigma} - 2\left(\frac{\mu+k+1}{\mu+k+2}\right)^{\sigma} \geq 2,$$

which follows from Theorem 1 by setting $x = \mu + k + 1$ and $a = 1$.

Example 2. Let $0 < \mu \leq 1$ and $\sigma \geq 1$. Then the sequence $\{c_k\}_{k=0}^{\infty}$ defined as

$$c_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{1}{2} \frac{\mu^{\sigma}}{(\mu+k)^{\sigma}} & \text{if } k \geq 1 \end{cases}$$

is a convex null sequence.

Proof. It is also obvious that $c_k \rightarrow 0$ as $k \rightarrow \infty$. Then we have to prove that $c_k - 2c_{k+1} + c_{k+2} \geq 0$, $k = 0, 1, \dots$. That is

$$\left(\frac{\mu}{\mu+k}\right)^{\sigma} - 2\left(\frac{\mu}{\mu+k+1}\right)^{\sigma} + \left(\frac{\mu}{\mu+k+2}\right)^{\sigma} \geq 0$$

In other words we have to prove that

$$(3) \quad \left(\frac{\mu+k+1}{\mu+k} \right)^{\sigma} - 2 \left(\frac{\mu+k+1}{\mu+k+2} \right)^{\sigma} \geq 2.$$

If $k \geq 1$, (3) follows from Theorem 1 by setting $x = \mu + k + 1$ and $a = 1$. It remains to prove that $c_0 - 2c_1 + c_2 \geq 0$. That is

$$(4) \quad \frac{2}{3} \left(\frac{\mu+1}{\mu} \right)^{\sigma} + \left(\frac{\mu+1}{\mu+2} \right)^{\sigma} \geq 2.$$

Take $x = \mu + 1 \in (1, 2]$ and $\alpha = 2/3$, $\beta = 1$, $a = 1$ and $b = 2$. The choice of $b = 2$ is due to $x = \mu + 1 \in (1, 2]$ since $0 < \mu \leq 1$. Thus the left hand side of equation (4) is greater or equal to $22/9$, which is greater than 2.

REMARKS

The Bernoulli inequality could have been helpful in problems like those given in the above examples (see also [1]) except for the open interval $0 < \sigma < 1$. The inequalities described in this paper similarly omit the interval $-1 < \sigma < 0$. We thus describe the inequalities as complementary Cousins of the Bernoulli inequality. We believe that these simple inequalities and the Bernoulli inequality itself would be able to strengthen one another, thus filling gaps in many areas of mathematical inequalities and applications.

The case $a = 1$, $x > 1$ and $\sigma \geq 0$ of Theorem 1 was first reported and applied in [1] in proving some important results in the theory of geometric functions.

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