

BOUNDS ON THE COEFFICIENTS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract. For the real number $\alpha > 1$, we use a technique due to Nehari and Netanyahu and an application of certain integral iteration of Caratheodory functions to find the best-possible upper bounds on the coefficients of functions of the class $T_n^\alpha(\beta)$ introduced in [4] by Opoola.

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1. INTRODUCTION

Let A denote the class of functions:

$$(1) \quad f(z) = z + a_2 z^2 + \dots$$

which are analytic in the unit disk $E = \{z \in \mathcal{C} : |z| < 1\}$. In [4], Opoola introduced the subclass $T_n^\alpha(\beta)$ consisting of functions $f \in A$ which satisfy:

$$(2) \quad \operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta.$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, $D^n (n \in N_0 = \{0, 1, 2, \dots\})$ is the Salagean derivative operator defined as: $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$ and powers in (2) meaning principal determinations only. The geometric condition (2) slightly modifies the one given originally in [4] (see [1]).

The object of the present work is the extension of some earlier results regarding the bounds on the coefficients, a_k , of functions belonging to the class $T_n^\alpha(\beta)$. Babalola and Opoola have begun to solve this problem in [2]. They determined sharp bounds on a_k for some $\alpha > 0$ and gave some rough estimate of the general coefficient bounds using the logarithmic coefficient approach. The sharp bounds were stated as follows:

THEOREM 1. *Let $f \in T_n^\alpha(\beta)$. Define*

$$B_m = \frac{2^m (1 - \beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1 - j\alpha)}{m!},$$

$$A_k(n, \alpha, \beta) = \sum_{m=1}^{k-1} B_m Q_{k-1}^{(m)},$$

$$A_{k-1}(n, \alpha, \beta) = \sum_{m=1}^{k-2} B_m Q_{k-1}^{(m)},$$

where for each $m = 1, 2, \dots, Q_{k-1}^{(m)}$ is defined by the power series

$$\left(\sum_{k=1}^{\infty} \frac{z^k}{(\alpha + k)^n} \right)^m = Q_m^{(m)} z^m + Q_{m+1}^{(m)} z^{m+1} + Q_{m+2}^{(m)} z^{m+2} + \dots$$

Also let

$$\begin{aligned} \Omega_1 &= \{\alpha | 0 < \alpha < (k-2)^{-1}, k = 2, 3, \dots\}, \\ \Omega_2 &= \{\alpha | (k-2)^{-1} \leq \alpha \leq (k-3)^{-1}, k = 4, 6, \dots\}, \\ \Omega_3 &= \{\alpha | (k-2)^{-1} \leq \alpha < (k-3)^{-1}, k = 3, 5, \dots\}. \end{aligned}$$

Then

$$|a_k| \leq \begin{cases} A_k & \text{if } \alpha \in \Omega_1 \cup \Omega_2, \\ A_{k-1} & \text{if } \alpha \in \Omega_3. \end{cases}$$

The inequalities are sharp. Equalities are attained for $f(z)$ satisfying

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \begin{cases} \frac{1+(1-2\beta)z}{1-z} & \text{if } \alpha \in \Omega_1 \cup \Omega_2, \\ \frac{1+(1-2\beta)z^2}{1-z^2} & \text{if } \alpha \in \Omega_3. \end{cases}$$

The rough estimate was also given as:

THEOREM 2. Let $f \in T_n^\alpha(\beta)$, $n \in N$. Suppose

$$\frac{f(z)^\alpha}{z^\alpha} = \sum_{k=0}^{\infty} A_{k+1}(\alpha) z^k, A_1(\alpha) = 1.$$

Then

$$|A_{k+1}(\alpha)| < \exp \left\{ 0.624\alpha^2 + \left(2\alpha^2 - \frac{1}{2} \right) \sum_{j=1}^k \frac{1}{j} \right\}.$$

From Theorem 1, which is best-possible, it is obvious that the problem has only been solved completely for a_2 and a_3 for all values of the index α . For $k \geq 4$, the problem has remained open for all $\alpha > (k-3)^{-1}$. In this article we proceed with the proof of the sharp bounds on the coefficients of functions in the class $T_n^\alpha(\beta)$, $\alpha > 1$. Our result is the following:

THEOREM 3. Let $f \in T_n^\alpha(\beta)$. If $\alpha > 1$, we have the sharp inequalities

$$(3) \quad |a_k| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+k-1)^n}, \quad k = 2, 3, 4, \dots$$

Equalities are attained for $f(z)$ satisfying

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 + (1-2\beta)z^{k-1}}{1 - z^{k-1}}.$$

For $k = 2, 3$, the above result is contained in Theorem 1 above. The proof here is however new. Also Singh [6] proved the same result for $k = 2, 3, 4$ for the particular case $n = 1$ and $\beta = 0$. In our proof we combine a method of classical analysis due to Nehari and Netanyahu [3] (also used by Singh [6]) with an application of certain integral iteration of the Caratheodory functions [1]. That is presented in Section 3. In the next section we state and prove some preliminary lemmas.

2. PRELIMINARY LEMMAS

Let P be the class of functions

$$(4) \quad p(z) = 1 + b_1 z + b_2 z^2 + \cdots$$

which are analytic in E and have positive real part. In [1] the following integral iteration of each $p \in P$ was identified:

DEFINITION 1. [1] Let $p \in P$ and $\alpha > 0$ be real. The n th iterated integral transform of $p(z)$, $z \in E$ is defined as

$$p_n(z) = \frac{\alpha}{z^\alpha} \int_0^z t^{\alpha-1} p_{n-1}(t) dt, \quad n \geq 1,$$

with $p_0(z) = p(z)$. The family of the n th integral iteration of $p \in P$ was denoted by P_n . Functions $p_n(z)$ in P_n have series expansion:

$$(5) \quad p_n(z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha + k)^n} b_k z^k.$$

Furthermore if $\operatorname{Re} p_n(z) > \beta$, ($0 \leq \beta < 1$), we denote by $P_n(\beta)$ the family of such functions given by:

$$p_n(z) = 1 + (1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha + k)^n} b_k z^k.$$

In the proof of our main result we require the following lemmas:

LEMMA 1. [3] If $p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ and $q(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ belong to P , then $r(z) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} b_k c_k z^k$ also belongs to P .

LEMMA 2. [3] Let $h(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ and $1 + G(z) = 1 + \sum_{k=1}^{\infty} b'_k z^k$ be functions in P . Set

$$(6) \quad \gamma_m = \frac{1}{2^m} \left[1 + \frac{1}{2} \sum_{\mu=1}^m \binom{m}{\mu} d_\mu \right], \quad \gamma_0 = 1.$$

If A_k is defined by

$$\sum_{m=1}^{\infty} (-1)^{m+1} \gamma_{m-1} G_1^m(z) = \sum_{k=1}^{\infty} A_k z^k.$$

Then

$$|A_k| \leq 2, \quad k = 1, 2, \dots$$

If, in the proof of the above lemma (as contained in [3]), we define $h_n(z)$ as the n -th iterated integral transform of $h_0(z) = h(z)$ we immediately obtain the following corollary.

COROLLARY 1. *Let $h_n(z)$ be the n th integral iteration of $h_0(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ with $\operatorname{Re} h_n(z) > \beta$, and $1 + G(z) = 1 + \sum_{k=1}^{\infty} b'_k z^k$ be functions in P . Define γ_m as in (6) and*

$$(7) \quad \eta_m = \frac{(1 - \beta)\alpha^n}{(\alpha + m)^n} \gamma_m, \quad \eta_0 = 1 - \beta.$$

If A_k is defined by

$$(8) \quad \sum_{m=1}^{\infty} (-1)^{m+1} \eta_{m-1} G_1^m(z) = \sum_{k=1}^{\infty} A_k z^k.$$

Then

$$(9) \quad |A_k| \leq \frac{2(1 - \beta)\alpha^n}{(\alpha + k)^n}, \quad k = 1, 2, \dots$$

LEMMA 3. [2] *Let $G(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series. Then the m th integer product of $G(z)$ is $G^m(z) = \sum_{k=0}^{\infty} c_k^{(m)} z^k$ where $c_k^{(1)} = c_k$ and $c_k^{(m)} = \sum_{j=0}^k c_j c_{k-j}^{(m-1)}$, $m \geq 2$.*

We now turn to the proof of the main result.

3. PROOF OF THE MAIN RESULT

Let $f \in T_n^\alpha(\beta)$. Then there exists an analytic functions $p_n \in P_n$ such that

$$(10) \quad f(z)^\alpha = z^\alpha [\beta + (1 - \beta)p_n(z)],$$

where $p_n(z)$ given by (5) is the n th iterated integral transform of an analytic function $p \in P$ defined by (4) (see Lemma 4.2 of [1]). The first part of the proof involves obtaining expression for the coefficients, a_k , of $f(z)$ in terms of the coefficients, b_k , of the function $p \in P$. This is contained in [2] and adapted here for completeness. We have, using (5) in (10),

$$(11) \quad f(z) = z \left(1 + (1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha + k)^n} b_k z^k \right)^{\frac{1}{\alpha}}.$$

Equation (11) expands binomially as

$$(12) \quad \frac{f(z)}{z} = 1 + \frac{(1-\beta)\alpha^n}{\alpha} \sum_{k=1}^{\infty} \frac{b_k z^k}{(\alpha+k)^n} + \frac{(1-\beta)^2 \alpha^{2n} (1-\alpha)}{2! \alpha^2} \left(\sum_{k=1}^{\infty} \frac{b_k z^k}{(\alpha+k)^n} \right)^2 + \dots + \frac{(1-\beta)^m \alpha^{mn} \prod_{j=0}^{m-1} (1-j\alpha)}{m! \alpha^m} \left(\sum_{k=1}^{\infty} \frac{b_k z^k}{(\alpha+k)^n} \right)^m + \dots$$

Using Lemma 3 in (12) we get

$$\frac{f(z)}{z} = 1 + \sum_{k=1}^{\infty} \tilde{B}_1 C_k^{(1)} z^k + \dots + \sum_{k=1}^{\infty} \tilde{B}_m C_k^{(m)} z^k + \dots,$$

where

$$\tilde{B}_m = \frac{(1-\beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1-j\alpha)}{m!}$$

and $C_k^{(m)}$, $m = 1, 2, \dots$; $k = m, m+1, \dots$ is defined by

$$\left(\sum_{k=1}^{\infty} \frac{b_k z^k}{(\alpha+k)^n} \right)^m = \sum_{k=1}^{\infty} C_k^{(m)} z^k,$$

having the general form

$$(13) \quad C_k^{(m)} = \sum_{j=1}^k C_j \prod_{l=1}^m \frac{b_l^{\rho_l}}{(\alpha+l)^{n\rho_l}}$$

for some nonnegative constants C_j , $j = 1, 2, \dots, k$ and indices ρ_l , $l = 1, 2, \dots, m$ taking values in the set $M = \{0, 1, 2, \dots, m\}$ such that $\rho_1 + \rho_2 + \dots + \rho_m = m$ (see page 10 of [2]). From (12) we write

$$(14) \quad f(z) = z + \sum_{k=2}^{\infty} \tilde{A}_k^{(m)} z^k,$$

where

$$\tilde{A}_k^{(m)} = \sum_{m=1}^{k-1} \tilde{B}_m C_{k-1}^{(m)}, \quad k = 2, 3, \dots$$

Comparing coefficients in (1) and (14), we see that $a_k = \tilde{A}_k^{(m)}$ which gives

$$(15) \quad a_k = \sum_{m=1}^{k-1} \frac{(1-\beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1-j\alpha)}{m!} \left(\sum_{j=1}^k C_j \prod_{l=1}^m \frac{b_l^{\rho_l}}{(\alpha+l)^{n\rho_l}} \right).$$

Now we compute the leading coefficients, A_v , in the expression (8). From (8) we have

$$(16) \quad \sum_{m=1}^{\infty} (-1)^{m+1} \eta_{m-1} G_1^m(z) = G_1(z) - \eta_1 G_1^2(z) + \cdots = \sum_{v=1}^{\infty} A_v z^v$$

with $G_1(z) = \sum_{v=1}^{\infty} b'_v z^v$. Using Lemma 3 again we have

$$(17) \quad G_1^m(z) = \sum_{v=m}^{\infty} C_v^{(m)} z^v, \quad m = 1, 2, \dots,$$

where

$$(18) \quad C_v^{(m)} = \sum_{j=1}^v C_j \prod_{l=1}^m b'_l{}^{\rho_l}$$

with C_j and indices ρ_l as already defined for (13). Using (17) and (18) in (16) we get

$$\begin{aligned} \sum_{m=1}^{\infty} (-1)^{m+1} \eta_{m-1} G_1^m(z) &= \sum_{v=1}^{\infty} \left(\sum_{m=1}^v (-1)^{m+1} \eta_{m-1} C_v^{(m)} \right) z^v \\ &= \sum_{v=1}^{\infty} \left(\sum_{m=1}^v (-1)^{m+1} \eta_{m-1} \left(\sum_{j=1}^v C_j \prod_{l=1}^m b'_l{}^{\rho_l} \right) \right) z^v \\ &= \sum_{v=1}^{\infty} A_v z^v, \end{aligned}$$

so that

$$A_v = \sum_{m=1}^v (-1)^{m+1} \eta_{m-1} \left(\sum_{j=1}^v C_j \prod_{l=1}^m b'_l{}^{\rho_l} \right) z^v.$$

By corollary 1, these coefficients, A_v , satisfy the inequality (9) if $1 + G(z) = 1 + b'_1 z + b'_2 z^2 + \cdots$ is a function of the class P , and by Lemma 1 we may set $b'_l = \frac{1}{2} b_l c_l$ where $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is the function (4) and $H(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an arbitrary function in P . Then

$$(19) \quad |A_v| = \left| \sum_{m=1}^v (-1)^{m+1} \frac{\eta_{m-1}}{2^m} \left(\sum_{j=1}^v C_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\beta)\alpha^n}{(\alpha+v)^n}.$$

Using (7) together with the fact that $\alpha > 0$, we can write (19) equivalently as

$$(20) \quad |A_v| = \left| \sum_{m=1}^v (-1)^{m+1} \frac{(1-\beta)\alpha^n}{(\alpha+m-1)^n} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^v C_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+v)^n}.$$

Since for each $m = 1, 2, \dots$ and any $\alpha > 0$,

$$\prod_{l=1}^m \frac{\alpha^{n\rho_l}}{(\alpha + l)^{n\rho_l}} \leq \prod_{l=1}^m \frac{\alpha^{n\rho_l}}{(\alpha + 1)^{n\rho_l}} = \frac{\alpha^{mn}}{(\alpha + 1)^{mn}} \leq \frac{\alpha^n}{(\alpha + m - 1)^n},$$

it is evident that for each $v = 1, 2, \dots$

$$\begin{aligned} \sum_{m=1}^v (-1)^{m+1} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^v C_j \prod_{l=1}^m \frac{(1-\beta)^{\rho_l} \alpha^{n\rho_l}}{(\alpha + l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \leq \\ \sum_{m=1}^v (-1)^{m+1} \frac{(1-\beta)\alpha^n}{(\alpha + m - 1)^n} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^v C_j \prod_{l=1}^m b_l^{\rho_l} c_l^{\rho_l} \right). \end{aligned}$$

Therefore by (20), we have

$$\left| \sum_{m=1}^v (-1)^{m+1} \frac{\gamma_{m-1}}{2^m \alpha} \left(\sum_{j=1}^v C_j \prod_{l=1}^m \frac{(1-\beta)^{\rho_l} \alpha^{n\rho_l}}{(\alpha + l)^{n\rho_l}} b_l^{\rho_l} c_l^{\rho_l} \right) \right| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha + v)^n}$$

that is,

$$(21) \quad \left| \sum_{m=1}^v (-1)^{m+1} \frac{(1-\beta)^m \alpha^{mn-1} \gamma_{m-1}}{2^m} \left(\sum_{j=1}^v C_j \prod_{l=1}^m \frac{b_l^{\rho_l} c_l^{\rho_l}}{(\alpha + l)^{n\rho_l}} \right) \right| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha + v)^n}.$$

Now comparing (15) and the term in the absolute value in (21) (with $v = k-1$), we would conclude that the inequalities (3) hold if we are able to find two members $h(z) = 1 + d_1 z + d_2 z^2 + \dots$ and $H(z) = 1 + c_1 z + c_2 z^2 + \dots$ of P which give rise to the constants γ_m (as required by (6)) and c_l . For $H \in P$, a natural choice of the Moebius function is suitable. That is, $H(z) = (1+z)/(1-z) = 1 + 2z + 2z^2 + \dots$. Thus we have $c_l = 2$, $l = 1, 2, \dots$. Using this in (21) we get

$$(22) \quad \left| \sum_{m=1}^{k-1} (-1)^{m+1} (1-\beta)^m \alpha^{mn-1} \gamma_{m-1} \left(\sum_{j=1}^{k-1} C_j \prod_{l=1}^m \frac{b_l^{\rho_l}}{(\alpha + l)^{n\rho_l}} \right) \right| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha + k - 1)^n}.$$

Comparing (15) and the term in the absolute value in (22) we find that

$$(-1)^{m+1} \frac{\gamma_{m-1}}{\alpha} = \frac{\prod_{j=0}^{m-1} (1 - j\alpha)}{m! \alpha^m}$$

that is

$$(23) \quad \gamma_{m-1} = \frac{\prod_{j=1}^{m-1} (j\alpha - 1)}{m! \alpha^{m-1}}, \quad \gamma_0 = 1.$$

Now with d_μ , $\mu = 1, 2, \dots, m-1$ defined by

$$(24) \quad \frac{1}{2^{m-1}} \left[1 + \frac{1}{2} \sum_{\mu=1}^{m-1} \binom{m-1}{\mu} d_\mu \right] = \frac{\prod_{j=1}^{m-1} (j\alpha - 1)}{m! \alpha^{m-1}},$$

we need to find $h(z)_k$, corresponding to each a_k , $k = 2, 3, 4, \dots$, such that the coefficients d_μ , $\mu = 1, 2, \dots, m-1$ of each $h(z)_k$ satisfy (24) and the proof would be complete. Observe from (22) that $m = 1, 2, \dots, k-1$ as we begin to implement our scheme for each $k = 2, 3, 4, \dots$.

$k = 2$: In this case $m = 1$. Hence by (23) $\gamma_0 = 1$ and we can therefore define $d_\mu = 0$ for all μ so that $h(z)_2 = 1$.

$k = 3$: Here we have $m = 1, 2$. Using (24) we get $d_1 = -\frac{2}{\alpha}$, so that

$$h(z)_3 = \frac{\alpha - 1}{\alpha} + \frac{1}{\alpha} \left(\frac{1 - z}{1 + z} \right) = 1 - \frac{2}{\alpha} z + \dots$$

$k = 4$: Now $m = 1, 2, 3$. From (24) we have

$$\frac{1}{4} \left(1 + \frac{1}{2} (2d_1 + d_2) \right) = \frac{(\alpha - 1)(2\alpha - 1)}{3! \alpha^2}.$$

Taking $d_1 = 0$, we get $\frac{d_2}{2} = \frac{\alpha^2 - 6\alpha + 2}{3\alpha^2}$ (note that $|d_2| \leq 2$), and we have

$$\begin{aligned} h(z)_4 &= \begin{cases} \frac{2(\alpha^2 + 3\alpha - 1)}{3\alpha^2} + \frac{\alpha^2 - 6\alpha + 2}{3\alpha^2} \left(\frac{1 + z^2}{1 - z^2} \right) & \text{if } \alpha \geq \alpha_0, \\ \frac{2(2\alpha^2 - 3\alpha + 1)}{3\alpha^2} + \frac{6\alpha - \alpha^2 - 2}{3\alpha^2} \left(\frac{1 - z^2}{1 + z^2} \right) & \text{if } 1 < \alpha \leq \alpha_0 \end{cases} \\ &= 1 + \frac{2(\alpha^2 - 6\alpha + 2)}{3\alpha^2} z^2 + \dots, \end{aligned}$$

where $\alpha_0 > 1$ is the solution of $\alpha^2 - 6\alpha + 2 = 0$.

$k = 5$: In this case $m = 1, 2, 3, 4$, and from (24) we get

$$\frac{1}{8} \left(1 + \frac{1}{2} (3d_1 + 3d_2 + d_3) \right) = \frac{(\alpha - 1)(2\alpha - 1)(3\alpha - 1)}{4! \alpha^3}.$$

Taking $d_1 = d_2 = 0$, we get $\frac{d_3}{2} = \frac{3\alpha^3 - 11\alpha^2 + 6\alpha - 1}{3\alpha^3}$ (note also that $|d_3| \leq 2$) and

$$\begin{aligned} h(z)_5 &= \begin{cases} \frac{11\alpha^2 - 6\alpha + 1}{3\alpha^3} + \frac{3\alpha^3 - 11\alpha^2 + 6\alpha - 1}{3\alpha^3} \left(\frac{1 + z^3}{1 - z^3} \right) & \text{if } \alpha \geq \alpha_0, \\ \frac{6\alpha^3 - 11\alpha^2 + 6\alpha - 1}{3\alpha^3} + \frac{1 - 6\alpha + 11\alpha^2 - 3\alpha^3}{3\alpha^3} \left(\frac{1 - z^3}{1 + z^3} \right) & \text{if } 1 < \alpha \leq \alpha_0 \end{cases} \\ &= 1 + \frac{2(3\alpha^3 - 11\alpha^2 + 6\alpha - 1)}{3\alpha^3} z^3 + \dots \end{aligned}$$

In this case $\alpha_0 > 1$ is the solution of $3\alpha^3 - 11\alpha^2 + 6\alpha - 1 = 0$.

$k \geq 6$: In general, $m = 1, 2, 3, \dots, k-1$. In (24) we set $d_1 = \frac{-2}{m-1}$, $d_2 = d_4 = \dots = d_\xi = \sigma$ where ξ equals $m-1$ if $m-1$ is *even* and $m-2$ otherwise, and $d_3 = d_5 = \dots = d_\omega = 0$ where ω equals $m-1$ if $m-1$ is *odd* and $m-2$ otherwise. With this we get $\frac{\sigma}{2} = \frac{2^{m-1} \prod_{j=1}^{m-1} \left(\frac{j\alpha - 1}{j\alpha} \right)}{m \left(\binom{m-1}{2} + \binom{m-1}{4} + \dots + \binom{m-1}{\xi} \right)}$. In this

sense $|d_\mu| \leq 2$ for all $\mu = 1, 2, \dots, m-1$. Setting $m = k-1$, we now define $h(z)_k$, $k \geq 6$ as follows

$$h(z)_k = 1 - \frac{2}{k-2} - \frac{2^{k-2} \prod_{j=1}^{k-2} \left(\frac{j\alpha-1}{j\alpha} \right)}{(k-1) \left(\binom{k-2}{2} + \binom{k-2}{4} + \dots + \binom{k-2}{\xi} \right)} + \frac{2}{k-2} (1-z) \\ + \frac{2^{k-2} \prod_{j=1}^{k-2} \left(\frac{j\alpha-1}{j\alpha} \right)}{(k-1) \left(\binom{k-2}{2} + \binom{k-2}{4} + \dots + \binom{k-2}{\xi} \right)} \left(\frac{1+z^2}{1-z^2} \right).$$

In other words,

$$h(z)_k = 1 - \frac{2}{k-2} z + \frac{2^{k-2} \prod_{j=1}^{k-2} \left(\frac{j\alpha-1}{j\alpha} \right)}{(k-1) \left(\binom{k-2}{2} + \binom{k-2}{4} + \dots + \binom{k-2}{\xi} \right)} z^2 \\ + \frac{2^{k-2} \prod_{j=1}^{k-2} \left(\frac{j\alpha-1}{j\alpha} \right)}{(k-1) \left(\binom{k-2}{2} + \binom{k-2}{4} + \dots + \binom{k-2}{\xi} \right)} z^4 + \dots.$$

That these functions $h(z)_k$ belong to P follows from the fact that the function $\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_m f_m$ belongs to P if $f_1, f_2, \dots, f_m$ belong, $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_m = 1$. This completes the proof of the theorem.

REMARK 1. We compute $h(z)_6, \dots, h(z)_{10}$ for the purpose of illustration.

$$h(z)_6 = 1 - \frac{1}{2} z + \frac{4(\alpha-1)(2\alpha-1) \dots (4\alpha-1)}{105\alpha^4} z^2 \\ + \frac{4(\alpha-1)(2\alpha-1) \dots (4\alpha-1)}{105\alpha^4} z^4 + \dots,$$

$$h(z)_7 = 1 - \frac{2}{5} z + \frac{4(\alpha-1)(2\alpha-1) \dots (5\alpha-1)}{675\alpha^5} z^2 \\ + \frac{4(\alpha-1)(2\alpha-1) \dots (5\alpha-1)}{675\alpha^5} z^4 + \dots,$$

$$h(z)_8 = 1 - \frac{1}{3} z + \frac{8(\alpha-1)(2\alpha-1) \dots (6\alpha-1)}{10765\alpha^6} z^2 \\ + \frac{8(\alpha-1)(2\alpha-1) \dots (6\alpha-1)}{10765\alpha^6} z^4 + \dots,$$

$$h(z)_9 = 1 - \frac{2}{7} z + \frac{2(\alpha-1)(2\alpha-1) \dots (7\alpha-1)}{19845\alpha^7} z^2 \\ + \frac{2(\alpha-1)(2\alpha-1) \dots (7\alpha-1)}{19845\alpha^7} z^4 + \dots,$$

$$h(z)_{10} = 1 - \frac{1}{4}z + \frac{4(\alpha - 1)(2\alpha - 1) \dots (8\alpha - 1)}{360045\alpha^8}z^2 \\ \frac{4(\alpha - 1)(2\alpha - 1) \dots (8\alpha - 1)}{360045\alpha^8}z^4 + \dots$$

With this work the coefficient problem of functions in the class $T_n^\alpha(\beta)$ is settled for any $\alpha > 1$. Of course the case $\alpha = 1$ is trivial as this simply gives $|a_k| \leq \frac{2(1-\beta)}{(k+1)^n}$, $k \geq 2$, as can be seen easily from (12). Thus the problem only remains open for $(k-3)^{-1} \leq \alpha < 1$, $k \geq 5$. Finally, we note a humble attempt at this problem made by the authors in [5]. Their results depended wholly on the triangle inequality, and were not sharp.

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