

DERIVATION OF A GENERALIZED DISTRIBUTION FOR MATERNAL MORTALITY RATIO

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CERTIFICATION

This is to certify that this thesis has been read and approved as meeting the requirements of the Department of Statistics, University of Ilorin, Ilorin, Nigeria, for the Award of Doctor of Philosophy degree.

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DEDICATION

This research work is dedicated to my husband.

DECLARATION

I, Omekam, Ifeyinwa Vivian, hereby declare that this thesis entitled, “Derivation of a Generalized Distribution for Maternal Mortality Ratio”, is a record of my research work. It has neither been presented nor accepted in any previous application for a higher degree. All sources of information have been specifically acknowledged.

In addition, the research work has been ethically approved by the University Ethical Review Committee.

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ABSTRACT

Maternal mortality is an outcome of adverse health and remains a major challenge in some societies. Study on distribution fitting for global Maternal Mortality Ratio (MMR) is therefore needful. Analysis of data without a pre-knowledge of the distribution that describes data may lead to misleading or irrelevant results. Distribution fitting to data provides the best fitting distribution for data analysis. Limitations in characteristics of existing distributions motivate generalization of distributions in order to improve goodness of fit and introduce more flexibility. The aim of this study was to derive a generalized distribution for MMR while the specific objectives were to: (i) generate families of generalized distributions; (ii) illustrate the flexibility of generalized distributions; (iii) fit some existing distributions to MMR to determine the best fitting distribution; (iv) derive generalized distributions for the best fitted distribution; and (v) fit and assess the derived generalized distributions to MMR.

Five existing parameter induction methods were applied to generate families of generalized distributions with an additional parameter. Methods in permutations of these five existing parameter induction methods, namely: Lehmann Alternative 1; Lehmann Alternative 2, Marshall and Olkin Method, Power Transformation Method and α -Power Transformation Method taking two methods at a time were applied sequentially to obtain generalized families with two additional shape parameters. Same methods were also applied twice. Exponentiated families of Generalized Pareto Distribution (GPD) were generated and flexibility of generalized families was illustrated by showing effects of introduced parameters on the Probability Density and Hazard Function shapes. Histogram plot, Kolmogorov-Smirnov (K-S) distances and Akaike Information Criterion (AIC) were employed in determining the best fitting distribution for MMR. Using best fitted distribution as base distribution, its generalized distributions were

derived from families generated and subsequently fitted to MMR using AIC and K-S distances for selection.

Findings of the study were that:

- (i) seventeen distinct generalized families of distributions having two additional parameters were generated for commutativity and idempotency of some methods. Five families with one additional parameter were also generated;
- (ii) Lehmann type I GPD improved flexibility by introducing the unimodal and bathtub shapes in both Probability Density and Hazard Functions shapes;
- (iii) Frechet distribution provided the best fit for MMR amongst plausible distributions studied;
- (iv) four generalized Frechet distributions with an additional shape parameter and two generalized Frechet distributions with two additional shape parameters were generated; and
- (v) generalized Frechet distributions improved goodness of fit based on K-S distances with Lehman type II generalized Frechet giving the best fit while AIC selected Marshall-Olkin generalized Frechet distribution as a good fitting but parsimonious distribution for MMR.

In conclusion, families of generalized distribution introducing n parameters may be generated by sequentially applying methods in permutations of s ($s \geq n$) distinct parameter induction methods taking n methods at a time. Lehmann type I GPD introduced flexibility thereby illustrating flexibility of generalized distributions. Generalized Frechet distributions improved goodness of fit but for parsimony, Marshal-Olkin generalized Frechet distribution is recommended as an alternative reference distribution to the Frechet distribution for modelling MMR.

LIST OF ABBREVIATIONS AND NOTATIONS

WHO	WORLD HEALTH ORGANIZATION
MMR	MATERNAL MORTALITY RATIO
PM	ADULT LIFETIME RISK OF MATERNAL DEATHS AMONG DEATHS OF WOMEN OF REPRODUCTIVE AGE.
UN	UNITED NATIONS
MDGs	MILLENIUM DEVELOPMENT GOALS
SDGs	SUSTAINABLE DEVELOPMENT GOALS
K-S	KOLMOGOROV- SMIRNOV
MMEIG	MATERNAL MORTALITY ESTIMATION INTER AGENCY GROUP
ELD	EXPONENTIATED LOMAX DISTRIBUTION
LA1	LEHMANN ALTERNATIVE 1 METHOD
CDF	CUMMULATIVE DISTRIBUTION FUNCTION
PDF	PROBABILITY DENSITY FUNCTION
EWD	EXPONENTIATED WEIBULL DISTRIBUTION
SF	SURVIVAL FUNCTION
HF	HARZARD FUNCTION

RHF	REVERSE HAZARD FUNCTION
LA2	LEHMANN ALTERNATIVE 2 METHOD
PTM	POWER TRANSFORMED METHOD
M-OM	MARSHALL-OLKIN METHOD
APT	α -POWER TRANSFORMED METHOD
APE	α -POWER EXPONENTIAL DISTRIBUTION
AIC	AKAIKE INFORMATION CRITERION
BIC	BAYESIAN INFORMATION CRITERION
CAIC	CONSISTENT AKAIKE INFORMATION CRITERION
MOEFR	MARSHALL-OLKIN EXTENDED FRECHET DISTRIBUTION
GPD	GENERALIZED PARETO DISTRIBUTION
LIFD	LEHMANN TYPE I FRECHET DISTRIBUTION
LIIFD	LEHMANN TYPE II FRECHET DISTRIBUTION
MOFRD	MARSHALL-OLKIN FRECHET DISTRIBUTION
PTFD	POWER TRANSFORMED FRECHET DISTRIBUTION
LIPTFD	LEHMANN TYPE II- POWER TRANSFORMED FRECHET DISTRIBUTION
LIPTFD	LEHMANN TYPE I-POWER TRANSFORMED FRECHET DISTRIBUTION

MMRATE	MATERNAL MORTALITY RATE
EGGa	EXPONENTIATED GENERALIZED GAMMA
EGGu	EXPONENTIATED GENERALIZED GUMBEL
EGF	EXPONENTIATED GENERALIZED FRECHET
EGN	EXPONENTIATED GENERALIZED NORMAL
EGP	EXPONENTIATED GENERALIZED PARETO DISTRIBUTION
LIGPD	LEHMANN TYPE I GENERALIZED PARETO DISTRIBUTION
LIIGPD	LEHMANN TYPE II GENERALIZED PARETO DISTRIBUTION
WDI	WORLD DEVELOPMENT INDICATORS
UNSD	UNITED NATIONS STATISTICS DIVISION
PRHM	PROPORTIONAL REVERSED HAZARD MODEL
PHM	PROPORTIONAL HAZARD MODEL
ICD-10	INTERNATIONAL STATISTICS OF DISEASES AND RELATED HEALTH PROBLEM, 10 TH REVISION
MLE	MAXIMUM LIKELIHOOD ESTIMATE

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CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

World Health Organization (WHO) defined Maternal mortality in the 10th revision (ICD-10) of international statistics of diseases and related health problem, to be the death of a woman while pregnant or within forty two days of termination of pregnancy irrespective of duration and site of pregnancy, from any cause related to or aggravated by the pregnancy or its management but not from accidental or incidental causes. Some measures or indications of maternal mortality include Maternal Mortality Rate (MMRate), adult lifetime risk of maternal deaths among deaths of women of reproductive age (PM), and Maternal Mortality Ratio (MMR). MMR measures maternal mortality as the number of maternal deaths per 100 000 live births.

Globally, number of maternal deaths in 2015, was estimated to be 303,000 with an estimated MMR of 216. Developing regions' global maternal deaths in 2015 was 302, 000, approximately 99% of the year total. 66% of this total was accounted for by Sub-Saharan Africa while Southern Asia accounted for approximately 21.78% (66, 000). Complications in pregnancy can happen to any woman regardless of nationality but the probability of women in developing countries dying is higher as they are much less likely to receive prompt and adequate treatment. In 2000, United Nation (UN) member states initiated Millennium Development Goals (MDGS). MDG 5A targeted 75% diminution in 1990 MMR by the year 2015. For continuation, UN member states at the Sustainable Development Summit in September, 2015 adopted seventeen Sustainable Development Goals (SDGs) to be achieved by 2030. The target of SDG 3.1 is to cut down global

MMR by the year 2030 to less than 70 per 100, 000 live birth, with aim of ending preventable maternal mortality. The adoption of these development goals has led to a number of plans and strategies geared towards reducing MMR. Progress towards goals provided by studies on the trends in maternal mortality has also been monitored on a constant basis. Probability distribution provides extra information crucial when making plans and taking decisions, therefore, the selection of a reference distribution that will represent MMR becomes necessary.

Salami (2004) in the study of flow along Asa River recommended Gumbel extreme value type I and Log-Pearson Type III and probability distributions for predicting the maximum and minimum annual flow regimes respectively. Olukanmi and Salami (2008) on the basis of goodness of fit tests selected probability distribution models for predicting reservoir inflow at Kainji, Shiroro, and Jebba hydropower dams in Nigeria. Gerba et al. (2013) fitted probability distribution functions to Kaduna South Water Works' discharge variability of Kaduna River using Kolmogorov- Smirnov (K-S) goodness of fit criterion. Literature on distribution fitting abound in literature. Probability distributions are instruments for dealing with uncertainty. Information obtainable from probability distributions helps in making rational decisions, guiding the choice of optimal course of action.

Limitations in characteristics of existing probability distributions motivate generalization of distributions to improve goodness of fit and to add more flexibility (Omekam & Adejumo, 2017). Parameter induction into existing probability distributions is an approach to generalizing distributions that yield more flexible distributions that often improve goodness of fit. The exponentiated Pareto distribution was proposed by Gupta et al. (1998), and results from study showed its use in effective analysis of lifetime data. Gupta et al. (1999) introduced Exponentiated

exponential distribution which was observed as alternative to the two-parameter gamma or two-parameter Weibull distribution. Batal and Kareem (2014) generalized the exponentiated Lomax distribution obtaining a new five- parameter continuous distribution called Kumarawanmy exponentiated Lomax distribution. Likewise, this research work targets the selection of a generalized distribution for MMR.

1.2 Statement of the Problem

Maternal mortality is an outcome of adverse health and remains a major challenge in some society. The establishment of MDGs and SDGs has led to numerous studies on issues related to maternal health. Studies in recent times have focused on modelling causes of death (Foreman et al., 2012, Khah, 2006), measuring maternal mortality (Graham et al., 2008, Immpect, 2007), preventing maternal death (Harrison, 2012), reducing maternal mortality (Chang, 2008, Harrison, 2009). There has also been an increasing interest in maternal mortality trends (AbouZahr, 2011, United Nations, 2017). Although literature discussing maternal mortality and related issues abound, study on distribution fitting for global MMR is needful.

Distribution fitting is carried out to access fit of distribution(s) to data. Analysis of data without a pre knowledge of the statistical distribution that describes data may lead to misleading or irrelevant results. The need to find a distribution that represents MMR data for planning and good decision making therefore cannot be overemphasized.

On the other hand, some data may possess traits that are not accommodated by existing probability distributions of choice. These distributions are then generalized to improve flexibility and goodness of fit. The process of parameter induction into a probability distribution is a well-

known technique of generating families of generalized distributions for added flexibility and improvement of goodness of fit.

1.3 Justification for the Study

The high level of global maternal deaths led to the establishment of millennium and sustainable development goals which target reduction of preventable maternal death by 2030. A probability distribution that will describe MMR will provide extra information crucial for making decisions towards the achievement of goal. Study will provide a probability distribution for MMR that can also be incorporated into subsequent analyses. Generalization of distributions adds more flexibility and improves goodness of fit of distributions hence, a generalized distribution for MMR will provide a more flexible distribution that will improve goodness of fit.

1.4 Aim and Objectives of the Study

The aim of this research work is to derive a generalized distribution for MMR while the specific objectives are to:

- (i) generate families of generalized distributions;
- (ii) illustrate flexibility of generalized distribution;
- (iii) fit some existing distributions to MMR to determine the best fitting distribution;
- (iv) derive generalized distributions for the best fitted distribution; and
- (v) fit and assess the derived generalized distributions to MMR.

1.5 Scope of the Study

The focus of this research is the derivation of a generalized distribution for MMR and the study's

limitation is inadequacy and unavailability of maternal mortality data. Limitation could be seen in wide uncertainty range of MMR estimates provided by Maternal Mortality Inter-Agency Group (MMEIG). Data was therefore collected from fifty one countries for a period of eleven years (2004-2014). 99% of countries considered were those amongst the sixty three countries whose civil registration data were classified by MMEIG to be complete with good attribution of cause of death and the eighty five countries having registration or other types of data but whose data were characterized as not complete. There are different approaches to generalizing distributions but this study considers the parameter induction method of generating generalized distributions. Five existing parameter induction methods are studied and families of distributions with two additional parameters are derived from sequentially applying methods in permutations of the five distinct methods taking two methods at a time. Flexibility of generalized distributions is illustrated using a distribution that is limited in the shapes of its probability and hazard rate functions while families of generalized distributions are derived by introducing one or two parameters into the probability distributions of an existing distribution. Generalized distributions obtained using the best fitted existing distribution as base distribution are fitted to MMR data and the best selected to represent MMR.

1.6 Significance of the Study

The findings of study will fill existing knowledge gap by revealing the choice of a distribution for MMR that prospective researchers can incorporate in subsequent analyses. The predictive ability of recommended distribution will also be of benefit to maternal mortality trend monitoring agencies, groups and individuals in the assessment of the possibilities of achieving SDG 3.1. Resulting probability forecast of any near future outcome will provide extra information that will inform and offer guidance when plans are made and decisions are taken.

The study also provides additional information on the methods of generating families of generalized distributions with more than one additional parameter and their corresponding families. This will guide researchers in their search for suitable distribution for any variable of interest. Methods and generalized families of distributions will also be beneficial to academicians for generating flexible statistical distributions that are capable of improving goodness of fit and also useful in the analysis of different types of data.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Establishment of MDGs and SDGs has led to many studies centered on measurement, reduction, and prevention of maternal mortality. Literature on distribution fitting for global MMR appear to be scarce. This research work may be the first or one out of a few others that could not be accessed within the period of study. Introducing parameter (s) to an existing probability distribution to generate generalized distributions has received attention in recent years. Literature on this topic are so engaging with findings that reveal the interesting properties of generalized distributions and their usefulness in modelling of real life situations.

2.2 The Review and Appraisal

Luguterah (2015) proposed the application of survival analysis techniques in developing better estimates for maternal mortality. In the study, using data obtained from the Ghana Statistical Service from 2002 to 2007, author fitted the two- parameter Weibull distribution to maternal survival estimates of Ghana women and found it as the distribution that best describes maternal survival well with an R^2 estimate of 0.9714. The development goals target reduction in global MMR hence this study focusses on distribution fitting for global MMR.

Abdul-Moniem and Abdel-Hammed (2012) generated a new class of distribution called Exponentiated Lomax Distribution (ELD) by generalizing the two-parameter Lomax distribution with Cumulative Distribution Function (CDF);

$$F(x, \lambda, \theta) = 1 - (1 + \lambda x)^{-\theta} \quad \lambda, \theta > 0, x \geq 0 \quad (2.1)$$

and Probability Density function (PDF);

$$f(x, \lambda, \theta) = \theta \lambda (1 + \lambda x)^{-(\theta+1)} \quad (2.1a)$$

The shape parameter is θ and λ is the scale parameter.

The ELD has CDF to be

$$F(y, \lambda, \theta, \alpha) = [1 - (1 + \lambda y)^{-\theta}]^\alpha \quad (2.2)$$

and the corresponding PDF is given by

$$f(y, \lambda, \theta, \alpha) = \alpha \theta \lambda (1 + \lambda y)^{-(\theta+1)} [1 - (1 + \lambda y)^{-\theta}]^{\alpha-1} \quad (2.2a)$$

ELD has a new shape parameter ($\alpha > 0$) introduced by applying Lehmann Alternative 1 (LA1) parameter induction method.

Gupta et al. (1998) generalized the Pareto distribution with CDF and PDF denoted as

$$F(x) = 1 - (B/x)^k \quad (2.3)$$

$$\text{and } f(x) = kB^k / x^{k+1} \quad (2.3a)$$

respectively by introducing an extra parameter. The new distribution is called exponentiated Pareto distribution with the following CDF and PDF;

$$F(y) = (1 - (B/y)^k)^a, \quad (2.4)$$

$$f(y) = akB^k [1 - (B/y)^k]^{a-1} / y^{k+1} \quad (2.4a)$$

$(a,k) > 0$ are shape parameters and B is a scale parameter.

Another form of exponentiated Pareto distribution was obtained by authors from the modification of Pareto distribution of the second kind. CDF and PDF are given below.

$$F(y) = [1 - (1 + y)^{-\lambda}]^{\theta} \quad (2.5)$$

$$f(y) = \lambda \theta (1 + y)^{-(\lambda+1)} [1 - (1 + y)^{-\lambda}]^{\theta-1} \quad (2.5a)$$

These two distributions are also obtained from the application of LA1 parameter induction method.

Mudholkar and Scrivastava (1993) applied LA1 method introducing a new shape parameter to two-parameter Weibull distribution with the following CDF and PDF

$$F(x) = 1 - e[-(x/\lambda)^k] \quad (2.6)$$

$$f(x) = \frac{k}{\lambda(x/\lambda)^{k-1}} e[-(x/\lambda)^k] \quad (2.6a)$$

Where $(k, \lambda) > 0$ are respectively scale and shape parameters and $x > 0$. The proposed model was referred to as Exponentiated Weibull Distribution (EWD) with an additional shape parameter.

For $y > 0$, EWD's CDF and corresponding PDF are respectively

$$F(y) = [1 - e(-(y/\lambda)^k)]^{\alpha} \quad (2.7)$$

$$f(y) = \frac{\alpha k}{\lambda(y/\lambda)^{k-1}} (1 - e(-(y/\lambda)^k))^{\alpha-1} e[-(y/\lambda)^k] \quad (2.7a)$$

When the baseline distribution is Weibull distribution having CDF as

$$F(x) = 1 - e(-\lambda x)^k, \quad (2.7b)$$

CDF of the proposed EWD becomes

$$F(x) = (1 - e(-\lambda x)^k)^\alpha \quad (2.7c)$$

Rady et al. (2016) extended the two-parameter Lomax distribution using power transformation parameter induction method, $X=T^{1/B}$, and called it the power Lomax distribution.

The CDF of two-parameter Lomax distribution and PDF are;

$$F(x) = [1 - (1 + x / \lambda)^{-\alpha}] \quad (2.8)$$

$$f(x) = \alpha \lambda^{-1} (1 + x / \lambda)^{-(\alpha+1)} \quad x > 0, \alpha, \lambda > 0 \quad (2.8a)$$

While those of the power Lomax distribution are given as

$$F(y) = [1 - \lambda^\alpha (y^\beta + \lambda)^{-\alpha}] \quad (2.9)$$

$$f(y) = \alpha \beta y^{\beta-1} \lambda^\alpha (\lambda + y^\beta)^{-\alpha-1} \quad (2.9a)$$

Marshall and Olkin (1997) presented method for introducing parameter to an existing distribution. CDF, PDF, Survival Function (SF), Reversed Hazard Function (RHF) and Hazard Function (HF), of this class of generalized distributions for $x \in \mathfrak{R}$ and any $\theta \in (0, \infty)$ are respectively given below;

$$F(x; \theta) = \frac{G(x)}{1 - (1 - \theta)[1 - G(x)]} = \frac{G(x)}{\theta + (1 - \theta)F(x)} \quad (2.10)$$

$$f(x; \theta) = \frac{\theta g(x)}{(1 - (1 - \theta)[1 - G(x)])^2} = \frac{\theta g(x)}{[\theta + (1 - \theta)F(x)]^2} \quad (2.10a)$$

$$\bar{F}(x, \theta) = \frac{\theta \bar{G}(x)}{1 - (1 - \theta)[1 - G(x)]} = \frac{\theta \bar{G}(x)}{\theta + (1 - \theta)F(x)} \quad (2.10b)$$

$$h(x, \theta) = \frac{h(x)}{1 - (1 - \theta)[1 - G(x)]} = \frac{h(x)}{\theta + (1 - \theta)F(x)} \quad (2.10c)$$

$$r(x, \theta) = \frac{\theta r(x)}{1 - (1 - \theta)[1 - G(x)]} = \frac{\theta r(x)}{\theta + (1 - \theta)F(x)} \quad (2.10d)$$

The additional parameter is θ while $G(x)$ and $g(x)$ are the CDF and PDF of a base distribution.

Gupta and Kundu (2009) provided a new interpretation for this family of generalized distribution.

Y 's distribution function is proportional odds model of that of X , having α as the proportionality constant, if

$$\varphi_Y(x) = \alpha \varphi_X(x), \text{ for all } x \quad (2.11)$$

(2.11) can be written equivalently as

$$\varphi_Y(x) = \frac{\bar{F}_Y(x)}{F_Y(x)} = \alpha \frac{\bar{F}_X(x)}{F_X(x)} = \alpha \varphi_X(x) \quad (2.11a)$$

or

$$\frac{r_Y(x)}{h_Y(x)} = \alpha \frac{r_X(x)}{h_X(x)} \quad (2.11b)$$

Therefore, the PDF and CDF of Y are then

$$f_Y(x) = \frac{\alpha f_X(x)}{(F_X(x) + \alpha \bar{F}_X(x))^2} = \frac{\alpha f_X(x)}{[1 - (1 - \alpha)(1 - F_X(x))]^2} \quad (2.11c)$$

and

$$F_Y(x) = \frac{F_X(x)}{\{F_X(x) + \alpha \bar{F}_X(x)\}} = \frac{F_X(x)}{1 - (1 - \alpha)(1 - F_X(x))} \quad (2.11d)$$

Tahir and Nadarajah (2015) discussed some existing proven families of generalized distributions comprising of Marshall-Olkin extended family, exponentiated family, beta-generated family, Kumaraswamy-generalized family, and McDonald-generalized family. Amongst the generalized families of distributions studied are the exponentiated families having an additional parameter ($a > 0$) obtainable from Lehmann Alternatives (Lehmann, 1953). For any given baseline distribution with CDF and PDF represented as $G(t)$ and $g(t)$, the exponentiated family of distributions obtained by using Lehmann Alternative 1 (LA1) has the properties below;

$$PDF = f(t) = ag(t)G(t)^{a-1} \quad (2.12)$$

$$CDF = F(t) = G(t)^a \quad (2.12a)$$

$$SF = \bar{F}(t) = 1 - G(t)^a \quad (2.12b)$$

$$HF = h(t) = ag(t)G(t)^{a-1}(1 - G(t)^a)^{-1} \quad (2.12c)$$

$$RHF = r(t) = ag(t)G(t)^{-1} \quad (2.12d)$$

The functions of the exponentiated family of distributions obtained from applying Lehmann Alternative 2 (LA2) are as follows;

$$PDF = f(t) = ag(t)(1 - G(t))^{a-1} \quad (2.13)$$

$$CDF = F(t) = 1 - (1 - G(t))^a = 1 - \bar{G}(t)^a \quad (2.13a)$$

$$SF = \bar{F}(z) = (1 - G(t))^a = \bar{G}(t)^a \quad (2.13b)$$

$$HF = h(t) = ag(t)(1 - G(t))^{-1} \quad (2.13c)$$

$$RHF = r(t) = ag(t)(1 - G(t))^{a-1} [1 - (1 - G(t))^a]^{-1} \quad (2.13d)$$

Tahir & Nadarajah (2015) also provided list of literature on these proven existing generalized families of distributions. Authors also propounded some generalizations of Kumaraswamy generalized family of distributions and Marshall-Olkin extended family.

An extension of the Marshall and Olkin parameter induction method (MOM) was also proposed and authors claim to have obtained the new generalized Marshall-Olkin extended family of distributions by applying LA2 approach to the Survival Function of the Marshall-Olkin extended family.

The PDF, CDF and SF of new generalized family are as follows;

$$f^{GMO}(t) = \theta \left[\frac{ag(t)}{[1 - \bar{a}\bar{G}(t)]^2} \right] \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^{\theta-1} \quad (2.14)$$

Simplified by authors to be

$$f^{GMO}(t) = \frac{a\theta g(t)G(t)^{\theta-1}}{1 - \bar{a}\bar{G}(t)} = \frac{a\theta g(t)G(t)^{\theta-1}}{a + \bar{a}G(t)}, \quad (2.14a)$$

$$F^{GMO}(t) = \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta, \quad (2.14b)$$

$$\bar{F}^{GMO}(t) = 1 - \left[1 - \frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta \quad (2.14c)$$

$-\infty < t < \infty$, $a > 0$. The additional shape parameter is $\theta > 0$. Notwithstanding, review reveals that it was rather obtained by applying LA1 approach and that (2.14) simplifies rather to

$$f^{GMO}(t) = \frac{a\theta g(t)G(t)^{\theta-1}}{[1 - \bar{a}\bar{G}(t)]^{\theta+1}} = \frac{a\theta g(t)G(t)^{\theta-1}}{[a + \bar{a}G(t)]^{\theta+1}} \quad (2.14d)$$

Authors also gave the SF of an existing Marshall-Olkin generalized family obtained by Jayakuma & Matthew (2008) stating that it was obtained from applying the LA1 approach to be

$$\bar{F}^{GMO}(t) = \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta \quad (2.14e)$$

$a > 0$, $-\infty < t < \infty$, and the additional shape parameter is $\theta > 0$. The corresponding CDF and PDF are respectively

$$F^{GMO}(t) = 1 - \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^\theta, \quad (2.14f)$$

and

$$f^{GMO}(t) = \theta \left[\frac{a\bar{G}(t)}{1 - \bar{a}\bar{G}(t)} \right]^{\theta-1} \left\{ \frac{ag(t)}{[1 - \bar{a}\bar{G}(t)]^2} \right\} \quad (2.14g)$$

It was observed that this generalized distribution by Jayakuma and Mathew (2008) was rather obtained by applying LA2 approach. The new generalized Marshall-Olkin extended family of distributions proposed by Tahir and Nadarajah (2015) and that by Jayakuma and Matthew (2008) agree with those obtained in this study by sequential application of MOM and LA1 (former), and

also MOM and LA2 (later).

Gupta and Kundu (2009) presented alternative interpretations to the exponentiated families that can be obtained from applying LA1 and LA2, those also considered by Tahir and Nadarajah (2015), observing them to be Proportional Reversed Hazard Model (PRHM) and Proportional Hazard Model (PHM) respectively. The Power Transformed Method (PTM) of parameter induction was also discussed by authors. Given X to be a non-negative random variable, considering a new random variable Y with an additional parameter ($\alpha > 0$) such that $Y = X^{1/\alpha}$, the corresponding functions of generalized distribution derived from PTM method are as follows;

$$PDF = f_Y(x) = \alpha x^{\alpha-1} f_X(x^\alpha) \quad (2.15)$$

$$CDF = F_Y(x) = F_X(x^\alpha) \quad (2.15a)$$

$$SF = \bar{F}_Y(x) = 1 - F_X(x^\alpha) \quad (2.15b)$$

$$HF = h_Y(x) = \alpha x^{\alpha-1} h_X(x^\alpha) \quad (2.15c)$$

$$RHF = r_Y(x) = \alpha x^{\alpha-1} r_X(x) \quad (2.15d)$$

Three other models obtained from parameter induction were also discussed; proportional cumulants model, Azzalini's skewed model, and proportional odds model earlier discussed. Gupta and Kundu (2009) in the same study did a comprehensive discussion of structural properties of these six skewed distributions obtainable from parameter induction. They applied six methods to exponential distribution resulting to five distinct generalizations of the exponential distribution as the sixth model (PHM) obtained also belongs to the exponential

family. Authors observed that successive combination of two or more than two methods, may introduce more than one parameter

and thus obtained generalized exponential distributions with two shape parameters. This research generalizes some of these observations in methods and families of generalized distributions proposed and provides more generalized families that will suffice for present purposes.

Mahdavi and Kundu (2017) proposed the α -power transformation (APT) parameter induction method. Suppose the CDF of a continuous random variable X is given as $F(x)$, then the α -power transformation of $F(x)$ for $x \in \mathbb{R}$ and if $\alpha > 0$ and $\alpha \neq 1$, is defined as follow;

$$f_{APT}(x) = \frac{\log \alpha}{\alpha - 1} f(x) \alpha^{F(x)} \quad (2.16)$$

$$F_{APT}(y) = \frac{\alpha^{F(x)} - 1}{\alpha - 1} \quad (2.16a)$$

$$\bar{F}_{APT}(y) = \frac{\alpha}{\alpha - 1} (1 - \alpha^{F(x)-1}) = \frac{\alpha - \alpha^{F(x)}}{\alpha - 1} \quad (2.16b)$$

$$h_{APT}(y) = f(x) \frac{\alpha^{F(x)-1}}{1 - \alpha^{F(x)-1}} \log(\alpha) = \frac{\log(\alpha) f(x) \alpha^{F(x)}}{\alpha - \alpha^{F(x)}} \quad (2.16c)$$

$$r_{APT}(y) = \frac{\log(\alpha) f(x) \alpha^{F(x)}}{\alpha^{F(x)} - 1} \quad (2.16d)$$

For $\alpha = 1$, APT distribution reduces to the base distribution.

Using the exponential distribution as base distribution, authors applied the APT method and obtained a distribution called the two-parameter APE distribution. For $y > 0$ and $\alpha = 1$, PDF of a two-parameter APE distribution is

$$f(y; \alpha, \lambda) = \lambda e^{-\lambda y} \quad (2.16e)$$

For $\alpha \neq 1$, the PDF becomes

$$f(y; \alpha, \lambda) = \frac{\log \alpha}{\alpha - 1} \alpha^{1-e^{-\lambda y}} \lambda e^{-\lambda y} \quad (2.16f)$$

The CDF of Y for $y > 0$ and $\alpha = 1$ is given as

$$F(y; \alpha, \lambda) = 1 - e^{-\lambda y}$$

And for $\alpha \neq 1$, the CDF becomes

$$F(y; \alpha, \lambda) = \frac{\alpha^{(1-e^{-\lambda y})} - 1}{\alpha - 1} \quad (2.16g)$$

The MLEs of APE distributions' parameters could not be obtained explicitly but through non-linear optimization. To show performance of proposed method, authors fitted the APE distribution to intervals in days between 109 successive coal-mining disasters in Great Britain. For comparison purposes, four other two-parameter models were fitted to the same dataset (gamma distribution, Weibull distribution, exponentiated exponential distribution (Gupta & Kundu, 1999, Gupta et al., 1998), and weighted exponential distribution (Gupta & Kundu, 2009)). Based on log-likelihood value and K-S statistics, APE fitted dataset better than others and so was recommended to be a possible alternative to gamma, weibull or exponentiated exponential models.

Mahdavi & Kundu (2017) also introduced three-parameter APE distribution having PDF for $\alpha = 1$ and $\alpha \neq 1$ are respectively given as

$$f(y; \alpha, \lambda, \mu) = \lambda e^{-\lambda(y-\mu)} \quad (2.16h)$$

$$f(y; \alpha, \lambda, \mu) = \frac{\log \alpha}{\alpha - 1} \lambda e^{-\lambda(y-\mu)} \alpha^{1-e^{-\lambda(y-\mu)}} \quad (2.16i)$$

The CDF for $\alpha = 1$ and $\alpha \neq 1$ are also respectively given as

$$F(y; \alpha, \lambda, \mu) = 1 - e^{-\lambda(y-\mu)} \quad (2.16j)$$

$$F(y; \alpha, \lambda, \mu) = \frac{\alpha^{1-e^{-\lambda(y-\mu)}} - 1}{\alpha - 1} \quad (2.16k)$$

The scale, shape, and location parameters are given as $\lambda > 0$, $\alpha > 0$, $-\infty < \mu < \infty$ respectively.

Three-parameter APE was fitted to the data set representing single carbon fibers' strength and impregnated 1000-carbon fiber tows. Three-parameter APE model provided a good fit to data set based on K-S statistics and corresponding P-value.

Cordeiro et al. (2013) proposed a method of introducing two parameters to a continuous distribution. They interpreted new class of exponentiated generalized distributions obtainable from the new method as a double construction of Lehmann alternatives. The PDF of the new class having two new parameters ($\alpha > 0$ and $\beta > 0$) is given as

$$f(x) = \alpha \beta \{1 - G(x)\}^{\alpha-1} [1 - \{1 - G(x)\}^\alpha]^{\beta-1} g(x) \quad (2.17)$$

Authors discussed some models obtainable from method which includes Exponentiated Generalized Gamma (EGGa), Exponentiated Generalized Gumbel (EGGu), Exponentiated Generalized normal (EGN), and Exponentiated Generalized Frechet (EGF) amongst others. Some mathematical properties of new class were derived and was detailed. Maximum likelihood

estimation method for model's parameters was investigated. Four of the special models obtained were fitted to four different real datasets and compared with those of three other sub-models.

For the ethylene data, EGN model was compared with exponentiated normal, normal and Lehmann II normal distributions. For the wheaton river data, EGGu was compared with exponentiated Gumbel, Lehmann II Gumbel, and Gumbel distributions. For the stress level data, EGF model was compared with the exponentiated Frechet, Lehmann II Frechet, and the Frechet distributions. For the carbon data, EGGa distribution was compared with exponentiated gamma, Lehmann II gamma, and gamma models. The results were in favour of proposed models based on Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and likelihood ratio test which led to the conclusion that exponentiated generalized distributions are very suitable to data. This new class of distributions identifies with one of the families obtained in present study by a sequential application of LA2 and LA1 parameter induction method.

Krishna et al. (2013) introduced the Marshall-Olkin extended Frechet distribution. Authors generalized the two-parameter Frechet distribution with CDF denoted as

$$F(x) = e^{-(\delta/x)^\beta} \quad (2.18)$$

β and δ are respectively the shape and scale parameters for $x > 0$, and $\delta, \beta > 0$.

Marshall-Olkin method (MOM) of parameter induction was applied to the two-parameter Frechet distribution to obtain the new distribution with an additional parameter (α) referred to as Marshall-Olkin Frechet (MOEFR) distribution and denoted by MOEFR (δ, β, α). The PDF and HF of MOEFR family are represented below:

$$g(x, \delta, \beta, \alpha) = \frac{\alpha\beta(\delta/x)^{\beta+1}e^{-(\delta/x)^{\beta}}}{\delta[\alpha + (1 - \alpha)e^{-(\delta/x)^{\beta}}]^2} \quad (2.18a)$$

$$h(x) = \frac{\beta(\delta/x)^{\beta+1}e^{-(\delta/x)^{\beta}}}{\delta(\alpha + (1 - \alpha)e^{-(\delta/x)^{\beta}})[1 - e^{-(\delta/x)^{\beta}}]} \quad (2.18b)$$

The PDF is unimodal and HF is upside-down bathtub shaped.

Suppose random variable X has MOEFR (δ, β, α) distribution, then random variable $Y = X/\delta$ has MOEFR ($1, \beta, \alpha$) distribution. Thus, the n^{th} moment of the the random variable Y after substituting $u = y^{\beta}$ can be represented thus;

$$E(Y^n) = \alpha\beta \int_0^{\infty} \frac{y^{n-\beta-1}e^{-(y^{-\beta})}}{(\alpha + (1 - \alpha)e^{-(y^{-\beta})})^2} dy = \alpha \int_0^{\infty} \frac{u^{-n/\beta}e^{-u}}{(\alpha + (1 - \alpha)e^{-u})^2} du \quad (2.18c)$$

If $\alpha > 1/2$, then using the expression

$$\frac{1}{(\alpha + (1 - \alpha)e^{-u})^2} = \frac{1}{\alpha^2} \sum_{i=1}^{\infty} i \left(\frac{\alpha - 1}{\alpha} \right)^{i-1} e^{-(i-1)u}, \quad (2.18d)$$

n^{th} moment of Y for $n < \beta$ and $\alpha > 1/2$ becomes

$$E(Y^n) = \frac{1}{\alpha} \sum_{i=1}^{\infty} (\alpha - 1/\alpha)^{i-1} i^{n/\beta} \Gamma(1 - n/\beta) \quad (2.18e)$$

$$= \alpha^{-1} (1 - n / \beta) \Phi(\alpha - 1 / \alpha, -n / \beta, 1), \quad (2.18f)$$

where Φ is Lench transcendental function defined as

$$\Phi(x, s, v) = \sum_{n=0}^{\infty} x^n / (n + v)^2 \quad \text{if } 0 < \alpha < 2, \quad (2.18g)$$

Then using expansion

$$\frac{1}{(\alpha + (1 - \alpha)e(-u))^2} = \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} (1 - e^{-u})^{i-1} \quad (2.18h)$$

$$= \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j e^{-ju}, \quad (2.18i)$$

Then for $n < \beta$ and $0 < \alpha < 2$, the n^{th} moment was obtained as

$$E(Y^n) = \alpha \Gamma(1 - n / \beta) \sum_{i=1}^{\infty} i(1 - \alpha)^{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j (1 + j)^{n / \beta - 1} \quad (2.18j)$$

MOEFR distribution q^{th} quantile is given by

$$x_q = G^{-1}(q) = \delta[\log(1 - (1 - \alpha)q / \alpha q)]^{-1 / \beta}, \quad 0 \leq q \leq 1, \quad (2.18k)$$

The inverse distribution function is $G^{-1}(\cdot)$ and median of the distribution is hence

$$\text{Median}(X) = \delta[\log(\alpha + 1 / \alpha)]^{-1 / \beta} \quad (2.18l)$$

Since the moments of a random variable with MOFR distribution could not be obtained in closed form, other methods were considered for estimating values of parameters upon which the maximum likelihood estimation method was then applied for estimating unknown values of the parameters of distribution.

The corresponding log-likelihood function of an observed sample, $x_1, x_2, \dots, x_n$ is

$$\begin{aligned} \log L = & n \log \alpha + n \log \beta + (\beta + 1) \sum_{i=1}^n \log(\delta / x_i) - \sum_{i=1}^n (\delta / x_i)^\beta - \\ & n \log \delta - 2 \sum_{i=1}^n \log(\alpha + (1 - \alpha)e^{-(\delta / x_i)^\beta}) \end{aligned} \quad (2.18m)$$

The normal equations are;

$$\begin{aligned} \frac{\partial \log L}{\partial \delta} = & \frac{n(\beta + 1)}{\delta} - \beta \delta^{\beta - 1} \sum_{i=1}^n x_i^{-\beta} - \frac{n}{\delta} + 2(1 - \alpha)\beta \delta^{\beta - 1} \times \\ & \sum_{i=1}^n \frac{e^{-(\delta / x_i)^\beta}}{x_i^\beta (\alpha + (1 - \alpha)e^{-(\delta / x_i)^\beta})} = 0 \end{aligned} \quad (2.18n)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \beta} = & \frac{n}{\beta} + \sum_{i=1}^n \log(\delta / x_i) - \delta^\beta \sum_{i=1}^n x_i^{-\beta} \log(\delta / x_i) + 2(1 - \alpha)\delta^\beta \times \\ & \sum_{i=1}^n \frac{e^{-(\delta / x_i)^\beta} \log(\delta / x_i)}{x_i^\beta (\alpha + (1 - \alpha)e^{-(\delta / x_i)^\beta})} = 0 \end{aligned} \quad (2.18o)$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1 - e(-(\delta / x_i)^\beta)}{\alpha + (1 - \alpha)e(-(\delta / x_i)^\beta)} = 0 \quad (2.18p)$$

Parameter estimates are obtained by solving these non-linear system of equations.

MOEFR distribution was compared with two-parameter Frechet distribution using data set consisting of 72 observations of survival times of injected guinea pigs with different doses of tubercle bacilli. The good fit provided by MOEFR based on K-S distances and associated P-values, AIC, and BIC informed its recommendation as a competitive model to the Frechet distribution.

Nadarajah and Kotz (2003) introduced exponentiated Frechet distribution that sought to generalize existing Frechet distribution with CDF

$$F(x) = \exp\{-(\sigma / x)^\lambda\} \quad \kappa > 0, \sigma, \lambda > 0. \quad (2.19)$$

The CDF and PDF of the exponentiated Frechet distribution with an additional parameter ($\alpha > 0$) are given below;

$$F(x) = 1 - [1 - \exp\{-(\sigma / x)^\lambda\}]^\alpha \quad (2.20)$$

$$f(x) = \alpha \lambda \sigma^\lambda [1 - \exp\{-(\sigma / x)^\lambda\}]^{\alpha-1} \frac{1}{x^{(1+\lambda)}} \exp\{-(\sigma / x)^\lambda\} \quad (2.20a)$$

(2.20a) is derived in this study as the PDF of Lehman Type II Frechet distribution from the application of LA2 parameter induction method.

Using the series representation below the PDF of the exponentiated Frechet distribution was

expressed in a mixture form.

$$(1 + z)^a = \sum_{j=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} \frac{z^j}{j!} \quad (2.20b)$$

The mixture form of PDF is

$$f(x) = \lambda \sigma^\lambda \Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{K! \Gamma(\alpha-k)} x^{-(1+\lambda)} \exp\{-(k+1)(\sigma/x)^\lambda\} \quad (2.20c)$$

Authors showed the unimodality of $f(x)$ at $x = x_0$ with $f(0)=f(\infty) = 0$, where x_0 is the solution of

$$\frac{\partial \log f(x)}{\partial x} = \frac{\lambda \sigma^\lambda}{x(1+\lambda)} \left[1 + \frac{1-\alpha}{\exp\{(\sigma/x)^\lambda\} - 1} \right] - \frac{1+\lambda}{x} = 0 \quad (2.20d)$$

Futhermore, $x_0 > [\alpha \lambda \sigma^\lambda / \{1 + \alpha \lambda\}]^{1/\lambda}$ if $0 < \alpha \leq 1$ and $x_0 < \sigma (\log \alpha)^{-1/\lambda}$ if $\alpha > 1$.

Exponentiated Frechet distribution hazard rate function takes the form

$$h(x) = \frac{\alpha \lambda \sigma^\lambda \exp[-(\sigma/x)^\lambda] x^{-(1+\lambda)}}{1 - \exp\{-(\sigma/x)^\lambda\}} \quad (2.20e)$$

(2.20e) exhibits a single mode at $x = x_0$ with $h(0) = h(\infty) = 0$.

The n^{th} moment of exponentiated Frechet distribution can be represented as follows:

$$E(X^n) = \int_0^\infty x^{n-1} [1 - F(x)] dx = \int_0^\infty x^{n-1} [1 - \exp\{-(\sigma/x)^\lambda\}]^\alpha dx \quad (2.20f)$$

On setting $y = (\sigma/x)^\lambda$, (2.89) is reduced to

$$E(X^n) = \frac{\sigma^n}{\lambda} \int_0^\infty y^{-(n/\lambda + 1)} \{1 - \exp(-y)\}^\alpha dy \quad (2.20g)$$

The integral in (2.20g) converges if $\alpha > n/\lambda$ but (2.20g) does not have a closed form.

For $\alpha = 1$, Exponentiated Frechet distribution reduces to the two-parameter Frechet distribution.

Authors considered the maximum likelihood method of parameter estimation. The log-likelihood for a random sample $x_1, \dots, x_n$ from (2.20a) is:

$$\begin{aligned} \log L(\sigma, \lambda, \alpha) = & n \log(\alpha \lambda \sigma^\lambda) + (\alpha - 1) \sum_{i=1}^n \log[1 - \exp\{-(\sigma/x_i)^\lambda\}] \\ & - (1 + \lambda) \sum_{i=1}^n \log x_i - \sigma^\lambda \sum_{i=1}^n x_i^{-\lambda} \end{aligned} \quad (2.20h)$$

The first order derivatives of (2.20h) with respect to the three parameters are as follows:

$$\frac{\partial \log L}{\partial \sigma} = \frac{n\lambda}{\sigma} + (\alpha - 1) \lambda \sigma^{\lambda-1} \sum_{i=1}^n \frac{\exp\{-(\sigma/x_i)^\lambda\}}{x_i^\lambda [1 - \exp\{-(\sigma/x_i)^\lambda\}]} - \lambda \sigma^{\lambda-1} \sum_{i=1}^n x_i^{-\lambda}, \quad (2.20i)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + n \log \sigma + (\alpha - 1) \sigma^\lambda \sum_{i=1}^n \frac{\log(\sigma/x_i) \exp\{-(\sigma/x_i)^\lambda\}}{x_i^\lambda [1 - \exp\{-(\sigma/x_i)^\lambda\}]} -$$

$$\sum_{i=1}^n \log x_i - \sum_{i=1}^n \log(\sigma / x_i)(\sigma / x_i)^\lambda, \quad (2.20j)$$

and

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log[1 - \exp\{-(\sigma / x_i)^\lambda\}] \quad (2.20k)$$

Setting (2.20i), (2.20j), and (2.20k) to zero and then solving simultaneous equations produces MLEs of parameters.

Mathematical properties of distribution were provided however, an application of distribution to data will serve to show performance of model.

Adeyemi & Adebajni (2004) introduced the Exponentiated Generalized Pareto (EGP) distribution as a three parameter generalized Pareto distribution. Given the two-parameter generalized Pareto distribution (GPD) with CDF given as:

$$F(x) = 1 - \left(1 - \frac{k}{\alpha} x\right)^{1/k}; \quad k \neq 0$$

$$= 1 - \exp(-(x / \alpha)); \quad k = 0 \quad (2.21)$$

For $0 \leq x \leq \infty$, $k \leq 0$ and $0 \leq x \leq \alpha / k$ for $k > 0$. Where α and k are respectively the scale and shape parameters. Adeyemi & Adebajni defined CDF of new EGP distribution by

$$F(x) = \left[1 - \left(1 - \frac{k}{\alpha} x\right)^{1/k}\right]^\beta \quad \text{for } \beta > 0, k \neq 0 \quad (2.21a)$$

$$= [1 - \exp(-x / \alpha)]^\beta; \quad \text{for } \beta > 0, k = 0 \quad (2.21b)$$

The corresponding PDF is given as:

$$f(x) = \frac{\beta}{\alpha} \left[1 - \left(1 - \frac{k}{\alpha} \right)^{1/k} \right]^{\beta-1} \left(1 - \frac{k}{\alpha} x \right)^{1/k-1}; \quad k \neq 0 \quad (2.21c)$$

$$= \frac{\beta}{\alpha} [1 - \exp(-x / \alpha)]^{\beta-1} \exp(-x / \alpha); \quad k = 0 \quad (2.21d)$$

The generalized Pareto is a special case of (2.21c and 2.21d) for $\beta = 1$

The moments, kurtosis, and moment generating functions of EGP were presented. The moment generating function is not in closed form and moments can be obtained by fixing k and β . Authors stated that the unimodality and multimodality of $f(x)$ and $h(x)$ respectively are verifiable from elementary calculations and therefore gave no mathematical proof nor diagrams to authenticate claim. Flexibility of EGP remains an interesting property and could be demonstrated by making comparisons between EGP and the base distribution.

Krishna et al. (2013) gave mathematical properties of MOEFR distribution and applied model to 72 observations of survival times of injected guinea pigs with different doses of tubercle bacilli and found it to be a competitive model to the Frechet distribution notwithstanding, application of model to other types of data will serve to buttress findings and set sufficient precedence to conclusions.

Gupta and Kundu (2009) and Tahir and Nadarajah (2015) discussed some existing parameter

induction methods but the former gave new interpretations to methods and also gave detailed structural properties of generalized families obtained from methods which included r^{th} - moment, and their corresponding L moments, probability density and hazard function shapes relative to those of the base distribution. Gupta and Kundu (2009) also gave basic inferential procedures for parameters of models obtainable from methods. Tahir and Nadarajah (2015) provided more lists of contributed literature on families of generalized distributions than Gupta and Kundu (2009) but both studies provided profound information on the subject.

All authors of literature reviewed proposed new generalized distributions but only Krishna et al., Mahdavi and Kundu (2017), and Cordeiro et al. (2017) applied proposed models to data. Krishna et al. (2013) applied MOEFR to one data set, Mahdavi and Kundu (2017) applied APE to two different data sets, while Cordeiro et al. (2013) applied special models derived from new proposed generalized family to four data sets. Results from applications of new models by these authors validated findings and give strong evidence of the need of introducing parameters in probability distributions for modelling of real life situations.

Other generalized families of distributions include Beta-Generated family of distributions/Beta Logit family (Jones (2004), Eugene et al., (2002)), Ferreira and Steel's Skewed family (Ferreira & Steel, 2006), Azzalini's Skewed family (Azzalini, 1985), transmuted Family (Shaw and Buckley, 2007), Gupta and Gupta's skewed family (Gupta & Gupta, 2008), Gamma-Generated (GG) family (Zografos & Balakrishnan, 2009), Transformed-Transformer (T-X) family (Alzaatreh et al., 2013) and Kumaraswamy (1980). For details on these generalized families, references should be made to Lee et al. (2013), Tahir & Nadarajah (2015), and Gupta & Kundu (2009).

Literature on generalized distributions include The generalized Gompert Distribution (El-Gohary,

et al., 2013), Generalized Rayleigh Distribution (Abd-Elfattah, 2011), Exponentiated Beta (Nadarajah, 2005), Exponentiated Gumbel Distribution (Nadarajah, 2006), Generalized Log-Logistic Distribution (Rao et al., 2013), Generalized Frechet Distribution (Ab-Elfattah et al., 2010), Marshall-Olkin Exponentiated Weibull Distribution (Tibork et al., 2015), Exponentiated-Weibull Lifetime Distribution (Cordeiro et al., 2014).

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Introduction

Study targets selecting a generalized distribution for MMR. The limitations encountered were non-availability and unreliability of maternal mortality data and also non convergence of optimization routines. Limitations were taken into account during data collection and while deciding the procedures to adopt.

3.2 Data Collection

Data on the number of maternal death and live birth were collected for fifty one countries including twenty one developing countries for a period of eleven years (2004 to 2014). Country inclusion criterion was civil registration data characterization by MMEIG. The goal of using the country inclusion criterion is to avoid using data with wide range of uncertainty. Forty two out of the fifty one countries included were countries whose civil registration data were classified as complete, and with good attribution of cause of death while seven were from countries classified as not having good complete registration data but where registration or other types of data are available. The remaining two countries (Cyprus and Malaysia) had no nationally representative data on maternal death but their live birth data for the years considered were complete. MMEIG estimates were used for the two countries. The study would have used a larger sample size but sample size was a function of criterion stated. Secondary data was used and the sources are given below:

Number of maternal death: World Development Indicators (WDI)

Number of live birth: Eurostat and United Nations Statistics Division (UNSD)

3.3 Methodology

Generalized families of distributions having one additional parameter were obtained by applying five parameter induction methods: LA1, LA2, MOM, PTM, and APT while the generalized families with two additional parameters were derived from sequentially applying methods in permutations of five distinct parameter induction methods (those aforementioned) taking two methods at a time. Each parameter induction method was also applied twice and observations made. Flexibility of generalized distributions was then illustrated by showing the effect of additional parameter in the shapes of probability density and hazard functions of two generalizations of GPD.

Next, the shape of data was ascertained using histogram plot. The shape revealed in the plot suggested plausible existing distributions that were then fitted to data. K-S goodness of fit test was employed in determining the goodness of fit of plausible distributions and AIC was subsequently applied to select the best model. The best fitted model was generalized to produce six generalized distributions belonging to some of the distinct generalized families earlier derived from parameter induction methods. Goodness of fit test was then carried out for the four generalized distributions with one additional parameter and two generalized distributions with two additional parameters obtained from generalizing the best fitted model. Generalized distribution for MMR was finally selected from these six generalized distributions using K-S distances and AIC. Easy Fit and R software were employed.

3.4 Methods of Data Analysis

Methods for data analysis are outlined in the following sub-sections in view of specific objectives earlier stated.

3.41 Generation of Families of Generalized Distributions

Let $f(x)$, $F(x)$, $s(x)$, $h(x)$, and $r(x)$ be respective denotations of the following functions of a base continuous random variable, X ;

Probability Density Function, Cumulative Distribution, Survival Function, Hazard Function, and the Reversed Hazard Function.

Given five parameter induction methods; LA1, LA2, MOM, PTM, and APT, consider a generalized continuous random variable Y , obtained by introducing a parameter ($c > 0$) to the distribution of the base random variable X , and let Z be another continuous variable belonging to a family of generalized distributions obtained by introducing another parameter ($t > 0$) to Y (Omekam & Adejumo, 2017). The functions of Y and Z will have the same notations as those given for X except for the interchange of letters (Y , and Z for X).

Method 1: Application of one parameter induction method

Family 1: Lehmann Type I Family

Applying (2.12) through (2.12d), statistical functions of family are given below:

$$f_Y(x) = cf(x)F(x)^{c-1} \quad (3.1)$$

$$F_Y(x) = F(x)^c \quad (3.1a)$$

$$\bar{F}_Y(x) = 1 - F(x)^c \quad (3.1b)$$

$$h_Y(x) = cf(x)F(x)^{c-1}(1-F(x)^c)^{-1} \quad (3.1c)$$

$$r_Y(x) = cf(x)F(x)^{-1} \quad (3.1d)$$

Family 2: Lehmann Type II Family

Functions of family derived from (2.13) through (2.13d) are as follows

$$f_Y(x) = cf(x)[1-F(x)]^{c-1} = cf(x)[\bar{F}(x)]^{c-1} \quad (3.2)$$

$$F_Y(x) = 1 - [1-F(x)]^c = 1 - \bar{F}(x)^c \quad (3.2a)$$

$$\bar{F}_Y(x) = [1-F(x)]^c = \bar{F}(x)^c \quad (3.2b)$$

$$h_Y(x) = cf(x)[1-F(x)]^{-1} = cf(x)[\bar{F}(x)]^{-1} \quad (3.2c)$$

$$r_Y(x) = cf(x)[1-F(x)]^{c-1}(1-[1-F(x)]^c)^{-1} \quad (3.2d)$$

Family 3: Marshall-Olkin Family

From (2.10) through (2.10d) functions are obtained as follows

$$F_Y(x) = \frac{F(x)}{1-(1-c)[1-F(x)]} \quad (3.3)$$

$$f_Y(x) = \frac{cf(x)}{(1-(1-c)[1-F(x)])^2} \quad (3.3a)$$

$$\bar{F}_Y(x) = \frac{c\bar{F}(x)}{1-(1-c)[1-F(x)]} \quad (3.3b)$$

$$h_Y(x) = \frac{h(x)}{1 - (1 - c)[1 - F(x)]} \quad (3.3c)$$

$$r_Y(x) = \frac{cr(x)}{1 - (1 - c)[1 - F(x)]} \quad (3.3d)$$

Family 4: Power Transformed Family

Applying (2.15) through (2.15d), statistical functions of family are given below:

$$f_Y(x) = cx^{c-1}f(x^c) \quad (3.4)$$

$$F_Y(x) = F(x^c) \quad (3.4a)$$

$$\bar{F}_Y(x) = 1 - F(x^c) = \bar{F}(x^c) \quad (3.4b)$$

$$h_Y(x) = cx^{c-1}h(x^c) \quad (3.4c)$$

$$r_Y(x) = cx^{c-1}r(x^c) \quad (3.4d)$$

Family 5: α -Power Transformed Family

Functions of family derived from (2.16) through (2.16d) are as follows

$$f_Y(x) = \frac{\log c}{c-1} f(x)c^{F(x)} \quad (3.5)$$

$$F_Y(x) = \frac{c^{F(x)} - 1}{c - 1} \quad (3.5a)$$

$$\bar{F}_Y(x) = \frac{c}{c-1}(1 - c^{F(x)-1}) = \frac{c - c^{F(x)}}{c-1} \quad (3.5b)$$

$$h_Y(x) = f(x) \frac{c^{F(x)-1}}{1-c^{F(x)-1}} \log(c) = \frac{\log(c) f(x) c^{F(x)}}{c - c^{F(x)}} \quad (3.5c)$$

$$r_Y(x) = \frac{\log(c) f(x) c^{F(x)}}{c^{F(x)} - 1} \quad (3.5d)$$

For $c = 1$, APT distribution reduces to the base distribution.

Method 2: Sequential Application of Methods in Permutations of Five Distinct Parameter

Induction Methods Taken Two Methods at a Time.

Methods in Permutation 1: LA1, LA2

Method: First apply LA1 followed by LA2

Family 1: Exponentiated Lehmann Type I Family

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.6)

$$f_Z(x) = ct[F(x)]^{c-1} f(x)(1-F(x)^c)^{t-1} \quad (3.6)$$

Substituting $\bar{F}_Y(x)$ in (3.1b) for $\bar{F}(x)$ in (3.2a)

$$F_Z(x) = 1 - (1 - [F(x)]^c)^t \quad (3.6a)$$

Substituting $\bar{F}_Y(x)$ in (3.1b) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_Z(x) = (1 - [F(x)]^c)^t \quad (3.6b)$$

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.2c)

$$h_Z(x) = ct[F(x)]^{c-1} f(x)(1 - [F(x)]^c)^{-1} \quad (3.6c)$$

Substituting $f_Y(\kappa)$ in (3.1) for $f(x)$ and $F_Y(\kappa)$ in (3.1a) for $F(x)$ in (3.2d)

$$r_z(x) = \frac{ct[F(x)]^{c-1} f(x)(1-[F(x)]^c)^{t-1}}{1-(1-[F(x)]^c)^t} \quad (3.6d)$$

Methods in Permutation 2: LA2, LA1

Method: First apply LA2 followed by LA1;

Family 2: Exponentiated Lehmann Type II Family

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.1)

$$f_z(x) = ct[\bar{F}(x)]^{c-1} f(x)(1-[\bar{F}(x)]^c)^{t-1} \quad (3.7)$$

Substituting $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.1a)

$$F_z(x) = (1-[\bar{F}(x)]^c)^t, \quad (3.7a)$$

Substituting $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.1b)

$$\bar{F}_z(x) = 1-(1-[\bar{F}(x)]^c)^t \quad (3.7b)$$

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.1c)

$$h_z(x) = \frac{ct[\bar{F}(x)]^{c-1} f(x)(1-[\bar{F}(x)]^c)^{t-1}}{1-(1-[\bar{F}(x)]^c)^t} \quad (3.7c)$$

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.1d)

$$r_z(x) = ct[\bar{F}(x)]^{c-1} f(x)(1-[\bar{F}(x)]^c)^{-1} \quad (3.7d)$$

Family 3: Power Transformed Lehmann Type II Family/Lemann Type II-Power Transformed Family

Methods in Permutation 3: LA2, PTM

Method: First apply LA2 followed by PTM

Substituting $f_Y(x)$ in (3.2) for $f(x)$ in (3.4)

$$f_Z(x) = c t x^{t-1} f(x^t) [\bar{F}(x^t)]^{c-1} \quad (3.8)$$

Substituting $F_Y(x)$ in (3.2a) for $F(x)$ in (3.4a)

$$F_Z(x) = 1 - [\bar{F}(x^t)]^c \quad (3.8a)$$

Substituting $F_Y(x)$ in (3.2a) for $F(x)$ in (3.4b)

$$\bar{F}_Z(x) = 1 - \{1 - [\bar{F}(x^t)]^c\} = [\bar{F}(x^t)]^c \quad (3.8b)$$

Substituting $h_Y(x)$ in (3.2c) for $h(x)$ in (3.4c)

$$h_Z(x) = c t x^{t-1} f(x^t) [\bar{F}(x^t)]^{-1} \quad (3.8c)$$

Substituting $r_Y(x)$ in (3.2d) for $r(x)$ in (3.4d)

$$r_Z(y) = c t x^{t-1} f(x^t) [\bar{F}(x^t)]^{c-1} \left(1 - [\bar{F}(x^t)]^c\right)^{-1} \quad (3.8d)$$

Methods in Permutation 4: PTM, LA2

Method: First apply PTM followed by LA2

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.4b) for $\bar{F}(x)$ in (3.2)

$$f_Z(x) = c t x^{c-1} f(x^c) [\bar{F}(x^c)]^{t-1} \quad (3.9)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.4b) for $\bar{F}(x)$ in (3.2a)

$$F_Z(x) = 1 - [\bar{F}(x^c)]^t \quad (3.9a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.4b) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_Z(x) = [\bar{F}(x^c)]^t \quad (3.9b)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.4b) for $\bar{F}(x)$ in (3.2c)

$$h_Z(x) = c t x^{c-1} f(x^c) [\bar{F}(x^c)]^{-1} \quad (3.9c)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.2d)

$$\begin{aligned} r_Z(x) &= c t x^{c-1} f(x^c) (1 - F(x^c))^{t-1} \left(1 - (1 - F(x^c))^t \right)^{-1} \\ &= c t x^{c-1} f(x^c) [\bar{F}(x^c)]^{t-1} (1 - [\bar{F}(x^c)]^t)^{-1} \end{aligned} \quad (3.9d)$$

From equations (3.8) through (3.8d) and equations (3.9) through (3.9d), it can be observed that PTM and LA2 are commutative hence application of methods in permutations 3 and 4 produced one family.

Methods in Permutation 5: LA2, APT

Method: First apply LA2 followed by APT

Family 4: α -Power Transformed Lehmann Type II Family

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.5)

$$: f_z(x) = \begin{cases} \frac{\log t}{t-1} c[\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c}, & \text{when}(t \neq 1) \\ c[\bar{F}(x)]^{c-1} f(x), & \text{when}(t = 1) \end{cases} \quad (3.10)$$

Substituting $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.5a)

$$F_z(x) = \begin{cases} t^{1-[\bar{F}(x)]^c} - 1 / t - 1, & \text{when}(t \neq 1) \\ 1 - [\bar{F}(x)]^c, & \text{when}(t = 1) \end{cases} \quad (3.10a)$$

Substituting $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.5b)

$$\bar{F}_z(x) = \begin{cases} t - t^{1-[\bar{F}(x)]^c} / t - 1, & \text{when}(t \neq 1) \\ [\bar{F}(x)]^c, & \text{when}(t = 1) \end{cases} \quad (3.10b)$$

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.5c)

$$h_z(x) = \begin{cases} c \log(t) [\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c} / t - t^{1-[\bar{F}(x)]^c}, & \text{when}(t \neq 1) \\ c f(x) [\bar{F}(x)]^{-1}, & \text{when}(t = 1) \end{cases} \quad (3.10c)$$

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.5d)

$$r_z(x) = \begin{cases} c \log(t) [\bar{F}(x)]^{c-1} f(x) t^{1-[\bar{F}(x)]^c} / t^{1-[\bar{F}(x)]^c} - 1, & \text{when}(t \neq 1) \\ c f(x) [\bar{F}(x)]^{c-1} [1 - (\bar{F}(x))^c]^{-1}, & \text{when}(t = 1) \end{cases} \quad (3.10d)$$

Methods in Permutation 6: APT, LA2

Method: First apply APT followed by LA2

Family 5: Lehmann Type II α -Power Transformed Family

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.2)

$$f_Z(x) = \begin{cases} t \left(c - c^{F(x)} / c - 1 \right)^{t-1} (\log c / c - 1) f(x) c^{F(x)}, & \text{when } (c \neq 1) \\ t [\bar{F}(x)]^{t-1} f(x), & \text{when } (c = 1) \end{cases} \quad (3.11)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.2a)

$$F_Z(x) = \begin{cases} 1 - (c - c^{F(x)} / c - 1)^t, & \text{when } (c \neq 1) \\ 1 - [\bar{F}(x)]^t, & \text{when } (c = 1) \end{cases} \quad (3.11a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_Z(x) = \begin{cases} (c - c^{F(x)} / c - 1)^t, & \text{when } (c \neq 1) \\ [\bar{F}(x)]^t, & \text{when } (c = 1) \end{cases} \quad (3.11b)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.2c)

$$h_Z(x) = \begin{cases} t (\log c / c - 1) f(x) c^{F(x)} (c - c^{F(x)} / c - 1)^{-1}, & \text{when } (c \neq 1) \\ t f(x) [\bar{F}(x)]^{-1}, & \text{when } (c = 1) \end{cases} \quad (3.11c)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.2d)

$$r_z(x) = \begin{cases} t(\log c / c - 1) f(x) c^{F(x)} (c - c^{F(x)} / c - 1)^{t-1} [1 - (c - c^{F(x)} / c - 1)^t]^{-1}, & \text{when } (c \neq 1) \\ tf(x) [\bar{F}(x)]^{t-1} [1 - (\bar{F}(x))^c]^t]^{-1}, & \text{when } (c = 1) \end{cases} \quad (3.11d)$$

Methods in Permutation 7: M-OM, LA2

Method: First apply M-OM followed by LA2

Family 6: Lehmann Type II Marshall-Olkin Family

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.3b) for $\bar{F}(x)$ in (3.2)

$$f_z(x) = \frac{c^t t (\bar{F}(x))^{t-1} f(x)}{(1 - (1 - c)[1 - F(x)])^{t+1}} \quad (3.12)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.3b) for $\bar{F}(x)$ in (3.2a)

$$F_z(x) = 1 - \left(\frac{c \bar{F}(x)}{1 - (1 - c)[1 - F(x)]} \right)^t \quad (3.12a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.3b) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_z(x) = \left(\frac{c \bar{F}(x)}{1 - (1 - c)[1 - F(x)]} \right)^t \quad (3.12b)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.3b) for $\bar{F}(x)$ in (3.2c)

$$h_z(x) = \frac{tf(x)}{(1 - (1 - c)[1 - F(x)]) \bar{F}(x)} = \frac{th(x)}{1 - (1 - c)[1 - F(x)]} \quad (3.12c)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.2d)

$$r_z(x) = \frac{c^t t (\bar{F}(x))^{t-1} f(x)}{(1 - (1 - c)(1 - F(x))) \left((1 - (1 - c)(1 - F(x)))^t - (t \bar{F}(x))^t \right)} \quad (3.12d)$$

Methods in Permutation 8: LA2, M-OM

Method: First apply LA2 followed by M-OM

Family 7: Marshall-Olkin Lehmann Type II Family

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.3a)

$$f_z(x) = \frac{ct [\bar{F}(x)]^{c-1} f(x)}{(1 - [\bar{F}(x)]^c + t [\bar{F}(x)]^c)^2} = \frac{ct (\bar{F}(x))^{c-1} f(x)}{(1 - (1 - t) [\bar{F}(x)]^c)^2} \quad (3.13)$$

Substituting $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.3)

$$F_z(x) = \frac{1 - (\bar{F}(x))^c}{1 - (\bar{F}(x))^c + t (\bar{F}(x))^c} = \frac{1 - (\bar{F}(x))^c}{1 - (1 - t) (\bar{F}(x))^c} \quad (3.13a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.2a) for $\bar{F}(x)$ and $\bar{F}_Y(\kappa)$ in (3.2b) for $\bar{F}(x)$ in (3.3b)

$$\bar{F}_z(x) = \frac{t (\bar{F}(x))^c}{1 - (\bar{F}(x))^c + t (\bar{F}(x))^c} = \frac{t (\bar{F}(x))^c}{1 - (1 - t) (\bar{F}(x))^c} \quad (3.13b)$$

Substituting $h_Y(\kappa)$ in (3.2c) for $h(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.3c)

$$h_z(x) = \frac{ct (\bar{F}(x))^{-1} f(x)}{1 - (\bar{F}(x))^c + t (\bar{F}(x))^c} = \frac{ct (\bar{F}(x))^{-1} f(x)}{1 - (1 - t) (\bar{F}(x))^c} \quad (3.13c)$$

Substituting $l_Y(\kappa)$ in (3.2d) for $r(x)$ and $F_Y(\kappa)$ in (3.2a) for $F(x)$ in (3.3d)

$$r_z(x) = \frac{ct(\bar{F}(x))^{c-1} f(x) (1 - (\bar{F}(x))^c)^{-1}}{1 - (\bar{F}(x))^c + t(\bar{F}(x))^c} \quad (3.13d)$$

Family 8: Power Transformed Lehmann Type I Family/Lehmann Type I-Power Transformed Family

Methods in Permutation 9: LA1, PTM

Method: First apply LA1 followed by PTM

Substituting $f_Y(\kappa)$ in (3.1) for $f(x)$ in (3.4)

$$f_z(x) = ct x^{t-1} f(x^t) [F(x^t)]^{c-1} \quad (3.14)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ in (3.4a)

$$F_z(x) = (F(x^t))^c \quad (3.14a)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ in (3.4a)

$$\bar{F}_z(x) = 1 - (F(x^t))^c \quad (3.14b)$$

Substituting $h_Y(\kappa)$ in (3.1c) for $h(x)$ in (3.4c)

$$h_z(x) = ct x^{t-1} f(x^t) [F(x^t)]^{c-1} (1 - [F(x^t)]^c)^{-1} \quad (3.14c)$$

Substituting $r_Y(\kappa)$ in (3.1d) for $r(x)$ in (3.4d)

$$r_z(x) = ct x^{t-1} f(x^t) (F(x^t))^{-1} = ct x^{t-1} r(x^t) \quad (3.14d)$$

Methods in Permutation 10: PTM, LA1

Method: First apply PTM followed by LA1

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.1)

$$f_Z(x) = c t x^{c-1} f(x^c) [F(x^c)]^{t-1} \quad (3.15)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.1a)

$$F_Z(x) = [F(x^c)]^t, \quad (3.15a)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.1b)

$$\bar{F}_Z(x) = 1 - [F(x^c)]^t \quad (3.15b)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.1c)

$$h_Z(x) = c t x^{c-1} f(x^c) [F(x^c)]^{t-1} (1 - [F(x^c)]^t)^{-1} \quad (3.15c)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.1d)

$$r_Z(x) = c t x^{c-1} f(x^c) [F(x^c)]^{-1} = c t x^{c-1} r(x^c) \quad (3.15d)$$

From equations (3.14) through (3.14d) and equations (3.15) through (3.15d), it can be observed that PTM and LA1 are commutative hence application of methods in permutations 9 and 10 produced one family.

Methods in Permutation 11: LA1, APT

Method: First apply LA1 followed by APT

Family 9: α -Power Transformed Lehmann Type I Family

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.5)

$$f_Z(x) = \begin{cases} c(\log t / t - 1) f(x) [F(x)]^{c-1} t^{[F(x)]^c}, & \text{when } (t \neq 1) \\ cf(x) [F(x)]^{c-1}, & \text{when } (t = 1) \end{cases} \quad (3.16)$$

Substituting $F_Y(x)$ in (3.1a) for $F(x)$ in (3.5a)

$$F_Z(x) = \begin{cases} t^{[F(x)]^c} - 1 / t - 1, & \text{when } (t \neq 1) \\ [F(x)]^c, & \text{when } (t = 1) \end{cases} \quad (3.16a)$$

Substituting $F_Y(x)$ in (3.1a) for $F(x)$ in (3.5b)

$$\bar{F}_Z(x) = \begin{cases} t - t^{[F(x)]^c} / t - 1, & \text{when } (t \neq 1) \\ 1 - [F(x)]^c, & \text{when } (t = 1) \end{cases} \quad (3.16b)$$

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.5c)

$$h_Z(x) = \begin{cases} c \log(t) f(x) [F(x)]^{c-1} t^{[F(x)]^c} / t - t^{[F(x)]^c}, & \text{when } (t \neq 1) \\ cf(x) [F(x)]^{c-1} [1 - (F(x))^c]^{-1}, & \text{when } (t = 1) \end{cases} \quad (3.16c)$$

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.5d)

$$r_Z(x) = \begin{cases} c \log(t) f(x) [F(x)]^{c-1} t^{[F(x)]^c} / t^{[F(x)]^c} - 1, & \text{when } (t \neq 1) \\ cf(x) [F(x)]^{-1}, & \text{when } (t = 1) \end{cases} \quad (3.16d)$$

Methods in Permutation 12: APT, LA1

Method: First apply APT followed by LA1

Family 10: Lehmann Type I α -Power Transformed Family

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.1)

$$f_Z(x) = \begin{cases} t(\log c / c - 1) f(x) c^{F(x)} (c^{F(x)} - 1 / c - 1)^{t-1}, & \text{when}(c \neq 1) \\ t f(x) [F(x)]^{t-1}, & \text{when}(c = 1) \end{cases} \quad (3.17)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.1a)

$$F_Z(x) = \begin{cases} (c^{F(x)} - 1 / c - 1)^t, & \text{when}(c \neq 1) \\ [F(x)]^t, & \text{when}(c = 1) \end{cases} \quad (3.17a)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.1b)

$$\bar{F}_Z(x) = \begin{cases} 1 - (c^{F(x)} - 1 / c - 1)^t, & \text{when}(c \neq 1) \\ 1 - [F(x)]^t, & \text{when}(c = 1) \end{cases} \quad (3.17b)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.1c)

$$h_Z(x) = \begin{cases} t(\log c / c - 1) f(x) c^{F(x)} (c^{F(x)} - 1 / c - 1)^{t-1} [1 - (c^{F(x)} - 1 / c - 1)^t]^{-1}, & \text{when}(c \neq 1) \\ t f(x) [F(x)]^{t-1} [1 - (F(x))^t]^{-1}, & \text{when}(c = 1) \end{cases} \quad (3.17c)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.1d)

$$r_z(x) = \begin{cases} t(\log c/c - 1)f(x)c^{F(x)}[(c^{F(x)} - 1/c - 1)]^{-1}, & \text{when}(c \neq 1) \\ tf(x)[F(x)]^{-1}, & \text{when}(c = 1) \end{cases} \quad (3.17d)$$

Methods in Permutation 13: M-OM, LA1

Method: First apply M-OM followed by LA1

Family 11: Lehmann Type I Marshall-Olkin Family

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.1)

$$f_z(x) = \frac{ctf(x)[F(x)]^{t-1}}{\{1 - (1-c)[1 - F(x)]\}^{t+1}} \quad (3.18)$$

Substituting $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.1a)

$$F_z(x) = \left(\frac{F(x)}{1 - (1-c)[1 - F(x)]} \right)^t \quad (3.18a)$$

Substituting $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.1b)

$$\bar{F}_z(x) = 1 - \left(\frac{F(x)}{1 - (1-c)[1 - F(x)]} \right)^t \quad (3.18b)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.1c)

$$h_z(x) = \frac{ctf(x)[F(x)]^{t-1}}{(1 - (1-c)[1 - F(x)])\{(1 - (1-c)[1 - F(x)])^t - [F(x)]^t\}} \quad (3.18c)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.1d)

$$r_z(x) = \frac{ctf(x)}{F(x)\{1-(1-c)[1-F(x)]\}} = \frac{ct \times r(x)}{1-(1-c)[1-F(x)]} \quad (3.18d)$$

Methods in Permutation 14: LA1, M-OM

Method: First apply LA1 followed by M-OM

Family 12: Marshall-Olkin Lehmann Type I Family

Substituting $f_Y(\kappa)$ in (3.1) for $f(x)$ and $F_Y(\kappa)$ in (3.1a) for $F(x)$ in (3.3a)

$$f_z(x) = \frac{ctf(x)[F(x)]^{c-1}}{\{F(x)^c + t(1-[F(x)]^c)\}^2} = \frac{ctf(x)[F(x)]^{c-1}}{\{t + (1-t)[F(x)]^c\}^2} \quad (3.19)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ in (3.3)

$$F_z(x) = \frac{[F(x)]^c}{[F(x)]^c + t(1-[\bar{F}(x)]^c)} = \frac{[F(x)]^c}{t + (1-t)[F(x)]^c} \quad (3.19a)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ and $\bar{F}_Y(\kappa)$ in (3.1b) for $\bar{F}(x)$ in (3.3b)

$$\bar{F}_z(x) = \frac{t\{1-[\bar{F}(x)]^c\}}{[F(x)]^c + t(1-[\bar{F}(x)]^c)} = \frac{t - t[\bar{F}(x)]^c}{t + (1-t)[F(x)]^c} \quad (3.19b)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ and $h_Y(\kappa)$ in (3.1c) for $h(x)$ in (3.3c)

$$h_z(x) = \frac{c[F(x)]^{c-1} f(x)}{\{t + (1-t)[F(x)]^c\}(1-[F(x)]^c)} \quad (3.19c)$$

Substituting $F_Y(\kappa)$ in (3.1a) for $F(x)$ and $r_Y(\kappa)$ in (3.1d) for $h(x)$ in (3.3d)

$$r_z(x) = \frac{ctf(x)}{(t + (1-t)[F(x)]^c)[F(x)]} \quad (3.19d)$$

Family 13: α -Power-Power Transformed Family

Methods in Permutation 15: PTM, APT

Method: First apply PTM followed by APT

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.5)

$$f_z(x) = \begin{cases} c(\log t / t - 1)x^{c-1}f(x^c)t^{F(x^c)}, & \text{when}(t \neq 1) \\ cx^{c-1}f(x^c), & \text{when}(t = 1) \end{cases} \quad (3.20)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.5a)

$$F_z(x) = \begin{cases} t^{F(x^c)} - 1 / t - 1, & \text{when}(t \neq 1) \\ F(x^c), & \text{when}(t = 1) \end{cases} \quad (3.20a)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.5b)

$$\bar{F}_z(x) = \begin{cases} t - t^{F(x^c)} / t - 1, & \text{when}(t \neq 1) \\ 1 - F(x^c), & \text{when}(t = 1) \end{cases} \quad (3.20b)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.5c)

$$h_z(x) = \begin{cases} c \log(t)x^{c-1}f(x^c)t^{F(x^c)} / t - t^{F(x^c)}, & \text{when}(t \neq 1) \\ cx^{c-1}h(x^c), & \text{when}(t = 1) \end{cases} \quad (3.20c)$$

Substituting $f_Y(\kappa)$ in (3.4) for $f(x)$ and $F_Y(\kappa)$ in (3.4a) for $F(x)$ in (3.5d)

$$r_z(x) = \begin{cases} c \log(t) x^{c-1} f(x^c) t^{F(x^c)} / t^{F(x^c)} - 1, & \text{when}(t \neq 1) \\ cx^{c-1} r(x^c), & \text{when}(t = 1) \end{cases} \quad (3.20d)$$

Methods in Permutation 16: APT, PTM

Method: First apply APT followed by PTM

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ in (3.4)

$$f_z(x) = \begin{cases} tx^{t-1} (\log c / c - 1) f(x^t) c^{F(x^t)}, & \text{when}(c \neq 1) \\ tx^{t-1} f(x^t), & \text{when}(c = 1) \end{cases} \quad (3.21)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.4a)

$$F_z(x) = \begin{cases} c^{F(x^t)} - 1 / c - 1, & \text{when}(c \neq 1) \\ F(x^t), & \text{when}(c = 1) \end{cases} \quad (3.21a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.4b)

$$\bar{F}_Z(x) = \begin{cases} 1 - (c^{F(x^t)} - 1 / c - 1), & \text{when}(c \neq 1) \\ 1 - F(x^t), & \text{when}(c = 1) \end{cases} \quad (3.21b)$$

Substituting $h_Y(\kappa)$ in (3.5c) for $h(x)$ in (3.4c)

$$h_z(x) = \begin{cases} tx^{t-1} \log(c) f(x^t) c^{F(x^t)} / c - c^{F(x^t)}, & \text{when}(c \neq 1) \\ tx^{t-1} h(x^t), & \text{when}(c = 1) \end{cases} \quad (3.21c)$$

Substituting $r_Y(\kappa)$ in (3.5d) for $r(x)$ in (3.4d)

$$r_z(x) = \begin{cases} tx^{t-1} \log(c) f(x^t) c^{F(x^t)} / c^{F(x^t)} - 1, & \text{when } (c \neq 1) \\ tx^{t-1} r(x^t), & \text{when } (c = 1) \end{cases} \quad (3.21d)$$

From equations (3.20) through (3.20d) and equations (3.21) through (3.12d), it can be observed that PTM and APT are commutative hence application of methods in permutations 15 and 16 produced one family.

Methods in Permutation 17: M-OM, APT

Method: First apply M-OM followed by APT

Family 14: α -Power Transformed Marshall-Olkin Family

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.5)

$$f_z(x) = \begin{cases} (\log t / t - 1) \{cf(x) / (1 - (1 - c)[1 - F(x)])\}^2 t^{(F(x)/(1 - (1 - c)[1 - F(x)])}, & \text{when } (t \neq 1) \\ cf(x) / \{1 - (1 - c)[1 - F(x)]\}^2, & \text{when } (t = 1) \end{cases} \quad (3.22)$$

Substituting $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.5a)

$$F_z(x) = \begin{cases} t^{(F(x)/(1 - (1 - c)[1 - F(x)])} - 1 / t - 1, & \text{when } (t \neq 1) \\ c\bar{F}(x) / 1 - (1 - c)[1 - F(x)], & \text{when } (t = 1) \end{cases} \quad (3.22a)$$

Substituting $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.5b)

$$\bar{F}_z(x) = \begin{cases} (t - t^{(F(x)/(1 - (1 - c)[1 - F(x)])} / t - 1, & \text{when } (t \neq 1) \\ c\bar{F}(x) / 1 - (1 - c)[1 - F(x)], & \text{when } (t = 1) \end{cases} \quad (3.22b)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.5c)

$$h_z(x) = \begin{cases} \frac{\log(t)\{cf(x)/(1-(1-c)[1-F(x)])^2\}t^{(F(x)/1-(1-c)[1-F(x)])}}{t - t^{(F(x)/1-(1-c)[1-F(x)])}}, & \text{when}(t \neq 1) \\ h(x)/1-(1-c)[1-F(x)], & \text{when}(t = 1) \end{cases} \quad (3.22c)$$

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.5d)

$$r_z(x) = \begin{cases} \frac{c \log(t)\{cf(x)/(1-(1-c)[1-F(x)])^2\}t^{(F(x)/1-(1-c)[1-F(x)])}}{t^{(F(x)/1-(1-c)[1-F(x)])} - 1}, & \text{when}(t \neq 1) \\ cr(x)/1-(1-c)[1-F(x)], & \text{when}(t = 1) \end{cases} \quad (3.22d)$$

Methods in Permutation 18: APT, M-OM

Method: First apply APT followed by M-OM

Family 15: Marshall-Olkin α -Power Transformed Family

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.3a)

$$f_z(x) = \begin{cases} t \log(c) f(x) c^{F(x)} / (c-1) [(c^{F(x)} - 1/c - 1) + t(c - c^{F(x)}/c - 1)]^2, & \text{when}(c \neq 1) \\ t f(x) / (1 - (1-t)[1-F(x)])^2, & \text{when}(c = 1) \end{cases} \quad (3.23)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.3)

$$F_z(x) = \begin{cases} (c^{F(x)} - 1/c - 1)(1/(c^{F(x)} - 1/c - 1) + t(c - c^{F(x)}/c - 1)), & \text{when}(c \neq 1) \\ F(x)/1 - (1-t)[1-F(x)], & \text{when}(c = 1) \end{cases} \quad (3.23a)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ and $\bar{F}_Y(\kappa)$ in (3.5b) for $\bar{F}(x)$ in (3.3b)

$$\bar{F}_z(x) = \begin{cases} t(c - c^{F(x)}/c - 1)(1/(c^{F(x)} - 1/c - 1) + t(c - c^{F(x)}/c - 1)), & \text{when}(c \neq 1) \\ t\bar{F}(x)/1 - (1-t)[1-F(x)], & \text{when}(c = 1) \end{cases} \quad (3.23b)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ and $h_Y(\kappa)$ in (3.5c) for $h(x)$ in (3.3c)

$$h_z(x) = \begin{cases} \log(c) f(x) c^{F(x)} / [(c - c^{F(x)}) \{(c^{F(x)} - 1/c - 1) + t(c - c^{F(x)} / c - 1)\}], & \text{when } (c \neq 1) \\ h(x) / (1 - (1 - t)[1 - F(x)]), & \text{when } (c = 1) \end{cases} \quad (3.23c)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ and $r_Y(\kappa)$ in (3.5d) for $r(x)$ in (3.3d)

$$r_z(x) = \begin{cases} t \log(c) f(x) c^{F(x)} / [(c^{F(x)} - 1) \{(c^{F(x)} - 1/c - 1) + t(c - c^{F(x)} / c - 1)\}], & \text{when } (c \neq 1) \\ tr(x) / (1 - (1 - t)[1 - F(x)]), & \text{when } (c = 1) \end{cases} \quad (3.23d)$$

Family 16: Power Transformed Marshall-Olkin Family

Methods in Permutation 19: M-OM, PTM

Method: First apply M-OM followed by PTM

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ in (3.4)

$$f_z(x) = \frac{ctx^{t-1} f(x^t)}{(1 - (1 - c)[1 - F(x^t)])^2} \quad (3.24)$$

Substituting $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.4a)

$$F_z(x) = \frac{F(x^t)}{1 - (1 - c)[1 - F(x^t)]} \quad (3.24a)$$

Substituting $\bar{F}_Y(\kappa)$ in (3.3b) for $\bar{F}(x)$ in (3.4b)

$$\bar{F}_z(x) = \frac{t\bar{F}(x^t)}{1 - (1-c)[1 - F(x^t)]} \quad (3.24b)$$

Substituting $h_y(\kappa)$ in (3.3c) for $h(x)$ in (3.4c)

$$h_z(x) = \frac{tx^{t-1}h(x^t)}{1 - (1-c)[1 - F(x^\beta)]} \quad (3.24c)$$

Substituting $r_y(\kappa)$ in (3.3d) for $r(x)$ in (3.4d)

$$r_z(x) = \frac{ctx^{t-1}r(x^t)}{1 - (1-c)[1 - F(x^\beta)]} \quad (3.24d)$$

Methods in Permutation 20: PTM, M-OM

Method: First apply PTM followed by M-OM

Substituting $f_y(\kappa)$ in (3.4) for $f(x)$ and $F_y(\kappa)$ in (3.4a) for $F(x)$ in (3.3a)

$$f_z(x) = \frac{ctx^{c-1}f(x^c)}{\left(1 - (1-t)[1 - F(x^c)]\right)^2} \quad (3.25)$$

Substituting $F_y(\kappa)$ in (3.4a) for $F(x)$ in (3.3)

$$F_z(x) = \frac{F(x^c)}{1 - (1-t)[1 - F(x^c)]} \quad (3.25a)$$

Substituting $F_y(\kappa)$ in (3.4a) for $F(x)$ and $\bar{F}_y(\kappa)$ in (3.4b) for $\bar{F}(x)$ in (3.3b)

$$\bar{F}_z(x) = \frac{t\bar{F}(x^c)}{1 - (1-t)[1 - F(x^c)]} \quad (3.25b)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ and $h_Y(\kappa)$ in (3.4c) for $h(x)$ in (3.3c)

$$h_z(x) = \frac{cx^{c-1}h(x^c)}{1 - (1-t)[1 - F(x^c)]} \quad (3.25c)$$

Substituting $F_Y(\kappa)$ in (3.4a) for $F(x)$ and $r_Y(\kappa)$ in (3.4d) for $r(x)$ in (3.3d)

$$r_z(x) = \frac{ctx^{c-1}r(x^c)}{1 - (1-t)[1 - F(x^c)]} \quad (3.25d)$$

From equations (3.24) through (3.24d) and equations (3.25) through (3.25d), it can be observed that PTM and MOM are commutative hence application of methods in permutations 19 and 20 produced one family.

Method 3: Application of Same Parameter Induction Method Twice

Method: Apply MOM twice

Substituting $f_Y(\kappa)$ in (3.3a) for $f(x)$ and $F_Y(\kappa)$ in (3.3) for $F(x)$ in (3.3a)

$$\begin{aligned} f_z(x) &= \frac{ctf(x)}{(F(x) + c\bar{F}(x))^2} \div \left(\frac{F(x)}{F(x) + c\bar{F}(x)} + \frac{ct\bar{F}(x)}{F(x) + c\bar{F}(x)} \right)^2 \\ &= \frac{ctf(x)}{[(F(x) + c\bar{F}(x))]^2} = \frac{ctf(x)}{\{1 - (1-ct)[1 - F(x)]\}^2} \end{aligned} \quad (3.26)$$

From (3.3a) and (3.26), it can be observed that M-OM is idempotent.

Method: Apply LA2 twice

Substituting $f_Y(\kappa)$ in (3.2) for $f(x)$ and $\bar{F}_Y(\kappa)$ in (3.2b) for $\bar{F}(x)$ in (3.2)

$$f_z(x) = t([\bar{F}(x)]^c)^{t-1} c[\bar{F}(x)]^{c-1} f(x) = ct[\bar{F}(x)]^{ct-1} f(x) \quad (3.27)$$

From (3.2) and (3.27), it can be observed that LA2 is idempotent.

Method: Apply LA1 twice

Substituting $f_Y(x)$ in (3.1) for $f(x)$ and $F_Y(x)$ in (3.1a) for $F(x)$ in (3.1)

$$f_z(x) = \beta(cf(x)[F(x)]^{c-1}) \times [(F(x)]^c)^{t-1} = ctF(x)^{ct-1} f(x) \quad (3.28)$$

From (3.1) and (3.28), it can be observed that LA1 is idempotent.

Method: Apply PTM twice

Substituting $f_Y(x)$ in (3.4) for $f(x)$ in (3.4)

$$f_z(x) = tx^{t-1} c(x^t)^{c-1} f(x^{ct}) = ct x^{ct-1} f(x^{ct}) \quad (3.29)$$

From (3.4) and (3.29), it can be observed that PTM is idempotent.

Family 17: α -Power² Transformed Family

Method: Apply APT twice

Substituting $f_Y(x)$ in (3.5) for $f(x)$ and $F_Y(x)$ in (3.5a) for $F(x)$ in (3.5)

$$f_z(x) = \begin{cases} \frac{\log(t)}{t-1} f(x) \frac{\log(c)}{c-1} c^{F(x)} t^{\frac{C^{F(x)}-1}{c-1}}, \text{ when } (t \neq 1) \\ \log(c)/c-1 f(x) c^{F(x)}, \text{ when } (t = 1) \\ f(x), \text{ when } (c = t = 1) \end{cases} \quad (3.30)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.5a)

$$F_Z(x) = \begin{cases} t^{c^{F(x)}-1/c-1} - 1/t - 1, & \text{when}(t \neq 1) \\ c^{F(x)} - 1/c - 1, & \text{when}(t = 1) \\ F(x), & \text{when}(c = t = 1) \end{cases} \quad (3.30a)$$

Substituting $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.5b)

$$\bar{F}_Z(x) = \begin{cases} \frac{t - t^{c^{F(x)}-1/c-1}}{t-1}, & \text{when}(t \neq 1) \\ \frac{c - c^{F(x)}}{c-1}, & \text{when}(t = 1) \\ \bar{F}(x), & \text{when}(c = t = 1) \end{cases} \quad (3.30b)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.5c)

$$h_X(x) = \begin{cases} \frac{\log(t) f(x) [\log(c)/c - 1] t^{c^{F(x)}-1/c-1} c^{F(x)}}{t - t^{c^{F(x)}-1/c-1}}, & \text{when}(t \neq 1) \\ \frac{\log(c) f(x) c^{F(x)}}{c - c^{F(x)}}, & \text{when}(t = 1) \\ h(x), & \text{when}(c = t = 1) \end{cases} \quad (3.30c)$$

Substituting $f_Y(\kappa)$ in (3.5) for $f(x)$ and $F_Y(\kappa)$ in (3.5a) for $F(x)$ in (3.5d)

$$r_z(x) = \begin{cases} \frac{\log(t) f(x) [\log(c)/c - 1] t^{c^{F(x)} - 1/c - 1} c^{F(x)}}{t^{c^{F(x)} - 1/c - 1} - 1}, & \text{when}(t \neq 1) \\ \frac{\log(c) f(x) c^{F(x)}}{c^{F(x)} - 1}, & \text{when}(t = 1) \\ r(x), & \text{when}(c = t = 1) \end{cases} \quad (3.30d)$$

From equations (3.5) through (3.5d) and equations (3.30) through (3.30d), it can be observed that APT is not idempotent.

3.42 Flexibility of Generalized Distributions

The GDP was introduced by Pickands (1975). The GPD is a skewed distribution parameterized with a shape, scale, and in some forms, location parameter therefore a suitable choice for illustrating flexibility. This sub-section gives mathematical and visual representation of shapes of probability density and hazard functions of two exponentiated GPDs obtained by applying LA1 and LA2 parameter induction methods. Comparison with those of the parent distribution (GPD) establishes flexibility of generalized distributions.

The Generalized Pareto Distribution (base distribution)

$$f(x) = \begin{cases} \sigma^{-1} (1 + kx/\sigma)^{-1-1/k}, & \text{when}(k \neq 0) \\ \sigma^{-1} \exp(-x/\sigma), & \text{when}(k = 0) \end{cases} \quad (3.31)$$

$$F(x) = \begin{cases} 1 - (1 + kx/\sigma)^{-1/k}, & \text{when}(k \neq 0) \\ 1 - \exp(-x/\sigma), & \text{when}(k = 0) \end{cases} \quad (3.31a)$$

$$\bar{F}(x) = \begin{cases} (1 + kx/\sigma)^{-1/k}, & \text{when } (k \neq 0) \\ \exp(-x/\sigma), & \text{when } (k = 0) \end{cases} \quad (3.31b)$$

$$h(x) = \begin{cases} \sigma^{-1}(1 + kx/\sigma)^{-1}, & \text{when } (k \neq 0) \\ 1/\sigma, & \text{when } (k = 0) \end{cases} \quad (3.31c)$$

$$r(x) = \begin{cases} \sigma^{-1} \left((1 + kx/\sigma)^{1/k} - 1 \right)^{-1} (1 + kx/\sigma)^{-1}, & \text{when } (k \neq 0) \\ \sigma^{-1} \exp(-x/\sigma) (1 - \exp(-x/\sigma))^{-1}, & \text{when } (k = 0) \end{cases} \quad (3.31d)$$

$0 < \kappa < \infty$ for $k \geq 0$, and $0 < \kappa < -\sigma / k$ for $k < 0$

Lehmann Type II GPD (LIIGPD)

Substituting $f(x)$ in (3.31) for $f(x)$ and $\bar{F}(x)$ in (3.31b) for $\bar{F}(x)$ in (3.2)

$$\begin{aligned} f(y) &= c \left((1 + ky/\sigma)^{-1/k} \right)^{c-1} \sigma^{-1} (1 + ky/\sigma)^{-1-1/k} \\ &= c \sigma^{-1} (1 + ky/\sigma)^{-1-c/k}, \quad k \neq 0 \end{aligned} \quad (3.32)$$

$$\begin{aligned} f(y) &= c (\exp(-y/\sigma))^{c-1} \sigma^{-1} \exp(-y/\sigma) \\ &= c \sigma^{-1} (\exp(-y/\sigma))^c, \quad k = 0 \end{aligned}$$

Substituting $\bar{F}(x)$ in (3.31b) for $\bar{F}(x)$ in (3.2a)

$$F(y) = 1 - (1 + ky/\sigma)^{-c/k}, \quad k \neq 0$$

$$F(y) = 1 - (\exp(-y/\sigma))^c, \quad k = 0 \quad (3.32a)$$

Substituting $\bar{F}(x)$ in (3.31b) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}(y) = (1 + ky/\sigma)^{-c/k}, \quad k \neq 0$$

$$\bar{F}(y) = (\exp(-y/\sigma))^c, \quad k = 0 \quad (3.32b)$$

Substituting $f(x)$ in (3.31) for $f(x)$ and $\bar{F}(x)$ in (3.31b) for $\bar{F}(x)$ in (3.2c)

$$\begin{aligned} h(y) &= c\sigma^{-1}(1 + ky/\sigma)^{-1-1/k}(1 + ky/\sigma)^{1/k} \\ &= c\sigma^{-1}(1 + ky/\sigma)^{-1}, \quad k \neq 0 \end{aligned} \quad (3.32c)$$

$$\begin{aligned} h(y) &= c\sigma^{-1}\exp(-y/\sigma)^{-1}\exp(-y/\sigma) \\ &= c\sigma^{-1}, \quad k = 0 \end{aligned}$$

Substituting $f(x)$ in (3.31) for $f(x)$ and $\bar{F}(x)$ in (3.31b) for $\bar{F}(x)$ in (3.2d)

$$\begin{aligned} r(y) &= c\sigma^{-1}(1 + ky/\sigma)^{-1-c/k}\left(1 - (1 + ky/\sigma)^{-c/k}\right)^{-1} \\ r(y) &= c\sigma^{-1}(\exp(-y/\sigma))^c\left(1 - (\exp(-y/\sigma))^c\right)^{-1}, \quad k = 0 \end{aligned} \quad (3.32d)$$

Some special cases of Lehmann Type II GPD

1. (3.32) parallels that of the GPD (base distribution) when $c = 1$
2. For $k = \sigma = 1$, (3.32) and PDF of beta type II distribution given below are identical

$$f(y; c) = c(1 + y)^{-(c+1)} \quad y > 0, \alpha > 0, \quad ,$$

3. (3.32) parallels that of uniform (0, c) for $k = -1$ and $\sigma = 1$
 4. (3.32) parallels the PDF of exponential distribution given below when $k=0$ and $c=1$
- $$f(y) = (1/\sigma)\exp(-y/\sigma)$$

5. For $k = 1$, (3.32) and the PDF of lomax distribution given below are identical

$$f(y; c, \sigma) = c\sigma^{-1}(1 + y/\sigma)^{-(c+1)} \quad y > 0, c, \sigma > 0$$

The shape of the PDF of LIIGPD

The scale parameter does not affect the PDF and HF shapes therefore σ is set to be equal to 1 ($\sigma = 1$)

Substituting $\sigma = 1$ into (3.32), the PDF of LIIGPD becomes

$$f(y) = c(1 + ky)^{-(c/k+1)}, \quad f(y) = c[\exp(-y)]^c \text{ if } k \neq 0 \text{ and } k = 0 \text{ respectively.}$$

$$f'(y) = -ck(c/k + 1)(1 + ky)^{-(c/k+2)}, \quad f'(y) = -c^2(e^{-y})^c. \text{ if } k \neq 0 \text{ and } k = 0 \text{ respectively.}$$

(i) If $k \geq 0$ and $c > 0$, $f'(y) < 0$. Thus, $f(y)$ is monotone decreasing.

(ii) If $k < 0$, and $c = -k$, $f'(y) = 0$, so, $f(y)$ is constant at c .

(iii) If $k < 0$, and $0 < c < -k$, $f'(y) > 0$, hence $f(y)$ is a monotone increasing function.

(iv) If $k < 0$, and $c > -k$, $f'(y) < 0$. Thus, $f(y)$ is monotone decreasing.

Therefore the shape of the PDF of LIIGPD is

(i) an increasing function if $0 < c < -k$ and $k < 0$

(ii) a decreasing function if $c > -k$ and $k < 0$ and when $c > 0$ and $k \geq 0$

(iii) Constant at c if $c = -k$ and $k < 0$

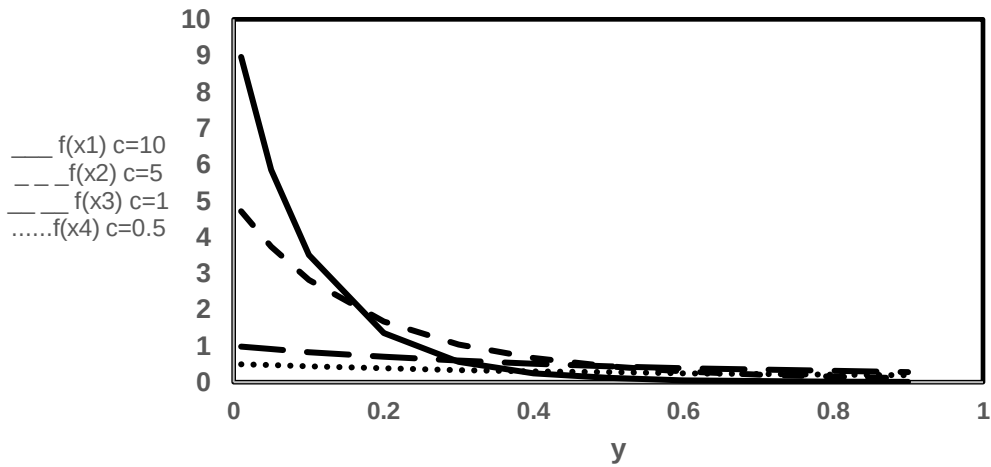


Figure3.1: PDF Shapes of LIIGPD ($k = 1, \sigma = 1$, varying c)

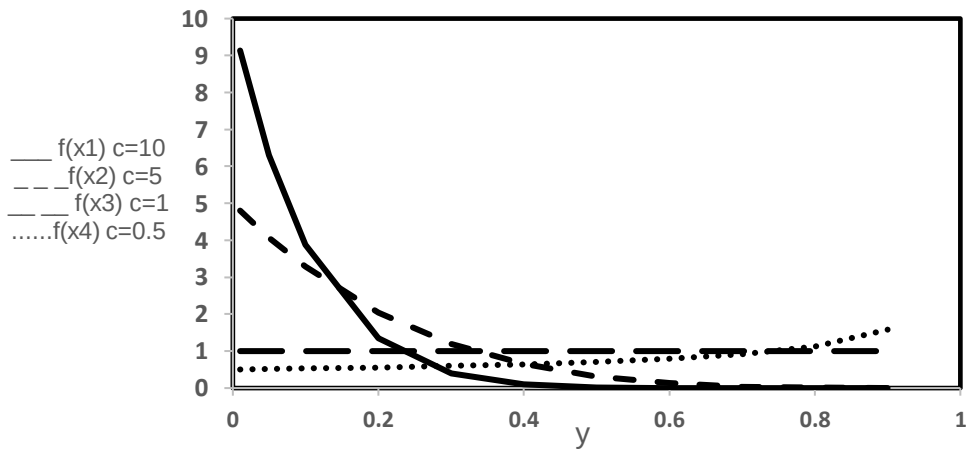
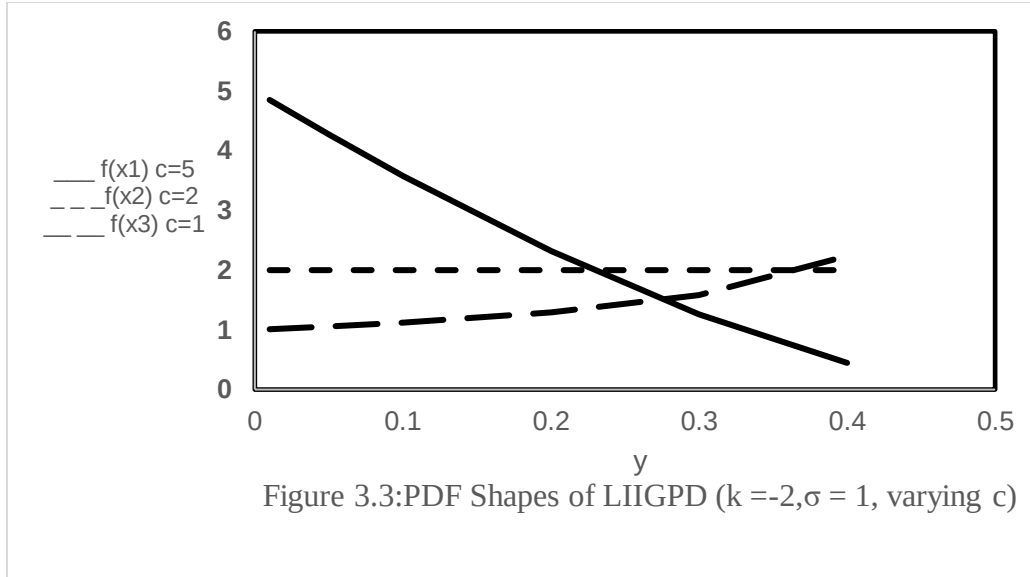


Figure 3.2: PDF Shapes of LIIGPD ($k = -1, \sigma = 1$, varying c)



The LIIGPD parallels GPD when $c = 1$, so at other values of c , effect of introducing c on PDF shape is represented in the figures above.

The shape of the HF of LIIGPD

Substituting $\sigma = 1$ into (3.32c), the HF of LIIGPD becomes

$$h(y) = c(1 + ky)^{-1}, \quad h(y) = c \quad \text{if } k \neq 0 \text{ and } k = 0 \text{ respectively.}$$

$$h'(y) = -ck(1 + ky)^{-2} \quad \text{if } k \neq 0 \quad \text{and for } k = 0,$$

$$h'(y) = 0$$

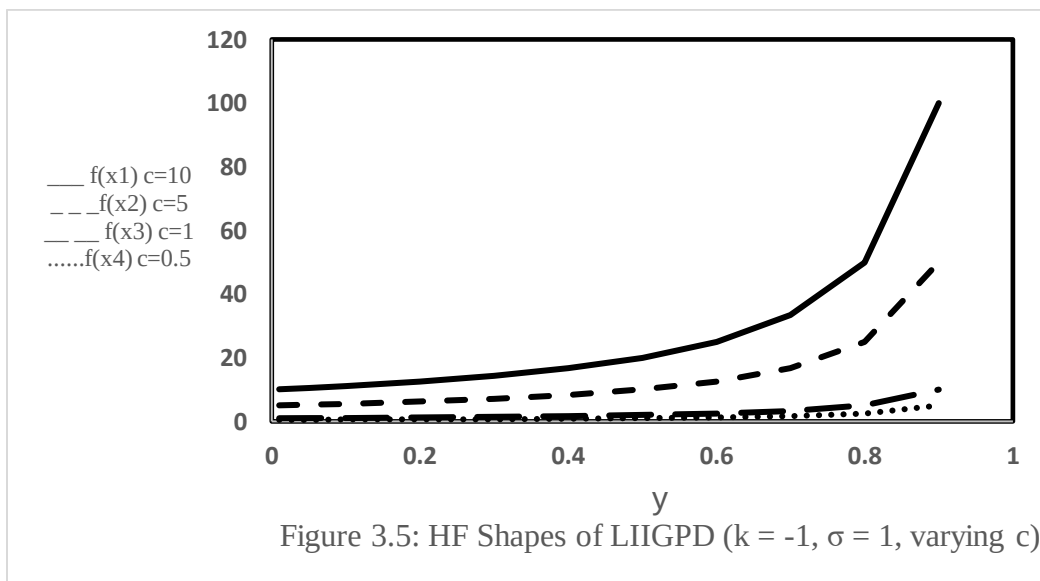
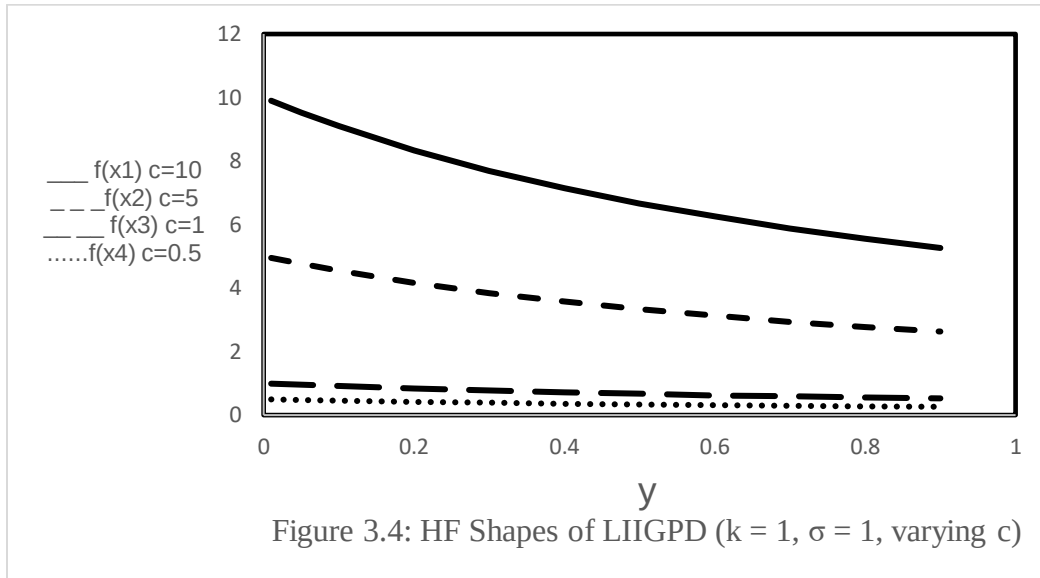
(i) If $k = 0$ and $c > 0$, $h'(y) = 0$, so, $h(y)$ is a constant function.

(ii) If $k < 0$, and $c > 0$, $h'(y) > 0$. Thus, $h(y)$ is a monotone increasing function.

(iii) If $k > 0$, and $c > 0$, $h'(y) < 0$, hence, $h(y)$ is monotone decreasing function.

Therefore the shape of HF of LIIGPD is

- (i) a decreasing function if $c > 0$ and $k > 0$
- (ii) an increasing function if $c > 0$ and $k < 0$
- (iii) Constant at c if $c > 0$ and $k = 0$.



The LIIGPD parallels GPD when $c = 1$, so at other values of c , effect of introduced c on HF shapes is represented in the figures above.

Lehmann Type I GPD (LIGPD)

Substituting $f(x)$ in (3.31) for $f(x)$ and $F(x)$ in (3.31a) for $F(x)$ in (3.1)

$$\begin{aligned}
 f(y) &= c\sigma^{-1}(1+ky/\sigma)^{-1-1/k} \left(1 - (1+(ky/\sigma))^{-1/k}\right)^{c-1} \\
 &= c\sigma^{-1}(1+ky/\sigma)^{-1-1/k} \left(1 - (1+(ky/\sigma))^{-1/k}\right)^{c-1} \\
 &= \frac{c}{\sigma} \times \frac{(1+ky/\sigma)^{-1} \times \left(1 - (1+ky/k)^{-1/k}\right)^c}{(1+ky/\sigma)^{1/k} - 1} \\
 &= \frac{c}{\sigma} \times \frac{\left((1+ky/\sigma)^{1/k} - 1\right) \left(1 - (1+ky/\sigma)^{-1/k}\right)^c}{(1+ky/\sigma) \left((1+ky/\sigma)^{1/k} - 1\right)} \\
 f(y) &= c\sigma^{-1} \left((1+ky/\sigma)^{1/k} - 1\right)^{c-1} (1+ky/\sigma)^{-1-c/k} \quad \text{if } k \neq 0 \tag{3.33}
 \end{aligned}$$

$$f(y) = c\sigma^{-1} \exp(-y/\sigma) (1 - \exp(-y/\sigma))^{c-1} \quad k = 0 \tag{3.33a}$$

Substituting $F(x)$ in (3.31a) for $F(x)$ in (3.1a)

$$F(y) = \left(1 - (1+ky/\sigma)^{-1/k}\right)^c \quad \text{if } k \neq 0 \tag{3.33b}$$

$$F(y) = (1 - \exp(-y/\sigma))^c \quad \text{if } k = 0 \tag{3.33c}$$

Substituting $F(x)$ in (3.31a) for $F(x)$ in (3.1b)

$$\bar{F}(y) = 1 - \left(1 - (1+ky/\sigma)^{-1/k}\right)^c \quad \text{if } k \neq 0 \tag{3.33d}$$

$$\bar{F}(y) = 1 - (1 - \exp(-y/\sigma))^c \quad \text{if } k = 0 \quad (3.33e)$$

Substituting $f(x)$ in (3.31) for $f(x)$ and $F(x)$ in (3.31a) for $F(x)$ in (3.1c)

$$h(y) = \frac{c\sigma^{-1} \left((1 + ky/\sigma)^{1/k} - 1 \right)^{c-1} (1 + ky/\sigma)^{-1-c/k}}{1 - \left((1 + ky/\sigma)^{1/k} - 1 \right)^c (1 + ky/\sigma)^{-c/k}}, \text{ if } (k \neq 0)$$

$$h(y) = \frac{c\sigma^{-1} \exp(-y/\sigma) (1 - \exp(-y/\sigma))^{c-1}}{1 - (1 - \exp(-y/\sigma))^c}, \text{ if } (k = 0) \quad (3.33f)$$

Substituting $f(x)$ in (3.31) for $f(x)$ and $F(x)$ in (3.31a) for $F(x)$ in (3.1d)

$$r(y) = c\sigma^{-1} \left((1 + ky/\sigma)^{1/k} - 1 \right)^{-1} (1 + ky/\sigma)^{-1} \quad \text{if } k \neq 0 \quad (3.33g)$$

$$r(y) = c\sigma^{-1} \exp(-y/\sigma) (1 - \exp(-y/\sigma))^{-1} \quad \text{if } k = 0 \quad (3.33h)$$

Some special cases of LIGPD

1. When $c=1$ LIGPD parallels the GPD (base distribution)
2. LIGPD parallels uniform $(0, \sigma)$ when $k = -1$ and $c = 1$
3. LIGPD and exponential distribution with following PDF are identical when $k = 0$ and $c = 1$

$$f(y) = \sigma^{-1} e^{(-y/\sigma)}$$

The shape of the PDF of LIGPD

The scale parameter does not affect the PDF and HF shapes, therefore σ is set to be equal to 1 ($\sigma = 1$). Substituting $\sigma = 1$ into (3.33) and (3.33a) and then taking first derivative wrt y ,

$$f'(y) = -(c+k)c(1+ky)^{-(c/k+2)}((1+ky)^{1/k} - 1)^{c-1} + (c-1)(1+ky)^{1/k} \\ \times ((1+ky)^{1/k} - 1)^{-1} c(1+ky)^{-(c/k+2)}((1+ky)^{1/k} - 1)^{c-1} \quad \text{if } k \neq 0$$

$$f'(y) = (c-1)e^{-y}(1-e^{-y})^{-1}ce^{-y}(1-e^{-y})^{c-1} - ce^{-y}(1-e^{-y})^{c-1} \quad \text{if } k = 0$$

(i) If $k \geq 0$ and $0 < c \leq 1$, $f'(y) < 0$ since for $0 < c < 1$, the first and second terms of $f'(y)$ are both less than zero and second term = 0 if $c = 1$, therefore $f(y)$ is monotone decreasing over the range of y .

(ii) If $k \geq 0$ and $c > 1$, $f(y)$ has one root, so $f'(y)$ changes sign once, therefore, $f(y)$ has a unimodal shape. The root of the function (z) is the critical point of $f(y)$ which is that for which $f'(y) = 0$. Setting $f'(y) = 0$ give

$$y = \left[-\left(\left[c - 1/c + k \right] - 1 \right) \right]^{-k} - 1/k \quad \text{for } k \neq 0, \text{ and for } k = 0,$$

$$y = -\ln(1/c)$$

From figures shown below $f'(y) > 0$ to the left of $y = z$ and $f'(y) < 0$ to the right of $y = z$, so $y = z$ is a relative maximum. $f(y)$ is increasing on $(0, z)$, and decreasing on (z, ∞) when $k \geq 0$ and $c > 1$.

(iii) The following are some shapes obtainable when $K < 0$

a. If $c=1$ and $k < -c$, $f'(y) > 0$ over the range of y , so $f(y)$ is a monotone increasing function.

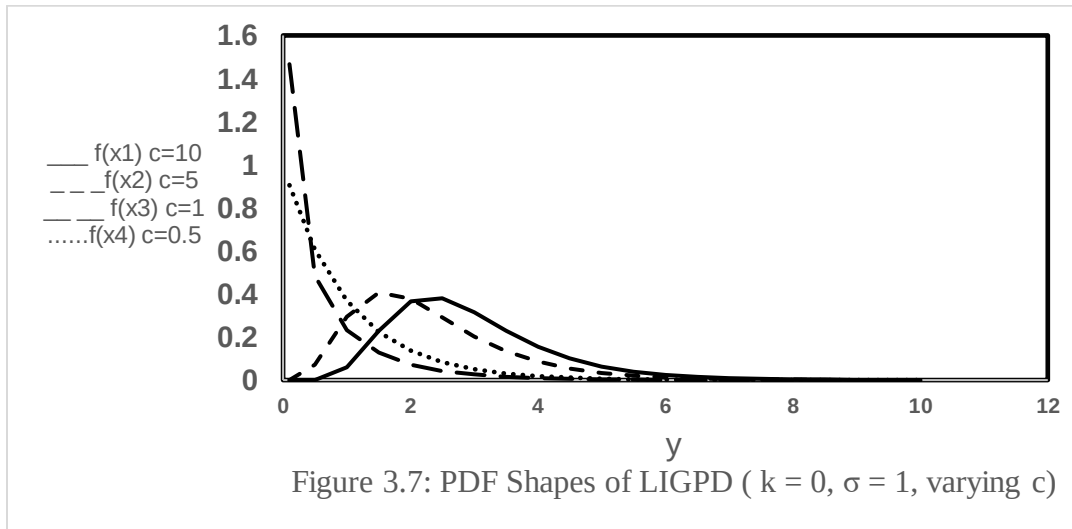
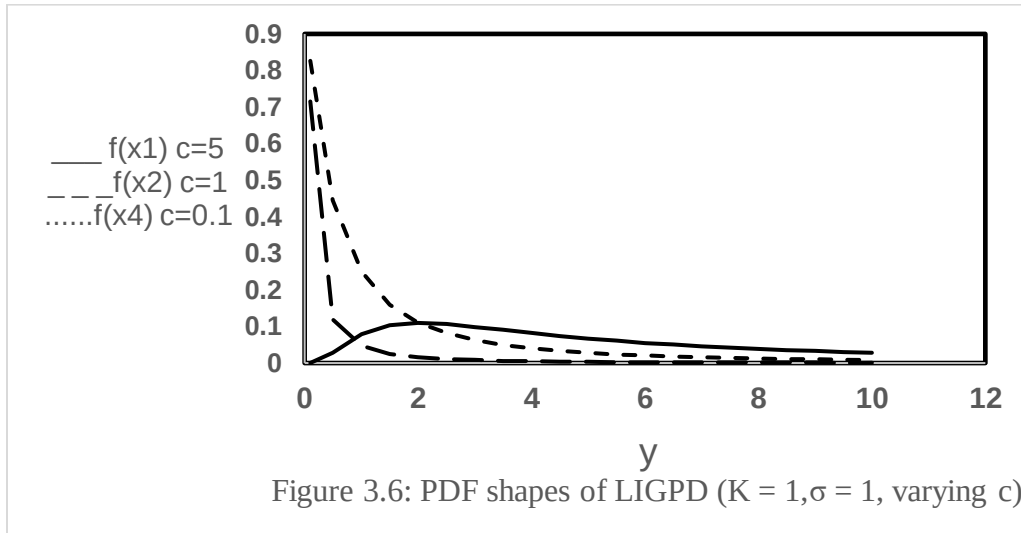
b. If $c=1$, and $k > -c$, $f'(y) < 0$ over the range of y , hence $f(y)$ is monotone decreasing.

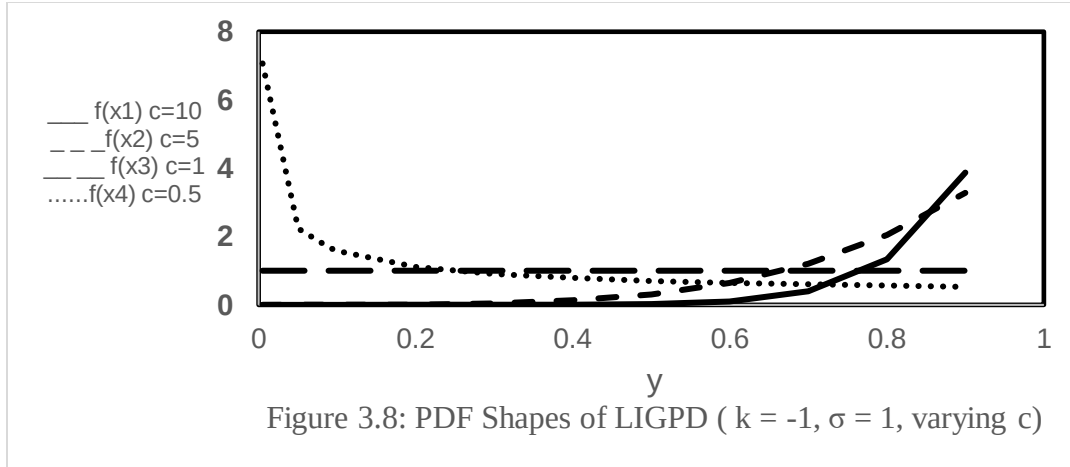
(iv) If $c=1$, and $k = -c$ $f'(y) = 0$ over the range of y . Thus, $f(y)$ is a constant function at c .

(v) If $c > 1$, and $k < -c$, $f'(y) > 0$ over the range of y , so $f(y)$ is monotone decreasing.

(vi) If $c < 1$, and $k > -c$, $f'(y) > 0$ over the range of y , so $f(y)$ is monotone on the interval.

PDF shapes of LIGPD for some values of parameters are shown in the figures below





The LIGPD parallels GPD when $c = 1$, so at other values of c , effect of introducing c on PDF shapes is represented in the figures above.

The shape of the HF of LIGPD

Taking the derivative of (3.33f), for $k = 0$, the shape of the HF is

- (i) a decreasing function (from ∞ to 1) if $0 < c < 1$
- (ii) Constant (at c) if $c = 1$
- (iii) An increasing function (from 0 to 1) if $c > 1$.

At other values of k , possible shapes of the hazard function are represented in the figures below.

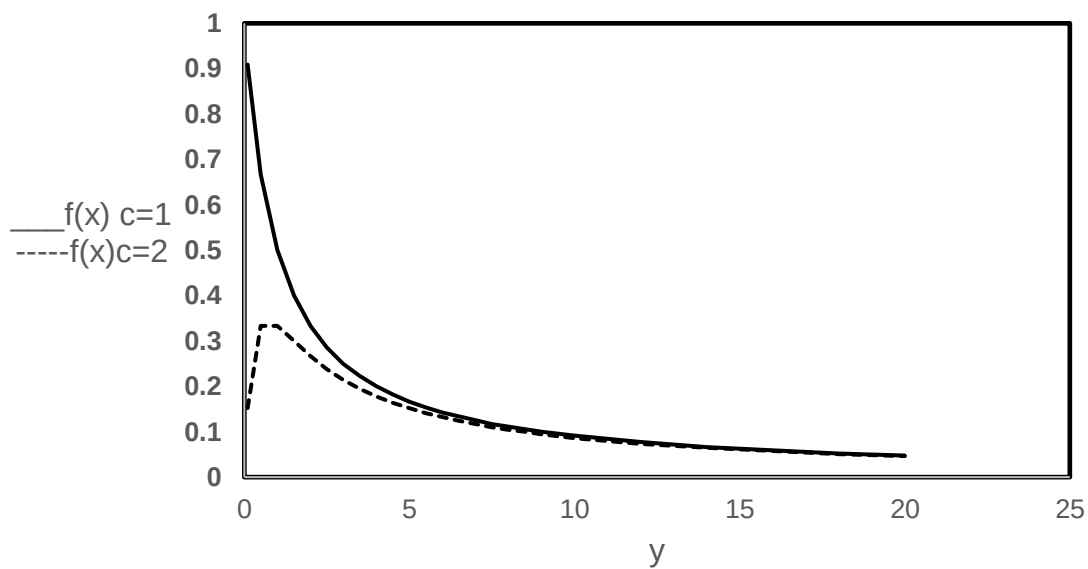


Figure 3.9: HF Shapes of LIGPD ($k = 1$, $\sigma = 1$, varying c)

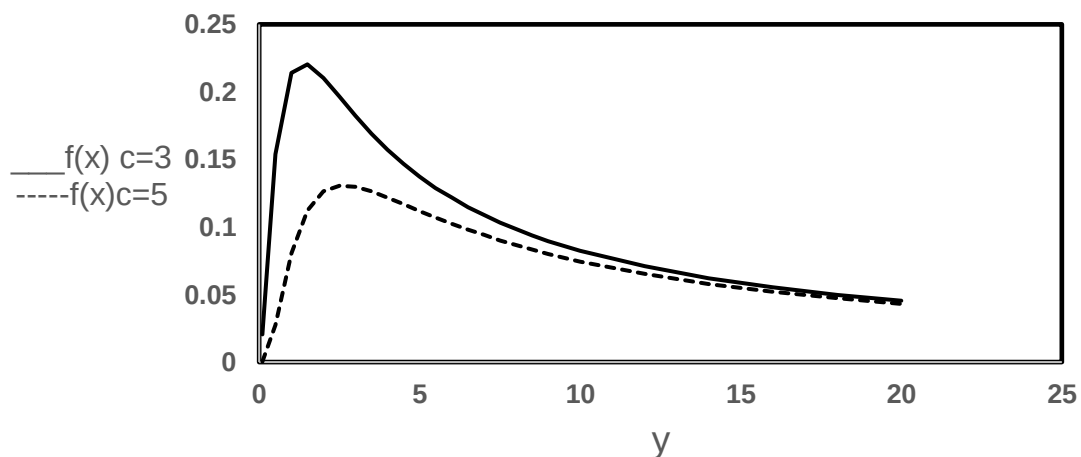
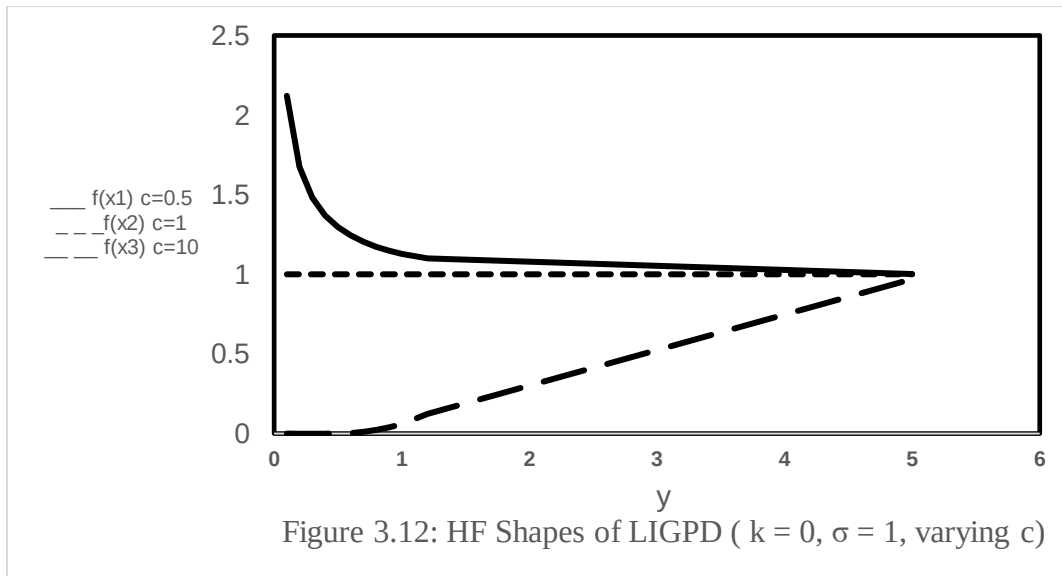
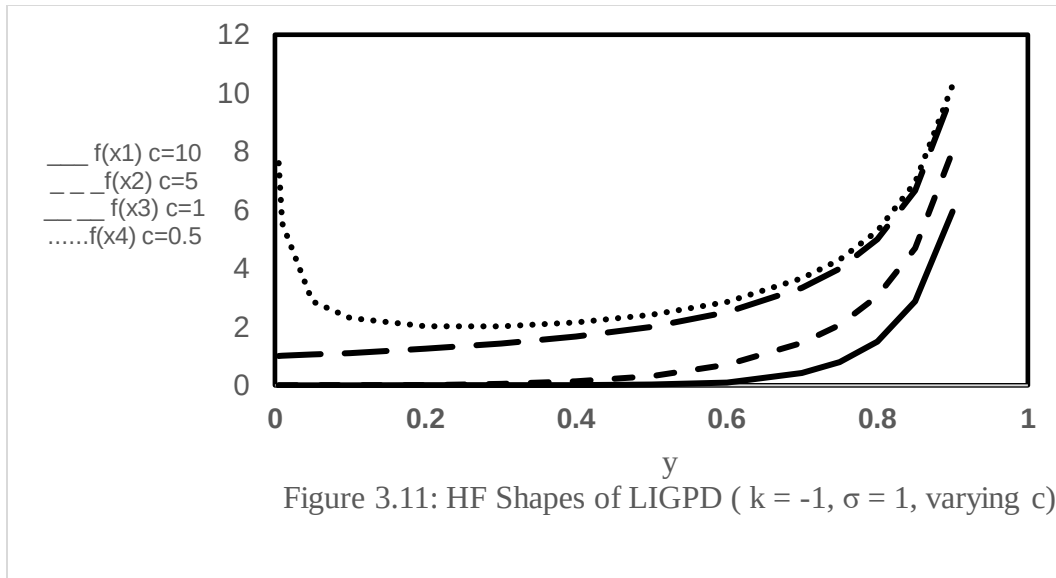


Figure 3.10: HF Shapes of LIGPD ($k = 1$, $\sigma = 1$, varying c)



The LIGPD parallels GPD when $c = 1$, so at other values of c , effect of introduced c on HF shapes is represented in the figures above.

3.43 Fitting of Distributions to Data

Fitting distribution involves choosing a probability distribution that models a given random variable and the estimation of distribution parameters. In other words, it is the process of

selecting a statistical distribution that provides the best fit to data or that represents a given data the most.

Histograms reveal skewness, kurtosis, tail behaviour, and existence of data outliers thereby indicating the shape of a data set. Statistical goodness of fit test as well as graphical displays are effective ways to determine whether the fitted distributions are consistent with the given set of observations (Stedinger et al.1993). Histograms are compared with the basic shapes associated with existing distributions to determine plausible distributions for distribution fitting to data. Therefore, plausible existing distributions for distribution fitting were selected after considering the following factors:

- (a) Variable type (discrete or continuous)
- (b) Bounds of variable, and
- (c) Shape of the data as represented in the histogram plot.

3.431 Goodness of Fit Test

Goodness of fit test is one of the most fundamental hypothesis testing problems (Lehmann & Romano, 2005). The extent to which a hypothesized probability distribution can describe a sample of data can be assessed using goodness of fit test. Goodness of fit tests include amongst others the K-S test, Cramer-Von Mises, Anderson-Darling test (Anderson and Darling, 1954) and Berk-Jones (Berk and Jones, 1979). K-S test has some desirable properties which include:

- (i) asymptotic consistency against any fixed alternative, good power against a shift in the median of the distribution (Janssen, 2000)
- (ii) it is an exact test

(iii) no restrictions on sample size

(iv) no assumption is made about the distribution of data and

(v) no grouping of data is required

K-S goodness of fit test was therefore adopted for goodness of fit assessment. The K-S test is a non-parametric test named after Andrey Kolmogorov and Nikola Smirnov. The K-S statistics quantifies a distance between the empirical distribution of the sample and the cumulative distribution function of the reference distribution, or between the empirical distribution functions of two samples.

H_0 : Sample is drawn from the reference distribution (one-sample case)

H_0 : Samples are drawn from the same distribution (two-sample case)

K-S test statistics is given as;

$$D_n = \max_{1 \leq i \leq n} \left(\left| F_n(y_i) - F_0(y_k) \right|, \left| F_n(y_k) - F_0(y_k) \right| \right) \quad (3.34)$$

Empirical distribution $F_n(y)$ is the fraction of sample observations less than or equal to the value y that is k / n when $k = 1, \dots, n$. $F_0(y_k)$ is the CDF of hypothesized distribution or theoretical CDF being tested. H_0 is rejected if $D_n >$ critical value obtained from table. There are many methods used in obtaining the distribution function of D_n and corresponding P-values for one-sided and two-sided K-S statistics. These methods can be found in Brown and Harvey (2008 a, b), Marsaglia et al. (2003), Durbin (1973), Noé (1972) and Birnham and Tingey (1951). The procedure of Marsaglia et al. provides K-S distribution function with far greater precision. (Marsaglia et

al.,2003).

Marsaglia et al.'s method is used in the study to obtain P-values for K-S statistics.

$$\text{K-S P-Value} = 2 * \exp(-(2.0000714 + .331 / \sqrt{n} + 1.409 / n) * D^*) \quad (3.35)$$

Where $D^* = n (K^2)$.

3.432 Model Selection

Model selection is the task of choosing a model from a set of candidate models given data. It involves estimating the performances of the different models in order to select the most appropriate.

Information loss is often the outcome of using another model to represent the “true” model. Information criteria are estimates of the information loss for a specified model and sample. AIC (Akaike 1974) is a criterion for estimating information loss in a model. Given a set of data and some statistical models, AIC is a measures of relative quality of the models. It measures the quality of each model, relative to each of the other models. AIC is therefore used for model selection. Akaike showed that information loss for a model was closely related to the likelihood and proportional to

$$- \ln(l) + k \quad (3.36)$$

l is the likelihood of the model given the data while k is the number of parameters in the model.

AIC is defined as;

$$AIC = -2 \ln(l) + 2k \quad (3.37)$$

Model selection by means of information criteria involves keeping information loss low and therefore selecting a model that minimizes an information criterion. Hence, the preferred model when using AIC is that with the minimum AIC values.

AIC can also show how much much more (or less) information is lost by one model than by the other model (Akaike, 1974). Given N candidate models with AIC values denoted by $AIC_1, AIC_2, \dots, AIC_N$ and the minimum of these AIC values given as $AIC_{\min}$, the relative likelihood of model i which is equivalent to

$$\exp((AIC_{\min} - AIC_i) / 2) \quad (3.38)$$

can be interpreted as being proportional to the probability that the model minimizes the estimated information loss (Akaike, 1974).

AIC produces good model selection in large samples (Shibata, 1980). The likelihood ratio test is valid only for nested models whereas AIC has no such restrictions (Boisbunon et al., 2014). AIC produces parsimonious models by including a penalty that discourages over fitting because studies show that when number of parameters are increased, goodness of fit is bound to improve in almost all the cases. Though AIC as some studies prove, penalizes the number of parameters less strongly than does the BIC, Burnham and Anderson (2004) suggested from findings that AIC tends to have practical performance advantages over BIC. AIC is derived from principles of information, BIC is not, despite its name, so, AIC has theoretical advantages over BIC (Burnham and Anderson, 2002). AIC was employed in the study for model selection.

3.433 Parameter Inference

The most commonly used method of parameter estimation is the maximum likelihood estimation

method. Maximum likelihood estimators have some desirable properties including consistency, asymptotic normality, and efficiency.

Given random variables $\kappa_1, \dots, \kappa_n$ having a joint density $f(\kappa_1, \kappa_2, \dots, \kappa_n | \theta)$ and observed values $X_i = \kappa_i$, where $i = 1, \dots, n$. The likelihood of θ as a function of $\kappa_1, \kappa_2, \dots, \kappa_n$ is defined as;

$$L(\theta) = f(\kappa_1, \kappa_2, \dots, \kappa_n | \theta) \quad (3.39)$$

$$L(\theta) = \prod_{i=1}^n f(X_i | \theta) \quad (3.40)$$

The log likelihood;

$$L(\theta) = \sum_{i=1}^n \log[f(X_i | \theta)] \quad (3.41)$$

is often maximized and gives the same result. When the first order derivatives of (3.41) with respect to parameters are set to zero and resulting equations are subsequently solved simultaneously, maximum likelihood estimators of parameters are obtained. Maximum likelihood estimators obtained for study model parameters could not be obtained in explicit forms, no close form expressions were obtained so model parameters were estimated numerically.

Many numerical optimization algorithms are in existence. The quasi-newton method is a numerical optimization routine that uses first order information only. It derives an approximate hessian from the sequence of function values and gradients from previous iteration. One advantage of this method is that it doesn't need to compute the hessian from scratch at every

iteration. Parameters of models in the study are bounded, hessians must be positive definite for variances of parameter

estimates to be positive. Therefore, constrained optimization was carried out. An R function by Fox et al., 1978 and Fox, 1997 that performs bound constrained quasi-newton optimization routine was then found appropriate for parameter estimation. R functions perform minimization so the negative log likelihood functions of models given data were minimized as minimization of $-f(x)$ is equivalent to maximization of $f(x)$.

Given $g(x)$ and $H(x)$ to be gradient and hessian functions respectively, the necessary condition for a local minimum is;

$\|g(x)\| = 0$ and $H(x)$ is positive semi definite (the eigenvalues of covariance matrix are positive or zero).

While the sufficient conditions for a strong local minimum is given as;

$\|g(x)\| = 0$ and $H(x)$ is positive definite (the eigenvalues of covariance matrix are strictly positive).

Mathematical or economic instinct about a variable can inform the choice of starting values for optimization algorithm, but sophisticated numerical representation and optimization algorithms that have to be carried out iteratively make it unpredictable whether a good starting point really leads to the global optimum or not (McCullough and Renfro, 2000, McCullough and Vinod, 2003). A run from a starting point close to the global optimum doesn't always guarantee

convergence. Rerunning an optimization with different starting values is a robustness check that should always be applied to optimization routines (Gilli and Schuman, 2010).

Maximum likelihood estimates of parameters of base distribution were used as initial guesses of parameter values. A set of starting values for each parameter was obtained by increasing or decreasing the initial guess by certain percentages. Optimization was rerun with the different starting values to ascertain if global optimum was reached at convergence point.

Hessian and gradients were calculated numerically using Richardson extrapolation method. The complex-step method of numerical differentiation is faster than Richardson extrapolation but Richardson is more accurate. The hessian and gradient functions provided in R package by Gilbert & Vardhan in 2015 was used to compute the hessian and gradients using Richardson extrapolation method. The covariance matrices were calculated from the inverse of corresponding hessian matrices or matrices of second order derivatives. A built-in R package function was used to estimate covariance and correlation matrices. The variances of parameter are the eigenvalues of the covariance matrix.

CHAPTER FOUR

VALIDATION OF RESULTS WITH DATA

4.1 Introduction

Let the continuous random variable, MMR, be represented by X . The functions of the distribution that describes X are given as follows:

Probability Density Function denoted by $f(X)$, Cumulative Distribution Function denoted by $F(x)$. If X is a lifetime random variable, then, the Survival Function is denoted by $s(x)$, the Hazard Function is denoted by $h(x)$, and the Reversed Hazard Function is denoted by $r(x)$.

Let Y be a generalized continuous random variable obtained by introducing a shape parameter ($c > 0$) to the distribution of the base random variable X , and let Z be another continuous variable belonging to a family of generalized distributions with two additional shape parameters ($c > 0$ and $t > 0$) also obtained from the distribution of X (Omekam & Adejumo, 2017).

The shape of MMR data is deduced from the histogram for X which will then suggest plausible existing distributions that will be subsequently fitted to data. Generalized distributions obtained for the best fitted existing distribution are then fitted to X . X is concluded to follow the generalized distribution with the best fit to data.

4.2 Distribution Fitting of Some Existing Distributions to Data

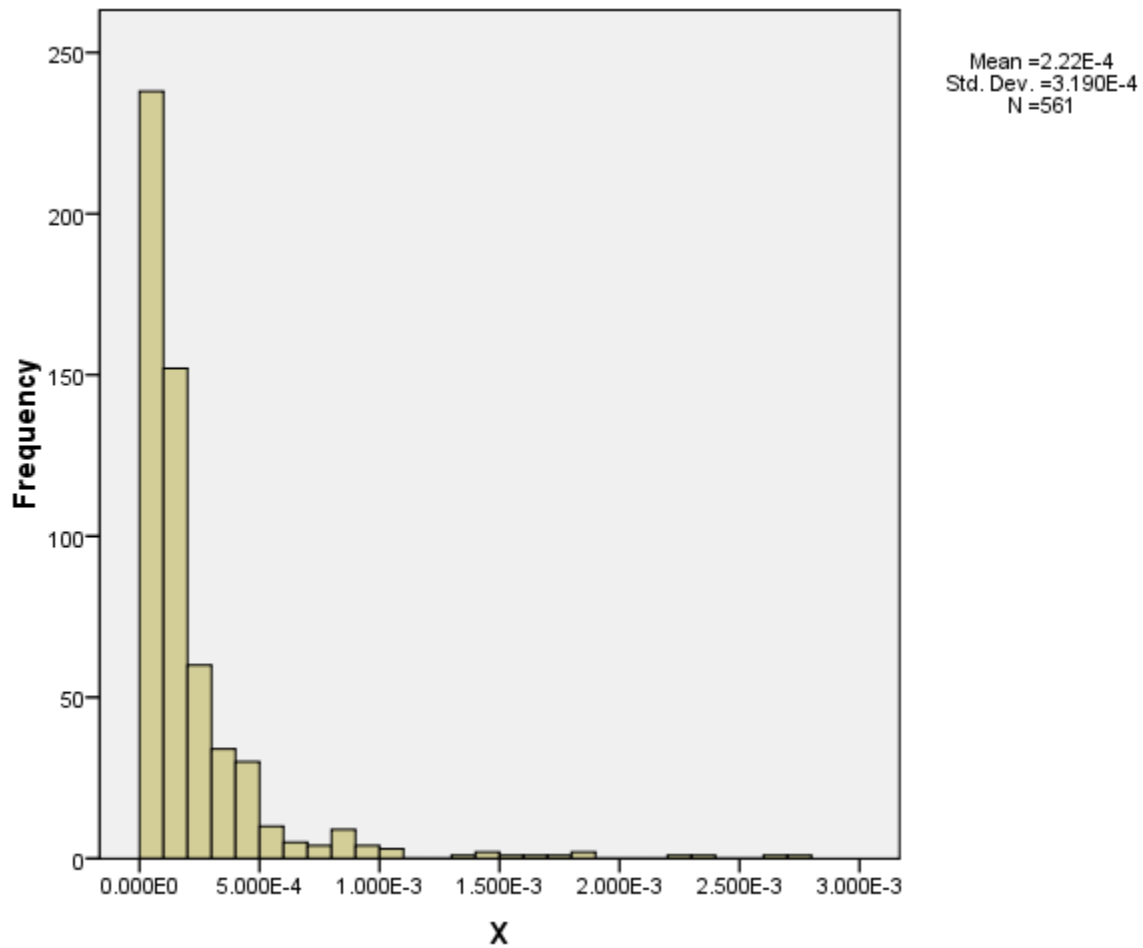


Figure 4.1: Histogram for MMR

Histogram shows a positive skew and unimodal shape.

Table 4.1; Goodness of Fit of some Existing Distributions to Data

Distribution	Kolmogorov Smirnov			AIC		
	Statistic	P-value	Rank	Statistic	Rank	Relative Likelihood
Beta	0.15969	0.00000				
Burr	0.05183	0.09483	3	-8546.016	3	0.00021029
Dagum	0.04357	0.23041	1	-8558.09	2	0.08803683
Exponential	0.14186	0.00000				
Fatigue Life	0.12951	0.00000				
Frechet	0.05164	0.09693	2	-8562.95	1	
Gamma	0.29941	0.00000				
Gen. Gamma	0.19235	0.00000				
Inverse Gaussian	0.10194	0.00000				
Pareto 2	0.16234	0.00000				
Rayleigh	0.33499	0.00000				
Rice	0.50072	0.00000				
Weibull	0.15485	0.00000				

The best fitted distribution based on K-S statistics and associated P-values is the three parameter Dagum distribution with a P-value of 0.23041 while the two-parameter Frechet distribution is

selected by AIC. The relative likelihood of Dagum distribution indicates that Dagum is 0.08803683 probable as the Frechet to minimize information loss, therefore, giving a strong

evidence for Frechet distribution as the best distribution for X.

4.3 Generalization of the Best Fitted Distribution

The Frechet Distribution

$$f(x) = \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \quad (4.1)$$

$$F(x) = \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \quad (4.2)$$

$$\bar{F}(x) = 1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \quad (4.3)$$

$$h(x) = \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right]^{-1} \quad (4.4)$$

$$r(x) = \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \quad (4.5)$$

$\alpha, \beta > 0, \kappa > 0$. α and β are shape and scale parameters respectively

Generalized Frechet Distributions

1. Lehmann Type I Frechet Distribution (LIFD)

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.1)

$$\begin{aligned}
 f_y(x) &= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^{c-1} \\
 &= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c
 \end{aligned} \tag{4.6}$$

Substituting $F(x)$ in (4.2) for $F(x)$ in (3.1a)

$$F_y(x) = \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \tag{4.7}$$

Substituting $\bar{F}(x)$ in (4.2) for $F(x)$ in (3.1b)

$$\bar{F}_y(x) = 1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \tag{4.8}$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.1c)

$$\begin{aligned}
 h_y(x) &= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^{c-1} \left(1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \right)^{-1} \\
 &= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \left(1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \right)^{-1}
 \end{aligned} \tag{4.9}$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.1d)

$$\begin{aligned}
r_y(x) &= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^{-1} \\
&= c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1}
\end{aligned} \tag{4.10}$$

2. Lehmann Type II Frechet Distribution (LIIFD)

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.2)

$$f_y(x) = c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right]^{c-1} \tag{4.11}$$

Substituting $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.2a)

$$F_y(x) = 1 - \left(1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right)^c \tag{4.12}$$

Substituting $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_y(x) = \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right]^c \tag{4.13}$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.2c)

$$h_y(x) = c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right]^{-1} \tag{4.14}$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.2d)

$$r_y(x) = c \frac{\alpha}{\beta} \left(\frac{\beta}{x} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right) \right]^{c-1} \left(1 - \left[1 - \exp \left(- \left(\frac{\beta}{x} \right)^{\alpha} \right)^c \right] \right)^{-1} \quad (4.15)$$

3. Marshall-Olkin Frechet Distribution (MOFRD)

Substituting $F(x)$ in (4.2) for $F(x)$ in (3.3)

$$F_y(x) = \frac{\exp(-(\beta/x)^{\alpha})}{1 - (1-c)[1 - \exp(-(\beta/x)^{\alpha})]} \quad (4.16)$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.3a)

$$f_y(x) = \frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1} \exp(-(\beta/x)^{\alpha})}{(1 - (1-c)[1 - \exp(-(\beta/x)^{\alpha})])^2} \quad (4.17)$$

Substituting $F(x)$ in (4.2) for $F(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.2b)

$$\bar{F}_y(x) = \frac{c[1 - \exp(-(\beta/x)^{\alpha})]}{1 - (1-c)[1 - \exp(-(\beta/x)^{\alpha})]} \quad (4.18)$$

Substituting $h(x)$ in (4.4) for $h(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.3c)

$$h_y(x) = \frac{\alpha\beta^{-1}(\beta/x)^{\alpha+1} \exp(-(\beta/x)^{\alpha})[1 - \exp(-(\beta/x)^{\alpha})]^{-1}}{1 - (1-c)[1 - \exp(-(\beta/x)^{\alpha})]} \quad (4.19)$$

Substituting $r(x)$ in (4.5) for $r(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.3d)

$$r_y(x) = \frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1}}{1 - (1-c)[1 - \exp(-(\beta/x)^{\alpha})]} \quad (4.20)$$

4. Power Transformed Frechet Distribution (PTFD)

Substituting $f(x)$ in (4.1) for $f(x)$ in (3.4)

$$f_y(x) = cx^{c-1} \frac{\alpha}{\beta} \left(\frac{\beta}{x^c} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right) \quad (4.21)$$

Substituting $F(x)$ in (4.2) for $F(x)$ in (3.4a)

$$F_y(x) = \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right) \quad (4.22)$$

Substituting $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.4b)

$$\bar{F}_y(x) = 1 - \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right) \quad (4.23)$$

Substituting $h(x)$ in (4.4) for $h(x)$ in (3.4c)

$$h_y(x) = cx^{c-1} \frac{\alpha}{\beta} \left(\frac{\beta}{x^c} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right) \left[1 - \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right) \right]^{-1} \quad (4.24)$$

Substituting $r(x)$ in (4.5) for $r(x)$ in (3.4d)

$$r_y(x) = cx^{c-1} \frac{\alpha}{\beta} \left(\frac{\beta}{x^c} \right)^{\alpha+1} \quad (4.25)$$

5. Lehmann Type II Power Transformed Frechet Distribution (LIPTFD)

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.9)

$$f_z(x) = c t x^{c-1} \alpha \beta^{-1} (\beta / x^c)^{\alpha+1} \exp(-(\beta / x^c)^\alpha) [1 - \exp(-(\beta / x^c)^\alpha)]^{t-1} \quad (4.26)$$

Substituting $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.9a)

$$F_z(x) = 1 - [1 - \exp(-(\beta / x^c)^\alpha)]^t \quad (4.27)$$

Substituting $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.9b)

$$\bar{F}_z(x) = [1 - \exp(-(\beta / x^c)^\alpha)]^t \quad (4.28)$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.9c)

$$h_z(x) = c t x^{c-1} \alpha \beta^{-1} (\beta / x^c)^{\alpha+1} \exp(-(\beta / x^c)^\alpha) [1 - \exp(-(\beta / x^c)^\alpha)]^{-1} \quad (4.29)$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (3.9d)

$$r_z(x) = c t x^{c-1} \alpha \beta^{-1} (\beta / x^c)^{\alpha+1} \exp(-(\beta / x^c)^\alpha) [1 - \exp(-(\beta / x^c)^\alpha)]^{t-1} (1 - [1 - \exp(-(\beta / x^c)^\alpha)]^t)^{-1} \quad (4.30)$$

6. Lehmann Type I-Power Transformed Frechet Distribution (LIPTFD)

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.15)

$$f_z(x) = c t x^{c-1} \alpha \beta^{-1} (\beta / x^c)^{\alpha+1} \exp(-(\beta / x^c)^\alpha) [\exp(-(\beta / x^c)^\alpha)]^{t-1} \quad (4.31)$$

Substituting $F(x)$ in (4.2) for $F(x)$ in (3.15a)

$$F_z(x) = \exp(-(\beta / x^c)^\alpha)^t \quad (4.32)$$

Substituting $F(x)$ in (4.2) for $F(x)$ in (3.15b)

$$\bar{F}_z(x) = 1 - \exp(-(\beta/x^c)^\alpha)^t \quad (4.33)$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $F(x)$ in (4.2) for $F(x)$ in (3.15c)

$$h_z(x) = ct x^{c-1} \alpha \beta^{-1} (\beta/x^c)^{\alpha+1} \exp(-(\beta/x^c)^\alpha)^t [1 - \exp(-(\beta/x^c)^\alpha)^t]^{-1} \quad (4.34)$$

Substituting $r(x)$ in (4.5) for $r(x)$ in (3.15d)

$$r_z(x) = ct x^{c-1} \alpha \beta^{-1} (\beta/x^c)^{\alpha+1} \quad (4.35)$$

4.4 Distribution Fitting of Generalized Distributions to Data

Table 4.2; Goodness of Fit of Generalized Distributions to Data

Distribution	Kolmogorov Smirnov			AIC		
	Statistic	P-value	Rank	Statistic	Rank	Relative Likelihood
Frechet	0.05164	0.09693	4	-8562.95	2	0.994018
LIIFD	0.04726	0.1598379	1	-8561.582	3	0.501576
MOFRD	0.04764	0.1534482	3	-8562.962	1	
PTFD	0.04741	0.1572908	2	-8561.572	4	
LIPTFD	0.04726	0.1598379	1	-8559.582	5	
LIPTFD	0.04741	0.1572908	2	-8559.572	6	
LIFD	0.04741	0.1572908	2	-8561.572	4	

The best fitted distribution based on K-S statistics and associated P-values is Lehmann Type II distribution with a P-value of 0.1598379 while Marshall-Olkin Frechet distribution is selected by AIC. The relative likelihood of Frechet distribution indicates that Frechet distribution is

0.994018 probable as MOFRD distribution to minimize information loss, therefore, MOFRD is selected as an alternative reference distribution to Frechet distribution for X. The ties in K-S statistics and

corresponding P-values are discussed in section (4.6).

4.5 Results

(ii) Sequentially applying methods in permutations of two distinct parameter induction methods (PTM inclusive) produced two generalized distributions belonging to the same family hence the two methods are said to be commutative and all methods derived except α -Power Transformed method were idempotent hence, only seventeen generalized families of distributions with two additional parameters were generated. Five families with one additional parameter were also derived.

(ii) Lehmann Type I Generalized Pareto Distribution introduced the unimodal and bathtub shapes in Probability Density and Hazard Functions illustrating flexibility of generalized distributions;

(iii) Two-parameter Frechet distribution provided the best fit for MMR amongst plausible distributions studied;

(iv) Four generalized Frechet distributions with an additional shape parameter and two generalized Frechet distributions with two additional shape parameters were generated;

(v) generalized Frechet distributions improved goodness of fit based on K-S distances with Lehman type II generalized Frechet giving the best fit while AIC selected Marshall-Olkin generalized Frechet distribution as a good fitting but parsimonious distribution for MMR.

4.6 Discussion of Results.

Gupta and Kundu (2009) observed that distributions with two additional parameters could be obtained by successive application of two parameter induction methods and thereafter obtained

generalized exponential distributions but observation was generalized to generate families of generalized distribution in present study. These families of generalized distributions proposed generalizes any given base distribution. The two exponentiated GPDs obtained generalized the GPD and demonstrated flexibility of generalized distributions. The lehmann Type I GPD has added flexibility for modelling unimodal and bath-tubed shaped data which is an advantage over GDP.

AIC was found to be negative. (Baguley, 2012, p.402) states that the common misconception is to think that the goal is to minimize the absolute value of AIC, but the arbitrary constant can (depending on data and model) produce negative values. Negative AIC indicates less information loss than a positive AIC therefore a better model. Moutulsky and Christopoulos,(2004, p.144) also states that an AIC value can be positive or negative and the sign of the AIC doesn't really tell anything (it can be changed by using different units to express your data). The ties observed in the K-S statistics and associated P-values is an indication that the simpler models (PTFD, LIFD, and LIIFD) give the same fits to data as the counterpart complex models (LIPTFD and LIIPTFD).

This may imply that there are more parameters in the complex models than are needed to represent MMR. The AIC ranks also show that the simplest model (Frechet distribution) is

preferred to other complex models except for MOFRD. This is also revealed in the correlation matrices of estimated parameters of the two more complex models.

Table 4.3; Correlation Statistics of Estimated Parameters of LIPTFD and LIPTFD

	LIPTFD					LIPTFD			
	c	t	α	β		c	t	α	β
c	1	-0.15580	0.155796	-0.8367	c	1	-0.0036	-0.0131	0.0056
t	-0.15580	1	-0.99998	-0.41058	t	-0.0036	1	-0.9999	-0.99998
α	0.155796	-0.99998	1	0.41057	α	-0.0131	-0.9999	1	0.9998
β	-0.8367	-0.41058	0.41057	1	β	0.0056	-0.99998	0.9998	1

Strong negative correlation between the additional parameter (t) and α in LIPTFD and LIPTFD and also between t and β in LIPTFD show interdependence of parameters. This indicates that a simpler model having fewer parameters (maybe without t) might fit MMR data almost as well.

The high standard errors of some parameter estimates obtained from optimization routines of (LIPTFD, LIPTFD, LIFD, and PTFD) can be attributed to flatness near the maximum. An optimization routine may terminate prematurely near the maximum point because changes in the log likelihood function may be too small to detect improvement. Though theoretically, these

parameter estimates may be considered less significant, global optimization checks showed global maximum was reached. The small P-Value corresponding to K-S statistics of the best fitted distribution may be due to the size of data. In practice, a relatively large number of data points is needed for K-S statistics to properly reject the null hypothesis.

The relative likelihood of the Frechet distribution led to the choice of Marshall-Olkin Frechet distribution as an alternative reference distribution to the Frechet distribution for MMR. Krishna et al. (2013) introduced and applied this model to 72 observations of survival time of injected guinea pigs with different doses of tubercle bacilli and found it to be a competitive model to the Frechet distribution. These two results strengthen the evidence for Marshall-Olkin Frechet distribution to perform as well as and even much better than the Frechet distribution. The properties of distribution are found in chapter two.

CHAPTER FIVE

SUMMARY, CONCLUSION, AND RECOMMENDATIONS

5.1 Introduction

This chapter contains the summary, conclusions, recommendations, and contribution to knowledge of the research work. The study revealed areas that may be valuable to prospective researchers for future research. These areas are also included in this chapter as suggestions for further research.

5.2 Summary

Sustainable Development Goal 3.1 targets reduction in the global MMR by the year 2030, to less than 70 per 100, 000 live births. Many studies have been focused on estimation and trends in maternal mortality but little or no work has been directed towards finding the distribution that describes global maternal mortality estimates. Probability distributions provide extra information crucial for making plans and taking of decisions.

For the purpose of finding a probability distribution to represent MMR, a measure of maternal mortality, study generated families of generalized distributions with additional parameter(s) by

applying five existing parameter induction methods and sequentially applying methods in permutations of these five distinct parameter induction methods to a base distribution taking two methods at a time. Each of the five parameter induction methods was also applied twice. The flexibility of these families of generalized distributions deal with the limitation on usage of some existing distributions for certain data analysis imposed by shapes of probability density and hazard functions. Generalized distributions do also improve goodness of fit.

Flexibility of generalized classes of distributions was illustrated by generating exponentiated families of GPD which enhanced its capability to accommodate analysis of data with unimodal and bathtub shapes of probability density and hazard functions. The shape of MMR data as was revealed by histogram plot suggested plausible existing distribution that were fitted to MMR data. Frechet distribution was found to provide the best fit amongst some plausible distributions based on AIC. Generalized Frechet distributions belonging to some of the generalized families of distributions earlier generated were subsequently fitted to MMR and results showed an improvement of fit based on K-S statistics. Marshall-Olkin generalized Frechet distribution was then selected by AIC as an alternative distribution to the Frechet distribution for MMR.

5.3 Conclusion

Generalized distributions introducing two parameters may be obtained from sequentially applying methods in permutations of $n \geq 2$ distinct parameter induction methods taken two methods at a time. In general, models introducing additional n parameters may be obtained by sequentially applying methods in permutation of s ($s \geq n$) distinct parameter induction methods taken n methods at a time. Only seventeen generalized families of distributions with two additional parameters were obtained from sequentially applying methods in permutations of five

distinct parameter induction methods taken two methods at a time and from the application of each of the five parameter induction methods twice due to commutativity and idempotency of some methods.

Generalized distributions improve flexibility and goodness of fit. Two modifications of GPD belonging to the exponentiated class of distributions were introduced for the purpose of demonstrating the flexibility of generalized distributions. Lehmann Type I GPD added flexibility by introducing the unimodal and bathtub shapes in the probability density and hazard functions, therefore, model can be used for analyzing different types of data. Frechet distribution was found to provide the best fit for MMR amongst some plausible existing distributions based on AIC. Generalized Frechet distributions improved goodness of fit but for parsimony, Marshall-Olkin generalized Frechet distribution is recommended as an alternative reference distributive to Frechet distribution for modelling MMR.

5.4 Recommendations

- (i) Marshall-Olkin generalized Frechet distribution as an alternative distribution to the Frechet distribution for MMR.
- (ii) Families of generalized distributions for use in modification of existing distributions.
- (iii) Lehmann Type I Generalized Pareto Distribution as a suitable distribution for modelling data with unimodal, bathtub, increasing, decreasing and constant Probability density and hazard functions.

5.5 Contribution to Knowledge

- (ii) Detailed procedures of generating families of generalized distributions

(ii) Families of generalized distributions for obtaining generalized classes of distributions

(iii) Extensions of the Generalized Pareto Distribution that can be used in analyzing different types of data.

(iv) A probability distribution to represent MMR

Suggestions for Further Research

(i) Competitive reference distribution to the Frechet distribution and Marshall-Olkin generalized Frechet distributions for MMR

(ii) Investigation of other properties of new families of generalized distribution

(iii) Application of new families of generalized distributions in subsequent analyses.

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Appendix A. Data Presentation

MM Data, LB Data and Corresponding MMR

Countries	2004 (MM)	2004 (LB)	2004 (MMR)	2005 (MM)	2005 (LB)	2005 (MMR)
Austria	3	78968	0.0000379901	3	78190	0.0000383681
Azerbaijan	50	131609	0.0003799130	56	141901	0.0003946410
Belgium	9	117295	0.0000767296	9	119622	0.0000752370
Bulgaria	11	69886	0.0001573990	11	71075	0.0001547660
Bosnia & Herz.	5	34167	0.0001463400	5	34627	0.0001443960
Belarus	15	88943	0.0001686470	12	90508	0.0001325850
Switzerland	5	73082	0.0000684163	5	72903	0.0000685843
Costa Rica	23	72247	0.0003183520	23	71548	0.0003214630
Cuba	53	127192	0.0004166930	55	120716	0.0004556150
Cyprus	1	8309	0.0001203510	1	8243	0.0001213150
Czech Republic	6	97664	0.0000614351	6	102211	0.0000587021
Germany	50	705622	0.0000708595	49	685795	0.0000714499
Denmark	6	64609	0.0000928663	5	64282	0.0000777823
Egypt, Arab Rep.	990	1779500	0.0005563360	970	1800972	0.0005385980
Spain	24	453172	0.0000529600	25	464811	0.0000537853
Estonia	2	13992	0.0001429390	2	14350	0.0001393730
Finland	3	57758	0.0000519409	2	57745	0.0000346350
France	80	800240	0.0000999700	80	807787	0.0000990360
United Kingdom	86	715996	0.0001201120	86	722549	0.0001190230

Greece	4	105655	0.0000378591	4	107545	0.0000371937
Hungary	13	95137	0.0001366450	14	97496	0.0001435960
Ireland	5	61972	0.0000806816	5	61372	0.0000814704
Israel	10	178723	0.0000559525	10	176427	0.0000566807
Italy	24	562599	0.0000426592	24	554022	0.0000433196
Japan	85	1110721	0.0000765269	81	1062530	0.0000762331
Kyrgyz Republic	88	109939	0.0008004440	96	109839	0.0008740070
Korea, Rep.	66	472761	0.0001396050	67	435031	0.0001540120
Lithuania	4	29769	0.0001343680	4	29510	0.0001355470
Luxembourg	1	5452	0.0001834190	1	5371	0.0001861850
Latvia	5	20551	0.0002432970	5	21879	0.0002285300
Moldova	16	38272	0.0004180600	17	37695	0.0004509880
Maldives	7	5220	0.0013409960	6	5543	0.0010824460
Macedonia, FYR	3	23361	0.0001284190	2	22482	0.0000889601
Montenegro	1	7849	0.0001274050	1	7352	0.0001360170
Mauritius	7	19102	0.0003664540	7	18578	0.0003767900
Malaysia	250	477768	0.0005232670	240	469204	0.0005115050
Netherlands	23	194007	0.0001185520	21	187910	0.0001117560
Norway	4	56951	0.0000702358	4	56756	0.0000704771
New Zealand	8	58073	0.0001377580	8	57745	0.0001385400
Poland	22	356131	0.0000617750	21	364383	0.0000576317
Portugal	13	109298	0.0001189410	13	109399	0.0001188310
Romania	77	216261	0.0003560510	71	221020	0.0003212380
Russian Fed.	650	1502477	0.0004326190	640	1457376	0.0004391450
Singapore	8	37292	0.0002145230	8	37492	0.0002133790
Serbia	16	78186	0.0002046400	15	72180	0.0002078140
Suriname	25	9062	0.0027587730	23	8657	0.0026568100
Slovenia	2	17961	0.0001113520	2	18157	0.0001101500
Sweden	6	100928	0.0000594483	5	101346	0.0000493359
Turkey	860	1222000	0.0007037640	760	1244000	0.0006109320
Ukraine	120	427259	0.0002808600	130	426086	0.0003051030
United States	540	4112052	0.0001313210	560	4138349	0.0001353200
	2006(MM)	2006(LB)	2006(MMR)	2007(MM)	2007(LB)	2007(MMR)
Austria	3	77914	0.0000385040	3	76250	0.0000393443
Azerbaijan	51	148946	0.0003424060	52	151963	0.0003421890
Belgium	9	122529	0.0000734520	10	124095	0.0000805834
Bulgaria	10	73978	0.0001351750	9	75349	0.0001194440
Bosnia & Herz.	5	34033	0.0001469160	5	33835	0.0001477760
Belarus	10	96721	0.0001033900	8	103626	0.0000772007
Switzerland	5	73371	0.0000681468	5	74494	0.0000671195
Costa Rica	24	71291	0.0003366480	21	73144	0.0002871050
Cuba	50	111323	0.0004491430	51	112472	0.0004534460
Cyprus	1	8731	0.0001145340	1	8575	0.0001166180

Czech Republic	6	105831	0.0000566942	6	114632	0.0000523414
Germany	47	672724	0.0000698652	46	684862	0.0000671668
Denmark	5	64984	0.0000769420	5	64082	0.0000780250
Egypt, Arab Rep.	950	1853746	0.0005124760	940	1949569	0.0004821580
Spain	25	481295	0.0000519432	25	491138	0.0000509022
Estonia	2	14877	0.0001344360	2	15775	0.0001267830
Finland	2	58840	0.0000339905	2	58729	0.0000340547
France	79	830288	0.0000951477	78	819605	0.0000951678
United Kingdom	85	748563	0.0001135510	84	772245	0.0001087740
Greece	3	112042	0.0000267757	4	111926	0.0000357379
Hungary	14	99871	0.0001401810	15	97613	0.0001536680
Ireland	5	65425	0.0000764234	5	71389	0.0000700388
Israel	10	171444	0.0000583281	10	170979	0.0000584867
Italy	23	560010	0.0000410707	23	563933	0.0000407850
Japan	75	1092674	0.0000686390	70	1089818	0.0000642309
Kyrgyz Republic	110	120737	0.0009110710	110	123251	0.0008924880
Korea, Rep.	60	451514	0.0001328860	66	493189	0.0001338230
Lithuania	4	29606	0.0001351080	4	30020	0.0001332450
Luxembourg	1	5514	0.0001813570	1	5477	0.0001825820
Latvia	5	22871	0.0002186170	5	23958	0.0002086990
Moldova	16	37602	0.0004255090	16	37973	0.0004213520
Maldives	6	5829	0.0010293360	6	6569	0.0009133810
Macedonia, FYR	2	22585	0.0000885544	2	22688	0.0000881523
Montenegro	1	7531	0.0001327840	1	7834	0.0001276490
Mauritius	7	17320	0.0004041570	8	16894	0.0004735410
Malaysia	230	465112	0.0004945050	220	456443	0.0004819880
Netherlands	19	185057	0.0001026710	17	181336	0.0000937486
Norway	4	58545	0.0000683235	4	58459	0.0000684240
New Zealand	8	59193	0.0001351510	8	64044	0.0001249140
Poland	20	374244	0.0000534411	19	387873	0.0000489851
Portugal	12	105449	0.0001137990	12	102492	0.0001170820
Romania	65	219483	0.0002961500	62	214728	0.0002887370
Russian Fed.	570	1479637	0.0003852300	540	1610122	0.0003353780
Singapore	8	38317	0.0002087850	7	39490	0.0001772600
Serbia	15	70997	0.0002112770	14	68102	0.0002055740
Suriname	22	9311	0.0023627970	22	9769	0.0022520220
Slovenia	2	18932	0.0001056410	2	19823	0.0001008930
Sweden	5	105913	0.0000472086	5	107421	0.0000465458
Turkey	630	1255000	0.0005019920	510	1289000	0.0003956560
Ukraine	130	460,368	0.0002823830	140	472657	0.0002961980
United States	570	4265555	0.0001336290	570	4316233	0.0001320600
	2008(MM	2008(LB)	2008(MMR)	2009(MM	2009(LB)	2009(MMR)
Austria	3	77752	0.0000385842	3	76344	0.0000392958

Azerbaijan	53	152086	0.0003484870	52	152139	0.0003417930
Belgium	10	127205	0.0000786133	10	127198	0.0000786176
Bulgaria	9	77712	0.0001158120	8	80956	0.0000988191
Bosnia & Herz.	5	34176	0.0001463010	5	34550	0.0001447180
Belarus	7	107876	0.0000648893	6	109263	0.0000549134
Switzerland	5	76691	0.0000651967	4	78286	0.0000510947
Costa Rica	22	75187	0.0002926040	23	75000	0.0003066670
Cuba	53	122569	0.0004324090	53	130036	0.0004075790
Cyprus	1	9205	0.0001086370	1	9608	0.0001040800
Czech Republic	6	119570	0.0000501798	6	118348	0.0000506979
Germany	46	682514	0.0000673979	47	665126	0.0000706633
Denmark	5	65038	0.0000768781	5	62818	0.0000795950
Egypt, Arab Rep.	920	2050704	0.0004486260	920	2217409	0.0004148990
Spain	25	518503	0.0000482157	24	493717	0.0000486108
Estonia	2	16028	0.0001247820	2	15763	0.0001268790
Finland	2	59530	0.0000335965	2	60430	0.0000330961
France	79	829311	0.0000952598	79	825564	0.0000956922
United Kingdom	84	794383	0.0001057420	82	790204	0.0001037710
Greece	4	118302	0.0000338118	4	117933	0.0000339176
Hungary	15	99149	0.0001512870	15	96442	0.0001555340
Ireland	6	75173	0.0000798159	6	75554	0.0000794134
Israel	9	166296	0.0000541204	10	166255	0.0000601486
Italy	22	576659	0.0000381508	23	568857	0.0000404320
Japan	67	1091156	0.0000614028	64	1070035	0.0000598111
Kyrgyz Republic	120	127332	0.0009424180	120	135494	0.0008856480
Korea, Rep.	66	465892	0.0001416640	70	444849	0.0001573570
Lithuania	4	31536	0.0001268390	3	32165	0.0000932691
Luxembourg	1	5596	0.0001786990	1	5638	0.0001773680
Latvia	4	24397	0.0001639550	4	22044	0.0001814550
Moldova	16	39018	0.0004100670	15	40803	0.0003676200
Maldives	7	6989	0.0010015740	7	7423	0.0009430150
Macedonia, FYR	2	22945	0.0000871650	2	23684	0.0000844452
Montenegro	1	8258	0.0001210950	1	8642	0.0001157140
Mauritius	7	16094	0.0004349450	8	15259	0.0005242810
Malaysia	220	487346	0.0004514250	230	496313	0.0004634170
Netherlands	16	184634	0.0000866579	15	184915	0.0000811184
Norway	4	60497	0.0000661190	3	61807	0.0000485382
New Zealand	8	64343	0.0001243340	8	65543	0.0001220570
Poland	18	414499	0.0000434259	17	417589	0.0000407099
Portugal	11	104594	0.0001051690	11	99491	0.0001105630
Romania	61	221900	0.0002748990	62	222388	0.0002787920
Russian Fed.	530	1713947	0.0003092280	510	1761687	0.0002894950
Singapore	7	39826	0.0001757650	6	39570	0.0001516300

Serbia	14	69083	0.0002026550	14	70299	0.0001991490
Suriname	19	10097	0.0018817470	18	9792	0.0018382350
Slovenia	2	21817	0.0000916716	2	21856	0.0000915081
Sweden	5	109301	0.0000457452	5	111801	0.0000447223
Turkey	420	1294000	0.0003245750	350	1263289	0.0002770550
Ukraine	150	510589	0.0002937780	140	512525	0.0002731570
United States	580	4247694	0.0001365450	600	4130665	0.0001452550
	2010(MM)	2010(LB)	2010(MMR)	2011(MM)	2011(LB)	2011(MMR)
Austria	3	78742	0.0000380991	3	78109	0.0000384079
Azerbaijan	54	165643	0.0003260020	52	176072	0.0002953340
Belgium	10	130100	0.0000768640	10	128705	0.0000776971
Bulgaria	8	75513	0.0001059420	8	70846	0.0001129210
Bosnia & Herz.	5	33528	0.0001491290	4	31875	0.0001254900
Belarus	6	108050	0.0000555298	6	109147	0.0000549717
Switzerland	5	80290	0.0000622743	5	80808	0.0000618751
Costa Rica	21	70922	0.0002961000	20	73459	0.0002722610
Cuba	54	127746	0.0004227140	52	133067	0.0003907810
Cyprus	1	9801	0.0001020300	1	9622	0.0001039280
Czech Republic	5	117153	0.0000426792	5	108673	0.0000460096
Germany	46	677947	0.0000678519	45	662685	0.0000679056
Denmark	5	63411	0.0000788507	5	58998	0.0000847486
Egypt, Arab Rep.	910	2261409	0.0004024040	900	2442094	0.0003685360
Spain	22	485252	0.0000453373	23	470553	0.0000488787
Estonia	1	15825	0.0000631912	1	14679	0.0000681245
Finland	2	60980	0.0000327976	2	59961	0.0000333550
France	72	833654	0.0000863668	69	824263	0.0000837111
United Kingdom	80	807271	0.0000990993	77	807776	0.0000953235
Greece	4	114766	0.0000348535	4	106428	0.0000375841
Hungary	14	90335	0.0001549790	15	88049	0.0001703600
Ireland	5	75174	0.0000665124	6	74033	0.0000810449
Israel	9	161042	0.0000558860	9	156923	0.0000573530
Italy	21	561944	0.0000373703	21	546585	0.0000384204
Japan	61	1071304	0.0000569400	61	1050806	0.0000580507
Kyrgyz Republic	120	146123	0.0008212260	130	149612	0.0008689140
Korea, Rep.	68	470171	0.0001446280	65	471265	0.0001379270
Lithuania	3	30676	0.0000977963	3	30268	0.0000991146
Luxembourg	1	5874	0.0001702420	1	6026	0.0001659480
Latvia	4	19781	0.0002022140	4	18825	0.0002124830
Moldova	15	40474	0.0003706080	11	39182	0.0002807410
Maldives	6	7115	0.0008432890	6	7180	0.0008356550
Macedonia, FYR	2	24296	0.0000823181	2	22770	0.0000878349
Montenegro	1	7418	0.0001348070	1	7215	0.0001386000
Mauritius	9	14831	0.0006068370	8	14626	0.0005469710

Malaysia	230	475816	0.0004833800	220	511594	0.0004300280
Netherlands	14	184397	0.0000759231	14	180060	0.0000777519
Norway	3	61442	0.0000488265	3	60220	0.0000498173
New Zealand	8	63897	0.0001252010	8	61403	0.0001302870
Poland	15	413300	0.0000362932	14	388416	0.0000360438
Portugal	11	101381	0.0001085020	9	96855	0.0000929224
Romania	61	212199	0.0002874660	60	196242	0.0003057450
Russian Fed.	500	1788948	0.0002794940	480	1796629	0.0002671670
Singapore	6	37967	0.0001580320	6	39654	0.0001513090
Serbia	15	68304	0.0002196060	15	65598	0.0002286660
Suriname	17	9712	0.0017504120	16	9703	0.0016489750
Slovenia	2	22343	0.0000895135	2	21947	0.0000911286
Sweden	5	115641	0.0000432373	5	111770	0.0000447347
Turkey	300	1256000	0.0002388540	270	1241412	0.0002174940
Ukraine	130	497689	0.0002612070	120	502595	0.0002387610
United States	580	3999386	0.0001450220	580	3953590	0.0001467020
	2012(MM)	2012(LB)	2012(MMR)	2013(MM)	2013(LB)	2013(MMR)
Austria	3	78952	0.0000379978	3	79330	0.0000378167
Azerbaijan	50	174469	0.0002865840	51	172671	0.0002953590
Belgium	9	128051	0.0000702845	9	125606	0.0000716526
Bulgaria	7	69121	0.0001012720	7	66578	0.0001051400
Bosnia & Herz.	4	32547	0.0001228990	4	30684	0.0001303610
Belarus	5	115893	0.0000431432	5	117997	0.0000423740
Switzerland	5	82164	0.0000608539	4	82731	0.0000483495
Costa Rica	19	73326	0.0002591170	19	70550	0.0002693130
Cuba	49	125674	0.0003898980	48	125880	0.0003813160
Cyprus	1	10161	0.0000984155	1	9341	0.0001070550
Czech Republic	5	108576	0.0000460507	5	106751	0.0000468380
Germany	44	673544	0.0000653261	44	682069	0.0000645096
Denmark	4	57916	0.0000690655	4	55873	0.0000715909
Egypt, Arab Rep.	890	2629769	0.0003384330	870	2621902	0.0003318200
Spain	21	453348	0.0000463220	21	424440	0.0000494770
Estonia	1	14056	0.0000711440	1	13531	0.0000739044
Finland	2	59493	0.0000336174	2	58134	0.0000344033
France	69	821844	0.0000839575	69	812343	0.0000849395
United Kingdom	77	812970	0.0000947144	76	778358	0.0000976414
Greece	3	100371	0.0000298891	3	94134	0.0000318695
Hungary	15	90269	0.0001661700	15	89524	0.0001675530
Ireland	6	71674	0.0000837124	5	68949	0.0000725174
Israel	9	151679	0.0000593358	9	148170	0.0000607410
Italy	20	534186	0.0000374401	19	514308	0.0000369428
Japan	61	1037231	0.0000588104	59	1029816	0.0000572918
Kyrgyz Republic	120	154918	0.0007746030	120	155520	0.0007716050

Korea, Rep.	60	484550	0.0001238260	56	436455	0.0001283060
Lithuania	3	30459	0.0000984931	3	29885	0.0001003850
Luxembourg	1	6115	0.0001635320	1	6070	0.0001647450
Latvia	4	19897	0.0002010350	4	20596	0.0001942120
Moldova	12	39435	0.0003042980	11	37871	0.0002904600
Maldives	6	7431	0.0008074280	5	7153	0.0006990070
Macedonia, FYR	2	23568	0.0000848608	2	23138	0.0000864379
Montenegro	1	7459	0.0001340660	1	7475	0.0001337790
Mauritius	8	14502	0.0005516480	8	13688	0.0005844540
Malaysia	220	508774	0.0004324120	210	503914	0.0004167380
Netherlands	13	175959	0.0000738808	13	171341	0.0000758721
Norway	3	60255	0.0000497884	3	58878	0.0000509528
New Zealand	7	61178	0.0001144200	7	58717	0.0001192160
Poland	13	386257	0.0000336563	13	369576	0.0000351754
Portugal	10	89841	0.0001113080	9	82787	0.0001087130
Romania	58	201104	0.0002884080	57	188599	0.0003022290
Russian Fed.	470	1896263	0.0002478560	460	1895822	0.0002426390
Singapore	6	42663	0.0001406370	5	39720	0.0001258810
Serbia	15	67257	0.0002230250	15	65554	0.0002288190
Suriname	15	10217	0.0014681410	16	10012	0.0015980820
Slovenia	2	21938	0.0000911660	2	21111	0.0000947373
Sweden	5	113177	0.0000441786	5	113593	0.0000440168
Turkey	250	1279864	0.0001953330	230	1283062	0.0001792590
Ukraine	120	520705	0.0002304570	120	503657	0.0002382570
United States	570	3952841	0.0001442000	570	3932181	0.0001449580
	2014(MM)	2014(LB)	2014(MMR)			
Austria	3	81722	0.0000367098			
Azerbaijan	50	170503	0.0002932500			
Belgium	9	125014	0.0000719919			
Bulgaria	7	67585	0.0001035730			
Bosnia & Herz.	4	29247	0.0001367660			
Belarus	5	118534	0.0000421820			
Switzerland	4	85287	0.0000469005			
Costa Rica	18	71793	0.0002507210			
Cuba	47	122643	0.0003832260			
Cyprus	1	9258	0.0001080150			
Czech Republic	5	109860	0.0000455125			
Germany	43	714927	0.0000601460			
Denmark	4	56870	0.0000703359			
Egypt, Arab Rep.	850	2720495	0.0003124430			
Spain	21	426076	0.0000492870			
Estonia	1	13551	0.0000737953			
Finland	2	57232	0.0000349455			

France	68	819328	0.0000829948			
United Kingdom	75	775908	0.0000966609			
Greece	3	92149	0.0000325560			
Hungary	15	93281	0.0001608040			
Ireland	5	67295	0.0000742997			
Israel	9	143913	0.0000625378			
Italy	19	502596	0.0000378037			
Japan	57	1003539	0.0000567990			
Kyrgyz Republic	120	161813	0.0007415970			
Korea, Rep.	53	435435	0.0001217170			
Lithuania	3	30369	0.0000987849			
Luxembourg	1	6115	0.0001635320			
Latvia	4	21746	0.0001839420			
Moldova	10	38616	0.0002589600			
Maldives	5	7245	0.0006901310			
Macedonia, FYR	2	23596	0.0000847601			
Montenegro	1	7529	0.0001328200			
Mauritius	8	13283	0.0006022740			
Malaysia	210	511865	0.0004102640			
Netherlands	12	175181	0.0000685006			
Norway	3	58976	0.0000508681			
New Zealand	7	57242	0.0001222880			
Poland	12	375160	0.0000319864			
Portugal	9	82367	0.0001092670			
Romania	57	193103	0.0002951790			
Russian Fed.	460	1913472	0.0002404010			
Singapore	5	42232	0.0001183940			
Serbia	15	66461	0.0002256960			
Suriname	15	10407	0.0014413380			
Slovenia	2	21165	0.0000944956			
Sweden	5	114907	0.0000435135			
Turkey	220	1337504	0.0001644850			
Ukraine	120	465882	0.0002575760			
United States	560	3988076	0.0001404190			

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Appendix B. R Codes for the Algorithms

Power Transformed Frechet Distribution (PTFD)

$$f_y(x) = cx^{c-1} \frac{\alpha}{\beta} \left(\frac{\beta}{x^c} \right)^{\alpha+1} \exp \left(- \left(\frac{\beta}{x^c} \right)^{\alpha} \right)$$

PTFD Log Likelihood Function

$$LLF = n \ln(c) + \sum_{i=1}^n (c-1) \ln(x) + n \ln(\alpha) - n \ln(\beta) + \sum_{i=1}^n (\alpha+1) \ln(\beta / x^c) - \sum_{i=1}^n (\beta / x^c)^{\alpha}$$

Declaration of PTFD Negative Log-Likelihood Function

```
PTFDLLF <- function(par){(-561 * log( par[1])) - sum(( par[1] - 1) * log( x)) -  
  (561 * log( par[2])) + (561 * log( par[3])) - sum(( par[2] + 1) *  
  log( par[3] /(x^ par[1]))) + sum(( par[3] /(x^ par[1]))^ par[2])}
```

Maximum Likelihood Estimate (MLE) is the estimate obtained from minimizing negative log likelihood.

Optimization of PTFD Log-Likelihood Function

```
PTFDLLFO <- nlmin b(c(c(initial), alpha(initial), beta(initial)), PTFDLLF, hessian = TRUE,  
  lower = (0.000000000001,0.000000000001,0.000000000001),  
  upper = (Inf, Inf, Inf))
```

Gradient Function

```
G1 <- -grad (PTFDLLF, c(0.710865040,1.892515954,0.001303365))
```

Hessian Function

$$H1 < -hesian(PTFDLLF, c(0.710865040), 1.892515954, 0.001303365))$$

Variance-Covariance Matrix

$$IH1 < -solve(H1)$$

Standard Errors of Estimates

$$STE1 < -sqrt(diag(IH1))$$

Marshall-Olkin Frechet Distribution (MOFD)

$$\begin{aligned} f_y(x) &= \frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1}\exp(-(\beta/x)^\alpha)}{(1-(1-c)[1-\exp(-(\beta/x)^\alpha)])^2} \\ &= \frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1}\exp(-(\beta/x)^\alpha)}{(\exp(-(\beta/x)^\alpha) + c(1 - [\exp(-(\beta/x)^\alpha)]))^2} \end{aligned}$$

MOFD Log Likelihood Function

$$\begin{aligned} LLF &= n \ln(c) + n \ln(\alpha) - n \ln(\beta) + \sum_{i=1}^n (\alpha + 1) \ln(\beta / x) + \sum_{i=1}^n -(\beta / x)^\alpha - \\ &\quad 2 * \ln(\exp(-(\beta / x)^\alpha) + c(1 - [\exp(-(\beta / x)^\alpha)])) \end{aligned}$$

Declaration of MOFD Negative Log-Likelihood Function

$$\begin{aligned} MOFDLLF &< -function(par){(-561 * log(par[1])) - (561 * log(par[2])) + (561 * log(par[3])) - \\ &\quad sum((par[2] + 1) * log(par[3] / x)) - sum(-(par[3] / x)^par[2]) + \\ &\quad sum(2 * log(exp(-(par[3] / x)^par[2]) + (par[1] * (1 - exp(-(par[3] / x)^par[2])))))} \end{aligned}$$

Optimization of MOFD Log-Likelihood Function

$MOFDLLFO < -nl \min b(c(c(initial), \alpha(initial), \beta(initial)), MOFDLLF, hessian = TRUE,$

$lower = (0.000000000001, 0.000000000001, 0.000000000001),$

$upper = (Inf, Inf, Inf))$

Gradient Function

$G2 < -grad(MOFDLLF, c(1.469365, 1.454048, 7.46626e - 05))$

Hessian Function

$H2 < -hessian(MOFDLLF, c(1.469365, 1.454048, 7.46626e - 05))$

Variance-Covariance Matrix

$IH2 < -solve(H2)$

Standard Errors of Estimates

$STE2 < -sqrt(diag(IH2))$

Lehmann Type II Frechet Distribution (LIIFD)

$$f_y(x) = c \left[1 - \exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right) \right]^{c-1} \frac{\alpha}{\beta} \left(\frac{\beta}{x}\right)^{\alpha+1} \exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)$$

LIIFD Log Likelihood Function

$$LLF = n \ln(c) + \sum_{i=1}^n (c-1) \ln(1 - \exp(-(\beta / x)^\alpha)) + n \ln(\alpha) - n \ln(\beta) +$$

$$\sum_{i=1}^n (\alpha + 1) \ln(\beta / x) + \sum_{i=1}^n [-(\beta / x)^\alpha]$$

Declaration of LIIFD Negative Log-Likelihood Function

$$\begin{aligned} LIIFDLLF < -function(par)\{(-561 * \log(par[1])) - sum((par[1] - 1) * \log(1 - (\exp(-((par[2] \\ / x)^\alpha par[3])))) - (561 * \log(par[3])) + (561 * \log(par[2])) - \\ sum((par[3] + 1) * \log(par[2] / x)) - sum(-((par[2] / x)^\alpha par[3]))\} \end{aligned}$$

Optimization of LIIFD Log-Likelihood Function

$$\begin{aligned} LIIFDLLFO < -nlminb(c(c(initial), \beta(initial), \alpha(initial)), LIIFDLLF, hessian = TRUE, \\ lower = (0.000000000001, 0.000000000001, 0.000000000001), \\ upper = (Inf, Inf, Inf)) \end{aligned}$$

Gradient Function

$$G3 < -grad(LIIFDLLF, c(9.758647e - 01, 8.603125e - 05, 1.364839))$$

Hessian Function

$$H3 < -hessian(LIIFDLLF, c(9.758647e - 01, 8.603125e - 05, 1.364839))$$

Variance-Covariance Matrix

$$IH3 < -solve(H3)$$

Standard Errors of Estimates

$$STE3 < -sqrt(diag(IH3))$$

Lehmann Type II Power Transformed Frechet Distribution (LIPTFD)

$$f_z(x) = tcx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^\alpha)[1-\exp(-(\beta/x^c)^\alpha)]^{t-1}$$

LIPTFD Log Likelihood Function

$$\begin{aligned} LLF = & n \ln(t) + n \log(c) + \sum_{i=1}^n (c-1) \ln(x) + n \ln(\alpha) - n \ln(\beta) + \\ & \sum_{i=1}^n (\alpha+1) \ln(\beta/x^c) + \sum_{i=1}^n [-(\beta/x^c)^\alpha] + \\ & \sum_{i=1}^n (t-1) \log(1 - \exp(-(\beta/x^c)^\alpha)) \end{aligned}$$

Declaration of LIPTFD Negative Log-Likelihood Function

$$\begin{aligned} LIPTFDLLF < -function(par){(-561 * \log(par[1])) - (561 * \log(par[2])) - sum((par[2] \\ & - 1) * \log(x)) - (561 * \log(par[3])) + (561 * \log(par[4])) - \\ & sum((par[3] + 1) * \log(par[4]/(x^par[2])) - sum(-((par[4]/(x^par[2]))^par[3])) - sum((par[1] - 1) * \log(1 - \exp(-((par[4] \\ & /(x^par[2]))^par[3]))))} \end{aligned}$$

Optimization of LIPTFD Log-Likelihood Function

$$\begin{aligned} LIPTFDLLFO < -nl \min \quad & b(c(t(initial), c(initial), \alpha(initial), \beta(initial)), LIPTFDLLF, \\ & hessian = TRUE, lower = (0.00000000001, 0.00000000001, \\ & 0.00000000001, 0.00000000001), upper = (Inf, Inf, Inf, Inf)) \end{aligned}$$

Gradient Function

$$G4 < -grad(LIPTFDLLF, c(0.09758650734, 0.7726897865, 1.7663482544, 0.0007223546))$$

Hessian Function

$$H4 < -hessian(LIPTFDLLF, c(0.09758650734, 0.7726897865, 1.7663482544, 0.0007223546))$$

Variance-Covariance Matrix

$$IH4 < -solve(H4)$$

Standard Errors of Estimates

$$STE4 < -sqrt(diag(IH4))$$

Lehmann Type I Power Transformed Frechet Distribution (LIPTFD)

$$f_z(x) = tcx^{c-1} \alpha \beta^{-1} (\beta / x^c)^{\alpha+1} \exp(-(\beta / x^c)^\alpha) [\exp(-(\beta / x^c)^\alpha)]^{t-1}$$

LIPTFD Log Likelihood Function

$$LLF = n \ln(t) + n \log(c) + \sum_{i=1}^n (c-1) \ln(x) + n \ln(\alpha) - n \ln(\beta) +$$

$$\sum_{i=1}^n (\alpha+1) \ln(\beta / x^c) + \sum_{i=1}^n [-(\beta / x^c)^\alpha] +$$

$$\sum_{i=1}^n (t-1)(-(\beta / x^c)^\alpha)$$

Declaration of LIPTFD Negative Log-Likelihood Function

$$LIPTFDLLF < -function(par){(-561 * log(par[1])) - (561 * log(par[2])) - sum((par[2]$$

$$- 1) * log(x)) - (561 * log(par[3])) + (561 * log(par[4])) -$$

$$\begin{aligned} & \text{sum}((\text{par}[3] + 1) * \log(\text{par}[4] / (x^{\text{par}[2]}))) - \text{sum}(-((\text{par}[4] / (x^{\text{par}[2]})^{\text{par}[3]}))) - \text{sum}((\text{par}[1] - 1) * (-((\text{par}[4] \\ & / (x^{\text{par}[2]})^{\text{par}[3]}))) \} \end{aligned}$$

Optimization of LIPTFD Log-Likelihood Function

$$\begin{aligned} \text{LIPTFDLLFO} &< -\text{nl min } b(c(t(\text{initial}), c(\text{initial}), \alpha(\text{initial}), \beta(\text{initial})), \text{LIPTFDLLF}, \\ & \text{hessian} = \text{TRUE}, \text{lower} = (0.000000000001, 0.000000000001, \\ & 0.000000000001, 0.000000000001), \text{upper} = (\text{Inf}, \text{Inf}, \text{Inf}, \text{Inf})) \end{aligned}$$

Gradient Function

$$G5 < -\text{grad}(\text{LIPTFDLLF}, c(0.03346370, 0.62567102, 2.15020884, 0.01402783))$$

Hessian Function

$$H5 < -\text{hessian}(\text{LIPTFDLLF}, c(0.03346370, 0.62567102, 2.15020884, 0.01402783))$$

Variance-Covariance Matrix

$$IH5 < -\text{solve}(H5)$$

Standard Errors of Estimates

$$\text{STE5} < -\text{sqrt}(\text{diag}(IH5))$$

Lehmann Type I Frechet Distribution (LIFD)

$$f_y(x) = c \frac{\alpha \left(\frac{\beta}{x}\right)^{\alpha+1}}{\beta} \exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)^c$$

LIFD Log Likelihood Function

$$LLF = n \ln(c) + n \log(\alpha) - n \ln(\beta) + \sum_{i=1}^n (\alpha + 1) \ln(\beta / x) + \sum_{i=1}^n c(-(\beta / x)^\alpha)$$

Declaration of LIFD Negative Log-Likelihood Function

$$\begin{aligned} LIFDLLF < -function(par){(-561 * \log(par[1])) - (561 * \log(par[2])) + (561 * \log(par[3])) - \\ sum((par[2] + 1) * \log((par[3] / x)) - sum((par[1]) * (-((par[3] \\ / x)^{par[2]})))} \end{aligned}$$

Optimization of LIFD Log-Likelihood Function

$$\begin{aligned} LIFDLLFO < -nlminb(c(c(initial), \alpha(initial), \beta(initial)), LIFDLLF, \\ hessian = TRUE, lower = (0.00000000001, 0.000000000001, \\ 0.000000000001), upper = (Inf, Inf, Inf)) \end{aligned}$$

Gradient Function

$$G6 < -grad(LIFDLLF, c(0.002953424, 1.345323591, 0.006637482))$$

Hessian Function

$$H6 < -hessian(LIFDLLF, c(0.002953424, 1.345323591, 0.006637482))$$

Variance-Covariance Matrix

$$IH6 < -solve(H6)$$

Standard Errors of Estimates

$$STE6 < -sqrt(diag(IH6))$$

