

Mahgoub Homotopy Analysis Method for the Solutions of Nonlinear Ordinary and Partial Differential Equations

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Abstract

Solving nonlinear problems involving both partial and ordinary differential equations is always more involving and mostly impossible through the well-established traditional analytical methods. We therefore proposed an analytical method that is derived through exploiting Mahgoub transform that is capable of solving variable coefficient linear differential equations. But this transform being unable to solve nonlinear problem, informed the integration of Homotopy analysis method to handle the nonlinear part. Thus, MahgoubHomotopy Analysis Method (MHAM) that is proposed reduces the volume of computation as well as producing the exact solution of the problem considered. Mahgoub of all the terms in the given equation is taken, and most importantly the differential coefficient. The Mahgoub terms independent of nonlinear term constitute the initial approximation, while the recurrence relation is developed by using the nonlinear terms. The two important deformation equation in the HAM are therefore obtained. The final solution is arrived at via taking the inverse Mahgoub of terms. Few questions on both ordinary and partial differential equations are used to validate the proposed method. The coding and general computations are carried out via Mathematica 13.3.

Keywords: Traditional methods, Mahgoub transform, Homotopy analysis method.

Introduction

Differential equations, such as ordinary, partial, and fractional differential equations, are used to mathematically model many physical, engineering, and biological processes. While integer-order derivatives are local in nature, fractional differential equations (FDEs), on the other hand, are defined by expanding the standard derivative to an arbitrary order, allowing for the inclusion of "memory effects" and "hereditary properties." (Podlubny, 1998).

Many real-world processes, such as heat transfer, fluid dynamics, viscoelastic materials, signal processing, and anomalous diffusion, are described by nonlinear fractional differential or integral equations. According to (Kilbas, Srivastava, and Trujillo, 2006), fractional differential equations are more effective than classical models because their power-law memory structure allows them to better capture the behavior of complex materials and processes. Furthermore, (Hilfer, 2000) extensively documented the transition from local classical mechanics to non-local fractional dynamics, demonstrating that fractional derivatives are essential for modeling time-dependent processes with long-range memory. Due to the complexity and nonlinearity of these models, obtaining exact solutions is often difficult or impossible. In recent years, researchers have developed various analytical, semi-analytical, and hybrid methods to obtain approximate solutions with acceptable accuracy.

Among these approaches are the application of integral transform to different forms of Differential Equation (DE) have been explored by different scientists, among them is the Laplace transform introduced by Pierre-Simon Laplace in the late eighteenth century, which is one of the oldest and most widely used transforms for solving linear differential equations with constant coefficients (Deakin, 1981). Fourier transform, introduced by Joseph Fourier in 1822 in *Théorie analytique de la chaleur*, has been extensively used for solving partial differential equations (PDEs) arising in heat conduction and wave propagation problems (Grattan-Guinness, 2005). The Sumudu transform, introduced by the Sri Lankan mathematician Watugala in the early 1990s, has been widely used to solve various ordinary differential equations (ODEs), PDEs, and integro-differential equations arising from problems in science and engineering, particularly in fields such as control systems, fluid mechanics, and mathematical physics (Watugala, 1993).

Several new integral transforms have been introduced in the literature to address these shortcomings. These include the Elzaki transform proposed by (Elzaki, 2011), the Shehu transform, and the Mahgoub transform. Out of the listed integral transforms employed, the Mahgoub transform in this work has drawn attention due to its wider range of applications, particularly when applied to differential equations with variable coefficients (Mahgoub, 2016).

In addition to the integral transform method approach, several semi-analytical methods have been developed to solve nonlinear differential and fractional differential equations without discretization. Prominent among these are Adomian decomposition method (ADM), the Variational Iteration Method (VIM) was introduced by (He, 1999). Despite its simplicity, VIM relies greatly on the proper selection of Lagrange multipliers, which is often problem specific. The Homotopy Analysis Method (HAM), introduced by Liao in 1992, is a powerful semi-analytical method that is independent of small or large parameters. The method introduces a convergence-control parameter h , which allows explicit control of the series convergence (Liao, 1992). HAM has been widely applied to nonlinear and fractional differential equations (Turq, 2020), and its flexibility in choosing the linear operator and initial approximation enhances both convergence and solution efficiency.

In the recent past, hybrid methods have been proposed to further enhance the accuracy and convergence. Integral transforms are integrated into semi-analytical methods, notable examples among these are the Laplace Adomian Decomposition Method (LADM) (Khan, Shah, Kumam, Baleanu, and Arif, 2020), the Shehu Transform Homotopy Analysis Method (STHAM) (Yisa and Baruwa, 2023), and the Laplace Homotopy Analysis Method (Vahidi, 2016). These methods reduce computational complexity and give greater convergence control compared to standalone methods.

In this work, the Mahgoub transform integrated with the Homotopy Analysis Method (HAM) is developed and proven to provide superior performance in solving variable-coefficient, strongly nonlinear, and fractional-order problems.

Brief Preliminaries of Mahgoub's Transform Method

This section will briefly review the definition and fundamental properties of the Mahgoub Transform.

Definition of Mahgoub Transform

The Mahgoub transform defined on the set of continuous function and of exponential order. We consider functions in the set B defined by:

$$M[f(\zeta)] = Q(\delta) = \delta \int_0^{\infty} f(\zeta) e^{-\delta\zeta} d\zeta, \quad \zeta \geq 0, \quad \xi_1 \leq \delta \leq \xi_2 .$$

The inverse Mahgoub Transform denoted by $M^{-1}[\cdot]$ is

$$f(\zeta) = M^{-1}[Q(\delta)] = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} e^{\delta\zeta} \frac{Q(\delta)}{\delta} d\delta ,$$

where δ is the Mahgoub transform variable, i is the imaginary unit, and k is a real constant.

Mahgoub transform of some mathematical functions

Function $f(\zeta)$	Mahgoub Transform $M[f(\zeta)] = Q(\delta)$
1	1
ζ^n	$\frac{n!}{\delta^n}$
$e^{a\zeta}$	$\frac{\delta}{\delta - a}$
$\sin(a\zeta)$	$\frac{a\delta}{\delta^2 + a^2}$
$\cos(a\zeta)$	$\frac{\delta^2}{\delta^2 + a^2}$

Mahgoub Transform of Derivative

Let $f(\zeta)$ be a function for which the derivative is to be determined using the Mahgoub transform. Applying the Mahgoub transform, we obtain

- i. $M[f'(\zeta)] = \delta Q(\delta) - \delta f(0)$
- ii. $M[f''(\zeta)] = \delta^2 Q(\delta) - \delta f'(0) - \delta^2 f(0)$
- iii. $M[f'''(\zeta)] = \delta^3 Q(\delta) - \delta f''(0) - \delta^2 f'(0) - \delta^3 f(0)$

The n th order derivative is given as

$$M[f^{(n)}(\zeta)] = \delta^n Q(\delta) - \sum_{k=0}^{n-1} \delta^{n-k} f^{(k)}(0)$$

On the Applications of Mahgoub Transform to IVPs

In this section, we demonstrate the effectiveness of the Mahgoub transform by solving number of differential equations.

Example 1

Consider the inhomogeneous constant IVP

$$y''(\zeta) - 3y'(\zeta) + 2y(\zeta) = 4e^{3\zeta}, \quad y(0) = -3, y'(0) = 5.$$

Solution

$$y''(\zeta) - 3y'(\zeta) + 2y(\zeta) = 4e^{3\zeta}$$

By taking the Mahgoub transform of both sides, to get

$$(\delta^2 - 3\delta + 2)Q(\delta) - \delta y'(0) - \delta^2 y(0) + 3\delta y(0) = \frac{4\delta}{\delta - 3}$$

Using the initial conditions, we have

$$\begin{aligned} (\delta^2 - 3\delta + 2)Q(\delta) - 5\delta + 3\delta^2 - 9\delta &= \frac{4\delta}{\delta - 3} \\ Q(\delta) &= \delta \left(\frac{-38 + 23\delta - 3\delta^2}{(\delta - 1)(\delta - 2)(\delta - 3)} \right) \\ Q(\delta) &= \frac{4\delta}{\delta - 2} + \frac{2\delta}{\delta - 3} - \frac{9\delta}{\delta - 1} \end{aligned}$$

Taking the inverse Mahgoub transform, we have

$$y(\zeta) = 4e^{2\zeta} + 2e^{3\zeta} - 9e^{\zeta}.$$

Brief Review of Homotopy Analysis Method (HAM)

This section will briefly review some fundamental properties of HAM

The Homotopy Derivatives

The homotopy-derivatives operator D_k is defined as

$$D_k[\varphi(\eta; \rho)] = \frac{1}{k!} \frac{\partial^k \varphi(\eta; \rho)}{\partial \rho} \Big|_{\rho=0}, \quad k \geq 0,$$

where $\rho \in [0,1]$ is the embedding parameter.

If

$$\varphi(\eta; \rho) = \sum_{k=0}^{\infty} \varphi_k \rho^k,$$

then

$$D_k[\varphi(\eta; \rho)] = \varphi_k.$$

The Zeroth and k th Order Deformation Equation

Let

$$N[u(x)] = 0$$

be the nonlinear problem. The zeroth order deformation equation is defined as

$$(1 - \rho)L[\varphi(\eta; \rho) - \varphi_0(\eta)] = \rho h N[\varphi(\eta; \rho)],$$

with the properties

$$\varphi(\eta; 0) = \varphi_0(\eta), \quad \varphi(\eta; 1) = \varphi(\eta),$$

where L is a linear operator, N is a nonlinear operator, $\varphi_0(\eta)$ is the initial approximation, h is the convergence-control parameter, and $\rho \in [0,1]$ is the embedding parameter.

Apply the k th order homotopy-derivative D_k to the zeroth-order equation gives k th order deformation equation

$$L[\varphi_k(\eta; \rho) - \chi_k \varphi_{k-1}(\eta; \rho)] = h D_{k-1} [N[\varphi(\eta; \rho)]],$$

where

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 0. \end{cases}$$

Problems Considered

The class of the problems considered in this work are as listed below.

Integer Order Nonlinear Ordinary Differential Equation

$$Ly(\eta) + N[y(\eta)] + Hy(\eta) = f(\eta), \quad (2.1)$$

where L denotes a linear differential operator, $N[y(\eta)]$ is the nonlinear term, $Hy(\eta)$ is the remaining linear term, and $f(\eta)$ is the given inhomogeneous source term.

Subject to initial conditions $y(\eta_0) = y_0, \quad y'(\eta_0) = y_1, \dots, y^{(n-1)}(\eta_0) = y_{n-1}$.

Integer Order Nonlinear Partial Differential Equation

$$Ly(\eta, \xi) + N[y(\eta, \xi)] + Hy(\eta, \xi) = f(\eta, \xi), \quad (2.2)$$

where L denotes a linear differential operator, $N[y(\eta, \xi)]$ is the nonlinear term, $Hy(\eta, \xi)$ is the remaining linear term, and $f(\eta, \xi)$ is a given nonhomogeneous source term, subject to the initial conditions

$$y(\eta, 0) = y_0(\eta), \quad y'(\eta, 0) = y_1(\eta), \dots, y^{(n-1)}(\eta, 0) = y_{n-1}(\eta).$$

The class of problems above is solved by employing the proposed method tagged MahgoubHomotopyAnalysis Method (MHAM).

Methodology

This section outlines the algorithm of our proposed methods for solving the class of problems listed above.

Proposed Algorithm for the Solution of Nonlinear ODEs

To solve these using MHAM, we first apply the Mahgoub transform of both sides of (2.1)

$$M[Ly(\eta)] + M[N[y(\eta)]] + M[Hy(\eta)] = M[f(\eta)]. \quad (3.1)$$

But

$$M[Ly(\eta)] = \delta^n Q(\delta) - \sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0). \quad (3.2)$$

Putting (3.2) in (3.1), we have

$$\delta^n Q(\delta) - \sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) + M[N[y(\eta)]] + M[Hy(\eta)] - M[f(\eta)] = 0. \quad (3.3)$$

Dividing (3.3) through by δ^n , we have

$$Q(\delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) + \frac{1}{\delta^n} [M[N[y(\eta)]] + M[Hy(\eta)] - M[f(\eta)]] = 0. \quad (3.4)$$

From (3.4), the resulting general nonlinear operator is defined as

$$N[y(\eta; \rho)] = Q(\delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) + \frac{1}{\delta^n} [M[N[y(\eta)]] + M[Hy(\eta)] - M[f(\eta)]] \quad (3.5)$$

The result in (3.5) shall be used in the k th order deformation equation

$$L[Q_k(\delta; \rho) - \chi_k Q(\delta; \rho)] = h D_{k-1} [N[y(\eta; \rho)]] \quad (3.6)$$

where ρ is the embedding parameter, and h is the convergence control parameter. The linear operator is defined as

$$L[Q_k(\delta; \rho) = Q_k(\delta),$$

thus

$$\begin{aligned} Q_k(\delta) - \chi_k Q_{k-1}(\delta) \\ = h D_{k-1} \left[Q(\delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) \right. \\ \left. + \frac{1}{\delta^n} [M[N[y(\eta)]] + M[Hy(\eta)] - M[f(\eta)]] \right] \end{aligned} \quad (3.7)$$

with the application of Homotopy Derivative D_{k-1} , (3.7) becomes

$$\begin{aligned} Q_k(\delta) - \chi_k Q_{k-1}(\delta) \\ = h \left[Q_{k-1}(\delta) - (1 - \bar{\chi}_{k-1}) \frac{1}{\delta^n} \left[\sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) + M[f(\eta)] \right] \right. \\ \left. + \frac{1}{\delta^n} [M[D_{k-1}[N[y(\eta)]]] + M[D_{k-1}[Hy(\eta)]]] \right] \end{aligned} \quad (3.8)$$

Taking $h = -1$, (3.8) becomes

$$\begin{aligned} Q_k(\delta) = -(1 - \chi_k) Q_{k-1}(\delta) + (1 - \bar{\chi}_{k-1}) \frac{1}{\delta^n} \left[\sum_{i=0}^{n-1} \delta^{n-i} y^{(i)}(0) + M[f(\eta)] \right] \\ - \frac{1}{\delta^n} [M[D_{k-1}[N[y(\eta)]]] + M[D_{k-1}[Hy(\eta)]]], \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \chi_k &= \begin{cases} 0, & k \leq 1, \\ 1, & k > 0 \end{cases} \\ \bar{\chi}_{k-1} &= \begin{cases} 0, & k-1 < 1, \\ 1, & k-1 \geq 1. \end{cases} \end{aligned}$$

But

$$y_k(\eta) = M^{-1}[Q(\delta)].$$

The series solution is obtained from (3.9) and the initial approximation $y_0(\eta)$ is derived from the Taylor series expansion using the initial condition prescribed.

Proposed Algorithm for the Solution of Nonlinear PDE

Let us consider the Integer order partial differential equation (PDE) in (2.2)

In order to solve these using MHAM, we first apply the Mahgoub transform of both sides of (2.2)

$$M[Ly(\eta, \xi)] + M[N[y(\eta, \xi)]] + M[Hy(\eta, \xi)] = M[f(\eta, \xi)] \quad (3.10)$$

but

$$M[Ly(\eta, \xi)] = \delta^n Q(\eta, \delta) - \sum_{i=0}^{n-1} \delta^{n-i} y_{\xi}^{(i)}(\eta, 0) \quad (3.11)$$

Putting (3.11) in (3.10), we have

$$\begin{aligned} \delta^n Q(\eta, \delta) - \sum_{i=0}^{n-1} \delta^{n-i} y_{\xi}^{(i)}(\eta, 0) + M[N[y(\eta, \xi)]] + M[Hy(\eta, \xi)] - M[f(\eta, \xi)] \\ = 0 \end{aligned} \quad (3.12)$$

Dividing (3.12) through by δ^n , we have

$$Q(\eta, \delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y_{\xi}^{(i)}(\eta, 0) + \frac{1}{\delta^n} [M[N[y(\eta, \xi)]] + M[Hy(\eta, \xi)] - M[f(\eta, \xi)]] = 0 \quad (3.13)$$

From (3.13), the resulting general nonlinear operator is defined as

$$\begin{aligned} N[y(\eta, \delta; \rho)] = Q(\eta, \delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y_{\xi}^{(i)}(\eta, 0) \\ + \frac{1}{\delta^n} [M[N[y(\eta, \xi)]] + M[Hy(\eta, \xi)] - M[f(\eta, \xi)]] \end{aligned} \quad (3.14)$$

The result above shall be implemented in k th order deformation equation

$$L[Q_k(\eta, \delta; \rho) - \chi_k Q_{k-1}(\eta, \delta; \rho)] = h D_{k-1} [N[y(\eta, \delta; \rho)]], \quad (3.15)$$

where ρ is the embedding parameter, and h is the convergence control parameter

The linear operator is obtained as

$$L[Q_k(\eta, \delta; \rho) = Q_k(\eta, \delta),$$

thus

$$\begin{aligned}
 Q_k(\eta, \delta) - \chi_k Q_{k-1}(\eta, \delta) &= h D_{k-1} \left[Q(\eta, \delta) - \frac{1}{\delta^n} \sum_{i=0}^{n-1} \delta^{n-i} y_\xi^{(i)}(0) \right. \\
 &\quad \left. + \frac{1}{\delta^n} [M[N[y(\eta, \xi)]] + M[Hy(\eta, \xi)] \right. \\
 &\quad \left. - M[f(\eta, \xi)] \right] \quad (3.16)
 \end{aligned}$$

With the application of Homotopy Derivative D_{k-1} , (3.16) becomes

$$\begin{aligned}
 Q_k(\eta, \delta) - \chi_k Q_{k-1}(\eta, \delta) &= h \left[Q_{k-1}(\eta, \delta) - (1 - \bar{\chi}_{k-1}) \frac{1}{\delta^n} \left[\sum_{i=0}^{n-1} \delta^{n-i} y_\xi^{(i)}(0) + M[f(\eta, \xi)] \right] \right. \\
 &\quad \left. + \frac{1}{\delta^n} [M[D_{k-1}[N[y(\eta, \xi)]]] + M[D_{k-1}[Hy(\eta, \xi)]] \right] \quad (3.17)
 \end{aligned}$$

Taking $h = -1$, (3.17) becomes

$$\begin{aligned}
 Q_k(\eta, \delta) &= -(1 - \chi_k) Q_{k-1}(\eta, \delta) + (1 - \bar{\chi}_{k-1}) \frac{1}{\delta^n} \left[\sum_{i=0}^{n-1} \delta^{n-i} y_\xi^{(i)}(0) + M[f(\eta, \xi)] \right] \\
 &\quad - \frac{1}{\delta^n} [M[D_{k-1}[N[y(\eta, \xi)]]] + M[D_{k-1}[Hy(\eta, \xi)]]], \quad (3.18)
 \end{aligned}$$

where

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 0 \end{cases}, \quad \bar{\chi}_{k-1} = \begin{cases} 0, & k - 1 < 1, \\ 1, & k - 1 \geq 1. \end{cases}$$

But

$$y_k(\eta, \xi) = M^{-1}[Q(\eta, \xi)].$$

The series solution is obtained from (3.18) and the initial approximation $y_0(\eta, \xi)$ is obtained from the Taylor series expansion of two variables using the initial condition prescribed.

On the Application of MHAM to ODE and PDE

Here, we solved examples employing the two algorithms explored in sections (3.1) and (3.2).

Examples on Ordinary Differential Equations

Problem 1 (Yisa and Baruwa, 2023) Consider the nonlinear homogeneous IVP

$$y''(\eta) + 2y(\eta)y'(\eta) = 0, \quad y(0) = 0, y'(0) = 1.$$

Solution

$$y''(\eta) + 2y(\eta)y'(\eta) = 0$$

We take the Mahgoub transform of both sides, to get

$$M[y''(\eta)] + 2M[y(\eta)y'(\eta)] = M[0] \\ \delta^2 Q(\delta) - \delta y'(0) - \delta^2 y(0) + 2M[y(\eta)y'(\eta)] = 0$$

Dividing through by δ^2 , we get

$$Q(\delta) - \frac{1}{\delta} y'(0) - y(0) + \frac{2}{\delta^2} M[y(\eta)y'(\eta)] = 0$$

Using the initial condition, we get

$$Q(\delta) - \frac{1}{\delta} + \frac{2}{\delta^2} M[y(\eta)y'(\eta)] = 0$$

The linear and nonlinear operators are obtained as

$$L[Q(\delta; \rho)] = Q(\delta)$$

$$N[y(\delta; \rho)] = Q(\delta) - \frac{1}{\delta} + \frac{2}{\delta^2} M[y(\eta)y'(\eta)]$$

Using the k th order deformation equation

$$L[Q_k(\delta; \rho) - \chi_k Q_{k-1}(\delta; \rho)] = h D_{k-1} [N[y(\delta; \rho)]] \\ Q_k(\delta) - \chi_k Q_{k-1}(\delta) = h D_{k-1} \left[Q(\delta) - \frac{1}{\delta} + \frac{2}{\delta^2} M[y(\eta)y'(\eta)] \right] \\ Q_k(\delta) = \chi_k Q_{k-1}(\delta) + h D_{k-1} \left[Q(\delta) - \frac{1}{\delta} + \frac{2}{\delta^2} M[y(\eta)y'(\eta)] \right]$$

With the application of Homotopy Derivative, we have

$$Q_k(\delta) = \chi_k Q_{k-1}(\delta) + h \left[Q_{k-1}(\delta) - (1 - \bar{\chi}_{k-1}) \frac{1}{\delta} + \frac{2}{\delta^2} M \left[\sum_{i=0}^{k-1} y_i \frac{dy_{k-1-i}}{d\eta} \right] \right]$$

Taking $h = -1$,

$$Q_k(\delta) = -(1 - \chi_k) Q_{k-1}(\delta) + (1 - \bar{\chi}_{k-1}) \frac{1}{\delta} - \frac{2}{\delta^2} M \left[\sum_{i=0}^{k-1} y_i \frac{dy_{k-1-i}}{d\eta} \right]$$

When $k = 1$:

$$Q_1(\delta) = -(1 - \chi_1)Q_0(\delta) + (1 - \bar{x}_0)\frac{1}{\delta} - \frac{2}{\delta^2}M[y_0y_0']$$

The initial approximation

$$y_0(\eta) = y(0) + \eta y'(0) + \frac{\eta^2}{2!}y''(0) + \frac{\eta^3}{3!}y'''(0) + \dots$$

We have, $y_0(\eta) = \eta$, $H_0(v) = \frac{1}{\delta}$.

$$Q_1(\delta) = -\frac{1}{\delta} + \frac{1}{\delta} - \frac{2}{\delta^2}M[x] = -\frac{2}{\delta^3}$$

Taking the inverse Mahgoub, we have $y_1(\eta) = -\frac{\eta^3}{3}$.

When $k = 2$:

$$Q_2(\delta) = -(1 - \chi_2)Q_1(\delta) + (1 - \bar{x}_1)\frac{1}{\delta} - \frac{2}{\delta^2}M\left[\sum_{i=0}^1 y_i \frac{dy_{1-i}}{d\eta}\right] = -\frac{2}{\delta^2}M[y_0y_1' + y_1y_0']$$

$$Q_2(\delta) = -\frac{2}{\delta^2}M\left[-\eta^3 - \frac{\eta^3}{3}\right]$$

$$Q_2(\delta) = \frac{16}{\delta^5}$$

Taking the inverse Mahgoub, we have

$$y_2(\eta) = \frac{2}{15}\eta^5.$$

When $k = 3$:

$$Q_3(\delta) = -(1 - \chi_3)Q_2(\delta) + (1 - \chi_2)\frac{1}{\delta} - \frac{2}{\delta^2}M\left[\sum_{i=0}^2 y_i \frac{dy_{2-i}}{d\eta}\right]$$

$$Q_3(\delta) = -\frac{2}{\delta^2}M[y_0y_2' + y_2y_0' + y_1y_1']$$

$$Q_3(\delta) = -\frac{2}{\delta^2}M\left[\frac{2}{3}\eta^5 + \frac{2}{15}\eta^5 + \frac{1}{5}\eta^5\right]$$

$$Q_3(\delta) = -\frac{2}{\delta^2}M[\eta^5]$$

$$H_3(v) = -\frac{240}{\delta^7}.$$

Taking the inverse Mahgoub, we have $y_3(\eta) = -\frac{\eta^7}{21}$.

The third- order approximation

$$y(\eta) = \sum_{k=0}^3 y_k(\eta) = \eta - \frac{1}{3}\eta^3 + \frac{2}{15}\eta^5 - \frac{1}{21}\eta^7$$

Problem 2(Yisa and Baruwa, 2023)

Consider the inhomogeneous nonlinear IVP

$$y''(\eta) - y'(\eta)^2 + y(\eta)^2 = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution

$$y''(\eta) - y'(\eta)^2 + y(\eta)^2 = 1$$

The procedure followed in the solution to Problem 1 leads to

Taking $h = -1$,

$$Q_k(\delta) = -(1 - \chi_k)Q_{k-1}(\delta) + (1 - \bar{\chi}_{k-1})\left(1 + \frac{1}{\delta^2}\right) + \frac{1}{\delta^2}M\left[\sum_{i=0}^{k-1} y'_i y'_{k-1-i}\right] - \frac{1}{\delta^2}M\left[\sum_{i=0}^{k-1} y_i y_{k-1-i}\right]$$

The initial approximation

$$y_0(\eta) = y(0) + \eta y'(0) + \frac{\eta^2}{2!} y''(0) + \frac{\eta^3}{3!} y'''(0) + \dots$$

$$y(\eta) = \sum_{k=0}^3 y_k(\eta)$$

$$y(\eta) = y_0(\eta) + y_1(\eta) + y_2(\eta) + y_3(\eta)$$

$$y(x) = 1 + 0 + 0 + 0$$

$$y(\eta) = 1.$$

Examples on Partial Differential Equations

Problem 1(Olayiwola, Morufu Oyedunsi and Yunus, Akeem Olarewaju and Alaje, Adedapo Ismaila and Adedeji, Joseph Adeleke, 2025)

Consider the nonlinear, nonlocal Fornberg-Whitham equation

$$\frac{\partial u}{\partial \xi} = \frac{\partial^3 u}{\partial \eta^2 \partial \xi} - \frac{\partial u}{\partial \eta} - u \frac{\partial u}{\partial \eta} + u \frac{\partial^3 u}{\partial \eta^3} + 3 \frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2}$$

$$\text{Subject to } u(\eta, 0) = \frac{4}{3} e^{\frac{\eta}{2}}.$$

Solution

$$\frac{\partial u}{\partial \xi} = \frac{\partial^3 u}{\partial \eta^2 \partial \xi} - \frac{\partial u}{\partial \eta} - u \frac{\partial u}{\partial \eta} + u \frac{\partial^3 u}{\partial \eta^3} + 3 \frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2}$$

Take the Mahgoub transform of both sides

$$\delta Q(\eta, \delta) - \delta u(\eta, 0) = M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] - M \left[\frac{\partial u}{\partial \eta} \right] - M \left[u \frac{\partial u}{\partial \eta} \right] + M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] + 3M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right]$$

$$Q(\eta, \delta) - u(\eta, 0) = \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] - \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] - \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] + \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right]$$

Using the initial condition, we have

$$Q(\eta, \delta) - \frac{4}{3} e^{\frac{\eta}{2}} = \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] - \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] - \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] + \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right]$$

$$Q(\eta, \delta) - \frac{4}{3} e^{\frac{\eta}{2}} - \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] + \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] - \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] - \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right] = 0$$

The linear and nonlinear operator is obtained as

$$L[Q(\eta, \delta; \rho)] = Q(\eta, \delta)$$

$$\begin{aligned} N[u(\eta, \delta; \rho)] &= Q(\eta, \delta) - \frac{4}{3} e^{\frac{\eta}{2}} - \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] + \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] - \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] \\ &\quad - \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right] \end{aligned}$$

Using the k th order deformation equation

$$L[Q_k(\eta, \delta; \rho) - \chi_k Q_{k-1}(\eta, \delta; \rho)] = h D_{k-1} [N[u(\eta, \delta; \rho)]]$$

$$\begin{aligned} Q_k(\eta, \delta) - \chi_k Q_{k-1}(\eta, \delta) &= h D_{k-1} \left[Q(\eta, \delta) - \frac{4}{3} e^{\frac{\eta}{2}} - \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] + \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] \right. \\ &\quad \left. - \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] - \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right] \right] \end{aligned}$$

$$\begin{aligned} Q_k(\eta, \delta) &= \chi_k Q_{k-1}(\eta, \delta) \\ &\quad + h D_{k-1} \left[Q(\eta, \delta) - \frac{4}{3} e^{\frac{\eta}{2}} - \frac{1}{\delta} M \left[\frac{\partial^3 u}{\partial \eta^2 \partial \xi} \right] + \frac{1}{\delta} M \left[\frac{\partial u}{\partial \eta} \right] + \frac{1}{\delta} M \left[u \frac{\partial u}{\partial \eta} \right] \right. \\ &\quad \left. - \frac{1}{\delta} M \left[u \frac{\partial^3 u}{\partial \eta^3} \right] - \frac{3}{\delta} M \left[\frac{\partial u}{\partial \eta} \frac{\partial^2 u}{\partial \eta^2} \right] \right] \end{aligned}$$

With the application of Homotopy Derivative, we have

$$\begin{aligned} Q_k(\eta, \delta) = & \chi_k Q_{k-1}(\eta, \delta) \\ & + h \left[Q_{k-1}(\eta, \delta) - (1 - \bar{\chi}_{k-1}) \frac{4}{3} e^{\frac{\eta}{2}} - \frac{1}{\delta} M \left[\frac{\partial^3 u_{k-1}}{\partial \eta^2 \partial \xi} \right] + \frac{1}{\delta} M \left[\frac{\partial u_{k-1}}{\partial \eta} \right] \right. \\ & \left. + \frac{1}{\delta} M \left[\sum_{i=0}^{k-1} u_i \frac{\partial u_{k-1-i}}{\partial \eta} \right] - \frac{1}{\delta} M \left[\sum_{i=0}^{k-1} u_i \frac{\partial^3 u_{k-1-i}}{\partial \eta^3} \right] - \frac{3}{\delta} M \left[\sum_{i=0}^{k-1} \frac{\partial u_i}{\partial \eta} \frac{\partial^2 u_{k-1-i}}{\partial \eta^2} \right] \right] \end{aligned}$$

Taking $h = -1$, we have

$$\begin{aligned} Q_k(\eta, \delta) = & -(1 - \chi_k) Q_{k-1}(\eta, \delta) + (1 - \bar{\chi}_{k-1}) \frac{4}{3} e^{\frac{\eta}{2}} + \frac{1}{\delta} M \left[\frac{\partial^3 u_{k-1}}{\partial \eta^2 \partial \xi} \right] - \frac{1}{\delta} M \left[\frac{\partial u_{k-1}}{\partial \eta} \right] \\ & - \frac{1}{\delta} M \left[\sum_{i=0}^{k-1} u_i \frac{\partial u_{k-1-i}}{\partial \eta} \right] + \frac{1}{\delta} M \left[\sum_{i=0}^{k-1} u_i \frac{\partial^3 u_{k-1-i}}{\partial \eta^3} \right] + \frac{3}{\delta} M \left[\sum_{i=0}^{k-1} \frac{\partial u_i}{\partial \eta} \frac{\partial^2 u_{k-1-i}}{\partial \eta^2} \right] \end{aligned}$$

When $k = 1$:

$$\begin{aligned} Q_1(\eta, \delta) = & -(1 - \chi_1) Q_0(\eta, \delta) + (1 - \bar{\chi}_0) \frac{4}{3} e^{\frac{\eta}{2}} + \frac{1}{\delta} M \left[\frac{\partial^3 u_0}{\partial \eta^2 \partial \xi} \right] - \frac{1}{\delta} M \left[\frac{\partial u_0}{\partial \eta} \right] \\ & - \frac{1}{\delta} M \left[u_0 \frac{\partial u_0}{\partial \eta} \right] + \frac{1}{\delta} M \left[u_0 \frac{\partial^3 u_0}{\partial \eta^3} \right] + \frac{3}{\delta} M \left[\frac{\partial u_0}{\partial \eta} \frac{\partial^2 u_0}{\partial \eta^2} \right] \end{aligned}$$

The initial approximation

$$u_0(\eta, \xi) = u(\eta, 0) + \xi \frac{\partial u}{\partial \xi}(\eta, 0) + \frac{\xi^2}{2!} \frac{\partial^2 u}{\partial \xi^2}(\eta, 0) + \dots$$

We have

$$\begin{aligned} u_0(\eta, \xi) = & \frac{4}{3} e^{\frac{\eta}{2}}, \quad Q_0(\eta, \delta) = \frac{4}{3} e^{\frac{\eta}{2}}. \\ Q_1(\eta, \delta) = & -\frac{4}{3} e^{\frac{\eta}{2}} + \frac{4}{3} e^{\frac{\eta}{2}} + \frac{1}{\delta} M[0] - \frac{1}{\delta} M \left[\frac{2}{3} e^{\frac{\eta}{2}} \right] - \frac{1}{\delta} M \left[\frac{8}{9} e^{\eta} \right] + \frac{1}{\delta} M \left[\frac{2}{9} e^{\eta} \right] \\ & + \frac{3}{\delta} M \left[\frac{2}{9} e^{\eta} \right] \\ Q_1(\eta, \delta) = & -\frac{2}{3\delta} e^{\frac{\eta}{2}}. \end{aligned}$$

Taking the inverse Mahgoub, we have

$$u_1(\eta, \xi) = -\frac{2}{3}e^{\frac{\eta}{2}}\xi.$$

When $k = 2$:

$$\begin{aligned} Q_2(\eta, \delta) &= -(1 - \chi_2)Q_1(\eta, \delta) + (1 - \bar{\chi}_1)\frac{4}{3}e^{\frac{\eta}{2}} + \frac{1}{\delta}M\left[\frac{\partial^3 u_1}{\partial \eta^2 \partial \xi}\right] - \frac{1}{\delta}M\left[\frac{\partial u_1}{\partial \eta}\right] \\ &\quad - \frac{1}{\delta}M\left[\sum_{i=0}^1 u_i \frac{\partial u_{1-i}}{\partial \eta}\right] + \frac{1}{\delta}M\left[\sum_{i=0}^1 u_i \frac{\partial^3 u_{1-i}}{\partial \eta^3}\right] + \frac{3}{\delta}M\left[\sum_{i=0}^1 \frac{\partial u_i}{\partial \eta} \frac{\partial^2 u_{1-i}}{\partial \eta^2}\right] \\ Q_2(\eta, \delta) &= \frac{1}{\delta}M\left[-\frac{1}{6}e^{\frac{\eta}{2}}\right] - \frac{1}{\delta}M\left[-\frac{1}{3}e^{\frac{\eta}{2}}\xi\right] - \frac{1}{\delta}M\left[-\frac{8}{9}e^{\eta}\xi\right] + \frac{1}{\delta}M\left[-\frac{2}{9}e^{\eta}\xi\right] \\ &\quad + \frac{3}{\delta}M\left[-\frac{2}{9}e^{\eta}\xi\right] \\ Q_2(\eta, \delta) &= -\frac{1}{6\delta}e^{\frac{\eta}{2}} + \frac{1}{3\delta^2}e^{\frac{\eta}{2}} \end{aligned}$$

Taking the inverse Mahgoub, we have

$$u_2(\eta, \xi) = \frac{1}{6}e^{\frac{\eta}{2}}(\xi^2 - \xi).$$

The second- order approximation

$$\begin{aligned} u(\eta, \xi) &= \sum_{k=0}^2 u_k(\eta, \xi) \\ u(\eta, \xi) &= \frac{4}{3}e^{\frac{\eta}{2}} - \frac{2}{3}e^{\frac{\eta}{2}}\xi + \frac{1}{6}e^{\frac{\eta}{2}}(\xi^2 - \xi). \end{aligned}$$

Discussion of the Results

The results obtained for Problems 1 and 2 on integer-order ordinary differential equations (ODEs), using the MahgoubHomotopy Analysis Method (MHAM), are in complete agreement with the results in the existing literature Yisa&Baruwa (2023), where the standard HAM and STHAM were employed respectively. Also, Problems 1 and 2, which involve integer-order PDEs, were previously solved by Olayiwola *et al.* (2025) using the Laplace Adomian Decomposition Method; notably, the MHAM employed in this work yielded the same results.

Conclusion

Mahgoub Homotopy Analysis Method (MHAM) has been successfully implemented to obtain approximate analytical solutions of some integer order differential equations. By integrating the Mahgoub transform with the Homotopy Analysis Method, we have developed a robust framework that simplifies the process of solving complex functional equations. This proposed method yields good results with less computation. Ultimately, MHAM proves to be a more efficient and reliable tool for solving integer order differential equations.

References

- Deakin, M. A. (1981). The development of the Laplace transform, 1737--1937: I. Euler to Spitzer, 1737--1880. *Archive for History of Exact sciences*, 343--390.
- Elzaki. (2011). The New Integral Transform 'Elzaki Transform'. *Global Journal of Pure and Applied Mathematics*, 57--64.
- Elzaki, T. M. (2012). Elzaki and Sumudu transforms for solving some differential equations. *Global Journal of Pure and Applied Mathematics*, 167--173.
- Grattan-Guinness, I. (2005). Joseph Fourier, Th{\'e}orie analytique de la chaleur (1822). In *Landmark Writings in Western Mathematics 1640-1940* (pp. 354--365). Elsevier.
- He, J.-H. (1999). Variational iteration method--a kind of non-linear analytical technique: some examples. *International journal of non-linear mechanics*, 699--708.
- Hilfer, R. (2000). *Applications of fractional calculus in physics*. World scientific.
- Khan, Shah, Kumam, Baleanu, and Arif. (2020). Laplace decomposition for solving nonlinear system of fractional order partial differential equations. *Advances in Difference Equations*, 375.
- Kilbas, Srivastava, and Trujillo. (2006). *Theory And Applications of Fractional Differential Equations*. Elsevier Science & Tech.
- Liao. (1992). The proposed homotopy analysis technique for the solution of nonlinear problems. *Ph. D. Thesis, Shanghai Jiao Tong University Shanghai*.
- Mahgoub, M. (2016). The new integral transform "Mahgoub Transform". *Advances in theoretical and applied mathematics*, 391--398.
- Olayiwola, Morufu Oyedunsi and Yunus, Akeem Olarewaju and Alaje, Adedapo Ismaila and Adedeji, Joseph Adeleke. (2025). Modified Laplace Adomian decomposition method: A fast yet efficient numerical approach for partial differential equations. *Discover Applied Sciences*, 1342.
- Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. elsevier.
- Turq, S. M. (2020). Homotopy type methods for nonlinear differential equations. *Palestine Polytechnic University*.
- Vahidi, J. (2016). The combined Laplace-homotopy analysis method for partial differential equations. *J. Math. Comput. Sci*, 88--102.

- Watugala, G. K. (1993). Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Integrated Education*, 35--43.
- Yisa and Baruwa. (2023). Shehu Transform Homotopy Analysis Method for the Solution of Nonlinear Initial Value Ordinary and Partial Differential Equations. *{Daffodil International University Journal of Science and Technology}*, 19-27.