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BOUNDS ON COEFFICIENTS OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

In this paper we obtain bounds on some early coefficients of analytic functions which have the form $f(z) = z + a^2z^2 + \ldots$ and which satisfy certain geometric conditions in the open unit disk $E = \{z \in C : |z| < 1\}$.

INTRODUCTION

Let A be the class of functions

$$f(z) = z + a_2 z^2 + \ldots$$

which are analytic in the unit disc $E = \{z \in C : |z| < 1\}$. We denote by S, the subclass of A consisting of univalent functions only. Let P be the class of functions.

$$P(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

which are also analytic in the unit disk E and satisfy $\mathrm{Re}\, p(z) > 0$, $z \in E$. A function f is said to be of bounded turning (whose derivative has positive real part) if $\mathrm{Re}\, f'(z) > 0$. A function f is said to be starlike if it maps the open unit disk E onto a starlike domain. The family of starlike functions is denoted by S^*. Analytically, $f \in S^*$ if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0.$$

Several subclasses of the class S have been introduced and investigated by researchers over the years. One of important such subclasses is the family of Bazilevic functions introduced by Bazilevic (1955), which has many other known sub-classes of univalent functions as particular cases. One such subclass of those (Bazilevic functions) is $B_1(\alpha)$ which was considered by Singh (1973). He said: a function $f \in A$ belongs to $B_1(\alpha)$ if and only if

$$\mathrm{Re}\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} > 0$$

where $\alpha \geq 0$ is a real number. If $\alpha = 0$, the class $B_1(\alpha)$ reduces to the class S^* of starlike functions, while for $\alpha = 1$, the class $B_1(\alpha)$ reduces to the class of functions whose derivatives have positive real parts in the open unit disk E.

In this paper we consider a class, J_α, of analytic functions $f \in A$ defined by the geometric condition

$$\mathrm{Re}\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0 \tag{1.1}$$

It is to be noted that J_0 consists of functions in A which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)^2 > 0$$

and are subclasses of S* as proved by Babalola (2013) while J_1 consists of analytic functions satisfying

$$\mathrm{Re}\ \frac{z(f'(z))^2}{f(z)} > 0.$$

We shall derive estimates for some early coefficients of functions in the class J_α.

The condition (1.1) suggests the existence of a function $p(z)$ in the class P of functions with positive real part such that

$$\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} = p(z).$$

We shall rely on this relationship in our investigations in this paper. In the next section we state the preliminary results we shall depend on. In Section 3, we state and prove the main result.

PRELIMNARY LEMMAS

In This section, we recall some Lemmas which are useful in proving our main results.

Lemma 1 (Duren, 1983).

If $p \in P$. Then $|c_k| \le 2, k = 1,2,...$

Equality is attained for the Moebius function

$$L_0(z) = \frac{1+z}{1-z}$$

Lemma 2 (Leutwiler & Schober, 1973).

Let $p \in P$. Then

$$\left|c_{2k} - \frac{c_k^{\,2}}{2}\right| \le 2 - \frac{|c_k|^2}{2}, \text{ k=1, 2,...}$$

The inequality is sharp.

Corrolary 1. *Let* $p \in P$. *Then for any real number* σ *we have the sharp inequalities*

$$\left|c_{2k} - \sigma\frac{c_k^{\,2}}{2}\right| \le \begin{cases} 2(1-\sigma) & if \sigma \le 0 \\ 2 & if 0 \le \sigma \le 2 \\ 2(\sigma - 1) & if \sigma \ge 2 \end{cases}$$

Proof. The proof follows as in case $k = 1$ proved by Babalola & Opóola (2008)

Lemma 3(Livingston, 1969).

If $p \in P$ *and s is a natural number, then* $|c_n - c_{n-s}c_s| \le 2, \ n \ge s, n = 1, 2, 3, \ldots$

Corollary 2. *Let* $p \in P$. *Then for any real number* σ *we have the sharp inequalities*

$$|c_n - \sigma c_{n-s}c_s| \le \begin{cases} 2(3-2\sigma) & if \sigma \le 1, \\ 2 & if \sigma = 1, \\ 2(2\sigma - 1) & if \sigma \ge 1. \end{cases}$$

Proof. We write $|c_n - \sigma c_{n-s}c_s|$ *as*

$$|c_n - \sigma c_{n-s}c_s| \le |c_n - c_{n-s}c_s + c_{n-s}c_s - \sigma c_{n-s}c_s|$$

So, by triangle inequality, we have

$$|c_n - \sigma c_{n-s}c_s| \le |c_n - c_{n-s}c_s| + |c_{n-s}c_s - \sigma c_{n-s}c_s|$$

Thus using the preceding lemma we have

$$|c_n - \sigma c_{n-s}c_s| \le 2 + 4|1 - \sigma|.$$

Now, if $|1 - \sigma| \ge 0$, we have the first inequality and if $1 - \sigma \not\le 0$ the third inequality results. The secon inequality is obtained if $1 - \sigma = 0$.

Lemma 4. *If* $p \in P$. *Then*

$$\left|c_3 - \sigma\frac{c_1^{\,3}}{2}\right| \le \begin{cases} 2(3-2\sigma) & if \sigma \le 0 \\ 6 & if 0 \le \sigma \le 2 \\ 2(2\sigma - 1) & if \sigma \ge 2. \end{cases}$$

Proof.
$$\left|c_3-\sigma\frac{c_1^3}{2}\right|=\left|c_3-c_1c_2+c_1c_2-\sigma\frac{c_1^3}{2}\right|$$
$$\le\left|c_3-c_1c_2\right|+\left|c_1\right|\left|c_2-\sigma\frac{c_1^2}{2}\right|.$$

Applying Lemma 1, Corollary 1 and Lemma 3 we obtain the result.

RESULTS

In this section, we present the main results of this paper.

Theorem 1. Let $f\in J_\alpha$. Then

$$|a_2|\le\frac{2}{\alpha+2}$$

$$|a_3|\le\begin{cases}\dfrac{2(\alpha+6)}{(\alpha+2)^2(\alpha+4)} & if\, 0<\alpha\le\dfrac{-3+\sqrt{17}}{2}\\[2ex] \dfrac{2}{\alpha+4} & if\, \alpha\ge\dfrac{-3+\sqrt{17}}{2}\end{cases}$$

$$|a_4|\le\begin{cases}\dfrac{52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288}{6(\alpha+2)^3(\alpha+4)(\alpha+6)} & if\, \alpha\le\dfrac{-5+\sqrt{33}}{2}\\[2ex] \dfrac{14\alpha^2+96\alpha+232}{3(\alpha+2)(\alpha+4)(\alpha+6)} & if\, \alpha\ge\dfrac{-5+\sqrt{33}}{2}\end{cases}$$

$$|a_5|\le\begin{cases}\dfrac{14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)} & if\, \alpha\le\dfrac{-7+\sqrt{57}}{2}\\[2ex] \dfrac{4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072}{(\alpha+2)(\alpha+4)^2(\alpha+6)(\alpha+8)} & if\, \alpha\ge\dfrac{-7+\sqrt{57}}{2}\end{cases}$$

Proof. Since

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}>0,$$

then for some $p\in P$ we have

$$\frac{zf'(z)}{f(z)}\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}=p(z).$$

Therefore

$$f'(z)^2f^{\alpha-1}(z)=z^{\alpha-2}f(z)p(z)$$

Equating the coefficients

$$(\alpha+3)a_2 = a_2 + c_1 \tag{3.1}$$

$$(\alpha+5)a_3 + \frac{\alpha^2+5\alpha+2}{2}a_2^{\,2} = a_3 + a_2c_1 + c_2 \tag{3.2}$$

$$(\alpha+7)a_4 + (\alpha^2+7\alpha+4)a_2a_3 + \frac{\alpha^3+6\alpha^2-\alpha-6}{6}a_2^{\,3} = a_4 + a_3c_1 + a_2c_2 + c_3 \tag{3.3}$$

$$(\alpha+9)a_5 + (\alpha^2+9\alpha+6)a_2a_4 + \frac{\alpha^3+8\alpha^2+\alpha-10}{2}a_2^{\,2}a_3 + \frac{\alpha^2+9\alpha+8}{2}a_3^{\,2} + \frac{\alpha^4+6\alpha^3-13\alpha^2-18\alpha+24}{24}a_2^{\,4} = a_5 + a_4c_1 + a_3c_2 + a_2c_3 + c_4 \tag{3.4}$$

Applying Lemma 1 to equation (3.1), we have

$$|a_2| \le \frac{2}{\alpha+2}$$

Eliminating a_2 from equation (3.2) using equation (3.1), we have

$$(\alpha+4)a_3 = c_2 - \frac{\alpha^2+3\alpha-2}{(\alpha+2)^2}\frac{c_1^{\,2}}{2}$$

so that

$$(\alpha+4)|a_3| = \left|c_2 - \frac{\alpha^2+3\alpha-2}{(\alpha+2)^2}\frac{c_1^{\,2}}{2}\right|$$

which can be written as

$$(\alpha+4)|a_3| = \left|c_2 - \sigma\frac{c_1^{\,2}}{2}\right|$$

where

$$\sigma = \frac{\alpha^2+3\alpha-2}{(\alpha+2)^2}$$

Therefore, applying Lemma 1 and the Corollary 1, we obtain

$$(\alpha+4)|a_3| \le \begin{cases} 2(1-\sigma) & if\, 0<\alpha\le\dfrac{-3+\sqrt{17}}{2} \\ 2 & if\, \alpha\ge\dfrac{-3+\sqrt{17}}{2} \end{cases}$$

which gives the second inequality.
Eliminating α_2 and α_3 from equation (3.3) using equations (3.1) and (3.2) we obtain

$$(\alpha+6)a_4 = c_3 - \frac{\alpha^2+5\alpha-2}{(\alpha+2)(\alpha+4)}c_1c_2 + \frac{2\alpha^4+17\alpha^3+31\alpha^2-8\alpha+12}{6(\alpha+2)^3(\alpha+4)}c_1^3. \quad (3.5)$$

Now, if $\alpha^2+5\alpha-2\le 0$, that is $\alpha \le \frac{-5+\sqrt{33}}{2}$, then we have

$$(\alpha+6)|a_4| \le |c_3| + \frac{2-\alpha^2-5\alpha}{(\alpha+2)(\alpha+4)}|c_1||c_2| + \frac{2\alpha^4+17\alpha^3+31\alpha^2-8\alpha+12}{6(\alpha+2)^3(\alpha+4)}|c_1^3|$$

Applying Lemma 1 we have,

$$|a_4| \le \left|\frac{52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288}{6(\alpha+2)^3(\alpha+4)(\alpha+6)}\right| \quad (3.6)$$

If $\alpha \ge \frac{-5+\sqrt{33}}{2}$, then rearranging (3.5) and using the triangle inequality y, we obtain

$$(\alpha+6)|a_4| \le |c_3 - c_1c_2| + \frac{\alpha+10}{(\alpha+2)(\alpha+4)}|c_1|\left|c_2 - \frac{3\alpha^3+41\alpha^2+110\alpha+52}{3(\alpha+2)^2(\alpha+10)}\frac{c_1^2}{2}\right|.$$

Appling Lemmas 1 and 3 with Corollary 1, we have

$$|a_4| \le \left|\frac{14\alpha^2+96\alpha+232}{3(\alpha+2)(\alpha+4)(\alpha+6)}\right| \quad (3.7)$$

Combining (3.6) and (3.7) gives the required result.

Using equations (3.1), (3.2) and (3.3) in (3.4) we have

$$(\alpha+8)a_5 = c_4 - \frac{\alpha^2+7\alpha}{2(\alpha+4)^2}c_2^2 - \frac{\alpha^2+7\alpha-2}{(\alpha+2)(\alpha+6)}c_1c_3$$
$$+ \frac{A(\alpha)}{2(\alpha+2)^2(\alpha+4)^2(\alpha+6)}c_1^2c_2 - \frac{B(\alpha)}{24(\alpha+2)^4(\alpha+4)^2(\alpha+6)}c_1^4 \quad (3.8)$$

where

$$A(\alpha) = 2\alpha^5+33\alpha^4+175\alpha^3+304\alpha^2+60\alpha+128$$

and

$$B(\alpha) = 6\alpha^7+121\alpha^6+892\alpha^5+2935\alpha^4+4258\alpha^3+3124\alpha^2+3448\alpha+768$$

If $\alpha^2+7\alpha-2\le 0$, that is, $\alpha \le \frac{-7+\sqrt{57}}{2}$, then equation (3.8) can be written as

$$(\alpha+8)a_5 = c_4 - \frac{\alpha^2+7\alpha}{2(\alpha+4)^2}c_2^2 + \frac{2-\alpha^2-7\alpha}{(\alpha+2)(\alpha+6)}c_1c_3$$
$$+ \frac{A(\alpha)}{2(\alpha+2)^2(\alpha+4)^2(\alpha+6)}c_1^2c_2 - \frac{B(\alpha)}{24(\alpha+2)^4(\alpha+4)^2(\alpha+6)}c_1^4$$

Rearranging, we have

$$(\alpha+8)|a_5| \le \left| c_4 - \frac{\alpha^2+7\alpha}{(\alpha+4)^2}\frac{c_2^{\,2}}{2} \right| + \frac{2-\alpha^2-7\alpha}{(\alpha+2)(\alpha+6)}|c_1||c_3|$$

$$+ \frac{A(\alpha)}{2(\alpha+2)^2(\alpha+4)^2(\alpha+6)}|c_1{}^2| \left| c_2 - \frac{B(\alpha)}{6(\alpha+2)^2 A(\alpha)}\frac{c_1^{\,2}}{2} \right|$$

Using Lemma 1 with Corollary 1, we obtain

$$(\alpha+8)|a_5| \le \frac{14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)}$$

which gives the first part of the inequality.

On the other hand if $\frac{\alpha^2+7\alpha-2}{(\alpha+2)(\alpha+6)} \ge 0$ that is $\alpha \ge \frac{-7+\sqrt{57}}{2}$ then equation (3.6) can be written as

$$(\alpha+8)|a_5| \le |c_4 - c_1c_3| + \frac{\alpha^2+7\alpha}{2(\alpha+4)^2}|c_2| \left| c_2 - \frac{A(\alpha)}{2(\alpha+2)^2(\alpha+6)(\alpha^2+7\alpha)}c_1^{\,2} \right|$$

$$+ \frac{\alpha\ \ +14}{(\alpha+2)\ \ (\alpha+6)}|c_1| \left| c_3 - \frac{B(\alpha)}{24(\alpha+2)^3(\alpha+4)^2(\alpha+14)}c_1^{\,3} \right|$$

Applying Lemmas 1 and 4 with Corollary 1 to the above, we obtain

$$(\alpha+8)|a_5| \le \frac{4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072}{(\alpha+2)\ (\alpha+4)^2(\alpha+6)}$$

leading to the second part of the inequality.

Corollary 3. *Let* $f \in J_0$. *Then*

$$|a_2| \le 1,\quad |a_3| \le \frac{3}{4},\quad |a_4| \le \frac{29}{18},\quad |a_5| \le 2$$

Corollary 4. *Let* $f \in J_1$. *Then*

$$|a_2| \le \frac{2}{3},\quad |a_3| \le \frac{2}{5},\quad |a_4| \le \frac{38}{35},\quad |a_5| \le \frac{218}{175}$$

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