

Third Hankel Determinant for Certain Subclass of Analytic Functions

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Abstract

The third Hankel determinant, $H_3(1)$ for subclass of analytic functions satisfying geometric condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0$$

for nonnegative real number α , in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ is derived in line with a method of classical analysis devised by Libera and Zlotkiewicz [9].

Keywords: Hankel determinant, caratheodory functions, product of geometric expression, analytic functions.

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1 Introduction

Let A denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. By S , S^* , C and R , we mean the well known subclasses of A which consist of univalent, starlike, convex and bounded turning functions respectively. In [8], Jimoh et-al introduced a subclass of analytic functions denoted by $\mathcal{J}_\alpha$ which satisfy the geometric condition:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}} > 0. \quad (1.2)$$

for non negative real number α , where estimates on the bounds of some coefficients were investigated. Also in [6], Ganiyu et-al obtained the bound on the second Hankel determinant, $H_2(2)$ for this same subclass of analytic functions, $\mathcal{J}_\alpha$. In [10], Noonan and Thomas defined the q th Hankel determinant of f for $q \geq 1$, $n \geq 0$ by:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

This determinant has been considered for specific choices of q and n by several authors with subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \rightarrow \infty$ to the determination of precise bounds on $H_q(n)$ for some subclasses of analytic functions. It is well known that the Fekete-Szegő functional is $|a_3 - a_2^2| = H_2(1)$. The

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second Hankel determinant defined by $H_2(2) = |a_2a_4 - a_3^2|$ also received a lot of attention by researchers among which is the notable work of Janteng, et-al, [7] where they obtained the second Hankel determinant for some subclasses of analytic functions. Other contributors in this regard include Abubaker [1], Al-Refai [2], Norlyda et-al [11], Vamshee [13].

Babalola [4], Shanmungam et-al [12], Vamshee et-al [14] have studied the third Hankel determinant, $H_3(1)$ for various classes of analytic and univalent functions. In the present investigation, our focus is on the third Hankel determinant, $H_3(1)$ for the subclass $\mathcal{J}_\alpha$ given by:

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

For $f \in A$, $a_1 = 1$ so that

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \quad (1.3)$$

In this paper, we seek to find the sharp upper bound on $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and $|H_3(1)|$ respectively for the functions belonging to the subclass $\mathcal{J}_\alpha$. We shall make use of our earlier results on the bounds on each of the coefficients and the functional $|a_2a_4 - a_3^2| = H_2(2)$.

2 Preliminary Lemmas.

To prove the main results in the next section, we need the following lemmas. Let P denote the class of Caratheodory functions $p(z) = 1 + c_1z + c_2z^2 + \dots$ which are analytic and satisfy $p(0) = 1$, $\operatorname{Re} p(z) > 0$ in open unit disk U .

Lemma 2.1. [5] Let $p \in P$. Then $|c_k| \leq 2, k = 1, 2, 3, \dots$. Equality is attained by the moebius function

$$L_0(z) = \frac{1+z}{1-z}.$$

Lemma 2.2. [9] Let $p \in P$, then

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.1)$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \quad (2.2)$$

for some value of x, z such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.3. [3] Let $p \in P$. Then we have sharp inequalities

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1 - \sigma), & \text{if } \sigma \leq 0, \\ 2, & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1), & \text{if } \sigma \geq 2. \end{cases}$$

Lemma 2.4. [6] Let $f \in \mathcal{J}_\alpha$. Then

$$|H_2(2)| \leq \frac{4}{(\alpha + 4)^2}$$

Lemma 2.5. [8] Let $f \in \mathcal{J}_\alpha$. Then

$$\begin{aligned} |a_2| &\leq \frac{2}{\alpha + 2} \\ |a_3| &\leq \begin{cases} \frac{2(\alpha+6)}{(\alpha+2)^2(\alpha+4)} & \text{if } 0 < \alpha \leq \frac{-3+\sqrt{17}}{2}, \\ \frac{2}{\alpha+4} & \text{if } \alpha \geq \frac{-3+\sqrt{17}}{2}. \end{cases} \\ |a_4| &\leq \begin{cases} \frac{52\alpha^4+472\alpha^3+1208\alpha^2+896\alpha+288}{6(\alpha+2)^3(\alpha+4)(\alpha+6)} & \text{if } \alpha \leq \frac{-5+\sqrt{33}}{2}, \\ \frac{14\alpha^2+96\alpha+232}{3(\alpha+2)(\alpha+4)(\alpha+6)} & \text{if } \alpha \geq \frac{-5+\sqrt{33}}{2}. \end{cases} \\ |a_5| &\leq \begin{cases} \frac{14\alpha^5+236\alpha^4+1348\alpha^3+2976\alpha^2+2160\alpha+1024}{(\alpha+2)^2(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \leq \frac{-7+\sqrt{57}}{2}, \\ \frac{4\alpha^4+74\alpha^3+584\alpha^2+2152\alpha+3072}{(\alpha+2)(\alpha+4)^2(\alpha+6)(\alpha+8)} & \text{if } \alpha \geq \frac{-7+\sqrt{57}}{2}. \end{cases} \end{aligned}$$

3 Main Results.

Theorem 3.1. *let $f \in J_\alpha$. Then we have*

$$|a_2a_3 - a_4| \leq \frac{2(\alpha^2 + 6\alpha + 16)}{3(\alpha + 2)(\alpha + 4)(\alpha + 6)} \sqrt{\frac{\alpha^3 + 8\alpha^2 + 28\alpha + 32}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}}, \quad 0 \leq \alpha < 1$$

Proof. Using the results obtained earlier in [8], we have that if $f \in \mathcal{J}_\alpha$, then

$$a_2 = \frac{c_1}{\alpha + 2} \quad (3.1)$$

$$a_3 = \frac{c_2}{\alpha + 4} - \frac{\alpha^2 + 3\alpha - 2}{2(\alpha + 2)^2(\alpha + 4)} c_1^2 \quad (3.2)$$

$$a_4 = \frac{c_3}{\alpha + 6} + \frac{2 - 5\alpha - \alpha^2}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} c_1 c_2 + \frac{2\alpha^4 + 17\alpha^3 + 31\alpha^2 - 8\alpha + 12}{6(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c_1^3 \quad (3.3)$$

so that

$$|a_2a_3 - a_4| = \left| \frac{\alpha^2 + 6\alpha + 4}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} c_1 c_2 - \frac{\alpha^4 + 10\alpha^3 + 29\alpha^2 + 20\alpha - 12}{3(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c_1^3 - \frac{c_3}{\alpha + 6} \right| \quad (3.4)$$

substituting c_2 and c_3 in Lemma 2.2 into equation (3.4), we have

$$|a_2a_3 - a_4| = \left| \frac{48 - 8\alpha - 32\alpha^2 - 10\alpha^3 - \alpha^4}{12(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c_1^3 - \frac{2(4 - c_1^2)}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} c_1 x + \frac{(4 - c_1^2)}{4(\alpha + 6)} c_1 x^2 - \frac{(4 - c_1^2)(1 - |x|^2)z}{2(\alpha + 6)} \right|$$

By Lemma 2.1, $|c_1| \leq 2$. Suppose that $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. By the use of triangle inequality with $\xi = |x|$ and noting that $48 - 8\alpha - 32\alpha^2 - 10\alpha^3 - \alpha^4 \geq 0$ for $0 \leq \alpha < 1$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{48 - 8\alpha - 32\alpha^2 - 10\alpha^3 - \alpha^4}{12(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c^3 + \frac{2(4 - c^2)}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} c\xi \\ &\quad + \frac{(c - 2)(4 - c^2)}{4(\alpha + 6)} \xi^2 + \frac{4 - c^2}{2(\alpha + 6)} \\ &= F(c, \xi) \end{aligned} \quad (3.5)$$

we assume the upper bound for equation (3.5) occurs at an interior point of the set $\{(\xi, c) : \xi \in [0, 1] \text{ and } c \in [0, 2]\}$. Differentiating $F(c, \xi)$ partially with respect to ξ , we get

$$F'(c, \xi) = \frac{2(4 - c^2)c}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} + \frac{(c - 2)(4 - c^2)\xi}{2(\alpha + 6)}$$

For $0 < \xi < 1$ and for fixed c with $0 < c < 2$, we observe that $F'(c, \xi) > 0$. Therefore, $F'(c, \xi)$ is an increasing function of ξ , which contradicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\xi, c) : \xi \in [0, 1] \text{ and } c \in [0, 2]\}$. Also for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \xi \leq 1} F(c, \xi) = F(c, 1) = G(c), \text{ say}$$

replacing ξ by 1 in equation (3.5), we obtain

$$G(c) = F(c, 1) = \frac{\alpha^2 + 6\alpha + 16}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} c - \frac{\alpha^4 + 10\alpha^3 + 41\alpha^2 + 68\alpha + 36}{3(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c^3$$

so that

$$G'(c) = \frac{\alpha^2 + 6\alpha + 16}{(\alpha + 2)(\alpha + 4)(\alpha + 6)} - \frac{\alpha^4 + 10\alpha^3 + 41\alpha^2 + 68\alpha + 36}{(\alpha + 2)^3(\alpha + 4)(\alpha + 6)} c^2$$

$G'(c) = 0$ implies

$$c = \pm \sqrt{\frac{(\alpha + 2)(\alpha^2 + 6\alpha + 16)}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}}$$

since $c \in [0, 2]$, we have $c = \sqrt{\frac{\alpha^3 + 8\alpha^2 + 28\alpha + 32}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}}$ as the maximum point of $G(c)$. Therefore

$$G(c) \leq \frac{2(\alpha^2 + 6\alpha + 16)}{3(\alpha + 2)(\alpha + 4)(\alpha + 6)} \sqrt{\frac{\alpha^3 + 8\alpha^2 + 28\alpha + 32}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}}$$

That is the upper bound of equation (3.5) corresponds to $\zeta = 1$ and $c = \sqrt{\frac{\alpha^3 + 8\alpha^2 + 28\alpha + 32}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}}$. □

Theorem 3.2. Let $f \in \mathcal{J}_\alpha$. Then

$$|a_3 - a_2^2| \leq \frac{2}{\alpha + 4}$$

Proof. Using equations (3.1) and (3.2),

$$|a_3 - a_2^2| = \frac{1}{\alpha + 4} \left| c_2 - \left(\frac{\alpha + 3}{\alpha + 2} \right) \frac{c_1^2}{2} \right|$$

Applying Lemma 2.3, with $\sigma = \frac{\alpha + 3}{\alpha + 2}$, we obtain

$$\left| c_2 - \left(\frac{\alpha + 3}{\alpha + 2} \right) \frac{c_1^2}{2} \right| \leq 2$$

hence the result. □

Corollary 3.1. Let $f \in \mathcal{J}_\alpha$. Then

$$|H_3(1)| \leq \begin{cases} \frac{J_1 + J_2 \sqrt{J_5}}{(\alpha + 2)^2 J_6}, & \text{if } 0 \leq \alpha \leq \frac{-3 + \sqrt{17}}{2}, \\ \frac{J_3 + J_4 \sqrt{J_5}}{J_6}, & \text{if } \alpha \geq \frac{-3 + \sqrt{17}}{2}. \end{cases}$$

where,

$$\begin{aligned} J_1 &= 252\alpha^{11} + 8784\alpha^{10} + 134316\alpha^9 + 1188072\alpha^8 + 6737328\alpha^7 \\ &\quad + 25615584\alpha^6 + 66411072\alpha^5 + 117846144\alpha^4 + 143325504\alpha^3 \\ &\quad + 41925888\alpha^2 + 64143360\alpha + 16920576, \end{aligned}$$

$$\begin{aligned} J_2 &= 52\alpha^8 + 1408\alpha^7 + 15944\alpha^6 + 99248\alpha^5 + 369248\alpha^4 + 818240\alpha^3 \\ &\quad + 997120\alpha^2 + 569344\alpha + 147456, \end{aligned}$$

$$\begin{aligned} J_3 &= 72\alpha^8 + 2304\alpha^7 + 31176\alpha^6 + 233442\alpha^5 + 1057248\alpha^4 + 2945178\alpha^3 \\ &\quad + 4854420\alpha^2 + 4258872\alpha + 1496880, \end{aligned}$$

$$J_4 = 28\alpha^6 + 696\alpha^5 + 7280\alpha^4 + 42144\alpha^3 + 143744\alpha^2 + 276480\alpha + 237568,$$

$$J_5 = \alpha^6 + 16\alpha^5 + 117\alpha^4 + 474\alpha^3 + 1100\alpha^2 + 1304\alpha + 576,$$

$$J_6 = 9(\alpha + 2)^2(\alpha + 4)^3(\alpha + 6)^2(\alpha + 8)(\alpha^3 + 8\alpha^2 + 25\alpha + 18).$$

Proof. By equation (1.3), we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|$$

using Lemma 4, the first inequality of the result in Lemma 5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8(\alpha+6)}{(\alpha+2)^2(\alpha+4)^3} + \frac{2(14\alpha^5 + 236\alpha^4 + 1348\alpha^3 + 2976\alpha^2 + 2160\alpha + 1024)}{(\alpha+2)^2(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \left(\frac{(\alpha^2 + 6\alpha + 16)(52\alpha^4 + 472\alpha^3 + 1208\alpha^2 + 896\alpha + 288)}{9(\alpha+2)^4(\alpha+4)^2(\alpha+6)^2(\alpha^3 + 8\alpha^2 + 25\alpha + 18)} \right) \\ & \left(\sqrt{(\alpha^3 + 8\alpha^2 + 28\alpha + 32)(\alpha^3 + 8\alpha^2 + 25\alpha + 18)} \right) \end{aligned}$$

simplifying, we have the first inequality.

Also by using Lemma 2.4, the second inequality of the result in Lemma 2.5 together with the results obtained in Theorems 3.1 and 3.2,

$$\begin{aligned} |H_3(1)| \leq & \frac{8}{(\alpha+4)^3} + \frac{2(4\alpha^4 + 74\alpha^3 + 584\alpha^2 + 2152\alpha + 3072)}{(\alpha+2)(\alpha+4)^3(\alpha+6)(\alpha+8)} \\ & + \frac{2(\alpha^2 + 6\alpha + 16)(14\alpha^2 + 96\alpha + 232)}{9(\alpha+2)^2(\alpha+4)^2(\alpha+6)^2} \sqrt{\frac{\alpha^3 + 8\alpha^2 + 28\alpha + 32}{\alpha^3 + 8\alpha^2 + 25\alpha + 18}} \end{aligned}$$

By simplification, we obtain the other inequality.

□

4 Conclusion

We have been able to find the sharp upper bound on functionals $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ and the third Hankel determinant, $|H_3(1)|$ for the functions belonging to the subclass $\mathcal{J}_\alpha$.

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