

Order Of Schlichtness Of Certain Linear Sums*

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Abstract

In this paper, we find the largest real number ρ such that the real part of certain geometric quantity greater than ρ implies schlichtness of certain linear sums.

1 Introduction

Let $\mathbb{H}$ denote the class of functions:

$$f(z) = z + a_2 z^2 + \dots \quad (1)$$

which are holomorphic in the unit disk $E = \{z \in \mathbb{C}: |z| < 1\}$. Let $f, g \in \mathbb{H}$, we say that $f(z)$ is subordinate to $g(z)$ (written as $f \prec g$) if there exists a holomorphic function $w(z)$ (not necessarily schlicht) in E , satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in E$. For example, it is well known that any Caratheodory function $p(z) = 1 + c_1 z + \dots$ is subordinate to the Möbius function $L_0(z) = (1+z)/(1-z)$.

In [2], Babalola proved that holomorphic functions satisfying

$$Re \left\{ (1-\lambda) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} + \lambda \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \beta, \quad z \in E, \quad (2)$$

also satisfy

$$Re \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > 0, \quad z \in E,$$

for all λ such that $\lambda \leq \beta < 1$, and that if $\beta = \frac{1}{2}$, then for all $\lambda \geq 0$,

$$Re \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \frac{1}{2}, \quad 0 < \alpha \leq 1.$$

D^n is defined as $D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))'$ and for $n \geq 1$, schlichtness is implied by both results.

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DEFINITION. We say $f \in T_{n,\lambda}^\alpha(\beta)$ if and only if $f(z)$ satisfies the inequality (2).

In this paper, we shall find the largest real number ρ such that

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho, \quad z \in E$$

given that $f \in T_{n,\lambda}^\alpha(\beta)$ for $0 \leq \lambda \leq \beta < 1$. To do this we shall employ the technique of Briot-Bouquet differential subordination. A holomorphic function $p(z)$ is said to satisfy Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec h(z), \quad z \in E \quad (3)$$

for complex constants η and μ , and a complex function $h(z)$ with $h(0) = 1$ such that $\operatorname{Re} [\eta h(z) + \mu] > 0$ in E . If the differential equation

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = h(z), \quad q(0) = 1 \quad (4)$$

has the schlicht solution $q(z)$ in E , then

$$p(z) \prec q(z) \prec h(z)$$

and $q(z)$ is the best dominant.

A schlicht function $\tilde{q}(z)$ is said to be the best dominant of the differential subordination (3) if $q(z)$ and $\tilde{q}(z)$ are dominants of (3) and $\tilde{q}(z) \prec q(z)$ for all the dominants $q(z)$ of (3). For more on the technique of differential subordination, see [3,4,5,7].

We state and prove our main result in Section 3.

2 Preliminary Lemmas

In this section, we present Lemmas 1–4 as follows.

LEMMA 1 ([1]). Let $f \in \mathbb{H}$ and $\alpha > 0$ be real, if

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha}$$

is independent of n for $z \in E$, then

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} = \alpha \frac{D^{n+1}f(z)}{D^n f(z)}.$$

LEMMA 2 ([3]). Let η and μ be complex constants and $h(z)$ be a convex schlicht function in E satisfying $h(0) = 1$ and $\operatorname{Re} [\eta h(z) + \mu] > 0$. Suppose $p \in P$ satisfies the differential subordination (3). If the differential equation (4) has schlicht solution $q(z)$

in E , then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (3). Furthermore, the formal solution of (4) is given as

$$q(z) = \frac{zF'(z)}{F(z)} = \frac{\eta + \mu}{\eta} \left(\frac{H(z)}{F(z)} \right)^\eta - \frac{\mu}{\eta} \quad (5)$$

where

$$F(z)^\eta = \frac{\eta + \mu}{z^\mu} \int_0^z t^{\mu-1} H(t)^\eta dt$$

and

$$H(z) = z \exp \left(\int_0^z \frac{h(t) - 1}{t} dt \right).$$

LEMMA 3 ([6]). Let v be a positive measure on $[0, 1]$ and let h be a complex valued function defined on $E \times [0, 1]$ such that $h(\cdot, t)$ is holomorphic in E for each $t \in [0, 1]$ for all $z \in E$. Suppose that $\operatorname{Re}[h(z, t)] > 0$, $h(-r, t)$ is real and

$$\operatorname{Re} \left[\frac{1}{h(z, t)} \right] \geq \frac{1}{h(-r, t)} \text{ for } |z| \leq r < 1 \text{ and } t \in [0, 1].$$

If

$$h(z) = \int_0^1 h(z, t) dv(t),$$

then

$$\operatorname{Re} \left[\frac{1}{h(z)} \right] \geq \frac{1}{h(-r)}.$$

For real or complex numbers a, b, c with $c \neq 0, -1, -2, \dots$, the hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{a.b}{c} \cdot \frac{z}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots \quad (6)$$

We note that the series converges absolutely for $z \in E$ and hence represents an holomorphic function in E . The following identities associated with the hypergeometric series are well known.

LEMMA 4 ([6]). For real numbers a, b, c ($c \neq 0, -1, -2, \dots$), we have

$$\int_0^1 t^{b-1} \frac{(1-t)^{c-b-1}}{(1-tz)^a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z), \quad (c > b > 0),$$

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$$

and

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right).$$

3 Main Result

In this section, we present our main result.

THEOREM 1. Let $f \in T_{n,\lambda}^\alpha(\beta)$. If $\beta \geq \lambda \geq 0$, then

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}$$

where

$$q(z) = \frac{\lambda(1-z)^{\frac{2\alpha(\beta-1)}{\lambda}}}{\alpha(1-\lambda) \int_0^1 s^{\frac{\alpha}{\lambda}-\alpha-1} (1-sz)^{\frac{2\alpha(\beta-1)}{\lambda}} ds} = 1 + q_1 z + \dots$$

is the best dominant. Furthermore

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \rho$$

where

$$\rho = \left[{}_2F_1 \left(1, \frac{2\alpha(1-\beta)}{\lambda}; \frac{\alpha}{\lambda} - \alpha + 1; \frac{1}{2} \right) \right]^{-1}.$$

PROOF. Since $f \in T_{n,\lambda}^\alpha(\beta)$, we see that

$$(1-\lambda) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} + \lambda \frac{D^{n+1} f(z)}{D^n f(z)} = B_n(f; \lambda) \prec \frac{1 + (1 - 2\beta)z}{1 - z}. \quad (7)$$

Let

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = p(z). \quad (8)$$

Then $p(z)$ is holomorphic in E with $p(0) = 1$. So taking the logarithmic differentiations in both sides of (8), we have

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha + \frac{z p'(z)}{p(z)}$$

Using Lemma 1 and then simplify, we have

$$\frac{D^{n+1} f(z)}{D^n f(z)} = 1 + \frac{z p'(z)}{\alpha p(z)} \quad (9)$$

With (8) and (9) in (7), we have

$$\lambda + (1-\lambda) \left[p(z) + \frac{z p'(z)}{\alpha \left(\frac{1-\lambda}{\lambda} \right) p(z)} \right] = B_n(f; \lambda) \prec \frac{1 + (1 - 2\beta)z}{1 - z}.$$

If

$$\lambda + (1-\lambda) \left[q(z) + \frac{z q'(z)}{\alpha \left(\frac{1-\lambda}{\lambda} \right) q(z)} \right] = \frac{1 + (1 - 2\beta)z}{1 - z},$$

then

$$q(z) + \frac{zq'(z)}{\alpha(\frac{1-\lambda}{\lambda})q(z)} = \frac{1-\lambda+(1+\lambda-2\beta)z}{(1-\lambda)(1-z)} = h(z). \quad (10)$$

Then $h(0) = 1$ and we take

$$\eta = \frac{\alpha(1-\lambda)}{\lambda} \quad \text{and} \quad \mu = 0.$$

It is easily verified that $\eta h(z) + \mu$ has positive real part given that $0 \leq \lambda \leq \beta$. Therefore, by Lemma 2, $p(z)$ satisfies the differential subordination (3) and hence

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q(z) \prec h(z)$$

where $q(z)$ is the solution of differential equation (10) obtained as follows:

$$H(z) = z(1-z)^{\frac{2(\beta-1)}{(1-\lambda)}}.$$

Also, we have

$$F(z) = \left[\frac{\alpha(1-\lambda)}{\lambda} \int_0^z t^{\frac{\alpha}{\lambda}-\alpha-1} (1-t)^{\frac{2\alpha(\beta-1)}{\lambda}} dt \right]^{\frac{\lambda}{\alpha(1-\lambda)}}.$$

Now from (5), we have

$$q(z) = \frac{H^\eta}{F^\eta} = \frac{\lambda}{\alpha(1-\lambda)Q(z)}$$

where

$$Q(z) = \int_0^1 s^{\frac{\alpha}{\lambda}-\alpha-1} \left(\frac{1-sz}{1-z} \right)^{\frac{2\alpha(\beta-1)}{\lambda}} ds$$

Next, we show that

$$\inf_{|z|<1} \{\operatorname{Re}(q(z))\} = q(-1), \quad z \in E, \quad (11)$$

To prove (11), we show that

$$\operatorname{Re} \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-1)}$$

and by Lemma 4 with some simplification, we get

$$Q(z) = \frac{\Gamma(b)}{\Gamma(c)} {}_2F_1 \left(1, a; c; \frac{z}{z-1} \right)$$

where

$$a = \frac{2\alpha(1-\beta)}{\lambda}, \quad b = \frac{\alpha}{\lambda} - \alpha \quad \text{and} \quad c = \frac{\alpha}{\lambda} - \alpha + 1.$$

Hence, we see that

$$Q(z) = \int_0^1 h(z, s) dv(s)$$

where

$$h(z, s) = \frac{1 - z}{1 - (1 - s)z}, \quad 0 \leq s \leq 1$$

and

$$dv(s) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} s^{a-1}(1-s)^{c-a-1} ds$$

which is a positive measure on $[0, 1]$.

It will be noted that $\operatorname{Re}\{h(z, s)\} > 0$, $h(-r, s)$ is real for $0 \leq r < 1$, and

$$\operatorname{Re}\left\{\frac{1}{h(z, s)}\right\} = \operatorname{Re}\left\{\frac{1 - (1 - s)z}{1 - z}\right\} \geq \frac{1 + (1 - s)r}{1 + r} = \frac{1}{h(-r, s)}$$

for $|z| \leq r < 1$ and $s \in [0, 1]$. So using Lemma 3, we have

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-r)},$$

and letting $r \rightarrow 1^-$, we obtain

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)}.$$

Hence we have

$$\operatorname{Re}\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > q(-1),$$

that is,

$$\operatorname{Re}\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \rho$$

where

$$\rho = \frac{\lambda}{\alpha(1 - \lambda)Q(-1)} = \frac{\lambda}{\alpha(1 - \lambda)} \left[2^a \int_0^1 s^{b-1}(1+s)^{-a} ds \right]^{-1}.$$

By Lemma 4, we see that

$$\rho = \frac{\lambda}{\alpha(1 - \lambda)} \left[\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1\left(1, a; c; \frac{1}{2}\right) \right]^{-1}$$

can be simplify as

$$\rho = \left[{}_2F_1\left(1, \frac{2\alpha(1-\beta)}{\lambda}; \frac{\alpha(1-\lambda)}{\lambda} + 1; \frac{1}{2}\right) \right]^{-1}.$$

The bound ρ is the best possible.

4 Remark

We first remark that for $n \geq 1$, our result shows that holomorphic function $f(z)$, satisfying the geometric condition (2) defined by linear sum, is schlicht of order ρ in the unit disk. Then by (6), we can write ρ in series form:

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^k \frac{2\alpha(1-\beta) + j\lambda}{\alpha(1-\lambda) + (j+1)\lambda},$$

from which it follows that $\lambda = 0$ yields $\rho = \beta$ for all α . The series can be rewritten as

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^k \left(\frac{\alpha(1-\beta) + (j+1)\lambda}{\alpha(1-\lambda) + (j+1)\lambda} + \frac{\alpha(1-\beta) - \lambda}{\alpha(1-\lambda) + (j+1)\lambda} \right).$$

Thus, if $\lambda \leq \alpha(1-\beta)$, we have

$$\frac{1}{\rho} > 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^k \frac{\alpha(1-\beta) + (j+1)\lambda}{\alpha(1-\lambda) + (j+1)\lambda} > 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} > 2.$$

It follows that $\rho > 1/2$ whenever $0 < \alpha \leq \frac{\beta}{1-\beta}$ which guarantees that $\lambda \leq \alpha(1-\beta)$ given that $\lambda \leq \beta$.

This deduction shows some improvement on the earlier result of Babalola [2] for λ in the real interval $[0, \beta]$.

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