

On the Number of Order-Preserving Alternating Semigroups

¹Bakare G.N., ²Ibrahim G.R. and ¹Makanjuola S.O.

¹Department of Mathematics, University of Ilorin, Kwara State.

²Department of Statistics and Mathematical Sciences, Kwara State University, Malete.

Abstract

A Semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. A transformation on X is a function from X to itself. Transformation semigroups are one of the most fundamental mathematical objects. They occur in theoretical computer science, where properties of language depend on algebraic properties of various transformation semigroups related to them.

Let be C_n the symmetric inverse semigroup on $X_n = \{1, 2, \dots, n\}$, A_n^c be the alternating semigroups on n -object, and OA_n^c be its subsemigroup of order preserving alternating semigroup of X_n . The set $E(OA_n^c)$ and $N(OA_n^c)$ be its idempotent and nilpotent element of order preserving alternating semigroups of X_n . In this paper, we obtain and discuss formulae for the number of OA_n^c , $E(OA_n^c)$ and $N(OA_n^c)$ respectively. We also characterize the Green's relation on OA_n^c .

Keywords: Semigroups, Symmetric Inverse Semigroup, Alternating Semigroup, order preserving, idempotent and nilpotent elements.

1.0 Introduction

Let $X_n = \{1, 2, \dots, n\}$. Then a (partial) transformation $\alpha : \text{Dom} \alpha \subseteq X_n \rightarrow \text{Im} \alpha$ is said to be full or total if $\text{Dom} \alpha = X_n$, otherwise is called Strictly Partial. The set of all partial transformations on n -objects form a semigroup under the usual composition of transformation. It is denoted by P_n when it is partial, T_n when it is full or total and C_n when it is partial one-to-one. The elements C_n are usually called Chart. All these are the three fundamental transformation semigroups which were introduced in [1]. The semigroup C_n is the main object of the study in this research work.

The idea of an even permutation has recently been generalized via path notation, of one-to-one partial transformation (charts) C_n . A transformation α in C_n is said to be even if it can be expressed as a product of an even number of transpositional or a product of any number of circuit/path of odd length. The set of all even charts of X_n called **Alternating Semigroups** is usually denoted by A_n^c . Combinatorial and algebraic properties of transformation semigroups have been studied over a long period and interesting results have emerged. Many researchers have studied combinatorial and algebraic properties on different classes of transformation semigroups. Kehinde [2] studied Some Algebraic and Combinatorial Properties of Semigroup of Injective Partial Contraction Mappings and Isometrics of a Finite Chain and many results were obtained. Adeniji [3] studied Identity Difference Transformation Semigroups and obtained several results.

Also Adeshola [4] studied Some Semigroups of Full Contraction Mapping on a Finite Chain and many results were obtained as well. Work have started on investigating Algebraic and Combinatorial Properties on new class of transformation semigroups called Alternating Semigroups (even chart in symmetric inverse semigroup) introduced in [5] and some results have been obtained [6].

2.0 Combinatorial Results

Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformation. Many numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers [1, pp.42 & 96], the factorial [7,8], the binomial [9,10], the Fibonacci number [11], Catalan numbers [12] and Lah numbers [13,14], etc., have all featured in these enumerated problems. For a nice survey article concerning combinatorial problems in the symmetric inverse

Corresponding author: Bakare G.N., E-mail: bakaregattanaimat@gmail.com, Tel.: +2348036459002, 2348052784033.

semigroup and some of its subsemigroups we refer the reader to [15,16,17]. The enumerated problems leads to many numbers in on line Sloane's encyclopaedia of integer sequences [18]. Let $\alpha \in OA_n^c$, then the height of α is denoted by $h(\alpha) = |\text{Im}\alpha|$, fix of α is denoted by $f(\alpha) = |\text{F}(\alpha)|$, where $\text{F}(\alpha) = \{x \in X_n : x\alpha = x\}$.

Definition 2.1: $F(n; p) = |\{\alpha \in S : h(\alpha) = |\text{Im}\alpha| = p\}|$, for natural number $n \geq p \geq 0$. Where S is any subsemigroup of A_n^c .

Definition 2.2: Also, an element ε in C_n is idempotent ($\varepsilon^2 = \varepsilon$) if and only if $|\text{Im}\varepsilon| = F(\varepsilon)$.

Definition 2.3: More also, it is well known that in a transformation semigroup on X_n , α is a nilpotent element if no subset of X_n is fixed by α or an element α in a semigroup C_n with zero, is nilpotent if there exists a positive integer k such that $\alpha^k = 0$.

(a) Order Preserving Alternating Semigroups

Lemma 2.4: Let $S = OA_n^c$. Then $F(n; 0) = 1$ for all n .

Proof: It is obvious, since only the empty mapping will produce zero height.

Lemma 2.5: Let $S = OA_n^c$. Then $F(n; n) = 1$ for all n .

Proof: Clearly there is exactly one element of order-preserving alternating semigroup of height n , i.e. $\begin{pmatrix} 12 \dots n \\ 12 \dots n \end{pmatrix}$ (identity).

Lemma 2.6: Let $S = OA_n^c$. Then $F(n; 1) = \begin{cases} 2 & \text{when } n = 2 \\ n^2 & \text{when } n \geq 3 \end{cases}$

Proof: For $n = 2$, the 2 elements of height 1 are order preserving elements, i.e. $F(2; 1) = 2$. For $n \geq 3$, it can be seen that any charts of rank below $n - 1$ are all members of alternating semigroups which are also order preserving element.

Theorem 2.7: Let $S = OA_n^c$ on $[n]$ and $F(n; p)$ be as defined in (2.1). $F(n; p) = |OA_n^c| = \binom{2n}{n} - U_n$ with initial conditions $U_1 = 1, U_2 = 2$ and $U_3 = 4$. And for $n \geq 4, U_n$ satisfies the recurrence relation

$$U_{n+3} = \begin{cases} 2U_{n+2} - U_{n+1} & \text{if } n \text{ is odd} \\ 2U_{n+1} - U_{n+1} + 2 & \text{if } n \text{ is even} \end{cases}$$

Proof: We first observe that the cardinality of order-preserving partial one-one $|S| = |OA_n^c| = \binom{2n}{n}$, obtained in [9] was $|S| = |OA_n^c| = \binom{2n}{n}$, hence, the first part of the statement of the theorem follows immediately. The second part of the statement on U_n for $n \geq 4$ was obtained from on-line sloane's encyclopaedia of integer sequences [18] which satisfies the recurrence relation $U_{n+3} = 2U_{n+2} - U_{n+1}$, if n is odd, and $U_{n+3} = 2U_{n+1} - U_{n+1} + 2$, if n is even, with the initial conditions $U_1 = 1, U_2 = 2, U_3 = 4$.

Remark 2.8: The triangle of number $F(n; p)$ is as at the time of writing this paper not yet listed in [18]. And for $p=0$ the concepts of order-preserving is always one for all values of n in all transformation semigroups. For some computed values of $F(n; p)$ for $|OA_n^c|$, See Table 1.

Table 1: Elements of Order-Preserving Alternating Semigroup(OA_n^c)

n/p	0	1	2	3	4	5	6	$\sum F(n; p) = OA_n^c $
0	1							1
1	1							1
2	1	2	1					4
3	1	9	5	1				16
4	1	16	36	8	1			62
5	1	25	100	100	13	1		240
6	1	36	225	400	225	18	1	906

(b) Idempotent element in Order Preserving Alternating Semigroups

Lemma 2.9: Let $S = E(OA_n^c)$. Then $F(n; 0) = 1$ for all n .

Proof: This follows directly from lemma 2.4.

Lemma 2.10: Let $S = E(OA_n^c)$. Then $F(n; 0) = 1$, and $F(n; 1) = n$, for all $n \geq 2$.

Proof: The first statement follows by lemma 2.5 and for the second statement, it is obvious to see that the map of height one is n .

Lemma 2.11: Let $S = E(OA_n^c)$. Then $|E(OA_n^c)| = 2^n$ for all $n \geq 2$, where $|E(OA_1^c)| = 1$.

Proof: It is obvious to see that all the idempotents in C_n are in A_n^c and they are all order preserving element.

Remark 2.8: The triangle of number $F(n; p)$ has already been listed in [18]. For some computed values of $F(n; p)$ for $|E(OA_n^c)|$, See Table 2.

Table 2. Idempotent Elements of $E(OA_n^c)$.

n/p	0	1	2	3	4	5	6	$\sum F(n; p) = E(OA_n^c) $
0	1							1
1	1							1
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
6	1	6	15	20	15	6	1	64

(c) Nilpotent element in Order Preserving Alternating Semigroups

Lemma 2.13: Let $S = N(OA_n^c)$ be the set of nilpotent element in order preserving alternating semigroup, then $F(n; 0) = 1$ for all n .

Proof: It follows directly from lemma 2.4.

Theorem 2.14: Let $S = N(OA_n^c)$ be the set of nilpotent elements in order preserving alternating semigroup on $[n]$, then

$$|N(OA_n^c)| = \begin{cases} b_n & \text{if } n \text{ is odd} \\ b_n - 2 & \text{if } n \text{ is even} \end{cases}$$

Where b_n , satisfies the recurrence relation

$$(n+1)b_{n+1} = 2(4n+1)b_n - 3(5n-3)b_{n-1} - 2(2n-1)b_{n-2}$$

With the initial condition $b_0 = 1 = b_1$.

Proof: It is clearly observed that whenever n is odd, the nilpotent elements in order-preserving partial one-one are also in order-preserving alternating semigroups [16] and hence the first statement follows. For charts in C_n of rank $n-1$, it has been found that exactly half of these charts are members of $A_n^c[5]$ and this shows that whenever n is even, the nilpotent elements in OA_n^c of rank $n-1$ decreases by 2 elements and these 2 elements are proper paths of length n which are not contained in A_n^c .

Remark 2.15: The triangle of number $F(n; p)$ as at the time of writing this paper not yet listed in Sloane [18]. For some computed values of $F(n; p)$ for $|N(OA_n^c)|$, See Table 3.

Table 3. Nilpotent Element in Order preserving Alternating Semigroup.

n/p	0	1	2	3	4	5	6	$\sum F(n; p) = N(OA_n^c) $
0	1							1
1	1							1
2	1							1
3	1	6	2					9
4	1	12	14					27
5	1	20	50	24	2			97
6	1	30	129	135	36			331

3.0 Green's relations .

The notion of ideals is natural to the consideration of certain equivalence relations on a semigroup. These equivalences have played a fundamental role in the development of semigroup theory. For the definition of Green's relation we refer the reader to [1, chapter 2]. It is customary that when a new class of transformation semigroup is discovered, one of the questions that is often asked concerns the characterization of Green's relations. Each D -class in a semigroup S is a union of L -classes and R -classes. The intersection of an L -class and an R -class is either empty or is an H -class. Hence, it is convenient to visualize a D -class as eggbox, in which each row represents an R -class, each column represents an L -class and each cell represents an H -class. (It is of course possible for the eggbox to contain a single row or a single column of cells, or even to contain only cell). One can then analyse a semigroup, by finding these uniform blocks and describing connections between them. By definition of order-preserving alternating semigroup and proposition 2.4.2 and exercise 2.6.16 of [1], we can deduce the following lemma.

Lemma 3.1: Let $\alpha, \beta \in OA_n^c$. Then

- I. $\alpha L \beta$ if and only if $Im \alpha = Im \beta$
- II. $\alpha R \beta$ if and only if $Dom \alpha = Dom \beta$

- III. $\alpha D \beta$ if and only if $|Im \alpha| = |Im \beta|$
 IV. $D_\alpha = J_\alpha$
 V. $\alpha H \beta$ if and only if $Dom \alpha = Dom \beta$ and $Im \alpha = Im \beta$.

Theorem 3.2: Let $\alpha \in OA_n^e$. Then the following result hold.

- I. $|L_\alpha| = 2^n$ for all $n \geq 2$.
 II. $|R_\alpha| = 2^n$ for all $n \geq 2$.
 III. $|D_\alpha| = n + 1$ for all $n \geq 2$.
 IV. $|H_\alpha| = 2^n$ for all $n \geq 2$.

Proof: Follows directly from Ganyushkin and Mazorchuk [15].

Remark 3.3: It has been observed that the cardinality of L -classes, R -classes, D -classes and H -classes in order-preserving Alternating semigroup are the same as in partial-one-to-one transformations semigroup for all $n \geq 2$. For some computed values of $F(n; p)$ for $|L_\alpha|, |R_\alpha|, |H_\alpha|$ and $|D_\alpha|$, See Tables 4 and 5.

Table 4: Number of Elements of $|L_\alpha|, |R_\alpha|, |H_\alpha|$, in Order preserving Alternating Semigroup.

n/p	0	1	2	3	4	5	6	$\sum F(n; p) = L_\alpha $
0	1							1
1	1							1
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
6	1	6	15	20	15	6	1	64

Table 5: Number of Elements of $|D_\alpha|$ in Order preserving Alternating Semigroup.

n/p	0	1	2	3	4	5	6	$\sum F(n; p) = D_\alpha $
0	1							1
1	1							1
2	1	1	1					3
3	1	1	1	1				4
4	1	1	1	1	1			5
5	1	1	1	1	1	1		6
6	1	1	1	1	1	1	1	7

Example 3.4: For $n = 4, OA_4^e$ has 62 elements. These elements were arranged by their height and within each height by their image sets and kernel classes. The idempotent and nilpotent elements were marked (*) and (+) respectively.

$ Im \alpha = 4$	
$Ker \alpha \backslash Im \alpha$	$\{1, 2, 3, 4\}$
1/2/3/4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$ Im \alpha = 3$				
$Ker \alpha \backslash Im \alpha$	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$
1/2/3	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$		$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$	
1/2/4		$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$		$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \end{pmatrix}$
1/3/4	$\begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$		$\begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix}$	
2/3/4		$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}$		$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$

$|Im\alpha| = 2$

$Ker\alpha \backslash Im\alpha$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$1/2$	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} *$	$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} +$	$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} +$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} +$
$1/3$	$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} *$	$\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} +$	$\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} *$
$1/4$	$\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} *$	$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} +$	$\begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$
$2/3$	$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} +$	$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} +$	$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} *$	$\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} +$
$2/4$	$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} +$	$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} *$	$\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$
$3/4$	$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} +$	$\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} +$	$\begin{pmatrix} 3 & 4 \\ 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} *$

 $|Im\alpha| = 1$

$Ker\alpha \backslash Im\alpha$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix} +$	$\begin{pmatrix} 1 \\ 3 \end{pmatrix} +$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix} +$
2	$\begin{pmatrix} 2 \\ 1 \end{pmatrix} +$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix} *$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix} +$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix} +$
3	$\begin{pmatrix} 3 \\ 1 \end{pmatrix} +$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix} +$	$\begin{pmatrix} 3 \\ 3 \end{pmatrix} *$	$\begin{pmatrix} 3 \\ 4 \end{pmatrix} +$
4	$\begin{pmatrix} 4 \\ 1 \end{pmatrix} +$	$\begin{pmatrix} 4 \\ 2 \end{pmatrix} +$	$\begin{pmatrix} 4 \\ 3 \end{pmatrix} +$	$\begin{pmatrix} 4 \\ 4 \end{pmatrix} *$

 $|Im\alpha| = 0$

$Ker\alpha \backslash Im\alpha$	$\{0\}$
$\emptyset$	$\begin{pmatrix} * \\ + \end{pmatrix}$

From the above example the following result were obtained.

We have 16 idempotent elements, 27 nilpotent elements and $|L_\alpha|, |R_\alpha|, |H_\alpha|$ has 16 elements in each class. While $|D_\alpha|$ class has 5 elements for $n=4$, in OA_4^c .

4.0 Conclusion

In this paper, combinatorial properties of subsemigroup of new class of transformations called order-preserving alternating semigroups have been studied. The two parameters $F(n; p)$ were used to carry out the cardinality of order-preserving alternating semigroups $|OA_n^c|$, idempotent and nilpotent elements in order-preserving alternating semigroups $(|E(OA_n^c)| \& |N(OA_n^c)|)$ respectively. More also Green's relation are characterized. Based on this paper, we have strong belief that many other subsemigroups can as well be studied such as order-decreasing, order preserving or order reversing, orientation preserving subsemigroups etc. The theory of semigroup has its scope widened to embrace many aspects of theoretical computer sciences, such as: automata theory, coding theory, computational theory and formal languages as well as applications in the sciences. It can also assist people and (Computers) in sorting data and designing better networks.

5.0 References

- [1] Howei, J. M. (1995), Fundamentals of Semigroup Theory. Oxford: Clarendon Press.
- [2] Kehinde, R. (2012), Some Algebraic and Combinatorial Properties of Injective partial Contraction Mappings and Isometrics of a Finite Chain. Ph.D. Thesis, University of Ilorin (unpublished).
- [3] Adeniji, A.O. (2012), Identity Difference in Transformation Semigroups. Ph.D. Thesis, University of Ilorin (unpublished).

- [4] Adeshofa, A.D.(2013), Some Semigroup of Full Contraction Mapping on a Finite Chain. Ph.D. Thesis, University of Ilorin (unpublished).
- [5] Lingscomb, S. (1996), Symmetric Inverse Semigroups, Mathematical Surveys of The American mathematical Society, no 46 providence, R L.
- [6] Bakare, G.N And Makanjuola, S.O.(2013), Some Combinatorial Results on Alternating Semigroups. Journal of Science, Technology, Mathematics and Education. (JOSTMED) 9(3), 98-102.
- [7] Munn, W. D. (1957), The Characters of the Symmetric Inverse Semigroup. Proc. Cambridge Philos. Soc. 53, 13-18.
- [8] UMAR, A. (1993), On the semigroup of partial one-to-one order-decreasing finite transformations, Proc. Roy. Soc. Edinburgh, 123A, 355-363.
- [9] Garba, G.U.(1994). Nilpotents in semigroups of partial one-to-one order-preserving Mappings. Semigroup Forum, 48, 37-49.
- [10] Howie, J.M, Lusk, E.L. And Mcfadden, R.B.(1990), Combinatorial results relating to product of idempotents in finite full transformation semigroups. Proc. Roy. Soc. Edinburgh Sect. A 115, 289-299.
- [11] Howie, J. M. (1971), Products of idempotents in certain semigroups of transformations. Proc. Edinburgh Math. Soc. 17, 223-236.
- [12] Ganyushkin, E. And Mazorchuk, V.(2003). On the structure of IO_n . Semigroup Forum 66, 455-483.
- [13] Janson, S. And Mazorchuk, V. (2005). Some remarks on the combinatorics of IS_n . Semigroup Forum. 109, 391-405.
- [14] Laradji, A. And Umar, A. (2004). On the number of nilpotents in the partial symmetric semigroup. Comm. Algebra, 32, 3017-3023.
- [15] Ganyushkin, E. And Mazorchuk, V.(2009), Classical Finite Transformation Semigroups: An introduction, Springer, London.
- [16] Laradji, A. And Umar, A. (2007). Combinatorial Results for the Symmetric Inverse Semigroup. Semigroup Forum. 75, 221-236.
- [17] Umar, A. (2010). Some Combinatorial Problems in the Theory of Symmetric Inverse Semigroups. Mathematics J. Algebra And Discrete Math. 9(2), 115-126.
- [18] Sloane, N. J.A. (Ed.), The On-Line Encyclopaedia of Integer Sequences, available at <http://oeis.org/>