ANALYTICAL APPROXIMATE SOLUTION OF NONLINEAR INITIAL VALUE PROBLEMS BY VARIATIONAL ITERATION METHOD AND ADOMIAN DECOMPOSITION METHOD

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Abstract
In this paper, variational iteration method and Adomian decomposition method are applied to certain initial value problems with varied nonlinearities. The two methods are equally applied to linear problems just to demonstrate their effectiveness in handling such category of problems too. The two methods give identical results which closely estimate the exact solution whenever such exist, in all cases, for both linear and nonlinear problems.

Keywords: Nonlinear, Decomposition, Iteration, Correction Functional, Lagrange Multiplier

Introduction

Most real world problems, when modeled mathematically, results in ordinary differential equations which are mostly nonlinear in nature. The task of solving such equations is enormous to the extent that it mostly defies the well known analytical methods of solution approaches. Thus, numerical methods are resulted into. Example of such numerical approaches is that discussed in Issa, et al. (2017) which involves linearization of the nonlinear problem before applying the Tau method to solve it.

Nonetheless, analytic approximation approaches (otherwise referred to as semi – analytic methods) provide better approximations to the exact solutions, whenever exact solution exists in closed form. Of all the analytic approximation techniques, the present work concerns itself with the Adomian decomposition method (ADM) introduced by Adomian (1994), and variational iteration method (VIM) introduced by He (2000). These two methods generate solutions to nonlinear problems in sequences which easily converge. The determination of the components of the sequence is robustly carried out with little or no stress. Unlike other strictly numerical methods, semi – analytic methods give closed form solutions, whenever such exists.

Statement of Problem

The class of problem considered in this work is given as

$$Ly + Ry + N(y) = f(x), \qquad (2.1a)$$

with the associated initial conditions

$$y(0) = \propto_0,\ y'(0) = \propto_1, \dots,\ y^{(n-1)}(0) = \propto_{n-1} \qquad (2.1b)$$

where L is the highest order linear differential operator, R is the remaining linear term, $N(y)$ is the nonlinear term and $f(x)$ is the inhomogeneous source term.

The nonlinear problem given in (2.1a) and (2.1b) is solved in this work tailoring the solution after Wazwaz [(2011), (1999) and (1997)], He (2000) and Hermann and Saravi (2016). The major task accomplish here is the application of the algorithm in the references cited above to problems of various category of nonlinearities.

Adomian Decomposition Method

This method entails decomposition of (2.1a) into its linear and nonlinear components, with the existence of the inverse of the highest order linear differential operator L as L^{-1} applied to both sides of (2.1a). Here,

$$y(x) = y_0(x) - L^{-1}(Ry) - L^{-1}\big(N(y)\big) \quad (2.2)$$

where the first member of the sequence is $y_0(x)$ and is given by Hermann and Saravi (2016) as

$$y_0(x) = \psi_0(x) + g(x), \quad (2.3)$$

where

$$\psi_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \ldots \quad (2.4)$$

and

$$g(x) = L^{-1}\{f(x)\} \quad (2.5)$$

The other components of the infinite sequence is obtained from

$$y_n(x) = -L^{-1}\big(Ry_{n-1}(x)\big) - L^{-1}\big(A_{n-1}(x)\big), \quad (2.6)$$

where $A_{n-1}(x)$ are the Adomian polynomials corresponding to the nonlinearity in (2.1a). This can be generated recursively as stated in Wazwaz (2011), Hermann and Saravi (2016) and many other authors as

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{k=0}^{\infty} y_k\,\lambda^k\right)\right], \quad n = 0,1,2,\ldots \quad (2.7)$$

The applications of ADM to nonlinear problems are shown in the sequel.

Numerical Experiments Using ADM for Nonlinear IVPs

In this section, implementation of ADM on selected nonlinear problems is demonstrated.

Problem 1 (Issa *et al.*, 2017)

Consider the variable coefficients nonlinear homogeneous IVP

$$y'(x) + (2x-1)y(x)^2 = 0, \quad y(0) = 1, \quad x\epsilon[0,1] \quad (3.1)$$

Solution

Here,

$Ly \equiv \frac{dy}{dx}$, $Ry \equiv 0$, $N(y) \equiv (2x-1)y^2$ and $f(x) \equiv 0$.

From (3.1) the ADM solution is of the form

$$y = \psi_0(x) + g(x) - \int_0^x (2\tau - 1)N(y(\tau))d\tau \quad (3.2)$$

where

$\psi_0(x) = 1$

and

$$g(x) = \int_0^x f(\tau)\,d\tau = 0.$$

Then,

$$y_n(x) = -\int_0^x (2\tau-1)A_{n-1}(\tau)\,d\tau, \quad n = 1,2,3,\ldots., \quad (3.3)$$

where $A_{n-1}(x)$, $n = 1,2,3,\ldots$ are the Adomian polynomials corresponding to the nonlinearity $N(y) = (2x-1)y^2$, and can easily be obtained from (2.7) as:

$$A_0 = (2x-1)y_0^2, \quad A_1 = 2(2x-1)y_0y_1, \quad A_2 = (2x-1)(2y_0y_2 + y_1^2),$$
$$A_3 = (2x-1)(2y_0y_3 + 2y_1y_2),$$

etc.

$$y_0(x) = 1,$$
$$y_1(x) = -\int_0^x (2\tau - 1)A_0(\tau)\, d\tau$$

That is

$$y_1(x) = -\int_0^x (2\tau - 1)y_0(\tau)^2\, d\tau$$
$$y_1(x) = x - x^2$$
$$y_2(x) = -\int_0^x 2(2\tau - 1)y_0(\tau)\, y_1(\tau) d\tau$$
$$y_2(x) = x^2 - 2x^3 + x^4$$

Following similar procedure, we get

$$y_3(x) = x^3 - 3x^4 + 3x^5 - x^6$$
$$y_4(x) = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$

The final solution is given as

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

which is

$y(x) = 1 + x - x^3 - x^4 + x^6 + x^7$ ....

This result eventually converges to $\frac{1}{1-x+x^2}$ which is the exact solution.

Problem 2 (Hermann and Saravi, 2016)

Consider the third order nonlinear inhomogeneous IVP

$y'''(x) + y''(x)^2 + y'(x)^2 = 1 - Cos(x)$ (3.4)

with the initial conditions $y(0) = y''(0) = 0$ and $y'(0) = 1$.

Solution

$Ly \equiv \frac{d^3y}{dx^3}$, $N_1(y) \equiv (y'')^2$, $N_2(y) \equiv (y')^2$, $Ry \equiv 0$ and $f(x) \equiv 1 - Cos(x)$,

where L is the highest order linear operator, $N_1(y)$ and $N_2(y)$ are the nonlinear terms, R is the remaining linear term, and $f(x)$ is the inhomogeneous source term.

Now the ADM is set up as follows:

$$y(x) = \psi_0(x) + g(x) - \int_0^x \int_0^\tau \int_0^t N_1(\xi) d\xi dt d\tau - \int_0^x \int_0^\tau \int_0^t N_2(\xi) d\xi dt d\tau \quad (3.5)$$

The triple integral sign in (3.5) is as a result of the fact that the highest order linear differential operator in the given problem is of order three. Following the earlier explanation in Problem 1, $\psi_0(x) = x$.

Also,

$$g(x) = \int_0^x \int_0^\tau \int_0^t (1 - Cos(\xi))\, d\xi dt d\tau$$

This gives

$$g(x) = \frac{x^3}{6} + Sin(x) - x.$$

Thus,

$$y_0(x) = x + \frac{x^3}{6} + Sin(x) - x\,,$$

which gives

$$y_0(x) = \frac{x^3}{6} + Sin(x).$$

$$y_n(x) = -\int_0^x \int_0^\tau \int_0^t A_{n-1}(\xi)d\xi dtd\tau - \int_0^x \int_0^\tau \int_0^t B_{n-1}(\xi)d\xi dtd\tau, \qquad n = 1,2,\ldots \quad (3.6)$$

To simplify matters, $y_0(x)$ is separated into two components $h_1(x) = Sin(x)$ and $h_2(x) = \frac{x^3}{6}$ such that

$y(x) =$
$h_1(x) + h_2(x) -$
$\int_0^x \int_0^\tau \int_0^t A_{n-1}(\xi)d\xi dtd\tau\ -$
$\int_0^x \int_0^\tau \int_0^t B_{n-1}(\xi)d\xi dtd\tau,$
$n = 1,2,\ldots$ (3.7)

This adjustment is done whenever the 'noise – term' (that is the occurrence of two identical terms with opposite signs) is noticed while implementing the ADM. Thus, the new equation we have is adopting the implementation of Wazwaz (1999) called modified Adomian decomposition method (mADM).

The Adomian polynomials corresponding to the nonlinearity $N_1(y) = (y')^2$ are:
$A_0 = (y_0')^2,\ A_1 = 2y_0'y_1',\ A_2 = 2y_0'y_2' + (y_1')^2\,,\ A_3 = 2y_0'y_3' + 2y_1'y_2'$, etc.

The Adomian polynomials corresponding to the nonlinearity $N_2(y) = (y')^2$ are:
$B_0 = (y_0')^2,\ B_1 = 2y_0'y_1',\ B_2 = 2y_0'y_2' + (y_1')^2\,,\ B_3 = 2y_0'y_3' + 2y_1'y_2'$, etc.

Thus, $y_0(x) = h_1(x) = Sin(x)$,

$y_1(x) = h_2(x) - \int_0^x \int_0^\tau \int_0^t A_0(\xi)d\xi dtd\tau\ - \int_0^x \int_0^\tau \int_0^t B_0(\xi)d\xi dtd\tau,$

$y_1(x) = \frac{x^3}{6} - \int_0^x \int_0^\tau \int_0^t (y_0'(\xi))^2 d\xi dtd\tau\ - \int_0^x \int_0^\tau \int_0^t (y_0'(\xi))^2 d\xi dtd\tau,$

$y_1(x) = \frac{x^3}{6} - \int_0^x \int_0^\tau \int_0^t (-Sin(\xi))^2 d\xi dtd\tau\ - \int_0^x \int_0^\tau \int_0^t (Cos(\xi))^2 d\xi dtd\tau,$

$y_1(x) = \frac{x^3}{6} + \frac{x}{8} - \frac{x^3}{12} - \frac{1}{8}Cos(x)Sin(x) - \frac{x}{8} - \frac{x^3}{12} + \frac{1}{8}Cos(x)Sin(x)$

$y_1(x) = \frac{x^3}{6} - \frac{x^3}{6}$

$y_1(x) = 0$

$y_2(x) = 0$

Therefore,

$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$
$y(x) = Sin(x) + 0 + 0 + \cdots$
$y(x) = Sin(x)$

Problem 3 (Hermann and Saravi, 2016)
Consider the second order nonlinear IVP

$y''(x) + Cos\big(y(x)\big) = 0,\ y(0) = 0,\ y'(0) = \frac{\pi}{2}$ (3.8)

Solution

The nonlinearity involved here is a trigonometric function, $Cos(y)$. To handle this effectively, we shall use Taylor's series expansion of the function, and this gives

$$Cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \cdots \quad (3.9)$$

Now, (3.8) becomes

$$y''(x) + \frac{y^2}{2} - \frac{y^4}{24} = 1, \quad y(0) = 0,\ y'(0) = \frac{\pi}{2}, \quad (3.10)$$

making use of the first three terms of the expansion.

From (3.10), we obtain

$Ly \equiv y''$, $Ry \equiv 0$, $N(y) \equiv \frac{y^2}{2} - \frac{y^4}{24}$ and $f(x) \equiv -1$.

Thus, we have

$$\psi_0(x) = \frac{\pi}{2}x$$

and

$g(x) = \int_0^x \int_0^\tau f(t)\, dtd\tau$

$$g(x) = \int_0^x \int_0^\tau (1)\, dtd\tau$$

$$g(x) = -\frac{x^2}{2}.$$

Therefore,

$$y_0(x) = \psi_0(x) + g(x) = \frac{\pi}{2}x - \frac{x^2}{2}.$$

The $y_1(x)$, $y_2(x)$, $y_3(x)$ and so on, are obtained from the recurrence relation

$y_1(x) = \frac{1}{2}\int_0^x \int_0^\tau N_1(y(t))\, dtd\tau - \frac{1}{24}\int_0^x \int_0^\tau N_2(y(t))\, dtd\tau,$ (3.11)

where $N_1(y)$ and $N_2(y)$ represent the nonlinearity $y(x)^2$ and $y(x)^4$ respectively.

The Adomian polynomials corresponding to the respective nonlinearities are:

$N_1(y) \equiv y^2$: $A_0 = y_0^2$, $A_1 = 2y_0y_1$, $A_2 = 2y_0y_2 + y_1^2$, $A_3 = 2y_0y_3 + 2y_1y_2$, etc.

$N_2(y) \equiv y^4$: $A_0 = y_0^4$, $A_1 = 4y_0^3y_1$, $A_2 = 4y_0^3y_2 + 6y_1^2y_0^2$, $A_3 = 4y_0^3y_3 + 12y_1y_2y_0^2 + 4y_1^3y_0$, etc.

Using these polynomials in (3.11), we have

$y_1(x) = \frac{1}{2}\int_0^x \int_0^\tau y_0(t)^2\, dtd\tau - \frac{1}{24}\int_0^x \int_0^\tau y_0(t)^4\, dtd\tau$

This gives

$$y_1(x) = \frac{\pi^2x^4}{96} - \frac{\pi x^5}{80} + \frac{x^6}{240} - \frac{\pi^4x^6}{11520} + \cdots$$

In a similar manner, the subsequent results are obtained as

$$y_2(x) = \frac{\pi^3x^7}{8064} - \frac{11\pi^2x^8}{53760} + \frac{\pi x^9}{8640} - \frac{\pi^5x^9}{276480} + \cdots,$$

$$y_3(x) = \frac{\pi^4x^{10}}{774144} - \frac{19\pi^3x^{11}}{7096320} + \frac{1363\pi^2x^{12}}{63866880} \ldots$$

And finally, the solution is obtained from

$y(x) = \sum_{n=0}^{\infty} y_n(x)$

Thus,

$$y(x) = \frac{\pi x}{2} - \frac{x^2}{2} + \frac{\pi^2x^4}{96} - \frac{\pi x^5}{80} + \frac{x^6}{240} - \frac{\pi^4x^6}{11520} + \frac{\pi^3x^7}{2688} \ldots.$$

Numerical Experiments Using ADM for Linear IVPs

In this section, ADM is implemented on some selected linear initial value problems, for the sake of completeness.

Problem 1(Yisa and Adeniyi, 2014)
Consider the third order inhomogeneous constant coefficient IVP

$$y'''(x) + 2y''(x) - 9y'(x) - 18y(x) = -18x^2 - 18x + 22$$

Subject to the initial conditions $y(0) = -2,\ y'(0) = -8$ and $y''(0) = -12.$

Solution
The solution is obtained from the ADM process as follows

$$y(x) = \psi_0(x) + g(x) - 2\int_0^x\int_0^\tau\int_0^t y''(\xi)d\xi\, dtd\tau + 9\int_0^x\int_0^\tau\int_0^t y''(\xi)d\xi\, dtd\tau + 18\int_0^x\int_0^\tau\int_0^t y''(\xi)d\xi\, dtd\tau$$

The triple integral here is the inverse of the third order linear differential operator contained in the problem. Thus, it follows that

$\psi_0(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0).$

$\psi_0(x) = -2 - 8x - 6x^2.$

And

$$g(x) = \int_0^x\int_0^\tau\int_0^t y''(\xi)d\xi\, dtd\tau$$

$g(x) = \int_0^x\int_0^\tau\int_0^t(-18\xi^2 - 18\xi + 22)\, d\xi\, dtd\tau$

$$g(x) = \frac{11}{3}x^3 - \frac{3}{4}x^4 - \frac{3}{10}x^5.$$

From the foregoing, $y_0(x),$ is derived as

$$y_0(x) = \psi_0(x) + g(x)$$

Therefore,

$y_0(x) = -2 - 8x - 6x^2 + \frac{11}{3}x^3 - \frac{3}{4}x^4 - \frac{3}{10}x^5.$

The other results are obtained from the recurrence relation

$$y_n(x) = \int_0^x\int_0^\tau\int_0^t(-2y''_{n-1}(\xi) + 9y'_{n-1}(\xi) + 18y_{n-1}(\xi))d\xi\, dtd\tau$$

$$y_1(x) = \int_0^x\int_0^\tau\int_0^t(-2y_0''(\xi) + 9y_0'(\xi) + 18y_0(\xi))\, d\xi dtd\tau$$

Thus,

$$y_1(x) = -14x^3 - \frac{37}{3}x^4 + \frac{3}{20}x^5 + \frac{17}{40}x^6 - \frac{9}{70}x^7 - \frac{9}{560}x^8$$

Subsequent iterations give

$$y_2(x) = 7x^4 - \frac{41}{30}x^5 - \frac{117}{20}x^6 - \frac{321}{280}x^7 + \frac{243}{2240}x^8 + \frac{3}{1120}x^9 \ldots$$

$$y_3(x) = -\frac{14}{5}x^5 + \frac{23}{9}x^6 + \frac{277}{140}x^7 - \frac{407}{560}x^8 - \frac{843}{2240}x^9 \ldots$$

The solution

$$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

gives

$$y(x) = -2 - 8x - 6x^2 - \frac{31}{3}x^3 - \frac{73}{12}x^4 - \frac{259}{60}x^5 - \frac{697}{360}x^6 \ldots$$

Problem 2 (Yisa and Adeniyi, 2014)
Consider the second order variable coefficient homogeneous IVP

$$y''(x) - (1-x)y'(x) + y(x) = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution
Following the algorithm demonstrated in the solution to problem 1, we have

$$y_0(x) = 1 + x.$$

$$y_n(x) = \int_0^x \int_0^\tau ((1-t)y'_{n-1}(t) - y_{n-1}(t))\, dtd\tau, \quad n = 1,2,3,\ldots \tag{3.12}$$

From (3.12),

$$y_1(x) = \int_0^x \int_0^\tau ((1-t)y_0'(t) - y_0(t))\, dtd\tau$$

$$y_1(x) = -\frac{x^3}{3},$$

$$y_2(x) = -\frac{x^4}{12} + \frac{x^5}{15},$$

$$y_3(x) = -\frac{x^5}{60} + \frac{x^6}{40} - \frac{x^7}{105},$$

etc.
The final solution is obtained as

$$y(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{20} + \frac{x^6}{45} - \frac{x^7}{252} \ldots,$$

which closely estimates the exact solution

$y(x) = e^{x - \frac{x^2}{2}}$.

Variational Iteration Method
The algorithm of variational iteration method (VIM) entails partioning of the given nonlinear problem into three. These include, the linear differential operator part, the nonlinear part and the inhomogeneous source term. After which the correction functional is developed for it as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)(Ly_n(\tau) + N(\bar{y}_n(\tau)) - f(\tau))\, d\tau, \quad n = 0,1,2,\ldots \tag{4.1}$$

where L and $N$are the linear and nonlinear operators respectively, $f(x)$ is the inhomogeneous term, and λ is the Lagrange multiplier which was rigorously established by Wazwaz (2011), Hermann and Saravi (2016) and the references cited in the two works, as

$\lambda(x) = \frac{(-1)^m}{(m-1)!}(\tau - x)^{m-1}, \quad m = 1,2,3,\ldots,$ (4.2)

where m is the order of the given differential equation.
Also, $y_0(x)$ in (4.1) is also in the authorities cited shortly as

$$y_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \ldots \tag{4.3}$$

The principle of restricted variation is applied in (4.1) to get the recurrence relation, from which the solution is given as

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{4.4}$$

The algorithm is implemented on some nonlinear problems in what follows.

Numerical Experiments Using VIM for Nonlinear IVPs

In this section, VIM is used to analytically approximate the solution of selected nonlinear problems that were solved earlier in 3.0 using ADM. The reason for using the same problems is just to facilitate easy comparison.

Problem 1(Issa, et al., 2017)

Consider the variable coefficients nonlinear homogeneous IVP

$$y'(x) + (2x-1)y(x)^2 = 0, \qquad y(0) = 1, \ x\epsilon[0,1] \tag{4.5}$$

Solution

$Ly \equiv \frac{dy}{dx}$, $N(y) \equiv (2x-1)y^2$ and $f(x) \equiv 0$.

The corresponding correction functional is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)(y_n'(\tau) + (2\tau-1)y_n(\tau)^2)\,d\tau \tag{4.6}$$

The optimal value of the Lagrange multiplier is obtained from (4.2) as $\lambda(x) = -1$. Also, the starting value $y_0(x)$ is also obtained from (4.3) as $y_0(x) = 1$.

To kick start the iteration, we make use of the $\lambda(x)$ and $y_0(x)$ in the correction functional (4.6), hence we have

$$y_1(x) = y_0(x) - \int_0^x (y_0'(\tau) + (2\tau-1)y_0(\tau)^2)\,d\tau$$

$$y_1(x) = 1 - \int_0^x (2\tau-1)\,d\tau$$

$$y_1(x) = 1 + x \ldots$$

The results of other iterations are as follows

$$y_2(x) = 1 + x - x^3 \ldots,$$

$$y_3(x) = 1 + x - x^3 - x^4 \ldots,$$

$$y_4(x) = 1 + x - x^3 - x^4 + x^6 \ldots,$$

$$y_5(x) = 1 + x - x^3 - x^4 + x^6 + x^7 \ldots,$$

$(n+1)$terms are used for subsequent iterations , where n is the number of iteration. Thus, the solution is

$$y(x) = \lim_{n\to\infty} y_n(x)$$

That is,

$$y(x) = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + x^{12} + x^{13} \ldots$$

And this result readily captures the exact solution

$$y(x) = \frac{1}{1 - x + x^2}.$$

Problem 2 (Hermann and Saravi, 2016)

Consider the third order inhomogeneous nonlinear initial value problem

$y'''(x) + y''(x)^2 + y'(x)^2 = 1 - \cos(x)$,

with the initial conditions $y(0) = y''(0) = 0$ and $y'(0) = 1$.

Solution

The correction functional corresponding to the problem is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)(y_n'''(\tau) + y_n''(\tau)^2 + y_n'(\tau)^2 - 1 + \cos(\tau))\,d\tau, \tag{4.7}$$

where n in (4.7) admits positive integral values only.

Following the algorithm as we did in Problem 1, we have $\lambda(x) = -\frac{(\tau-x)^2}{2}$ and $y_0(x) = x$.
Subsequent iterations give

$$y_1(x) = \sin(x),$$
$$y_2(x) = \sin(x),$$
$$y_3(x) = \sin(x),$$

etc.
Thus,

$$y(x) = \lim_{n\to\infty} y_n(x) = \sin(x).$$

Problem 3 (Hermann and Saravi, 2016)
Consider the nonlinear IVP

$$y''(x) + \cos\big(y(x)\big) = 0, \qquad y(0) = 0, \qquad y'(0) = \frac{\pi}{2}.$$

Solution
For effective handling of the given IVP, the series expansion of the nonlinear term is given thus:

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \cdots.$$

Here, we shall retain the first three terms of the expansion, since additional terms in the expansion do not adversely affect the desired accuracy. In addition, it facilitates effective comparison with the earlier result from ADM.
The IVP now becomes

$$y''(x) - \frac{y(x)^2}{2} + \frac{y(x)^4}{24} = -1, \qquad y(0) = 0,\ y'(0) = \frac{\pi}{2}.$$

Thus,
$Ly \equiv \frac{d^2y}{dx^2},\ N(y) \equiv -\frac{y^2}{2} + \frac{y^4}{24}$ and $f(x) = -1.$
Going through the entire process as explained earlier gives

$$y(x) = \frac{\pi}{2}x - \frac{x^2}{2} + \frac{\pi^2x^4}{96} - \frac{\pi x^5}{80} + \frac{x^6}{240} - \frac{\pi^4x^6}{11520} + \frac{\pi^3x^7}{2688} \cdots .$$

Numerical Experiment Using VIM for the Solution of Linear IVPs
Here again, for the sake of completeness, VIM is applied to some linear IVPs that were earlier considered in section 3.1, to facilitate easy comparison. The aim of applying VIM to linear IVPs is emphasize that the method equally performs creditably well in handling such category of problems too.

Problem 1(Yisa and Adeniyi, 2014)
Consider the third order constant coefficient inhomogeneous IVP

$$y'''(x) + 2y''(x) - 9y'(x) - 18y(x) = -18x^2 - 18x + 22,$$

subject to the initial conditions $y(0) = -2,\ y'(0) = -8$ and $y''(0) = -12.$

Solution
The correction functional corresponding to the given IVP is
$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\tau)\big(y_n'''(\tau) + 2y_n''(\tau) - 9y_n'(\tau) - 18y_n(\tau) + 18\tau^2 + 18\tau - 22\big)d\tau.$
The optimal value of the Lagrange multiplier is

$$\lambda(x) = -\frac{(\tau - x)^2}{2},$$

and

$y_0(x) = -2 - 8x - 6x^2.$

The result obtained is

$$y(x) = -2 - 8x - 6x^2 - \frac{31}{3}x^3 - \frac{73}{12}x^4 - \frac{259}{60}x^5 \ldots .$$

And this result agrees reasonable with the exact solution

$$y(x) = e^{-2x} - 1 - 2e^{3x} + x^2.$$

Problem 2 (Yisa and Adeniyi, 2014)
Consider the second order variable coefficient homogeneous IVP
$y''(x) - (1-x)y'(x) + y(x) = 0,\ y(0) = 1,\ y'(0) = 1.$

Solution
Following the procedure highlighted in the earlier examples, we arrive at

$$y(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{20} + \frac{x^6}{45} - \frac{x^7}{252} - \frac{11x^8}{3360} + \cdots ,$$

which closely estimate the exact solution

$$y(x) = e^{x - \frac{x^2}{2}}.$$

Discussion of Results
The two analytic approximation methods considered in this work for the solutions of nonlinear initial value problems of different categories of nonlinearities have proven to be very efficient and mostly estimate the exact solutions whenever such solutions exist in closed form. The applications of both ADM and VIM to linear problems have also shown to produce very accurate results as shown in the numerical experiments in sections 3.2 and 4.2. The two methods do not require any form of linearity of the nonlinear problems before solving them, thereby preserving computer memory and computation time.

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