

ON COEFFICIENT BOUNDS OF A SUBCLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let $T_n^\alpha(\beta)$ denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, univalent and analytic in the unit disk $U = \{z : |z| < 1\}$ such that $\operatorname{Re}(D^n[f(z)^\alpha]/z^\alpha) > \beta$, $z \in U$, $n \in \{0\} \cup \mathbb{N}$, $\alpha > 0$, $0 \leq \beta < 1$ and D^n is the Salagean differential operator, in this paper, we establish some coefficient bounds for functions of the class $T_n^\alpha(\beta)$.

1. Introduction.

Let A denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are analytic and univalent in the unit disk U . Also let P denote the class consisting of analytic functions $p(z)$ of the form:

$$(1.2) \quad p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k,$$

such that $\operatorname{Re} p(z) > 0$, $z \in U$. Furthermore, let $T_n^\alpha(\beta)$ denote the class of functions in A such that

$$(1.3) \quad \frac{D^n[f(z)^\alpha]}{z^\alpha} > \beta, \quad z \in U$$

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$n = 0, 1, 2, \dots, \alpha > 0, 0 \leq \beta < 1$. D^n is the Salagean differential operator defined as follows:

$$D^n f(z) = L(D^{n-1} f(z)) = z[D^{n-1} f(z)]$$

where $D^n f(z) = f(z)$.

The class $T_n^\alpha(\beta)$ was studied by Opoolá [3] as well as by Ntatin [4]. Some interesting properties of the class were established. Among other things, it was shown that if $f(z)$ belongs to the class $T_n^\alpha(\beta)$, then

$$(1.4) \quad |\alpha_2| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n}, \quad \alpha > 0$$

$$(1.5) \quad |\alpha_3| \leq \begin{cases} \frac{2(1-\beta)\alpha^{n-1}[(\alpha+1)^{2n} - (\alpha-1)\alpha^{n-1}(1-\beta)(\alpha+2)^n]}{(\alpha+2)^n(\alpha+1)^{2n}} & 0 < \alpha < 1 \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} & \alpha \geq 1 \end{cases}$$

In this paper, we obtain further properties of functions of the class. In particular we obtain bounds for $|a_4|$ and $|a_5|$.

2. Main Results

In this section, we state and prove our main results.

Theorem 2.1 Let $f(z) \in T_n^\alpha(\beta)$, then

$$|a_4| \leq \begin{cases} F_1 + F_2 + F_3, & \alpha \in (0, 1) \\ F_1 + F_3 + F_4, & \alpha \in [1, 2) \\ F_1 + F_3, & \alpha \in [2, \infty) \end{cases}$$

where

$$F_1 = 2(1-\beta)(\alpha+3)^{-n}\alpha^{n-1}$$

$$F_2 = 4(1-\beta)^2(1-\alpha)(\alpha+1)^{-n}(\alpha+2)^{-n}\alpha^{2n-2}$$

$$F_3 = 4(1-\beta)^3(\alpha-1)^2(\alpha+1)^{-3n}\alpha^{3n-3}$$

$$F_4 = \frac{4}{3}(1-\beta)^3(\alpha-1)(2-\alpha)(\alpha+1)^{-3n}\alpha^{3n-3}$$

and

$$|a_5| \leq \begin{cases} F_1 + F_2 + F_4 + F_5 + F_6 + F_8, & \alpha \in (0, 1) \\ F_1 + F_2 + F_7, & \alpha \in [1, 2) \\ F_1 + F_2 + F_3 + F_8, & \alpha \in [2, 3) \\ F_1 + F_2 + F_8, & \alpha \in [3, \infty) \end{cases}$$

where

$$\begin{aligned} F_1 &= 2(1-\beta)(\alpha+4)^{-n}\alpha^{n-1} \\ F_2 &= 12(1-\beta)^2(\alpha-1)(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3} \\ F_3 &= \frac{20}{3}(1-\beta)^4(\alpha-1)^2(\alpha-2)(\alpha+1)^{-4n}\alpha^{4n-4} \\ F_4 &= 10(1-\beta)^4(1-\alpha)^3(\alpha+1)^{-4n}\alpha^{4n-4} \\ F_5 &= 4(1-\beta)^2(1-\alpha)(\alpha+1)^{-n}(\alpha+3)^{-n}\alpha^{2n-2} \\ F_6 &= 2(1-\beta)^2(1-\alpha)(\alpha+2)^{-2n}\alpha^{2n-2} \\ F_7 &= 4(1-\beta)^3(\alpha-1)(2-\alpha)(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3} \\ F_8 &= \frac{2}{3}(1-\beta)^4(\alpha-1)(\alpha-2)(3-\alpha)(\alpha+1)^{-4n}\alpha^{4n-4} \end{aligned}$$

Proof Let $f \in T_n^\alpha(\beta)$, define

$$(2.1) \quad p(z) = \frac{D^n |f(z)^\alpha| \alpha^n z^\alpha - \beta}{1 - \beta} = 1 + p_1 z + p_2 z^2 + \dots$$

$$(2.2) \quad f(z)^\alpha = z^\alpha \left[1 + \sum_{j=1}^{\infty} \alpha_j (a_2 z + a_3 z^2 + \dots)^j \right]$$

where in (2.2),

$$\alpha_j = \binom{n}{j}, \quad j = 1, 2, 3, \dots$$

hence

$$(2.3) \quad \begin{aligned} p(z) &= 1 + \alpha_1 \frac{A_1}{B} a_2 z + [\alpha_1 a_3 + \alpha_2 a_2^2] \frac{A_2}{B} z^2 + [\alpha_1 a_4 + 2\alpha_2 a_2 a_3 + \alpha_3 a_2^3] \frac{A_3}{B} z^3 \\ &\quad + [\alpha_1 a_5 + \alpha_2 (2a_2 a_4 + a_3^2) + 3\alpha_3 a_2^2 a_3 + \alpha_4 a_2^4] \frac{A_4}{B} z^4 + \dots \end{aligned}$$

In (2.3), α_j is as defined in (2.2), $A_j = (\alpha+j)^n$, $j = 1, 2, 3, \dots$ and $B = \alpha^n(1-\beta)$. Comparing coefficients in (2.1) and (2.3), we obtain

$$(2.4) \quad a_2 = \frac{B p_1}{\alpha_1 A_1}$$

$$(2.5) \quad a_3 = \frac{Bp_2}{\alpha_1 A_2} - \frac{\alpha_2 B^2 p_1^2}{\alpha_1^3 A_1^2}$$

$$(2.6) \quad a_4 = \frac{Bp_3}{\alpha_1 A_3} - \frac{2B^2 p_1 p_2 \alpha_2}{\alpha_1^3 A_1 A_2} + \frac{2\alpha_2^2 B^3 p_1^3}{\alpha_1^5 A_1^3} - \frac{\alpha_3 B^5 p_1^3}{\alpha_1^4 A_1^3}$$

and

$$(2.7) \quad a_5 = \frac{Bp_4}{\alpha_1 A_4} + \frac{6\alpha_2^2 B^3 p_1^2 p_2}{\alpha_1^5 A_1^2 A_2} + \frac{5\alpha_2 \alpha_3 B^4 p_1^4}{\alpha_1^7 A_1^4} - \frac{2\alpha_2 B^2 p_1 p_3}{\alpha_1^3 A_1 A_3} \\ - \frac{\alpha_2 B^2 p_2^2}{\alpha_1^3 A_2^2} - \frac{3\alpha_3 B^3 p_1^2 p_2}{\alpha_1^4 A_1^2 A_2} - \frac{\alpha_4 B^4 p_1^4}{\alpha_1^5 A_1^4}$$

It is known that if $p \in P$ then

$$(2.8) \quad |p_i| \leq 2, \quad i = 1, 2, 3, \dots$$

Hence we have,

$$(2.9) \quad |a_2| \leq 2(1 - \beta)(\alpha + 1)^{-n} \alpha^{n-1}, \quad \alpha > 0$$

$$(2.10) \quad |\alpha_3| \leq \begin{cases} 2(1 - \beta)[(\alpha + 2)^{-2n} + (1 - \alpha)(\alpha + 1)^{-2n} \alpha^{n-1}] \alpha^{n-1}, & \alpha \in (0, 1) \\ 2(1 - \beta)(\alpha + 2)^{-n} \alpha^{n-1}, & \alpha \in [1, \infty) \end{cases}$$

which are results of Opoola and Ntatin in [4]. Now substituting for B , α_j and A_j in (2.6) we obtain

$$\alpha_n = F_4^{(1)} - F_4^{(2)} + F_4^{(3)} - F_4^{(4)}$$

where

$$F_4^{(1)} = (1 - \beta)(\alpha + 3)^{-n} \alpha^{n-1} p_3$$

$$F_4^{(2)} = (1 - \beta)^2 (\alpha - 1)(\alpha + 1)^{-n} (\alpha + 2)^{-n} \alpha^{2n-2} p_1 p_2$$

$$F_4^{(3)} = \frac{1}{2} (1 - \beta)^3 (\alpha - 1)^2 (\alpha + 1)^{-3n} \alpha^{3n-3} p_1^3$$

$$F_4^{(4)} = \frac{1}{6} (1 - \beta)^3 (\alpha - 1)(\alpha - 2)^{-n} (\alpha + 1)^{-3n} \alpha^{3n-3} p_1^3.$$

We consider the intervals $(0, 1)$, $[1, 2)$ and $[2, \infty)$ for the estimation of a_j and by applying (2.8), we have

$$(2.11) \quad |a_4| \leq \begin{cases} F_1 + F_2 + F_3, & \alpha \in (0, 1) \\ F_1 + F_3 + F_4, & \alpha \in [1, 2) \\ F_1 + F_3, & \alpha \in [2, \infty) \end{cases}$$

where

$$F_1 = 2(1 - \beta)(\alpha + 3)^{-n} \alpha^{n-1}$$

$$F_2 = 4(1 - \beta)^2 (1 - \alpha)(\alpha + 1)^{-n} (\alpha + 2)^{-n} \alpha^{2n-2}$$

$$F_3 = 4(1-\beta)^3(\alpha-1)^2(\alpha+1)^{-3n}\alpha^{3n-3}$$

$$F_4 = (1-\beta)^3(\alpha-1)(2-\alpha)(\alpha+1)^{-3n}\alpha^{3n-3}$$

as required and in (2.7)

$$a_5 = F_5^{(1)} + F_5^{(2)} + F_5^{(3)} - F_5^{(4)} - F_5^{(5)} - F_5^{(6)} - F_5^{(7)} - F_5^{(8)}$$

where,

$$F_5^{(1)} = (1-\beta)(\alpha+4)^{-n}\alpha^{n-1}p_4$$

$$F_5^{(2)} = \frac{3}{2}(1-\beta)^3(\alpha-1)^2(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3}p_1^2p_2$$

$$F_5^{(3)} = \frac{5}{12}(1-\beta)^4(\alpha-1)^2(\alpha-2)(\alpha+1)^{-4n}\alpha^{4n-4}p_1^4$$

$$F_5^{(4)} = \frac{5}{8}(1-\beta)^4(\alpha-1)^3(\alpha+1)^{-4n}\alpha^{4n-4}p_1^4$$

$$F_5^{(5)} = (1-\beta)^2(\alpha-1)(\alpha+1)^{-n}(\alpha+3)^{-n}\alpha^{2n-2}p_1p_3$$

$$F_5^{(6)} = \frac{1}{2}(1-\beta)^2(\alpha-1)(\alpha+2)^{-2n}\alpha^{2n-2}p_2^2$$

$$F_5^{(7)} = \frac{1}{2}(1-\beta)^3(\alpha-1)(\alpha-2)(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3}p_1^2p_2$$

$$F_5^{(8)} = \frac{1}{24}(1-\beta)^4(\alpha-1)(\alpha-2)(\alpha-3)(\alpha+1)^{-4n}\alpha^{4n-4}p_1^4.$$

The coefficient bound $|a_5|$ thus depends on α in the intervals $(0, 1)$, $[1, 2)$, $[2, 3)$ and $[3, \infty)$, hence applying (2.8) we have

$$(2.12) \quad |a_5| \leq \begin{cases} F_1 + F_2 + F_4 + F_5 + F_6 + F_8, & \alpha \in (0, 1) \\ F_1 + F_2 + F_7, & \alpha \in [1, 2) \\ F_1 + F_2 + F_3 + F_8, & \alpha \in [2, 3) \\ F_1 + F_2 + F_8, & \alpha \in [3, \infty) \end{cases}$$

where

$$F_1 = 2(1-\beta)(\alpha+4)^{-n}\alpha^{n-1}$$

$$F_2 = 12(1-\beta)^3(\alpha-1)^2(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3}$$

$$F_3 = \frac{20}{3}(1-\beta)^4(\alpha-1)^2(\alpha-2)(\alpha+1)^{-4n}\alpha^{4n-4}$$

$$F_4 = 10(1-\beta)^4(1-\alpha)^3(\alpha+1)^{-4n}\alpha^{4n-4}$$

$$F_5 = 4(1-\beta)^2(1-\alpha)(\alpha+1)^{-n}(\alpha+3)^{-n}\alpha^{2n-2}$$

$$F_6 = 2(1-\beta)2(1-\alpha)(\alpha+2)^{-2n}\alpha^{2n-2}$$

$$F_7 = 4(1-\beta)^3(\alpha-1)(2-\alpha)(\alpha+1)^{-2n}(\alpha+2)^{-n}\alpha^{3n-3}$$

$$F_8 = \frac{2}{3}(1-\beta)^4(\alpha-1)(\alpha-2)(3-\alpha)(\alpha+1)^{-4n}\alpha^{4n-4}$$

and this completes the proof

3. Corollaries

We consider some special cases of equivalent classes of $T_n^\alpha(\beta)$.

3.1 The class $S_0 \equiv G_0 \equiv T_0^1(0)$. Let $f \in T_0^1(0)$, then $|a_k| \leq 2$, $k = 2, 3, 4, 5$.

3.2 The class $B(\beta) \equiv T_0^1(\beta)$. Let $f \in T_0^1(\beta)$, then $|a_k| \leq 2(1-\beta)$, $0 \leq \beta < 1$, $k = 2, 3, 4, 5$.

3.3 The class $\delta(\beta) \equiv T_1^1(\beta)$. Let $f \in T_1^1(\beta)$, then $|a_k| \leq \frac{2(1-\beta)}{k}$, $0 \leq \beta < 1$, and $k = 2, 3, 4, 5$.

3.4 The class $B_1(\alpha)$ of Bazilevic functions which are equivalent to $T_1^\alpha(0)$. Let $f \in T_1^\alpha(0)$, then $|a_2| \leq \frac{2}{(1+\alpha)}$.

The above corollaries are easy to see from (2.9) to (2.12). In particular (3.1) and (3.4) are in agreement with earlier results of Kanas et al [2].

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