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INTEGRAL MEANS OF ANALYTIC MAPPINGS BY ITERATION OF JANOWSKI FUNCTIONS

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Abstract. In this short note we apply certain iteration of the Janowski functions to estimate the integral means of some analytic and univalent mappings of $|z| < 1$. Our method of proof follows an earlier one due to Leung [4].

1. Introduction

Let A be the class of normalized analytic functions $f(z) = z + a_2z^2 + ...$ in the unit disk $|z| < 1$. In [2], among others, we added a new generalization class, namely; $T_n^\alpha[a, b]$, $\alpha > 0$, $-1 \leq b < a \leq 1$ and $n \in \mathbb{N}$; to the large body of analytic and univalent mappings of the unit disk $|z| < 1$. This consists of functions in $|z| < 1$ satisfying the geometric conditions

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \in P[a, b] \tag{1}$$

where $P[a, b]$ is the family of Janowski functions $p(z) = 1 + c_1z + \cdots$ which are subordinate to $L_0(a, b : z) = (1 + az)/(1 + bz)$, $-1 \leq b < a \leq 1$, in $|z| < 1$. The operator D^n, defined as $D^n f(z) = z[D^{n-1)} f(z)]'$ with $D^0 f(z) = f(z)$, is the well known Salagean derivative [5].

In Section 2 of the paper [2] we extended certain integral iteration of the class of Caratheodory functions (which we developed in [1]) to $P[a, b]$ via which the new class, $T_n^\alpha[a, b]$, was studied. The extension was obtained simply by choosing

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the analytic function $p(z) = 1+c_1z+\cdots$, $\operatorname{Re} p(z) > 0$ from $P[a,b]$ in the iteration defined in [1] as:

$$p_n(z) = \frac{\alpha}{z^\alpha}\int_0^z t^{\alpha-1}p_{n-1}(t)dt, \ \ n \geq 1,$$

with $p_0(z) = p(z)$.

We will denote this extention by $P_n[a,b]$ in this note. We had remarked (in [2]) that the statements (i) $p(z) \prec L_0(a,b:z)$, (ii) $p \in P[a,b]$, (iii) $p_n(z) \in P_n[a,b]$ and (iv) $p_n \prec L_n(a,b:z)$ are all equivalent. Thus we also remarked that (1) is equivalent to $f(z)^\alpha/z^\alpha \in P_n[a,b]$. This new equivalent geometric condition will lead us to the following interesting results regarding the integral means of functions in $T_n^\alpha[a,b]$ for $0 < \alpha \leq 1$ and $n \geq 1$.

Theorem 1.1. *Let Φ be a convex non-decreasing function Φ on $(-\infty, \infty)$. Then for $f \in T_n^\alpha[a,b]$, $\alpha \in (0,1]$, $n \geq 1$ and $r \in (0,1)$*

$$\int_{-\pi}^{\pi} \Phi\left(\log|f'(re^{i\theta})|\right)d\theta \leq \int_{-\pi}^{\pi}\Phi\left(\log\left|\frac{L_{n-1}(a,b:re^{i\theta})k'(re^{i\theta})^{1-\alpha}}{L_0(re^{i\theta})^{1-\alpha}}\right|\right)d\theta \qquad (2)$$

where

$$L_n(a,b:z) = \frac{\alpha}{z^\alpha}\int_0^z t^{\alpha-1}L_{n-1}(a,b:t)dt, \quad n \geq 1$$

and $k(z) = z/(1-z)^2$ is the Koebe function.

Theorem 1.2. *With the same hypothesis as in Theorem 1, we have*

$$\int_{-\pi}^{\pi} \Phi\left(-\log|f'(re^{i\theta})|\right)d\theta \leq \int_{-\pi}^{\pi}\Phi\left(-\log\left|\frac{L_{n-1}(a,b:re^{i\theta})k'(re^{i\theta})^{1-\alpha}}{L_0(re^{i\theta})^{1-\alpha}}\right|\right)d\theta.$$

The above inequalities represent the integral means of functions of the class $T_n^\alpha[a,b]$ for $\alpha \in (0,1]$ and $n \geq 1$. Our method of proof follows an earlier one due to Leung [4] using the equivalent geometric relations $f(z)^\alpha/z^\alpha \in P_n[a,b]$ for $f \in T_n^\alpha[a,b]$.

It is worthy of note that very many particular cases of the above results can be obtained by specifying the parameters n, α, a and b as appropriate. In particular, the following special cases of $P[a,b]$ are well known: $P[1,-1]$; $P[1-2\beta,-1]$, $0 \leq \beta < 1$; $P[1, 1/\beta - 1]$, $\beta > 1/2$; $P[\beta,-\beta]$, $0 < \beta \leq 1$ and $P[\beta, 0]$, $0 < \beta \leq 1$ (see [2]). Thus several cases of $T_n^\alpha[a,b]$ may also be deduced.

2. Fundamental Lemmas

The following results are due to Baernstein [3] and Leung [4]. Let $g(x)$ be a real-valued integrable function on $[-\pi,\pi]$. Define $g^*(x) = \sup_{|E|=2\theta}\int_E g$, $(0 \leq \theta \leq \pi)$ where $|E|$ denotes the Lebesgue measure of the set E in $[-\pi,\pi]$. Further details can be found in the Baernstein's work [3].

Lemma 2.1 ([3]). *For $g, h \in L^1[-\pi,\pi]$, the following statements are equivalent:*

(i) *For every convex non-decreasing function Φ on $(-\infty, \infty)$,*

$$\int_{-\pi}^{\pi}\Phi(g(x))dx \leq \int_{-\pi}^{\pi}\Phi(h(x))dx.$$

(ii) *For every* $t \in (-\infty, \infty)$,

$$\int_{-\pi}^{\pi} [g(x) - t]^+ dx \le \int_{-\pi}^{\pi} [h(x) - t]^+ dx.$$

(iii) $g^*(\theta) \le h^*(\theta)$, $(0 \le \theta \le \pi)$.

Lemma 2.2 ([3])**.** *If* f *is normalized and univalent in* $|z| < 1$, *then for each* $r \in (0, 1)$, $(\pm \log |f(re^{i\theta}|)^* \le (\pm \log |k(re^{i\theta}|)^*$.

Lemma 2.3 ([4])**.** *For* $g, h \in L^1[-\pi, \pi]$, $[g(\theta) + h(\theta)]^* \le g^*(\theta) + h^*(\theta)$. *Equality holds if* g, h *are both symmetric in* $[-\pi, \pi]$ *and nonincreasing in* $[0, \pi]$.

Lemma 2.4 ([4])**.** *If* g, h *are subharmonic in* $|z| < 1$ *and* g *is subordinate to* h, *then for each* $r \in (0, 1)$, $g^*(re^{i\theta}) \le h^*(re^{i\theta})$, $(0 \le \theta \le \pi)$.

Corollary 2.5. *If* $p \in P_n[a, b]$, *then*

$$\left(\pm \log |p_n(re^{i\theta}|\right)^* \le \left(\pm \log |L_n(a, b : re^{i\theta}|\right)^*, \quad 0 \le \theta \le \pi.$$

Proof. Since $p_n(z)$ and $L_n(a, b : z)$ are analytic, $\log |p_n(z)|$ and $\log |L_n(a, b : z)|$ are both subharmonic in $|z| < 1$. Furthermore, since $p_n \prec L_n(a, b : z)$, there exists $w(z)$ $(|w(z)| < 1)$, such that $p_n(z) = L_n(a, b : w(z))$. Thus we have $\log p_n(z) = \log L_n(a, b : w(z))$ so that $\log p_n(z) \prec \log L_n(a, b : z)$. Hence by Lemma 3 we have the first of the inequalities.

As for the second, we also note from the above that $1/p_n(z) = 1/L_n(a, b : w(z))$ so that $-\log p_n(z) = -\log L_n(a, b : w(z))$ and thus $-\log p_n(z) \prec -\log L_n(a, b : z)$. Also $\log |1/p_n(z)|$ and $\log |1/L_n(a, b : z)|$ are both subharmonic in $|z| < 1$ since $1/p_n(z)$ and $1/L_n(a, b : z)$ are analytic there. Thus by Lemma 3 again, we have the desired inequality. ☐

3. Proofs of Main Results

We begin with

Proof of Theorem 1. Since $f \in T_n^\alpha[a, b], \alpha \in (0, 1]$, then there exists $p_n \in P_n[a, b]$, such that $f(z)^\alpha / z^\alpha = p_n(z)$. Then $f'(z) = p_{n-1}(z)(f(z)/z)^{1-\alpha}$ so that

$$\begin{aligned} \log |f'(z)| &= \log |p_{n-1}(z)| + \log \left|\frac{f(z)}{z}\right|^{1-\alpha} \\ &= \log |p_{n-1}(z)| + (1 - \alpha) \log \left|\frac{f(z)}{z}\right| \end{aligned} \tag{3}$$

so that, by Lemma 3,

$$(\log |f'(z)|)^* = (\log |p_{n-1}(z)|)^* + \left(\log \left|\frac{f(z)}{z}\right|^{1-\alpha}\right)^*.$$

For $n \geq 1$, $f(z)$ is univalent (see [2]), so that by Lemma 2 and Corollary 1 we have

$$\begin{aligned}(\log|f'(z)|)^* &= \big(\log|L_{n-1}(a,b:re^{i\theta})|\big)^* + \left(\log\left|\frac{k(re^{i\theta})}{r}\right|^{1-\alpha}\right)^* \\ &= \left(\log\left|\frac{L_{n-1}(a,b:re^{i\theta})k'(re^{i\theta})^{1-\alpha}}{L_0(re^{i\theta})^{1-\alpha}}\right|\right)^*.\end{aligned}$$

Hence by Lemma 1, we have the inequality. If for some $r \in (0,1)$ and some strictly convex Φ, we consider the function $f_0(z)$ is defined by

$$e^{-i\alpha\gamma}\frac{f_0(ze^{i\gamma})^{\alpha}}{z^{\alpha}} = L_n(a,b:ze^{i\gamma}) \tag{4}$$

for some real γ. Then we have

$$\begin{aligned}e^{i\gamma(1-\alpha)}\frac{f_0(ze^{i\gamma})^{\alpha-1}f_0'(ze^{i\gamma})}{z^{\alpha-1}} &= L_n(a,b:ze^{i\gamma}) + \frac{ze^{i\gamma}L_n(a,b:ze^{i\gamma})}{\alpha} \\ &= L_{n-1}(a,b:ze^{i\gamma}),\end{aligned}$$

so that

$$|f_0'(ze^{i\gamma})| = |L_{n-1}(a,b:ze^{i\gamma})|\left|\frac{f_0(ze^{i\gamma})}{ze^{i\gamma}}\right|^{1-\alpha}.$$

Now equality in (2) can be attained by taking $|f_0(z)| = |k(z)|$. This completes the proof. □

Next we have

Proof of Theorem 2. From (3) we have

$$\log\frac{1}{|f'(z)|} = \log\frac{1}{|p_{n-1}(z)|} + (1-\alpha)\log\left|\frac{z}{f(z)}\right|.$$

Hence, by Lemmas 2, 3 and Corollary 1 again, we have

$$\begin{aligned}(-\log|f'(z)|)^* &\leq \left(\log\left|\frac{1}{L_{n-1}(a,b:re^{i\theta})}\right|\right)^* + \left(\log\left|\frac{r}{k(re^{i\theta})}\right|^{1-\alpha}\right)^* \\ &= \left(-\log\left|\frac{L_{n-1}(a,b:re^{i\theta})k'(re^{i\theta})^{1-\alpha}}{L_0(re^{i\theta})^{1-\alpha}}\right|\right)^*.\end{aligned}$$

Hence by Lemma 1, we have the inequality. Similarly if equality is attained for some $r \in (0,1)$ and some strictly convex Φ, then $f_0(z)$ given by (4) is the equality function. □

4. Particular cases

With the same hypothesis as in Theorem 1 except:

(i) $n = 1$, we have:

$$\int_{-\pi}^{\pi}\Phi\left(\pm\log|f'(re^{i\theta})|\right)d\theta \leq \int_{-\pi}^{\pi}\Phi\left(\pm\log\left|\frac{L_0(a,b:re^{i\theta})k'(re^{i\theta})^{1-\alpha}}{L_0(re^{i\theta})^{1-\alpha}}\right|\right)d\theta.$$

(ii) $\alpha = 1$, we have:

$$\int_{-\pi}^{\pi} \Phi\left(\pm \log |f'(re^{i\theta})|\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\pm \log |L_{n-1}(a, b : re^{i\theta})|\right) d\theta.$$

(iii) $n = \alpha = 1$, we have:

$$\int_{-\pi}^{\pi} \Phi\left(\pm \log |f'(re^{i\theta})|\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\pm \log |L_{0}(a, b : re^{i\theta})|\right) d\theta.$$

Remark 4.1. The case $n = 1, a = 1$ and $b = -1$ gives the estimate for the special case $s(z) = z$ of the Leung results [4].

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