

RELATIVITY IN NUMERICAL PERFORMANCES OF CERTAIN POLYNOMIALS IN TAU METHOD

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Abstract

This paper examines the order of numerical usefulness of the four kinds of Chebyshev polynomials, especially in the differential formulation of the Lanczos' Tau method. Their practical applications on selected numerical problems, which are constant and variable coefficients, homogeneous and nonhomogeneous problems of varied orders, showed correlation between the accuracy of first and fourth kinds. The second kind was next in performance, while the third kind performed least.

Keywords: Chebyshev Polynomials, Canonical Polynomials, Differential Formulation, Tau Approximant, Ordinary Differential Equation

Introduction

The usefulness of various types of orthogonal polynomials like Laguerre, Hermite, Legendre and Chebyshev polynomials can never be over – emphasized. This is evidenced in the works of many earlier scientists like Gauss, in his popular integration method called Gauss Quadrature formula, where he derived the nodes for any given problem by equating to zero the polynomial of equivalent order, mostly Legendre. Lanczos (1938, 1956) elucidated more on the properties of these polynomials and came to the conclusion that Chebyshev polynomial is the best, considering the mini – max property of the latter in the error propagation. Fox (1962) and Fox and Parker (1972) concentrated on the analysis of Chebyshev polynomials of the first and second kind.

In Yisa (2015a), it was reported that although Chebyshev polynomial has its usefulness as emphasized by Lanczos and some other scientists, but Legendre polynomial in certain instances could out-perform it. Yisa (2015a) was not the first to maintain this position, but rather it was the position maintained by Ortiz (1975). In the present work, the attention is focused on the relativity in accuracy of the results obtained when the four kinds of Chebyshev polynomials are used as bases functions within the same interval $[0, 1]$, that is when it was shifted from the usual interval $[-1, 1]$ to a more usable interval $[0, 1]$. Similar efforts were made in the Recursive formulation of the Tau method (see Yisa (2015b)). The beauty of extending it to the Differential Formulation is that of showing that the other kinds of Chebyshev polynomial can as well be used to get results with acceptable level of accuracy.

Application of the Four Kinds of Chebyshev Polynomials in the Differential Formulation of the Tau Method

The applications of these polynomials as bases functions were made to the class of general m th order ordinary differential equation (ODE)

$$\sum_{r=1}^m \left(\sum_{k=0}^{N_r} P_{r,k} x^k \right) y^{(r)}(x) = \sum_{r=0}^F f_r x^r \quad (2.1a)$$

with the supplementary conditions

$$L^* y(x_0) = y^{(k)}(x_0) = \alpha_k, \quad k = 0(1)m - 1 \quad (2.1b)$$

where m is the order of the ODE, N_r and F are non – negative integers and $x_0, \alpha_k, f_r, P_{r,k}$, are given real numbers.

Application of Chebyshev Polynomial of the First Kind

In solving (2.1), we consider an approximant

$$y_n = \sum_{r=0}^n a_r x^r, \quad n < +\infty \quad (2.2)$$

which is the exact solution of the corresponding perturbed problem

$$Ly_n(x) = \sum_{r=0}^F f_r x^r + H_n(x) \quad (2.3)$$

where the perturbation term

$$H_n(x) = \sum_{k=1}^m \tau_k T_{n-k+1}^*(x) \quad (2.4)$$

and

$$T_r^*(x) = \csc \left\{ r \cos^{-1} \left[\frac{2x - a - b}{b - a} \right] \right\} \equiv \sum_{k=0}^r C_k^{(r)} x^k \quad (2.5)$$

is the r th degree Chebyshev polynomial valid in the interval $[a, b]$ (assuming that (2.1) is defined in this interval).

After obtaining $T_0^*(x) = 1$ and $T_1^*(x) = 2x - 1$ (when the interval is narrowed in (2.5) to $[0, 1]$), the subsequent terms are generated recursively from

$$T_{n+1}^*(x) = 2T_1^*(x)T_n^*(x) - T_{n-1}^*(x), \quad n \geq 1 \quad (2.6)$$

Thus, using all these in (2.3), the tau system is now written as

$$Ly_n(x) = \sum_{r=0}^F f_r x^r + \sum_{k=1}^m \tau_k \sum_{r=0}^{n-k+1} C_r^{(k)} x^r \quad (2.7)$$

When expanded, (2.7) generates a system of equations from which the values of the unknown constants and those of the free tau parameters $\tau_r, r = 1(1)m$ are obtained.

Application of Chebyshev Polynomial of Second Kind

In this case, (2.4) takes the form

$$H_n(x) = \sum_{k=1}^m \tau_k U_{n-k+1}^*(x) \quad (2.8)$$

where $U_r(x)$ is defined as

$$U_r(x) = \frac{\sin(r+1)\theta}{\sin \theta} \quad (2.9)$$

as can be found in Mason and Handscomb (2013).

In (2.9) $x = \cos \theta$, $U_0(x) = 1$ and $U_1(x) = 2x$. But with the interval shifted to $[0, 1]$, $U_0^*(x) = 1$ and $U_1^*(x) = 4x - 2$. Subsequent members are obtained from the general recurrence formula

$$U_{n+1}^*(x) = 2(2x - 1)U_n^*(x) - U_{n-1}^*(x), \quad n \geq 1 \quad (2.10)$$

Equations (2.10) and (2.8) are used in equation (2.7) to derive the required Tau system. The Tau system thus obtained is solved for the unknowns, including the free tau parameters.

Application of Chebyshev Polynomial of Third Kind

Chebyshev polynomials of the third kind is generated in Mason et al.(2013) using the trigonometric relation

$$V_n^*(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta}, \quad n \geq 0 \quad (2.11)$$

which gives $V_0^*(x) = 1$ and $V_1^*(x) = 2x - 1$, and the equivalents in $[0, 1]$ are $V_0^*(x) = 1$ and $V_1^*(x) = 4x - 3$. The other members are generated recursively from

$$V_{n+1}^*(x) = 2(2x - 1)V_n^*(x) - V_{n-1}^*(x), \quad n \geq 0 \quad (2.12)$$

Equation (2.12) is now used in equation (2.7) to develop the required Tau system using Chebyshev polynomial of the third kind as the basis function. The Tau system thus developed will be solved as usual to get the values of the unknown constants and the free tau parameters.

Application of Chebyshev Polynomial of Fourth Kind

The differential formulation of the tau method equally admits the use of Chebyshev polynomial of the fourth kind as a basis function. This kind is generated in Mason and Handscomb (2013) from the trigonometric relation

$$W_n(x) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}, \quad n \geq 0 \quad (2.13)$$

from where we get $W_0(x) = 1$ and $W_1(x) = 2x + 1$. When the interval is shifted from $[-1, 1]$ to $[0, 1]$, $W_0^*(x)$ and $W_1^*(x)$ are now derived as

$$W_0^*(x) = 1 \text{ and } W_1^*(x) = 4x - 1 \quad (2.14)$$

and the general recursive form of generating other members as

$$W_{n+1}^*(x) = 2(2x - 1)W_n^*(x) - W_{n-1}^*(x), \quad n \geq 1 \quad (2.15)$$

This is again used in (2.7) to develop the required tau system in terms of Chebyshev polynomial of the fourth kind. The system thus developed is solved for the unknown constants and the free tau parameters.

Numerical Experiments

To demonstrate what was discussed in the preceding sections, four numerical problems were considered, namely: first order variable coefficient homogeneous initial value problem (IVP); first order constant coefficient non - homogeneous IVP; third order constant coefficient homogeneous IVP; and second order constant coefficient non - homogeneous IVP.

Problem 3.1

Consider the following first order variable coefficient initial value problem

$$(1 + x)y' + y = 0, \quad x \in [0, 1]$$

subject to $y(0) = 1$.

The exact solution of this problem is $y(x) = \frac{1}{1+x}$.

Note that this problem and the remaining three were handled with the aid of Mathematica 7.0.

This and the remaining three problems were solved for $y_n(x)$ (the approximate or numerical solution) with $n = 5(1)10$. The table of results presents the absolute errors given by

$$\text{Error} = |y(x) - y_n(x)| \text{ for } n = 5(1)10.$$

In each case, the approximate solution is written for all the problems when $n=5$, just to conserve space.

First Kind:

$$y_5(x) = 1 - \frac{2339}{2342}x + \frac{1132}{1171}x^2 - \frac{932}{1171}x^3 + \frac{512}{1171}x^4 - \frac{128}{1171}x^5$$

Second Kind:

$$y_5(x) = 1 - \frac{2441}{2450}x + \frac{1168}{1225}x^2 - \frac{944}{1225}x^3 + \frac{512}{1225}x^4 - \frac{128}{1225}x^5$$

Third Kind:

$$y_5(x) = 1 - \frac{29378}{29543}x + \frac{27728}{29543}x^2 - \frac{21568}{29543}x^3 + \frac{11008}{29543}x^4 - \frac{2560}{29543}x^5$$

Fourth Kind:

$$y_5(x) = 1 - \frac{19442}{19457}x + \frac{18992}{19457}x^2 - \frac{16192}{19457}x^3 + \frac{9472}{19457}x^4 - \frac{2560}{19457}x^5$$

Problem 3.2

Consider the following first order constant coefficient non – homogeneous IVP

$$y' - 2y = -2x^3 - 3, \quad x \in [0, 1]$$

subject to the initial condition $y(0) = 2$.

The exact solution is

$$y(x) = \frac{1}{4}(9 - e^{2x} + 6x + 6x^2 + 4x^3).$$

The approximate solution for $n = 5$ for the four kinds are as shown below.

First Kind:

$$y_5(x) = 2 + \frac{1327}{1328}x + \frac{169}{166}x^2 + \frac{48}{83}x^3 - \frac{16}{83}x^5$$

Second Kind:

$$y_5(x) = 2 + \frac{447}{448}x + \frac{43}{42}x^2 + \frac{4}{7}x^3 - \frac{4}{21}x^5$$

Third Kind:

$$y_5(x) = 2 + \frac{2411}{2422}x + \frac{2521}{2422}x^2 + \frac{635}{1211}x^3 + \frac{64}{1211}x^4 - \frac{256}{1211}x^5$$

Fourth Kind:

$$y_5(x) = 2 + \frac{2953}{2954}x + \frac{2983}{2954}x^2 + \frac{901}{1477}x^3 - \frac{64}{1477}x^4 - \frac{256}{1477}x^5$$

Problem 3.3

Consider the following third order constant coefficient homogeneous IVP

$$y''' - y'' - y' + y = 0, \quad x \in [0, 1]$$

Subject to the initial conditions $y(0) = 2, y'(0) = 1, y''(0) = 0$.

The exact solution is $y(x) = -e^x(x - 2)$.

The approximate solution for $n = 5$ for the four kinds are:

First Kind:

$$y_5(x) = 2 + x - \frac{27480800}{158536731}x^3 - \frac{8786176}{158536731}x^4 - \frac{7956608}{158536731}x^5$$

Second Kind:

$$y_5(x) = 2 + x - \frac{116422064}{647668587}x^3 - \frac{35224064}{647668587}x^4 - \frac{10727168}{21588529}x^5$$

Third Kind:

$$y_5(x) = 2 + x - \frac{37750544}{196575195}x^3 - \frac{1568384}{39315039}x^4 - \frac{10934528}{196575195}x^5$$

Fourth Kind:

$$y_5(x) = 2 + x - \frac{11859008}{698944299}x^3 - \frac{46329728}{698944299}x^4 - \frac{31322368}{698944299}x^5$$

Problem 3.4

Solve the following second order constant coefficient non – homogeneous IVP

$$y'' - 2y' + y = xe^x - x, \quad x \in [0, 1]$$

subject to the initial conditions $y(0) = 0, y'(0) = 0$.

The exact solution is $y(x) = \frac{1}{6}(-12 + 12e^x - 6x - 6xe^x + x^3e^x)$.

The exponential term on the right hand side of the IVP was expanded using Taylor series up to the term x^{10} . And unless that is done, it will be impossible to apply the tau method to solve it.

The approximate solution for the four kinds when $n = 5$ are:

First Kind:

$$y_5(x) = -\frac{10444}{1577829}x^2 + \frac{246607}{3681601}x^3 - \frac{8537003}{132537636}x^4 + \frac{46847939}{265075272}x^5$$

Second Kind:

$$y_5(x) = -\frac{1100100}{67109687}x^2 + \frac{15591755}{201329061}x^3 - \frac{52724095}{805316244}x^4 + \frac{279753527}{1610632488}x^5$$

Third Kind:

$$y_5(x) = -\frac{647828}{19668151}x^2 + \frac{7450859}{59004453}x^3 - \frac{9687637}{78672604}x^4 + \frac{92597431}{472035624}x^5$$

Fourth Kind:

$$y_5(x) = -\frac{223356}{75627781}x^2 + \frac{504209}{13346079}x^3 - \frac{16905437}{907533372}x^4 + \frac{282127621}{1815066744}x^5$$

Table I: Absolute Error for Problem 3.1

n	First Kind	Second Kind	Third Kind	Fourth Kind
5	1.111950E-4	9.555480E-5	1.688811E-4	7.720191E-5
6	1.671442E-5	1.667203E-5	2.934306E-5	1.277401E-5
7	2.778982E-6	2.883493E-6	5.099220E-6	2.135890E-6
8	4.831527E-7	5.003264E-7	8.763733E-7	3.581129E-7
9	8.157911E-8	8.470255E-8	1.501434E-7	6.069182E-8
10	1.361626E-8	1.487140E-8	2.595823E-8	1.027940E-8

Table II: Absolute Error for Problem 3.2

n	First Kind	Second Kind	Third Kind	Fourth Kind
5	1.543975E-4	2.421037E-4	5.889893E-4	1.076266E-4
6	8.538861E-6	1.874403E-5	3.900965E-5	6.597075E-6
7	4.861735E-7	1.093217E-6	2.391575E-6	4.110882E-7
8	2.490964E-8	6.394749E-8	1.272931E-7	2.097047E-8
9	1.163636E-9	3.055830E-9	6.325908E-9	1.047145E-9
10	4.957146E-11	1.439382E-10	2.792939E-10	4.498446E-11

Table III: Absolute Error for Problem 3.3

n	First Kind	Second Kind	Third Kind	Fourth Kind
5	2.769667E-3	2.111608E-3	5.840958E-3	9.502988E-4
6	5.946880E-4	5.946880E-4	3.046912E-4	3.456934E-5
7	3.541969E-6	5.898922E-6	1.392815E-5	1.276627E-6
8	1.078388E-7	1.078388E-7	5.485130E-7	4.099256E-8
9	2.503951E-9	8.427159E-9	1.887393E-8	1.229457E-9
10	7.845502E-11	2.602425E-10	5.738374E-10	3.265788E-11

Table IV: Absolute Error for Problem 3.4

n	First Kind	Second Kind	Third Kind	Fourth Kind
5	1.358168E -3	2.055562E -3	4.962947E -3	3.456127E -4
6	2.855565E -4	9.222670E -5	1.611892E -4	2.375255E -4
7	1.877631E -4	1.967308E -4	2.050510E -4	1.893681E -4
8	1.879043E -4	1.875189E -4	2.857980E -4	1.878417E -4
9	1.875322E -4	1.875466E -4	1.875603E -4	1.875341E -4
10	1.875111E -4	1.875106E -4	1.875102E -4	1.875111E -4

Discussion

In all the problems considered, the first and fourth kinds Chebyshev polynomials performed very well with errors of the same order, with the fourth kind performing relatively better than the first in most cases. Next to these two was the second kind which results were mostly closer but not as accurate. The tabulated results also exposed the weakness of Chebyshev polynomial of the third kind in terms of accuracy for all the problems considered. Problem 3.4 gave results with peculiarities. The errors were reducing as n increases up to $n = 8$. From that point, the error was still decreasing though, but the order remained the same for whatever values considered for n , even with increase in the number of terms retained in the Taylor's series expansion of the exponential term. The situation was similar for the four kinds of Chebyshev polynomials considered. This strange behavior was as a result of the exponential term in the non – homogeneous part of Problem 3.4, because the situation was not like that for Problem 3.2 despite it was also non – homogeneous type, but with the right hand side terms completely polynomial.

Conclusion

From the foregoing, it is very clear that the Chebyshev polynomials of the first and fourth kinds are the best in terms of accuracy, while the second kind is also good. The third kind performed least and may not be recommended for numerical purposes. And this corroborates the author's position in Yisa (2015b).

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