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THE AVERAGE WORK DONE BY A TRANSFORMATION SEMIGROUP

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ABSTRACT

The main object of study in this paper is the average work done by T_n (the full transformation semi group of X_n). We derived the formula for the total work $w(T_n)$ performed by set T_n which in turn leads to a formula for the explicit average work done $\overline{w}(T_n)$.

1.0 INTRODUCTION

The algebraic system (S, *) is called semigroup if;

i) S is closed under *

ii) * is associative in S

An example of semigroup is (Z, +) i.e the algebraic system consisting of the Z (integers) and the binary operation + (addition) on Z. In describing a particular semigroup one has to specify the binary operation as well as the set. The same set may be seen as many different semigroups depending on the binary operation.

BASIC DEFINITIONS

Commutative Semigroup:

A semigroup (S, *) is called a commutative semigroup if S has the additional property i.e $xy = yx \ \forall \ x, y \in S$.

Subsemigroup:

Suppose that (S, *) is a semigroup and that T is a subset of S, then T is subsemigroup of S if the algebraic system (T, *) is a semigroup where * is the same binary operation as defined on S.

Identity Element:

If there exist an element $I \in S$ such that XI=IX =X, it means that I is an identity element of S and that S is a semigroup with identity called monoid.

Zero Element:

If a semigroup S with at least two element contains an element 0 such that $\forall x \in S$ $x\,0 = 0x = 0$. Then, it implies that 0 is zero element of S and that S is a semigroup with zero.

Idempotent Element:

An element e in a semigroup S is an idempotent if $e^2=e$.

2.0 TRANSFORMATION SEMIGROUP

An important type of semigroup is obtained from the set of all transformation of a given set. A transformation on S is a function from S to S.

If α and β are transformation on S, then their product $\alpha\beta$ is defined (writing function on the right) by $(x)(\alpha\beta) =((x)\alpha))\beta$ i.e associative law holds for transformation on a set S.

With this definition, the set of all transformation on S becomes a semigroup (full transformation semigroup denoted by T_n). It should be noted that a semigroup of transformation is any subsemigroup of a full set of transformations.

As a special case of this type of semigroup, let S be a finite set i.e $X = (x_1, x_2.....x_n)$. Then, the transformation α which maps x_i to y_i (with $y_i \in S$) is often written as

$$\alpha = \begin{pmatrix} x_1 & x_2 \ldots x_n \\ y_1 & y_2 \ldots y_n \end{pmatrix}$$

Without any loss of generality, we often use the set $\{X_n = 1,2, \ldots n\}$ for all defined sets with n – element. If n = 4, then $\{X_n = 1,2,3,4\}$. With this notation, it is quite easy to calculate product, for example, $\{X_4 = 1, 2, 3, 4\}$,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

$$\text{then } \alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix}$$

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The full transformation semigroup T_n, the semigroup of partial one to one mapping I_n and the semigroup of partial transformation P_n are the three fundamental semigroup of transformation of X_n. The subsemigroups of semigroup of transformation include order preserving/order reversing, order-decreasing and order preserving + order decreasing.

TRANFORMATION	FULL	PARTIAL	PARTIAL ONE TO ONE
Transformation	T_n	P_n	I_n
Order Preserving	O_n	PO_n	IO_n
Order Decreasing	D_n	PD_n	ID_n
Order Preserving + order Decreasing	C_n	PC_n	IC_n

Let $x_n = 1, 2, \ldots n$. Then a (partial) transformation α: Dom $\alpha \subseteq x_n \rightarrow$ Im $\alpha \subseteq x_n$ is said to be full or total if Dom $\alpha = x_n$; otherwise it is called strictly partial.
A transformation α: is said to be order preserving (order-reversing) if

$$\forall x,y \in \text{Dom } \alpha \qquad x \leq y \Rightarrow x\alpha \leq y\alpha \ (x\alpha \geq y\alpha)$$

We shall call a partial transformation α Dom $\alpha \subseteq x_n \rightarrow x_n$ order decreasing (order increasing) if $x\alpha \leq x$ ($x\alpha \geq x$) for all x in the Dom α.
The parameters height, breadth, right (left) waist of α denoted by $h(\alpha)$, $b(\alpha)$, $w^+(\alpha)$, $w^-(\alpha)$, respectively are defined below:

$$h(\alpha) = |\text{Im}(\alpha)|$$

$$b(\alpha) = |\text{Dom}(\alpha)|$$

$$w^+(\alpha) = \max |\text{Im}(\alpha)|$$

$$w^-(\alpha) = \min |\text{Im}(\alpha)|$$

$$C(\alpha) = U(t\alpha^- : |t\alpha^-| \geq 2)$$

and the fix of α is denoted by $f(\alpha)$ and defined by

$f(\alpha) = |f(\alpha)| = (x \in X_n : x\alpha = x)$.

We shall illustrate the use of these parameters in full transformation semigroup T_n for n=1,2and 3.

We shall illustrate the use of these symbols by the following examples in T_n

T_n	α	$h(\alpha)$	$k(\alpha)$	$c(\alpha)$	$f(\alpha)$	$w^+(\alpha)$	$w^-(\alpha)$
T_1	$\begin{pmatrix}1\\1\end{pmatrix}$	1	1	-	1	1	1
T_2	$\begin{pmatrix}1&2\\1&1\end{pmatrix}$	1	2	1	1	1	1
	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	2	2	-	2	2	1
	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$	2	2	-	-	2	1
	$\begin{pmatrix}1&2\\2&2\end{pmatrix}$	1	2	1	1	2	2
T_3	$\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}$	1	3	1	1	1	1
	$\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}$	2	3	1	1	2	1
	$\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}$	2	3	1	2	3	1
	$\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}$	2	3	1	-	2	1

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T_α	α	$h(\alpha)$	$b(\alpha)$	$c(\alpha)$	$f(\alpha)$	$w'(\alpha)$	$w(\alpha)$
	$\begin{bmatrix}1&2&3\\2&1&2\end{bmatrix}$	2	3	1	-	2	1
	$\begin{bmatrix}1&2&3\\2&1&3\end{bmatrix}$	3	3	-	1	3	1
	$\begin{bmatrix}1&2&3\\1&2&1\end{bmatrix}$	2	3	1	2	2	1
	$\begin{bmatrix}1&2&3\\1&2&2\end{bmatrix}$	2	3	1	2	2	1
	$\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}$	3	3	1	3	3	1
	$\begin{bmatrix}1&2&3\\3&3&1\end{bmatrix}$	2	3	1	-	3	2
	$\begin{bmatrix}1&2&3\\3&3&2\end{bmatrix}$	2	3	1	-	3	2
	$\begin{bmatrix}1&2&3\\3&3&3\end{bmatrix}$	1	3	1	1	3	3
	$\begin{bmatrix}1&2&3\\1&3&1\end{bmatrix}$	2	3	1	1	3	1
	$\begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}$	3	3	-	1	3	1
	$\begin{bmatrix}1&2&3\\1&3&3\end{bmatrix}$	2	3	1	2	3	1
	$\begin{bmatrix}1&2&3\\2&2&1\end{bmatrix}$	2	3	1	1	2	1

T_n	α	$h(\alpha)$	$b(\alpha)$	$c(\alpha)$	$r(\alpha)$	$w^*(\alpha)$	$w(\alpha)$
	$\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}$	1	3	1	1	2	2
	$\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}$	2	3	1	2	3	2
	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	3	3	-	-	3	1
	$\begin{pmatrix}1&2&3\\2&3&2\end{pmatrix}$	2	3	1	-	3	2
	$\begin{pmatrix}1&2&3\\2&3&3\end{pmatrix}$	2	3	1	1	3	2
	$\begin{pmatrix}1&2&3\\3&1&1\end{pmatrix}$	2	3	1	-	3	1
	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	3	3	-	3	3	1
	$\begin{pmatrix}1&2&3\\3&1&3\end{pmatrix}$	2	3	1	1	3	1
	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$	3	3	-	1	3	1
	$\begin{pmatrix}1&2&3\\3&2&2\end{pmatrix}$	2	3	1	1	3	2
	$\begin{pmatrix}1&2&3\\3&2&3\end{pmatrix}$	2	3	1	2	3	2

Example 2: (for n=2)

Tn	Transformation	Work done
T_2	$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$	1
	$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$	0
	$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	2
	$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$	1

Adding up the work done by each transformation gives

$w(T_2) = 1 + 0 + 2 + 1 = 4$

The average work done $w(T_2) = \frac{w(T_n)}{|T_n|} = {}^4/_4 = 1$

Example 3: (for n=3)

Tn	Transformation	work done
T_3	$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$	3
	$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$	2
	$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$	1
	$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$	2

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Tn	Transformation	Work done
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	4
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$	3
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$	2
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}$	5
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$	4
	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$	3

The total work done by these 27 elements is

$w(T_3) = 72$ *Therefore* $\bar{w}(T_3) = {}^{72}/_{27} = 2^{2}/_{3}$

4. *GENERALIZATION*:

Definition: *Let* $S \subseteq T_n$ *and* $i \in N$

We set $w_i(S) = \sum w_i(\alpha)$ *for the total work performed by* S *in moving* i.

We also set

$$w(S) = \sum_{\alpha \in S}(\alpha) = \sum_{\alpha \in S}\sum_{i \in N} W_i(\alpha) = \sum_{i \in \alpha} W_i(S)$$

as total work done

Further, for $i, j \in N$. *Set*

$M_{ij}(S) = \left\{ \propto \in \frac{S}{ix} = j \right\}$ *as* the set of all elements of S *which moves* i *to* j *and write*

$$M_{ij}(S) = \left| M_{ij}(S) \right|$$

Note that $W_i(\propto) = \left| i - j \right| \forall \propto \in M_{ij}(S)$ *and so*

$$W_i(S) = \sum_{j \in n} \left| i - j \right| M_{ij}(S)$$

Thus we have the lemma below:

Lemma:-

Let $S \subseteq T_n$. *Then* $W(S) = \sum_{i,j \in n} \left| i - j \right| M_{ij}(S)$

if $S = T_n$ *then;*

$$W(T_n) = \sum_{i,j \in n} \left| i - j \right| M_{ij}(T_n)$$

For any $i, j \in n$ *we have*

$$M_{ij}(T_n) = \left\{ \propto \in \frac{T_n}{\square} i \propto = j \right\}$$

and it follows that $Mij(T_n) = n^{n-1}$

We have

$$W(T_n) = \sum_{i,j \in n} \left| i - j \right| M_{ij}(T_n)$$

$$= \sum_{i,j \in n} |i-j| n^{n-1}$$

$$= \frac{n^3 - n}{3} n^{n-1}$$

$$since \sum_{i,j \in n} \left| i-j \right| = \frac{n^3 - n}{3}$$

$$= \frac{n(n^2-1)}{3} n^n . \frac{1}{n}$$

$$= \frac{n^n(n^2-1)}{3}$$

We also have

$$\overline{W}(T_n) = \frac{W(T_n)}{T_n} = \frac{n^n(n^2-1)}{3.n^n} \qquad since\ T_n = n^n$$

$$= \frac{n^2-1}{3}$$

$$W(T_n) = \frac{n^n(n^2-1)}{3} \ and\ \overline{W}(T_n) = \frac{n^2-1}{3}$$

The table below gives the calculated values of $w(T_n)$ and $w(T_n)$ for values of n up to 10

N	1	2	3	4	5	6	7	8	9	10
$W(T_n)$	0	4	72	1280	2500	544320	13176688	352321536	10331213040	330000000000
$\overline{W}(T_n)$	0	1	$2^2/_3$	5	8	$11^2/_3$	16	21	$26^2/_3$	333

5. CONCLUSION

If $S = T_n$, then we define the (total) work performed by T_n to be

$$w(T_n) = \sum_{\alpha \in T_n} w(\alpha)$$

We also write

$$\overline{w}(T_n) = \frac{1}{T_n} w(T_n)$$

for the average work performed by an element of T_n.

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