HEAT AND MASS TRANSFER EFFECTS ON FREE CONVECTIVE COUETTEDISSIPATING FLUID FLOW THROUGH A POROUS MEDIUM WITH SLIPCONDITIONS

M. S. Dada, S. A. Agunbiadeand, A. Jimoh

Department of Mathematics, University of Ilorin, Ilorin, Nigeria
dadamsa@gmail.com, abdulwaheedjimoh@gmail.com, agunbiade1971@gmail.com

Abstract

In this work, the effects of heat and mass transfer on unsteady free convective couette dissipating fluid flow with slip conditions in presence of variable suction and radiative heat source are investigated. The governing equations of the model namely; continuity, momentum, energy and concentration equations are non-dimensionlised and solved using perturbation techniques. The numerical results are presented graphically and effects of material parameters have been discussed.

Keywords: Free convective, Couette flow, Variable suction, Permeability, Dissipating fluid, Mass transfer.

1.0. Introduction

More recently, the attention of many researchers has been drawn to Convective fluid flow with mass transfer along a vertical porous plate in the presence of magnetic field and variable suction due to its useful applications in different branches of science and technology such as cosmical and geophysical science, fire engineering, combustion modeling, nuclear power plants, gas turbines and various propulsion devices for aircrafts, missiles, satellite and space vehicles etc. One of the early studies on the problem was presented by Ostrach (1952) who carried out the analysis of laminar free convective flow and heat transfer about a flat plate parallel to direction of the generating body force using similarity solution method. Soundalgekar (1979) investigated unsteady free convection flow along vertical porous plate with different boundary conditions and viscous dissipation effect. Singh (1983) analyzed the MHD free convective flow past an accelerated vertical porous plate by finite difference method. Hossain and Begum (1984) presented the effects of mass transfer and free convection past a vertical porous plate while Crpeau and Clarksean (1997) studied similarity solution of natural convection with internal heat generation which decays exponentially. Kim (2000) investigated unsteady MHD convective flow and heat transfer past a semi-infinite vertical porous moving plate with variable suction. Makinde et al. (2003) reported unsteady free convective flow with suction on an accelerating porous plate.

Chamkha(2004) discussed unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. He assumed that the plate moved with a constant velocity in the direction of the fluid flow while the free stream velocity is assumed to follow the exponentially increasing small perturbation law. Singh and Gupta (2005) investigated the MHD free convective flow of a viscous fluid through a porous medium bounded by an oscillating porous plate in the slip flow regime with mass transfer.The effects of chemical reaction and radiation absorption on the unsteady MHD free convection flow past a semi-infinite vertical permeable moving plate with heat source and suction was studied by Ibrahim et al. (2008). Sreekanth et al. (2001) analyzed transient MHD free convection flow of an incompressible viscous dissipative fluid and obtained the solution employing explicit finite difference method. Chamkha (2003) studied MHD flow of a uniformly stretched vertical permeable surface in presence of heat generation/absorption and chemical reaction. Recently,Al-Odat and Al-Ghamdi(2012)investigated numerically Dufour and Sorets effects on unsteady MHD natural convection flow past vertical plate embedded in non-Darcy porous medium. Das (2009) reported the effects of suction and injection on MHD three dimensional

couetteflow and heat transfer through a porous medium. The stationary plate and the plate in uniform motion are respectively subject to transverse sinusoidal injection and uniform suction of the fluid. Das et al. (2010) discussed the hydromagnectic convective flow past a vertical porous plate through a porous medium with suction and heat source. The study concluded among others that the effect of increasing magnetic parameter slow down the transient velocity of the flow field at all points.

More recently, Das et al. (2012) analyze the effects of Radiative heat and mass transfer on Natural convection couette flow through a porous medium in the slip flow regime. The effect of magnetic field viscous dissipation is neglected in this study. In view of the importance of the parameter neglected, we are concerned in this study with the effects of heat and mass transfer on free convective couette flow with dissipative heat through a porous medium in the slip flow conditions.

NOMENCLATURE

$P_r = \frac{\mu c_p}{k}$ Prandtl number

$V = \frac{\mu}{\rho}$ kinematic viscosity

$G_r = \frac{v^7 g \beta q}{k v_0}$ Grashof number for heat transfer

α_1, α_2 Suction parameters

h_1, h_2 Slip flow parameters

F Radiation parameter

K_p Permeability parameter

S_c Schmidt number

U Dimensional velocity

T^1 The dimensional temperature

C^1 The concentration

β Volumetric coefficient of expansion for heat transfer

β^1 Volumetric coefficient of expansion for heat transfer

K The thermal conductivity

C_p Specific heat at constant pressure

DMolecular diffusivity

gAcceleration due to gravity

A and B Real positive constants

tThe time

E_cEckert number

MMagnetic parameter

$\in$a small positive number such that $\in A \ll 1$ and $\in B \ll 1$

2.0. Mathematical Analysis

We consider an unsteady free convective flow of a viscous incompressible fluid between two vertical parallel porous plates placed at a distance h apart in the slip flow regime in presence of variable suction and radiative heat source.Let the medium between the plate be filled with a porous material of permeability

$K^1(t^1) = -K_0^1(1 + \epsilon A e^{-w^1 t^1})$

and a time dependent suction

$$V^1(t^1) = -V_0^1(1 + \epsilon A e^{-w^1 t^1})$$

be applied at the plate $y = 0$ and the same injection velocity be applied at the plate $y = 1$ .We choose x-axis along the plate and y −axis normal to it.

The dimensional governing equations can be stated as follows

Continuity equation

$$\frac{\partial v^*}{\partial y^*} = 0 \qquad (1)$$

which resulted to the variable time suction as defined above , i.e

$$V^1 = -V_0^1(1 + \epsilon A e^{-w^1 t^1})$$

Momentum equation:

$$\frac{\partial u^1}{\partial t^1} - V_0^1(1 + \epsilon Ae^{-w^1 t^1})\frac{\partial u^1}{\partial y^1} = g\beta^*(T^1 - T_h^1) + g\beta^*(C^1 - C_h^1) + v\frac{\partial^2 u^1}{\partial y^{12}} - v\frac{u^1}{K^1(t^1)} + \frac{\sigma B_0^2 u^1}{\rho}$$

(2)

Energy equation:

$$\frac{\partial T^1}{\partial t^1} - V_0^1(1 + \epsilon Ae^{-w^1 t^1})\frac{\partial T^1}{\partial y^1} = \frac{K}{\rho C_p}\frac{\partial^2 T^1}{\partial y^{12}} - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial y^1} + \frac{\mu}{\rho C_p}\left(\frac{\partial u^1}{\partial y^1}\right)^2 \quad (3)$$

Concentration equation:

$$\frac{\partial C^1}{\partial t^1} - V_0^1(1 + \epsilon Ae^{-w^1 t^1})\frac{\partial C^1}{\partial y^1} = \frac{\partial^2 C^1}{\partial y^{12}} \quad (4)$$

with the following boundary conditions

$$C^1 - U_1 = L_1\frac{\partial u^1}{\partial y^1}, \frac{\partial T^1}{\partial y^1} = -\frac{q}{K}, \frac{\partial C^1}{\partial y^1} = -\frac{m}{C}, \text{at } y^1 = 0 \quad (5)$$

$$u^1 - U_2 = L_2\frac{\partial u^1}{\partial y^1}, T^1 = T_h^1, C^1 = C_h^1 \text{at} y^1 = h$$

where $L_1 = \frac{2-\mu_1}{\mu_1}$L ,L being the mean free path and μ_1 , the Maxwell's reflection coefficient.

The radiative heat flux q_r is given by

$$\frac{\partial q_r}{\partial y^1} = 4(T^1 - T_h^1)\text{I} \quad (6)$$

where I= $\int K_{\lambda\omega}\frac{\partial e_{b\lambda}}{\partial T}d\lambda$, $K_{\lambda\omega}$ is the absorption coefficient at the wall, $e_{b\lambda}$ is Planck's function and λ is the frequency.

3.0. Methods of Solutions

In order to write the governing equations and the boundary conditions in dimensionless form,the following non-dimensional quantities are introduced

$$y = \frac{y^1 V_0^1}{v}, t = \frac{t^1 V_0^{12}}{v}, \omega = \frac{v\omega^1}{V_0^{12}}, u = \frac{u^1}{V_0^1}, v = \frac{\Pi_0}{\rho}, K_p = \frac{V_0^2}{v^2},$$

$$\theta = kV_0^1(T^1 - T_h^1), C = \frac{DV_0^1(C^1 - C_h^1)}{V_m}, P_r = \frac{PvC_p}{k}, G_r = \frac{v^1 g\beta_q}{KV_0^{14}},$$

$$G_c = \frac{V^2 g\beta^* m}{DV_0^{14}}, K = \frac{\Lambda vI}{\rho C_p V_0^{12}}, S_c = \frac{v}{D}, \alpha_1 = \frac{U_1}{V_0}, \alpha_2 = \frac{U_2}{V_0}, R = \frac{V_0^1 h}{v},$$

$$h_1 = \frac{L_1 V_0^1}{v}, h_2 = \frac{L_2 V_0^1}{v}, m = \frac{\sigma B_0^2 v}{\rho V_0^{12}}, Ec = \frac{V_0^{12}}{C_p(T^1 - T_h^1)} \quad (7)$$

into the governing equations (1)-(4) which now reduce to :

Momentum equation

$$\frac{\partial u}{\partial t} - (1 + cAe^{\omega t})\frac{\partial u}{\partial y} = G_r\theta + G_c C + \frac{\partial^2 u}{\partial y^2} - \frac{u^1}{K_p(1+\epsilon\beta e^{-\omega t})} + mu \quad (8)$$

Energy equation

$$\frac{\partial \theta}{\partial t} - (1 + \epsilon Ae^{-\omega t})\frac{\partial \theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} - F\theta + Ec\left(\frac{\partial v}{\partial y}\right)^2 \quad (9)$$

Concentration equation

$$\frac{\partial C}{\partial t} \quad (1 + \epsilon Ae^{-\omega t})\frac{\partial C}{\partial y} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} \quad (10)$$

$$u = \alpha_1 + h_1\frac{\partial u}{\partial y}, \frac{\partial \theta}{\partial y} = -1, \frac{\partial C}{\partial y} = -1 \text{ at } y = 0, \quad (11)$$

$$u = \alpha_2 + \frac{\partial u}{\partial y}, \theta = 0, C = 0, \text{at } y = h \quad (12)$$

In order to reduce the above system of partial differential equations to a system of ordinary differential equations in dimensionless form, the velocities, momentum, temperature, and mass are represented as:

$$u(y,t) = u_0(y) + \epsilon e^{iwt}u_1(y) + 0(\epsilon^2) + \ldots \quad (13)$$

$$\theta(y,t) = \theta_0(y) + \epsilon e^{iwt}\theta_1(y) + 0(\epsilon^2) + \ldots \quad (14)$$

$$C(y,t) = C_0(y) + \epsilon e^{iwt}C_1(y) + 0(\epsilon^2) + \ldots \quad (15)$$

On substituting equations(13)-(15) into equations(8)-(10)and neglecting the coefficient of like powers of c, we get the following set of differential equations.

$$U_0^{11} + U_0^1 - \left(M + \frac{1}{k_p}\right) U_0 = -G_r\theta_0 - G_c C_0 \quad (16)$$

$$U_1^{11} + U_1^1 - \left(M + \frac{1}{k_p} - w\right) U_1 = -AU_0^1 - \frac{B}{k_p} U_0 - G_r\theta_1 - G_c C_1 \quad (17)$$

$$\frac{\theta_0^{11}}{P_r} + \theta_0^1 \quad F\theta_0 = \quad Ec(U_0^1)^2 \quad (18)$$

$$\frac{\theta_1^{11}}{P_r} + \theta_1^1 + (w - F)\theta_1 = -A\theta_0^1 - 2EcU_0^1 U_1^1 \quad (19)$$

$$\frac{C_0^{11}}{S_r} + C_0^1 = 0 \quad (20)$$

$$\frac{C_1^{11}}{S_r} + C_1^1 + wC_1 = -AC_0^1 \quad (21)$$

and the corresponding boundary conditions reduced to:

$$u_0(y) = u_{00}(y) + Ecu_{01}(y) + 0(Ec^2)$$
$$u_1(y) = u_{10}(y) + Ecu_{11}(y) + 0(Ec^2)$$
$$\theta_0(y) - \theta_{00}(y) + Ec\theta_{01}(y) + 0(Ec^2)$$
$$\theta_1(y) = \theta_{10}(y) + Ec\theta_{11}(y) + 0(Ec^2)$$
$$C_0(y) = C_{00}(y) + EcC_{01}(y) + 0(Ec^2)$$
$$C_1(y) = C_{10}(y) + EcC_{11}(y) + 0(Ec^2) \quad (24)$$

$$U_{00}^{11} + U_{00}^1 - \left(M + \frac{1}{k_p}\right) U_{00} - -G_r\theta_{00} - G_c C_{00} \quad (25)$$

$$U_{01}^{11} + U_{01}^1 - \left(M + \frac{1}{k_p}\right) U_{01} = -G_r\theta_{01} - G_c C_{01} \quad (26)$$

$$U_{10}^{11} + U_{10}^1 - \left(M + \frac{1}{k_p} - \omega\right) U_{10} = -AU_{00}^1 - \frac{BU_{00}}{K_p} - G_r\theta_{10} - G_c C_{10} \quad (27)$$

$$U_{11}^{11} + U_{11}^1 - \left(M + \frac{1}{k_p} - \omega\right) U_{11} = -AU_{01}^1 - \frac{BU_{01}}{K_p} - G_r\theta_{11} - G_c C_{11} \quad (28)$$

$$\frac{1}{P_r}\theta_{00}^{11} + \theta_{00}^1 - F\theta_{00} = 0 \quad (29)$$

$$\frac{1}{P_r}\theta_{01}^{11} + \theta_{01}^1 - F\theta_{01} = -(U_{00}^1)^2 \quad (30)$$

$$\frac{1}{P_r}\theta_{10}^{11} + \theta_{10}^1 + (M + F - \omega)\theta_{10} = -A\theta_{00}^1 \quad (31)$$

$$\frac{1}{P_r}\theta_{11}^{11} + \theta_{11}^1 + (M - F - \omega)\theta_{11} = -A\theta_{01}^1 - 2(U_{00}^1)(U_{10}^1) \quad (32)$$

$$C_{00}^{11} + ScC_{00}^1 - 0 \quad (33)$$

$$C_{01}^{11} + ScC_{01}^1 = 0 \quad (34)$$

$$\frac{1}{Sc}C_{10}^{11} + C_{10}^1 + \omega C_{10} = -AC_{00}^1 \quad (35)$$

$$\frac{1}{Sc}C_{11}^{11} + C_{11}^1 + \omega C_{11} = -AC_{01}^1 \quad (36)$$

With the following corresponding boundary conditions

$$U_{00} = \alpha_1 + h_1 U_{00}^1, U_{01} = h_1 U_{01}^1, U_{10} = h_1 U_{10}^1, U_{11} = h_1 U_{11}^1,$$

at $y = 0$ (37)

$$U_{00} = \alpha_2 \; | \; h_2 U_{00}^1, U_{01} = h_2 U_{01}^1, U_{10} = h_2 U_{10}^1, U_{11} = h_2 U_{11}^1,$$

at $y = 1$ (38)

$$\theta_{00}^1 = -1, \theta_{01}^1 = 0, \theta_{10}^1 = 0, and\ \theta_{11}^1 = 0, at\ y = 0 \quad (39)$$

$$\theta_{00} = 0, \theta_{01} = 0, \theta_{10} = 0, and\ \theta_{11} = 0,\ at\ y = 1 \quad (40)$$

$$C_{00}^1 = -1, C_{01}^1 = 0, C_{10}^1 = 0, and\ C_{11}^1 = 0, at\ y = 0 \quad (41)$$

$$C_{00} = 0, C_{01} = 0, C_{10} = 0, and\ C_{11} = 0,\ at\ y = 1 \quad (42)$$

The solutions of equations (25)-(36) subject to the boundary conditions (37) and (42) are respectively:

$$U(y,t)=A_1e^{m_5y} + A_2e^{m_6y} + L_1e^{m_3y} + L_2e^{m_4y} + k_1 + L_3e^{scy} + Ec(A_{19}e^{m_{21}y} + A_{20}e^{m_{22}y} + L_{94}e^{m_7y} + L_{95}e^{m_8y} + L_{96}e^{2m_5y} + L_{97}e^{(m_6-m_5)y} + L_{98}e^{(m_3+m_5)y} + L_{99}e^{(m_4+m_5)y} + L_{100}e^{(r_2+m_5)y} + L_{101}e^{2m_6y} + L_{102}e^{(m_3+m_6)y} + L_{103}e^{2m_3y} + L_{104}e^{(m_4+m_3)y} + L_{105}e^{(r_2-m_3)y} + L_{106}e^{2m_4y} + L_{107}e^{(r_2+m_4)y} + L_{108}e^{(m_4-m_6)y} + L_{109}e^{(r_2+m_3)y} + L_{110}e^{2r_2y}) + \varepsilon e^{-\omega t}[A_{15}e^{m_{17}y} + A_{16}e^{m_{18}y} + L_{22}e^{m_5y} + L_{23}e^{m_6y} + L_{24}e^{m_3y} + L_{25}e^{m_4y} + L_{26}e^{r_2y} + L_{27}e^{m_9y} + L_{28}e^{m_{10}y} + L_{29}e^{m_{11}y} + L_{30}e^{m_{12}y} + Ec\ \{ A_{21}e^{m_{23}y} + A_{22}e^{m_{24}y} + L_{111}e^{m_{21}y} + L_{112}e^{m_{22}y} + L_{113}e^{m_7y} + L_{114}e^{m_8y} + L_{115}e^{2m_5y} + L_{116}e^{(m_6+m_5)y} + L_{117}e^{(m_3+m_5)y} + L_{118}e^{(m_4+m_5)y} + L_{119}e^{(r_2-m_5)y} + L_{120}e^{2m_6y} + L_{121}e^{(m_6+m_3)y} + L_{122}e^{2m_3y} + L_{123}e^{(m_4+m_3)y} + L_{124}e^{(r_2+m_3)y} + L_{125}e^{2m_4y} + L_{126}e^{(r_2+m_4)y} + L_{127}e^{(m_6-m_4)y} + L_{128}e^{(m_6+r_2)y} + L_{129}e^{2m_2y} + L_{130}e^{m_{19}y} + L_{131}e^{m_{20}y} + L_{132}e^{(m_{17}-m_5)y} + L_{133}e^{(m_{18}+m_5)y} + L_{134}e^{(m_9+m_5)y} + L_{135}e^{(m_{10}+m_5)y} + L_{136}e^{(m_{11}+m_5)y} + L_{137}e^{(m_{12}+m_5)y} + L_{138}e^{(m_6+m_{17})y} + L_{139}e^{(m_6+m_{18})y} + L_{140}e^{(m_9+m_6)y} + L_{141}e^{(m_{10}+m_6)y} + L_{142}e^{(m_6+m_{11})y} + L_{143}e^{(m_6+m_{12})y} + L_{144}e^{(m_3+m_{17})y} + L_{145}e^{(m_3+m_{18})y} + L_{146}e^{(m_3+m_9)y} + L_{147}e^{(m_3+m_{10})y} + L_{148}e^{(m_3+m_{11})y} + L_{149}e^{(m_3+m_{12})y} + L_{150}e^{(m_4+m_{17})y} + L_{151}e^{(m_4+m_{18})y} + L_{152}e^{(m_4-m_9)y} + L_{153}e^{(m_4+m_{10})y} + L_{154}e^{(m_4-m_{11})y} + L_{155}e^{(m_4+m_{12})y} + L_{156}e^{(r_2+m_{17})y} + L_{157}e^{(r_2+m_{18})y} + L_{158}e^{(r_2+m_9)y} + L_{159}e^{(r_2+m_{10})y} + L_{160}e^{(r_2+m_{11})y} + L_{161}e^{(r_2+m_{12})y} + L_{162}e^{m_{13}y} + L_{163}e^{m_{14}y} + L_{164}e^{m_{11}y} + L_{165}e^{m_{12}y} + L_{166}e^{r_2y}\}] \quad (43)$$

$$\theta(y,t) = C_5e^{m_5y} + C_6e^{m_6y} + Ec\ (A_3e^{m_7y} + A_4e^{m_8y} + L_{31}e^{2m_5y} + L_{32}e^{(m_6+m_5)y} + L_{33}e^{(m_3+m_5)y} + L_{34}e^{(m_4-m_5)y} + L_{35}e^{(r_2+m_5)y} + L_{36}e^{2m_6y} + L_{37}e^{(m_6+m_3)y} + L_{38}e^{2m_3y} + L_{39}e^{(m_3+m_4)y} + L_{40}e^{(m_3+r_2)y} + L_{41}e^{2m_4y} + L_{42}e^{(m_4-r_2)y} + L_{43}e^{(m_4+m_6)y} + L_{44}e^{(m_6+r_2)y} + L_{45}e^{2r_2y}) + \varepsilon e^{-\omega t}[A_5e^{m_9y} + A_6e^{m_{10}y} + L_9e^{m_3y} + L_{10}e^{m_4y} + Ec\ (A_{17}e^{m_{19}y} + A_{18}e^{m_{20}y} + L_{46}e^{m_7y} + L_{47}e^{m_8y} + L_{48}e^{2m_5y} + L_{49}e^{(m_6+m_5)y} + L_{50}e^{(m_3+m_5)y} + L_{51}e^{(m_4+m_5)y} + L_{52}e^{(r_2+m_5)y} + L_{53}e^{2m_6y} + L_{54}e^{(m_6+m_3)y} + L_{55}e^{2m_3y} + L_{56}e^{(m_4+m_3)y} + L_{57}e^{(r_2+m_3)y} + L_{58}e^{2m_4y} + L_{59}e^{(m_4+r_2)y} + L_{60}e^{(m_6+m_4)y} + L_{61}e^{(m_6-r_2)y} + L_{62}e^{2r_2y} + L_{63}e^{(m_5-m_{17})y} + L_{64}e^{(m_5+m_{18})y} + L_{65}e^{(m_5+m_9)y} + L_{66}e^{(m_5-m_{10})y} + L_{67}e^{(m_5+m_{11})y} + L_{68}e^{(m_5-m_{12})y} + L_{69}e^{(m_{17}+m_6)y} + L_{70}e^{(m_{18}+m_6)y} + L_{71}e^{(m_9+m_6)y} + L_{72}e^{(m_{10}+m_6)y} + L_{73}e^{(m_{11}+m_6)y} + L_{74}e^{(m_{12}+m_6)y} + L_{75}e^{(m_3+m_{17})y} + L_{76}e^{(m_{18}+m_3)y} + L_{77}e^{(m_3+m_9)y} + L_{78}e^{(m_{10}+m_3)y} + L_{80}e^{(m_{12}-m_3)y} + L_{81}e^{(m_4+m_{17})y} + L_{82}e^{(m_4+m_{18})y} + L_{83}e^{(m_4-m_9)y} + L_{84}e^{(m_4+m_{10})y} + L_{85}e^{(m_4-m_{11})y} + L_{86}e^{(m_4+m_{12})y} + L_{87}e^{(r_2-m_{17})y} + L_{88}e^{(r_2-m_{18})y} + L_{89}e^{(r_2+m_6)y} + L_{90}e^{(r_2+m_9)y} + L_{91}e^{(r_2-m_{10})y} + L_{92}e^{(r_2+m_{11})y} + L_{93}e^{(r_2+m_{12})y})] \quad (44)$$

$$C(y,t) = C_1 + C_2e^{-Scy} + \varepsilon e^{-\omega t}\ (\ A_7e^{m_{11}y} + A_8e^{m_{12}y} + L_{11}e^{r_2y} + Ec\ (\ A_9e^{m_{13}y} + A_{10}e^{m_{14}y} + L_{12}e^{m_{11}y} + A_{13}e^{m_{12}y} + L_{14}e^{r_2y}\ )\ ] \quad (45)$$

Skin-friction coefficient is expressed as follows:

$$c_f = \left|\frac{\tau_\omega}{\rho U_0 V_0}\right| = \left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\partial u_0(y)}{\partial y} + \varepsilon e^{nt}\frac{\partial u_1(y)}{\partial y}\right)_{y=0}$$

$$\tau = A_1m_5 + A_2m_6 + L_1m_3 + L_2m_4 + L_3r_2 + Ec\ (A_{10}m_{21} + A_{20}m_{22} + L_{94}m_7 + L_{95}m_5 + L_{96}2m_5 + L_{97}(m_5+m_6) + L_{98}(m_5+m_3) + L_{99}(m_5+m_4) + L_{100}(m_5+r_2) + L_{101}2m_6 + L_{102}(m_3+m_6) + L_{103}2m_3 + L_{104}(m_4+m_3) + L_{105}(m_5+r_2) + L_{106}2m_4 + L_{107}(m_4+r_2) + L_{108}(m_4+m_6) + L_{97}(m_5+m_6) + L_{109}(r_2+m_6) + L_{110}2m_2 + \varepsilon e^{-\omega t}[\ A_{15}m_{17} + A_{16}m_{18} + L_{22}m_5 + L_{23}m_6 + L_{24}m_3 + L_{25}m_4 + L_{26}r_2 + L_{27}m_9 + L_{28}m_{10} + L_{29}m_{11} + L_{30}m_{12} + Ec\ \{A_{21}m_{23} + A_{22}m_{24} + L_{111}m_{21} + L_{112}m_{22} + L_{113}m_7 + L_{114}m_8 + L_{115}2m_5 + L_{116}(m_5+m_6) + L_{117}(m_5+m_3) + L_{118}(m_5+m_4) + L_{119}(m_5+r_2) + L_{120}2m_6 + L_{121}(m_3+m_6) + L_{122}2m_3 + L_{123}(m_4+m_3) + L_{124}(r_2+m_3) + L_{125}2m_4 + L_{126}(m_4+r_2) + L_{127}(m_4+m_6) + L_{128}(m_4+r_2) + L_{129}2r_2 + L_{130}m_{19} + L_{131}m_{20} + L_{132}(m_5+m_{17}) + L_{133}(m_5+m_{18}) + L_{134}(m_5+m_9) + L_{135}(m_5+m_{10}) + L_{136}(m_5+m_{11}) + L_{137}(m_5+m_{12}) + L_{138}(m_{17}+m_6) + L_{139}(m_{18}+m_6) + L_{140}(m_9+m_6) + L_{141}(m_{10}+m_6) + L_{142}(m_{11}+m_6) + L_{143}(m_{12}+m_6) + L_{144}(m_3+m_{17}) + L_{145}(m_3+m_{18}) + L_{146}(m_9+m_3) + L_{147}(m_{10}+m_3) + L_{148}(m_{11}+m_3) + L_{149}(m_{12}+m_3) + L_{150}(m_4+m_{17}) + L_{151}(m_4+m_{18}) + L_{152}(m_4+m_9) + L_{153}(m_4+m_{10}) + L_{154}(m_4+m_{11}) + L_{155}(m_4+m_{12}) + L_{156}(r_2+m_{17}) + L_{157}(r_2+m_{18}) + L_{158}(r_2+m_9) + L_{159}(r_2+m_{10}) + L_{160}(r_2+m_{11}) + L_{161}(r_2+m_{12}) + L_{162}m_{10} + L_{163}m_{14} + L_{164}m_{11} + L_{165}m_{12} + L_{166}r_2\ \}\ ] \quad (46)$$

The rate of heat transfer (i.e) the heat flux at the wall is given by;

$$Nu=\left(\frac{\partial\theta}{\partial y}\right)_{y=0}=\left(\frac{\partial\theta_{01}}{\partial y}\right)_{y=0}\mathrm{Ec}\left(\frac{\partial\theta_{01}}{\partial y}\right)_{y=0}\varepsilon e^{-\omega t}\left[\left(\frac{\partial\theta_{1}}{\partial y}\right)_{y=0}+Ec\left(\frac{\partial\theta_{1}}{\partial y}\right)_{y=0}\right]$$

$$Nu = C_5m_3 + C_6m_4 + Ec(A_3m_7 +A_4m_8 + L_{31}2m_5 + L_{32} (m_6 + m_5) + L_{33} (m_3 + m_5) + L_{34} (m_4 + m_5) + L_{35} (r_2 + m_5) + L_{36} 2m_6 + L_{37} (m_6 + m_3)+ L_{38}2m_3 + L_{39}(m_4+ m_3) + L_{40}(m_3+ r_2) + L_{41}2m_4 + L_{42}(m_4+ r_2) + L_{43}(m_6 + m_4) +$$

$$L_{44}(m_6 + r_2) + L_{45}2r_2) +\varepsilon e^{-\omega t} [A_5m_9 +A_6m_{10} + L_9m_3 +L_{10}m_4 + Ec \{ A_{17}m_{19}+A_{18}m_{20} + L_{46}m_7 +L_{48}2m_5 +$$

$$L_{49}(m_6 + m_5) + L_{50}(m_3+ m_5) + L_{51}(m_4+ m_5) + L_{52}(r_2+ m_5) + L_{53} 2m_6 + L_{54} (m_6 + m_3) + L_{55} 2m_3 + L_{56} (m_4 + m_3) + L_{57} (m_3 + r_2) + L_{58} 2m_4 + L_{59} (m_4 + r_2) + L_{60} (m_6 + m_4) + L_{61} (m_6 + r_2) + L_{62} 2r_2 + L_{63} (m_{17}+ m_5) + L_{64} (m_{18} + m_5) + L_{65} (m_9+ m_5) + L_{66}(m_{10}+ m_5) + L_{67}(m_{11}+ m_5) + L_{68}(m_{12}+ m_5) + L_{69}(m_6 + m_{17}) + L_{70}(m_6 + m_{18}) + L_{71}(m_6 + m_9) + L_{71}(m_6 + m_{10}) + L_{73}(m_6 + m_{11}) + L_{74}(m_6 + m_{12}) + L_{75}(m_3+ m_{17}) + L_{76}(m_3+ m_{18}) + L_{77}(m_3+ m_9) + L_{78} (m_3 + m_{10}) + L_{79} (m_3 + m_{11}) + L_{80} (m_3 + m_{12}) + L_{81} (m_4 + m_{17}) + L_{82} (m_4 + m_{18}) + L_{83} (m_4 + m_9) + L_{84} (m_4 + m_{10}) +$$

$$L_{85} (m_4 + m_{11}) + L_{86} (m_4 + m_{12}) + L_{87} (r_2 + m_{17}) + L_{88} (r_2 + m_{18}) + L_{89} (r_2 + m_6) + L_{90} (r_2 + m_9) + L_{91} (r_2 + m_{10}) +$$

$$L_{92} (r_2 + m_{11}) + L_{93} (r_2 + m_{12}) \}] \qquad (47)$$

Where $r_1 = 0$, $r_2 = Sc$, $m_3 = \frac{-Pr-\sqrt{Pr^2+4Pr}}{2}$, $m_4 = \frac{-Pr-\sqrt{Pr^2+4Pr}}{2}$,

$$m_5 = \frac{-1+\sqrt{1+4a}}{2}, m_6 = \frac{-1-\sqrt{1+4a}}{2}, m_7 = \frac{-Pr+\sqrt{Pr^2+4Pr}}{2},$$

$$m_8 = \frac{-Pr-\sqrt{Pr^2+4Pr}}{2}, m_9 = m_{19} = \frac{-Pr+\sqrt{Pr^2+4\,\mathrm{Pr}(f-\omega)}}{2},$$

$$m_{10} = m_{20} = \frac{-Pr-\sqrt{Pr^2+4\,\mathrm{Pr}(f-\omega)}}{2}, m_{11} = m_{13} = \frac{-Sc+\sqrt{Sc^2-4Sc\omega}}{2}, m_{12} = m_{14} = \frac{-Sc-\sqrt{Sc^2-4Sc\omega}}{2},$$

$$m_{15} = 0, m_{16} = -Sc, m_{17} = m_{23} = \frac{-1-\sqrt{1+4\left(\frac{1}{K_p}-\omega-m\right)}}{2}, m_{18} = m_{24} = \frac{-1-\sqrt{1+4\left(\frac{1}{K_p}-\omega-m\right)}}{2}, m_{21} = \frac{-1+\sqrt{1+4\left(\frac{1}{K_p}-m\right)}}{2}, m_{22} = \frac{-1-\sqrt{1+4\left(\frac{1}{K_p}-m\right)}}{2}, C_1 = \frac{-e^{-Sc}}{Sc}, C_2 = -\frac{1}{r_2}$$

$$C_5 = \frac{-e^{m_4}}{m_3e^{m_4}-m_4e^{m_3}}, C_6 = \frac{-C_5e^{m_3}}{e^{m_4}}, a = \frac{1}{K_p} - m, K_1 = \frac{-G_rC_1}{\frac{1}{K_p}-m}, K_2 = -\left(AA_1m_5 + \frac{B}{K_p}A_1\right), K_3 = \left(AA_2m_6 + \frac{B}{K_p}A_2\right), K_4 = \left(AL_1m_3 + \frac{B}{K_p}L_1 + G_rL_9\right), K_5 = \left(AL_2m_4 + \frac{B}{K_p}L_2 + G_rL_{10}\right),$$

$$K_6 = -\left(AL_3r_2 + \frac{B}{K_p}L_3 + G_cL_{11}\right), L_1 = \frac{-G_rC_5}{m_3^2+m_3\left(\frac{1}{K_p}-m\right)}, L_2 = \frac{-G_rC_6}{m_4^2+m_4\left(\frac{1}{K_p}-m\right)}, L_3 = \frac{-G_rC_2}{Sc^2-Sc\left(\frac{1}{K_p}-m\right)},$$

$$L_9 = \frac{PrAC_5m_3}{m_3^2+P_rm_3-P_r(f-\omega)}, L_{10} = \frac{PrAC_6m_4}{m_4^2+P_rm_4-P_r(f-\omega)}, L_{11} = \frac{AScC_2r_2}{r_2^2+r_2Sc-Sc\omega}, L_{12} = \frac{AScA_1m_{11}}{m_{11}^2+Scm_{11}-Sc\omega}, L_{13} = \frac{AScA_2m_{12}}{m_{12}^2+Scm_{12}-Sc\omega}, L_{14} = \frac{AScL_{11}r_2}{r_2^2+Scr_2+Sc\omega}, L_{22} = \frac{K_2}{m_5^2+m_5-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{23} = \frac{K_3}{m_6^2+m_6\left(\frac{1}{K_p}-\omega-m\right)}, L_{24} = \frac{K_4}{m_3^2+m_3\left(\frac{1}{K_p}-\omega-m\right)}, L_{25} = \frac{K_5}{m_4^2+m_4\left(\frac{1}{K_p}-\omega-m\right)}, L_{26} = \frac{K_6}{r_2^2+r_2\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{27} = \frac{-G_rA_5}{m_9^2+m_9-\left(\frac{1}{K_p}-\omega-m\right)}, L_{28} = \frac{-G_rA_8}{m_{10}^2+m_{10}-\left(\frac{1}{K_p}-\omega-m\right)}, L_{29} = \frac{-G_cA_7}{m_{11}^2+m_{11}-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{30} = \frac{-G_cA_8}{m_{12}^2+m_{12}\left(\frac{1}{K_p}-\omega-m\right)}, L_{31} = \frac{n_1}{4m_5^2-2P_rm_5-P_r}, L_{32} = \frac{n_2}{(m_6+m_5)^2+(m_6+m_5)P_r-P_r},$$

$$L_{33} = \frac{n_3}{(m_3+m_5)^2-(m_3+m_5)P_r-P_r}, L_{34} = \frac{n_4}{(m_4+m_5)^2+(m_4+m_5)P_r-P_r},$$

$$L_{35} = \frac{n_5}{(m_5+r_2)^2+(m_5+r_2)P_r-P_r}, L_{36} = \frac{n_6}{4m_6^2+2m_6P_r-P_r}, L_{37} = \frac{n_7}{(m_6+m_3)^2+(m_6+m_3)P_r-P_r},$$

$$L_{38} = \frac{n_9}{(m_4+m_3)^2-(m_4+m_3)P_r-P_r}, L_{39} = \frac{n_{10}}{(m_3+r_2)^2+(m_3+r_2)P_r-P_r}, L_{40} = \frac{n_8}{4m_3^2+2m_3P_r-P_r},$$

$$L_{41}=\frac{n_{11}}{4m_4^2+2m_4P_r-P_r},L_{42}=\frac{n_{12}}{(m_1-r_2)^2-(m_1-r_2)P_r-P_r},L_{43}=\frac{n_{13}}{(m_1-m_6)^2+(m_2+m_6)P_r-P_r},$$

$$L_{44}=\frac{n_{14}}{(m_8+r_2)^2+(m_8+r_2)P_r-P_r},L_{45}=\frac{n_{15}}{4r_2^2+2r_2P_r-P_r},L_{46}=\frac{t_1}{m_7^2+P_rm_7-P_r(f-\omega)},$$

$$L_{47}=\frac{t_2}{m_8^2-P_rm_8-P_r(f-\omega)},L_{48}=\frac{t_3}{4m_5^2+2P_rm_5-P_r(f-\omega)},L_{49}=\frac{t_4}{(m_6+m_5)^2+P_r(m_6-m_5)-P_r(f-\omega)},$$

$$L_{50}=\frac{t_5}{(m_3+m_5)^2-P_r(m_3-m_5)-P_r(f-\omega)},L_{51}=\frac{t_6}{(m_1-m_5)^2+P_r(m_2+m_5)-P_r(f-\omega)}$$

$$L_{52}=\frac{t_7}{(r_2+m_5)^2+P_r(r_2-m_5)-P_r(f-\omega)},L_{53}=\frac{t_8}{4m_6^2+2P_rm_6-P_r(f-\omega)}L_{54}=\frac{t_9}{(m_3+m_6)^2+P_r(m_3+m_6)-P_r(f-\omega)},$$

$$L_{55}=\frac{t_{10}}{4m_3^2+2P_rm_3-P_r(f-\omega)},L_{56}=\frac{t_{11}}{(m_4+m_3)^2+P_r(m_2+m_3)-P_r(f-\omega)},L_{57}=\frac{t_{12}}{(r_2-m_3)^2+P_r(r_2+m_3)-P_r(f-\omega)},$$

$$L_{58}=\frac{t_{13}}{4m_4^2+2P_rm_4-P_r(f-\omega)},\quad L_{59}=\frac{t_{14}}{(m_4-r_2)^2-P_r(m_4+r_2)-P_r(f-\omega)},$$

$$L_{60}=\frac{n_{15}}{(m_4+m_6)^2-P_r(m_4-m_6)-P_r(f-\omega)},L_{61}=\frac{t_{16}}{(m_6+r_2)^2+P_r(m_6+r_2)-P_r(f-\omega)}L_{62}=\frac{t_{17}}{4r_2^2-2P_rr_2-P_r(f-\omega)},$$

$$L_{63}=\frac{t_{18}}{(m_{17}+m_3)^2+P_r(m_{17}+m_3)-P_r(f-\omega)},$$

$$L_{64}=\frac{t_{19}}{(m_{18}+m_3)^2+P_r(m_{18}+m_3)-P_r(f-\omega)},L_{65}=\frac{t_{20}}{(m_5+m_9)^2+P_r(m_5+m_9)-P_r(f-\omega)},$$

$$L_{66}=\frac{t_{21}}{(m_5+m_{10})^2+P_r(m_5+m_{10})-P_r(f-\omega)},$$

$$L_{67}=\frac{t_{22}}{(m_5+m_{11})^2+P_r(m_5+m_{11})-P_r(f-\omega)},\quad L_{68}=\frac{t_{23}}{(m_5+m_{12})^2+P_r(m_5+m_{12})-P_r(f-\omega)},$$

$$L_{69}=\frac{t_{24}}{(m_6+m_{17})^2-P_r(m_6+m_{17})-P_r(f-\omega)},\quad L_{70}=\frac{t_{25}}{(m_6+m_{18})^2+P_r(m_6+m_{18})-P_r(f-\omega)},$$

$$L_{71}=\frac{t_{26}}{(m_6+m_9)^2+P_r(m_6+m_9)-P_r(f-\omega)},\quad L_{72}=\frac{t_{27}}{(m_6+m_{10})^2+P_r(m_6+m_{10})-P_r(f-\omega)},$$

$$L_{73}=\frac{t_{28}}{(m_6+m_{11})^2+P_r(m_6+m_{11})-P_r(f-\omega)},L_{74}=\frac{t_{29}}{(m_6+m_{12})^2+P_r(m_6+m_{12})-P_r(f-\omega)},$$

$$L_{75}=\frac{t_{30}}{(m_3+m_{17})^2+P_r(m_3+m_{17})-P_r(f-\omega)},L_{76}=\frac{t_{31}}{(m_3+m_{18})^2+P_r(m_3+m_{18})-P_r(f-\omega)},$$

$$L_{77}=\frac{t_{32}}{(m_3+m_9)^2+P_r(m_3+m_9)-P_r(f-\omega)},L_{78}=\frac{t_{33}}{(m_3+m_{10})^2+P_r(m_3+m_{10})-P_r(f-\omega)},$$

$$L_{79}=\frac{t_{34}}{(m_3+m_{11})^2-P_r(m_3+m_{11})-P_r(f-\omega)},L_{80}=\frac{t_{35}}{(m_3+m_{12})^2+P_r(m_3+m_{12})-P_r(f-\omega)},$$

$$L_{81}=\frac{t_{36}}{(m_3+m_{17})^2+P_r(m_3+m_{17})-P_r(f-\omega)},L_{82}=\frac{t_{37}}{(m_4+m_{18})^2+P_r(m_4+m_{18})-P_r(f-\omega)},$$

$$L_{83}=\frac{t_{38}}{(m_4+m_9)^2+P_r(m_4+m_9)-P_r(f-\omega)},L_{84}=\frac{t_{39}}{(m_4+m_{10})^2+P_r(m_4+m_{10})-P_r(f-\omega)},$$

$$L_{85}=\frac{t_{40}}{(m_4+m_{11})^2+P_r(m_4+m_{11})-P_r(f-\omega)},L_{86}=\frac{t_{41}}{(m_4+m_{12})^2+P_r(m_4+m_{12})-P_r(f-\omega)},$$

$$L_{87}=\frac{t_{42}}{(r_2-m_{17})^2+P_r(r_2+m_{17})-P_r(f-\omega)},L_{88}=\frac{t_{43}}{(r_2+m_{18})^2+P_r(r_2+m_{18})-P_r(f-\omega)},$$

$$L_{89}=\frac{t_{44}}{(r_2+m_9)^2+P_r(r_2-m_9)-P_r(f-\omega)},L_{90}=\frac{t_{45}}{(r_2+m_9)^2+P_r(r_2-m_9)-P_r(f-\omega)},$$

$$L_{91}=\frac{t_{46}}{(r_2+m_{10})^2+P_r(r_2+m_{10})-P_r(f-\omega)},L_{92}=\frac{t_{47}}{(r_2-m_{11})^2-P_r(r_2+m_{11})-P_r(f-\omega)},$$

$$L_{93}=\frac{t_{48}}{(r_2+m_{12})^2+P_r(r_2+m_{12})-P_r(f-\omega)},L_{94}=\frac{-G_rA_3}{m_7^2+m_7-\left(\frac{1}{K_p}-m\right)},$$

$$L_{95}=\frac{-G_rA_4}{m_8^2+m_8-\left(\frac{1}{K_p}-m\right)},\quad L_{96}=\frac{-G_rL_{31}}{4m_5^2+m_5-\left(\frac{1}{K_p}-m\right)},$$

$$L_{97}=\frac{-G_rL_{32}}{(m_6+m_5)^2+(m_6+m_5)-\left(\frac{1}{K_p}-m\right)},L_{98}=\frac{-G_rL_{33}}{(m_3+m_5)^2-(m_3+m_5)-\left(\frac{1}{K_p}-m\right)},$$

$$L_{99}=\frac{-G_rL_{34}}{(m_4-m_5)^2+(m_4+m_5)-\left(\frac{1}{K_p}-m\right)},L_{100}=\frac{-G_rL_{35}}{(m_3+r_2)^2+(m_3+r_2)-\left(\frac{1}{K_p}-m\right)},$$

$$L_{101}=\frac{-G_rL_{36}}{4m_6^2+2m_6-\left(\frac{1}{K_p}-m\right)},L_{102}=\frac{-G_rL_{37}}{(m_6+m_3)^2+(m_6+m_3)-\left(\frac{1}{K_p}-m\right)},$$

$$L_{103}=\frac{-G_rL_{38}}{4m_3^2+2m_3-\left(\frac{1}{K_p}-m\right)},L_{104}=\frac{-G_rL_{39}}{(m_4+m_3)^2+(m_4+m_3)-\left(\frac{1}{K_p}-m\right)},$$

$$L_{105}=\frac{-G_rL_{40}}{(m_3+r_2)^2+(m_3+r_2)-\left(\frac{1}{K_p}-m\right)},L_{106}=\frac{-G_rL_{41}}{4m_4^2+2m_4-\left(\frac{1}{K_p}-m\right)},$$

$$L_{107}=\frac{-G_rL_{42}}{(m_2+r_2)^2+(m_2+r_2)-\left(\frac{1}{K_p}m\right)},\ L_{108}=\frac{-G_rL_{43}}{(m_1+m_6)^2+(m_2+m_6)-\left(\frac{1}{K_p}-m\right)},$$

$$L_{109}=\frac{-G_rL_{44}}{(m_6+r_2)^2+(m_6+r_2)-\left(\frac{1}{K_p}-m\right)},\ L_{110}=\frac{-G_rL_{45}}{4r_2^2+2r_2-\left(\frac{1}{K_p}-m\right)},$$

$$L_{111}=\frac{-AA_{19}m_{21}-\frac{B}{K_p}A_{19}}{m_{21}^2+m_{2}\left(\frac{1}{K_p}\ \omega\ m\right)},\ L_{112}=\frac{-A_{20}\left(Am_{22}+\frac{B}{K_p}\right)}{m_{22}^2\ m_{22}\left(\frac{1}{K_p}\ \omega\ m\right)}$$

$$L_{113}=\frac{-L_{94}\left(Am_7+\frac{B}{K_p}\right)-G_rL_{46}}{m_7^2+m_7-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{114}=\frac{-L_{95}\left(Am_8+\frac{B}{K_p}\right)-G_rL_{47}}{m_8^2+m_8-\left(\frac{1}{K_p}\omega-m\right)},$$

$$L_{115}=\frac{L_{96}\left(2Am_5+\frac{B}{K_p}\right)\ G_rL_{48}}{4m_5^2+2m_5-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{116}=\frac{L_{97}\left(A(m_6+m_5)+\frac{B}{K_p}\right)\ G_rL_{49}}{(m_5-m_6)^2+(m_5+m_6)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{117}=\frac{-AL_{98}(m_3-m_3)-\frac{B}{K_p}L_{98}-G_rL_{50}}{(m_3+m_5)^2\ \ (m_3\ \ m_5)\ \left(\frac{1}{K_p}\ \omega\ m\right)},\ L_{118}=\frac{-AL_{99}(m_5+m_4)-\frac{B}{K_p}L_{99}-G_rL_{51}}{(m_4+m_5)^2+(m_4+m_5)\ \left(\frac{1}{K_p}\ \omega\ m\right)},$$

$$L_{119}=\frac{-AL_{100}(m_5+r_2)-\frac{B}{K_p}L_{100}-G_rL_{52}}{(r_2+m_5)^2+(r_2+m_5)-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{120}=\frac{-2AL_{101}m_6-\frac{B}{K_p}L_{101}-G_rL_{53}}{(2m_6)^2+(2m_6)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{121}=\frac{-AL_{102}(m_6-m_3)-\frac{B}{K_p}L_{102}-G_rL_{54}}{(m_3+m_6)^2-(m_3-m_6)-\left(\frac{1}{K_p}\omega-m\right)},\ L_{122}=\frac{-2AL_{103}m_3-\frac{B}{K_p}L_{103}-G_rL_{55}}{(2m_3)^2-(2m_3)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{123}=\frac{-AL_{104}(m_4-m_3)-\frac{B}{K_p}L_{104}-G_rL_{56}}{(m_3+m_4)^2\ \ (m_3\ \ m_4)\ \left(\frac{1}{K_p}\ \omega\ m\right)},\ L_{124}=\frac{-AL_{105}(r_2+m_3)-\frac{B}{K_p}L_{105}-G_rL_{57}}{(m_3+r_2)^2+(m_3+r_2)\ \left(\frac{1}{K_p}\ \omega\ m\right)},$$

$$L_{125}=\frac{-2AL_{106}m_4-\frac{B}{K_p}L_{106}-G_rL_{58}}{(2m_4)^2+(2m_4)-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{126}=\frac{-AL_{107}(r_2-m_4)-\frac{B}{K_p}L_{107}-G_rL_{59}}{(m_4+r_2)^2+(m_4+r_2)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{127}=\frac{-AL_{108}(m_4-m_6)-\frac{B}{K_p}L_{108}-G_rL_{60}}{(m_6+m_4)^2-(m_6-m_6)-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{128}=\frac{-AL_{109}(r_2+m_6)-\frac{B}{K_p}L_{109}-G_r(L_{61}+L_{80})}{(m_6+r_2)^2+(m_6+r_2)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{129}=\frac{-2AL_{110}r_2-\frac{B}{K_p}L_{110}-G_rL_{62}}{(2r_2)^2\ \ (2r_2)\ \left(\frac{1}{K_p}\ \omega\ m\right)},\ L_{130}=\frac{-G_rA_{17}}{(m_{19})^2\ \ (m_{19})\ \left(\frac{1}{K_p}\ \omega\ m\right)},$$

$$L_{131}=\frac{G_rA_{18}}{(m_{20})^2+(m_{20})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{132}=\frac{G_rL_{63}}{(m_5+m_{17})^2+(m_5+m_{17})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{133}=\frac{-G_rL_{64}}{(m_3+m_{18})^2+(m_3+m_{18})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{134}=\frac{-G_rL_{65}}{(m_3+m_9)^2+(m_3+m_9)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{135}=\frac{G_rL_{66}}{(m_5+m_{10})^2+(m_5+m_{10})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{136}=\frac{G_rL_{67}}{(m_5+m_{11})^2+(m_5+m_{11})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{137}=\frac{-G_rL_{68}}{(m_3+m_{12})^2+(m_3+m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{138}=\frac{-G_rL_{69}}{(m_6+m_{17})^2+(m_6+m_{17})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{139}=\frac{-G_rL_{70}}{(m_6+m_{18})^2-(m_6+m_{18})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{140}=\frac{-G_rL_{71}}{(m_6+m_9)^2+(m_6+m_9)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{141}=\frac{-G_rL_{72}}{(m_6+m_{10})^2+(m_6+m_{10})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{142}=\frac{-G_rL_{73}}{(m_6+m_{11})^2+(m_6+m_{11})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{143}=\frac{-G_rL_{74}}{(m_6+m_{12})^2+(m_6+m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{144}=\frac{-G_rL_{75}}{(m_3+m_{17})^2+(m_3+m_{17})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{145}=\frac{-G_rL_{76}}{(m_3+m_{18})^2+(m_3+m_{18})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{146}=\frac{-G_rL_{77}}{(m_3+m_9)^2+(m_3+m_9)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{147}=\frac{-G_rL_{78}}{(m_3+m_{10})^2+(m_3+m_{10})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{148}=\frac{-G_rL_{79}}{(m_3+m_{11})^2+(m_3+m_{11})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{149}=\frac{-G_rL_{80}}{(m_3+m_{12})^2-(m_3+m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{150}=\frac{-G_rL_{81}}{(m_4+m_{17})^2-(m_4+m_{17})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{151}=\frac{-G_rL_{82}}{(m_2+m_{18})^2+(m_4+m_{18})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{152}=\frac{-G_rL_{83}}{(m_4+m_9)^2+(m_4+m_9)-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{153}=\frac{-G_rL_{84}}{(m_2+m_{10})^2+(m_4+m_{10})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{154}=\frac{-G_rL_{85}}{(m_4+m_{11})^2+(m_4+m_{11})-\left(\frac{1}{K_p}-\omega-m\right)}$$

$$,L_{155}=\frac{-G_rL_{86}}{(m_4+m_{12})^2+(m_4+m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{156}=\frac{-G_rL_{87}}{(r_2+m_{17})^2+(r_2-m_{17})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{157}=\frac{-G_rL_{88}}{(r_2+m_{18})^2+(r_2+m_{18})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{158}=\frac{-G_rL_{90}}{(r_2+m_9)^2-(r_2+m_9)-\left(\frac{1}{K_p}-\omega-m\right)}$$

$$,L_{159}=\frac{-G_rL_{91}}{(r_2+m_{10})^2+(r_2+m_{10})\left(\frac{1}{K_p}-\omega-m\right)},\ L_{160}=\frac{-G_rL_{92}}{(r_2+m_{11})^2+(r_2+m_{11})\left(\frac{1}{K_p}-\omega-m\right)}$$

$$,L_{161}=\frac{-G_rL_{93}}{(r_2+m_{12})^2+(r_2+m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{162}=\frac{-G_cA_9}{(m_{13})^2+(m_{13})-\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{163}=\frac{-G_cA_{10}}{(m_{14})^2+(m_{14})\left(\frac{1}{K_p}-\omega-m\right)},\ L_{164}=\frac{-G_cL_{12}}{(m_{11})^2+(m_{11})\left(\frac{1}{K_p}-\omega-m\right)},$$

$$L_{165}-\frac{-G_cL_{13}}{(m_{12})^2+(m_{12})-\left(\frac{1}{K_p}-\omega-m\right)},\ L_{166}-\frac{-G_cL_{14}}{(r_2)^2+(r_2)-\left(\frac{1}{K_p}-\omega-m\right)},\ A_1-\frac{P_2P_6-P_3P_5}{P_2P_4-P_1P_5},$$

$$A_2=\frac{P_3-A_1P_1}{P_2},\ A_3=\frac{-P_{25}-A_4m_8}{m_7},\ A_4=\frac{m_7P_{26}-P_{25}e^{m_7}}{m_8e^{m_7}-m_7e^{m_8}},\ A_5=\frac{-m_{10}P_{10}+P_9e^{m_{10}}}{m_{10}e^{m_9}-m_9e^{m_{10}}},$$

$$A_6=\frac{-P_9-A_5m_9}{m_{10}},\ A_7=\frac{-A_8m_{12}-L_{11}r_2}{m_{11}},\ A_8=\frac{L_{11}r_2e^{m_{11}}-L_{11}m_{11}e^{r_2}}{m_{11}e^{m_{12}}-m_{12}e^{m_{11}}},$$

$$A_9=\frac{m_{14}P_{12}-P_{11}e^{m_{14}}}{m_{13}e^{m_{14}}-m_{14}e^{m_{13}}},\ A_{10}=\frac{-P_{11}-A_9m_{13}}{m_{14}},\ A_{11}=0,\ A_{12}=0,$$

$$A_{15}=\frac{P_{20}P_{24}-P_{21}P_{23}}{P_{20}P_{22}-P_{19}P_{23}},\ A_{16}=\frac{P_{21}-A_{15}P_{19}}{P_{20}},\ A_{17}=\frac{-P_{27}-A_{18}m_{20}}{m_{19}},\ A_{18}=\frac{m_{19}P_{28}-P_{27}e^{m_{19}}}{m_{20}e^{m_{19}}-m_{19}e^{m_{20}}},\ A_{19}=$$

$$\frac{P_{31}-A_{20}P_{30}}{P_{29}},\ A_{20}=\frac{P_{29}P_{34}-P_{31}P_{32}}{P_{29}P_{33}-P_{30}P_{32}},$$

$$A_{21}-\frac{P_{37}P_{39}-P_{36}P_{40}}{P_{35}P_{39}-P_{36}P_{38}},A_{22}-\frac{P_{40}-A_{21}P_{38}}{P_{39}},\ P_1-1-h_1m_5,\ P_2-1-h_1m_6,$$

$P_3=e^{m_5}(1-h_2m_5)$, $P_4=e^{m_6}(1-h_2m_6)$, $P_5=\alpha-L_1(1-h_1m_3)-L_2(1-h_1m_4)-L_3(1+h_1Sc)-K_1$, $P_6=\alpha_2-L_1(1-h_2m_3)e^{m_3}-L_2e^{m_4}(1-h_2m_4)-L_3e^{-Sc}(1+h_2Sc)-K_1$, $P_9=-(L_9m_3+L_{10}m_4)$,$P_{10}=L_9e^{m_3}+L_{10}e^{m_4}$,$P_{11}=L_{12}m_{11}+L_{13}m_{12}+L_{14}r_2$,$P_{10}=L_{12}e^{m_{11}}+L_{13}e^{m_{12}}+L_{14}e^{r_2}$,$P_{19}=1-h_1m_{17}$,$P_{20}=1-h_2m_{18}$,

$P_{21}=L_{22}(h_1m_5-1)+L_{23}(h_1m_6-1)+L_{24}(h_1m_3-1)+L_{25}(h_1m_4-1)+L_{26}(h_1r_2-1)+L_{27}(h_1m_9-1)+L_{28}(h_1m_{10}-1)+L_{29}(h_1m_{11}-1)+L_{30}(h_1m_{12}-1)$,$P_{22}=e^{m_{17}}(1-h_2m_{17})$,$P_{23}=e^{m_{18}}(1-h_2m_{18})$,

$$P_{24}=L_{22}e^{m_5}(h_2m_5-1)+L_{23}e^{m_6}(h_2m_6-1)+L_{24}e^{m_3}(h_2m_3-1)+L_{25}e^{m_4}(h_2m_4-1)+L_{26}e^{r_2}(h_2r_2-1)+L_{27}e^{m_9}(h_2m_9-1)+L_{28}e^{m_{10}}(h_2m_{10}-1)+L_{29}e^{m_{11}}(h_2m_{11}-1)+L_{30}e^{m_{12}}(h_2m_{12}-1),$$

$P_{25}=2L_{31}m_5+L_{32}(m_6+m_5)+L_{33}(m_3+m_5)+L_{34}(m_4+m_5)+L_{35}(m_5+r_2)+2L_{36}m_6+L_{37}(m_3+m_6)+2L_{38}m_3+L_{39}(m_3+m_4)+L_{40}(m_3+r_2)+2L_{41}m_4+L_{42}(m_4+r_2)+L_{43}(m_6+m_4)+L_{44}(m_6+r_2)+2L_{45}r_2$,

$P_{26}=L_{31}e^{2m_5}+L_{32}e^{(m_6+m_5)}+L_{33}e^{(m_3+m_5)}+L_{34}e^{(m_4+m_5)}+L_{35}e^{(r_2+m_5)}+L_{36}e^{2m_6}+L_{37}e^{(m_6+m_3)}+L_{38}e^{2m_3}+L_{39}e^{(m_4-m_3)}+L_{40}e^{(m_3+r_2)}+L_{41}e^{2m_4}+L_{42}e^{(m_4+r_2)}+L_{43}e^{(m_6+m_4)}+L_{44}e^{(m_6-r_2)}+L_{45}e^{2r_2}$

,

$P_{27}=L_{46}m_7+L_{47}m_8+L_{48}m_5+L_{49}(m_5+m_6)+L_{50}(m_5+m_3)+L_{51}(m_5+m_4)+L_{52}(m_5+r_2)+L_{53}m_6+L_{54}(m_3+m_6)+L_{55}m_3+L_{56}(m_4+m_3)+L_{57}(m_3+r_2)+L_{58}m_4+L_{59}(m_4+r_2)+L_{60}(m_4+m_6)+L_{61}(r_2+m_6)+L_{62}2r_2+L_{63}(m_5+m_{17})+L_{64}(m_5+m_{18})+L_{65}(m_5+m_9)+L_{66}(m_5+m_{10})+L_{67}(m_5+m_{11})+L_{68}(m_5+m_{12})+$

$L_{69}(m_6+m_{17})+L_{70}(m_6+m_{18})+L_{71}(m_6+m_9)+L_{72}(m_{10}+m_6)+L_{73}(m_{11}+m_6)+L_{74}(m_{12}+m_6)+L_{75}(m_3+m_{17})+L_{76}(m_3+m_{18})+L_{77}(m_3+m_9)+L_{78}(m_3+m_{10})+L_{79}(m_3+m_{11})+L_{80}(m_3+m_{12})+L_{81}(m_4+m_{17})+L_{82}(m_4+m_{18})+L_{83}(m_4+m_9)+L_{84}(m_4+m_{10})+L_{85}(m_4+m_{11})+L_{86}(m_4+m_{12})+L_{87}(r_2+m_{17})+L_{88}(r_2+m_{18})+L_{89}(r_2+m_{17})+L_{90}(r_2+m_9)+L_{91}(r_2+m_{10})+L_{92}(r_2+m_{11})+L_{93}(r_2+m_{12})$,

$P_{28} = L_{46}\, e^{m_7} + L_{47}\, e^{m_8} + L_{48}\, e^{2m_3} + L_{49}\, e^{(m_6+m_5)} + L_{50} e^{(m_3+m_5)} + L_{51} e^{(m_4+m_5)} + L_{52} e^{(r_2+m_3)} + L_{53} e^{2m_6} + L_{54} e^{(m_6+m_3)} + L_{55} e^{2m_5} + L_{56} e^{(m_4-m_5)} + L_{57} e^{(m_5+r_2)} + L_{58} e^{2m_4} + L_{59} e^{(m_4+r_2)} + L_{60} e^{(m_6+m_4)} + L_{61} e^{(m_6+r_2)} + L_{62} e^{2m_2} + L_{63} e^{(m_{17}+m_5)} + L_{64} e^{(m_{18}+m_5)} + L_{65} e^{(m_6+m_5)} + L_{66} e^{(m_{10}+m_5)} + L_{67} e^{(m_{11}+m_5)} + L_{68} e^{(m_{12}+m_5)} + L_{69} e^{(m_6-m_{17})} + L_{70} e^{(m_6-m_{18})} + L_{71} e^{(m_6+m_9)} + L_{72} e^{(m_6+m_{10})} + L_{73} e^{(m_6+m_{11})} + L_{74} e^{(m_6+m_{12})} + L_{75} e^{(m_6-m_{17})} + L_{76} e^{(m_3+m_{18})} + L_{77} e^{(m_3+m_9)} + L_{78} e^{(m_3+m_{10})} + L_{79} e^{(m_3+m_{11})} + L_{80} e^{(m_3+m_{12})} + L_{81} e^{(m_4+m_{17})} + L_{82} e^{(m_4+m_{18})} + L_{83} e^{(m_4+m_9)} + L_{84} e^{(m_4+m_{10})} + L_{85} e^{(m_4+m_{11})} + L_{86} e^{(m_4+m_{12})} + L_{87} e^{(r_2+m_{17})} + L_{88} e^{(r_2+m_{18})} + L_{89} e^{(r_2+m_9)} + L_{90} e^{(r_2+m_{10})} + L_{91} e^{(r_2+m_{11})} + L_{92} e^{(r_2-m_{17})} + L_{93} e^{(r_2+m_{12})}$,

$P_{29} = 1 - h_1 m_{21}$, $P_{30} = 1 - h_1 m_{22}$, $P_{31} = L_{94}(h_1 m_7 - 1) + L_{95}(h_1 m_8 - 1) + L_{96}(2h_1 m_5 - 1) + L_{97}\{h_1(m_6 - m_5) - 1\} + L_{98}\{h_1(m_3 + m_5) - 1\} + L_{99}\{h_1(m_4 + m_5) - 1\} + L_{100}\{h_1(r_2 + m_5) - 1\} + L_{101}\{h_1(2m_6) - 1\} + L_{102}\{h_1(m_6 + m_3) - 1\} + L_{103}\{2h_1 m_3 - 1\} + L_{104}\{h_1(m_3 + m_4) - 1\} + L_{105}\{h_1(m_3 + r_2) - 1\} + L_{106}\{2h_1 m_4 - 1\} + L_{107}\{h_1(m_4 + r_2) - 1\} +$
$L_{108}\{h_1(m_6 + m_4) - 1\} + L_{109}\{h_1(m_6 + r_2) - 1\} + L_{110}(2h_1 r_2 - 1)$,
$P_{32} = (1 - h_2 m_{21})\, e^{m_{21}}$, $P_{33} = (1 - h_2 m_{22})\, e^{m_{22}}$, $P_{34} = L_{94} e^{m_7}(h_2 m_7 - 1) + L_{95} e^{m_8}(h_2 m_8 - 1) + L_{96} e^{2m_5}(2h_2 m_5 - 1) + L_{97} e^{(m_6-m_5)}\{h_2(m_6 + m_5) - 1\} + L_{98} e^{(m_3-m_5)}\{h_2(m_3 + m_5) - 1\} + L_{99} e^{(m_4+m_5)}\{h_2(m_4 + m_5) - 1\} + L_{100} e^{(r_2-m_5)}\{h_2(r_2 + m_5) - 1\} + L_{101} e^{2m_6}\{h_2(2m_6) - 1\} + L_{102} e^{(m_6+m_3)}\{h_2(m_6 + m_3) - 1\} + L_{103} e^{2m_3}\{2h_2 m_3 - 1\} + L_{104} e^{(m_3-m_4)}\{h_2(m_3 + m_4) - 1\} + L_{105} e^{(m_3+r_2)}\{h_2(m_3 + r_2) - 1\} + L_{106} e^{2m_4}\{2h_2 m_4 - 1\} + L_{107} e^{(m_4+r_2)}\{h_2(m_4 + r_2) - 1\} + L_{108} e^{(m_6+m_4)}\{h_2(m_6 + m_4) - 1\} + L_{109} e^{(m_6-r_2)}\{h_2(m_6 + r_2) - 1\} + L_{110} e^{2r_2}(2h_2 r_2 - 1)$,

$P_{35} = 1 - h_1 m_{23}$, $P_{36} = 1 - h_1 m_{24}$, $P_{37} = L_{111}(h_1 m_{21} - 1) + L_{112}(h_1 m_{22} - 1) + L_{113}(h_1 m_7 - 1) + L_{114}(h_1 m_8 - 1) + L_{115}(2h_1 m_5 - 1) + L_{116}\{h_1(m_6 + m_5) - 1\} + L_{117}\{h_1(m_3 + m_5) - 1\} + L_{118}\{h_1(m_4 + m_5) - 1\} + L_{119}\{h_1(m_5 + r_2) - 1\} + L_{120}(2h_1 m_6 - 1) + L_{121}\{h_1(m_6 + m_3) - 1\} + L_{122}(2h_1 m_3 - 1) + L_{123}\{h_1(m_4 + m_3) - 1\} + L_{124}\{h_1(r_2 + m_3) - 1\} + L_{125}(2h_1 m_4 - 1) + L_{126}\{h_1(r_2 + m_4) - 1\} + L_{127}\{h_1(m_6 + m_4) - 1\} + L_{128}\{h_1(r_2 + m_6) - 1\} + L_{129}(2h_1 r_2 - 1) +$
$L_{130}(h_1 m_{19} - 1) + L_{131}(h_1 m_{20} - 1) + L_{132}\{h_1(m_5 + m_{17}) - 1\} + L_{133}\{h_1(m_{18} + m_5) - 1\} + L_{134}\{h_1(m_5 + m_6) - 1\} + L_{135}\{h_1(m_{10} + m_5) - 1\} + L_{136}\{h_1(m_{11} + m_5) - 1\} + L_{137}\{h_1(m_6 + m_{12}) - 1\} + L_{138}\{h_1(m_6 + m_{17}) - 1\} +$
$L_{139}\{h_1(m_{18} + m_6) - 1\} + L_{140}\{h_1(m_6 + m_9) - 1\} + L_{141}\{h_1(m_{10} + m_6) - 1\} + L_{142}\{h_1(m_6 + m_{11}) - 1\} + L_{143}\{h_1(m_6 + m_{12}) - 1\} + L_{144}\{h_1(m_3 + m_{17}) - 1\} + L_{145}\{h_1(m_{18} + m_3) - 1\} + L_{146}\{h_1(m_3 + m_9) - 1\} + L_{147}\{h_1(m_3 + m_{10}) - 1\} + L_{148}\{h_1(m_{11} + m_3) - 1\} + L_{149}\{h_1(m_3 + m_{12}) - 1\} + L_{150}\{h_1(m_4 + m_{17}) - 1\} + L_{151}\{h_1(m_4 + m_{18}) - 1\} + L_{152}\{h_1(m_4 + m_9) - 1\} + L_{153}\{h_1(m_4 + m_{10}) - 1\} + L_{154}\{h_1(m_4 + m_{11}) - 1\} + L_{155}\{h_1(m_4 + m_{12}) - 1\} + L_{156}\{h_1(r_2 + m_{17}) - 1\} + L_{157}\{h_1(r_2 + m_{18}) - 1\} + L_{158}\{h_1(r_2 + m_9) - 1\} + L_{159}\{h_1(r_2 + m_{10}) - 1\} + L_{160}\{h_1(r_2 + m_{11}) - 1\} + L_{161}\{h_1(r_2 + m_{12}) - 1\} + L_{162}(h_1 m_{13} - 1) + L_{163}(h_1 m_{14} - 1) + L_{164}(h_1 m_{11} - 1\} + L_{165}(h_1 m_{12} - 1) + L_{166}\{h_1 r_2 - 1\}$,

$P_{38} = (1 - h_2 m_{23}) e^{m_{23}}$, $P_{39} = (1 - h_2 m_{24}) e^{m_{24}}$, $P_{40} = L_{111}(h_2 m_{21} - 1) e^{m_{21}} + L_{112}(h_2 m_{22} - 1)\, e^{m_{22}} + L_{113}(h_2 m_7 - 1)\, e^{m_7} + L_{114}(h_2 m_8 - 1)\, e^{m_8} + L_{115}(2h_2 m_5 - 1) e^{2m_5} + L_{116}\{h_2(m_6 + m_5) - 1\}\, e^{(m_6+m_5)} + L_{117}\{h_2(m_3 + m_5) - 1\} e^{(m_3+m_5)} + L_{118}\{h_2(m_4 + m_5) - 1\}\, e^{(m_4+m_5)} + L_{119}\{h_2(m_5 + r_2) - 1\}\, e^{(r_2+m_5)} + L_{120}(2h_2 m_6 - 1)\, e^{2m_6} + L_{121}\{h_2(m_6 + m_3) - 1\} e^{(m_6+m_3)} + L_{122}(2h_2 m_3 - 1) e^{2m_3} + L_{123}\{h_2(m_4 + m_3) - 1\}\, e^{(m_3+m_4)} + L_{124}\{h_2(r_2 + m_3) - 1\}\, e^{(m_3+r_2)} + L_{125}(2h_2 m_4 - 1)\, e^{2m_4} + L_{126}\{h_2(r_2 + m_4) - 1\} e^{(r_2-m_4)} + L_{127}\{h_2(m_6 + m_4) - 1\}\, e^{(m_6+m_4)} + L_{128}\{h_2(r_2 + m_6) - 1\} e^{(r_2+m_6)} + L_{129}(2h_2 r_2 - 1)\, e^{2r_2} + L_{130}(h_2 m_{19} - 1)\, e^{m_{19}} + L_{131}(h_2 m_{20} - 1)\, e^{m_{20}} + L_{132}\{h_2(m_5 + m_{17}) - 1\}\, e^{(m_5+m_{17})} + L_{133}\{h_2(m_{18} + m_5) - 1\}\, e^{(m_{18}+m_5)} + L_{134}\{h_2(m_5 + m_6) - 1\}\, e^{(m_6+m_5)} + L_{135}\{h_2(m_{10} + m_5) - 1\}\, e^{(m_{10}+m_5)} + L_{136}\{h_2(m_{11} + m_5) - 1\}\, e^{(m_{11}-m_5)} + L_{137}\{h_2(m_6 + m_{12}) - 1\}\, e^{(m_6-m_{12})} + L_{138}\{h_2(m_6 + m_{17}) - 1\}\, e^{(m_6+m_{17})} +$
$L_{139}\{h_2(m_{18} + m_6) - 1\} e^{(m_6+m_{18})} + L_{140}\{h_2(m_6 + m_9) - 1\}\, e^{(m_6+m_9)} + L_{141}\{h_2(m_{10} + m_6) - 1\} e^{(m_6+m_{10})} + L_{142}\{h_2(m_6 + m_{11}) - 1\}\, e^{(m_6+m_{11})} + L_{143}\{h_2(m_6 + m_{12}) - 1\}\, e^{(m_6+m_{12})} + L_{144}\{h_2(m_3 + m_{17}) - 1\}\, e^{(m_3+m_{17})} + L_{145}\{h_2(m_{18} + m_3) - 1\}\, e^{(m_3+m_{18})} + L_{146}\{h_2(m_3 + m_9) - 1\}\, e^{(m_3+m_9)} + L_{147}\{h_2(m_3 + m_{10}) - 1\}\, e^{(m_3+m_{10})} + L_{148}\{h_2(m_{11} + m_3) - 1\}\, e^{(m_3+m_{11})} + L_{149}\{h_2(m_3 + m_{12}) - 1\}\, e^{(m_3+m_{12})} + L_{150}\{h_2(m_4 + m_{17}) - 1\}\, e^{(m_4+m_{17})} +$
$L_{151}\{h_2(m_4 + m_{18}) - 1\}\, e^{(m_4-m_{18})} + L_{152}\{h_2(m_4 + m_9) - 1\}\, e^{(m_4+m_9)} + L_{153}\{h_2(m_4 + m_{10}) - 1\}\, e^{(m_4+m_{10})} + L_{154}\{h_2(m_4 + m_{11}) - 1\}\, e^{(m_4-m_{11})} + L_{155}\{h_2(m_4 + m_{12}) - 1\}\, e^{(m_4+m_{12})} + L_{156}\{h_2(r_2 + m_{17}) - 1\}\, e^{(r_2+m_{17})} + L_{157}\{h_2(r_2 + m_{18}) - 1\}\, e^{(r_2+m_{18})} + L_{158}\{h_2(r_2 + m_9) - 1\} e^{(r_2+m_9)} + L_{159}\{h_2(r_2 + m_{10}) - 1\}\, e^{(r_2+m_{10})} +$
$L_{160}\{h_2(r_2 + m_{11}) - 1\}\, e^{(r_2+m_{11})} + L_{161}\{h_2(r_2 + m_{12}) - 1\}\, e^{(r_2+m_{12})} + L_{162}(h_2 m_{13} - 1)\, e^{m_{13}} + L_{163}(h_2 m_{14} - 1)\, e^{m_{14}} + L_{164}(h_2 m_{11} - 1)\, e^{m_{11}} + L_{165}(h_2 m_{12} - 1) e^{m_{12}} + L_{166}\{h_2 r_2 - 1\}\, e^{r_2}$

$n_1 = -P_r A_1^2 m_3^2$, $n_2 = -2P_r A_1 A_2 m_5 m_6$, $n_3 = -2P_r A_1 L_2 m_5 m_3$, $n_4 = -2P_r A_1 L_2 m_5 m_4$,
$n_5 = -2P_r A_1 L_3 m_5 r_2$, $n_6 = -P_r A_2^2 m_6^2$, $n_7 = -2P_r A_2 L_2 m_6 m_3$, $n_8 = -P_r L_1^2 m_3^2$, $n_9 = -2P_r L_1 L_2 m_3 m_4$,
$n_{10} = -2P_r L_1 L_3 m_3 r_2$, $n_{11} = -P_r L_2^2 m_4^2$, $n_{12} = -2P_r L_3 L_3 m_4 r_2$, $n_{13} = -2P_r A_2 L_2 m_6 m_4$,
$n_{14} = -2P_r A_2 L_3 m_6 r_2$, $n_{15} = -P_r L_3^2 r_2^2$, $t_1 = -AP_r A_3 m_7$, $t_2 = -AP_r A_1 m_8$,
$t_3 = -P_r(2AL_{31}m_5 + 2m_5 A_1 m_5 L_{22})$, $t_4 = -P_r(A(m_6 + m_5)L_{32} + 2m_5 A_1 m_6 L_{23} + 2m_6 A_2 m_5 L_{22})$,
$t_5 = -P_r(A(m_3 + m_5)L_{33} + 2m_5 A_1 + 2m_3 L_1 m_5 L_{22})$,
$t_6 = -P_r(A(m_4 + m_5)L_{34} + 2m_5 A_1 m_4 L_{25} + 2m_6 A_2 m_5 L_{25} + 2m_4 L_2 m_5 L_{22})$, $t_7 = -P_r(A(r_2 + m_5)L_{35} + 2m_5 A_1 r_2 L_{26} + 2m_6 A_2 r_2 L_{26} + 2r_2 L_3 m_5 L_{22})$,
$t_8 = P_r(2Am_6 L_{36} + 2m_6^2 A_2 L_{23})$, $t_9 = P_r(A(m_3 + m_6)L_{37} + 2m_6 A_2 m_3 L_{24} + 2m_3 L_3 m_6 L_{23})$,
$t_{10} = P_r(2Am_3 L_{38} + 2m_3^2 L_1 L_{24})$, $t_{11} = P_r(A(m_3 + m_4)L_{39} + 2m_3 L_1 m_4 L_{25} + 2m_4 L_2 m_3 L_{24})$,
$t_{12} = P_r(A(m_3 + r_2)L_{10} + 2r_2 L_3 m_3 L_{24} + 2m_3 L_1 r_2 L_{26})$, $t_{13} = P_r(2Am_4^2 L_{41} + 2m_4^2 L_2 L_{25})$,
$t_{14} = -P_r(A(r_2 + m_4)L_{42} + 2r_2 L_3 m_4 L_{25} + 2m_4 L_2 r_2 L_{26})$,
$t_{15} = -P_r(A(m_4 + m_6)L_{43} + 2m_4 L_2 m_6 L_{23})$, $t_{16} = -P_r(A(r_2 + m_6)L_{44}$,
$t_{17} = -P_r(2Ar_2 L_{45} + 2r_2^2 L_3 L_{26})$, $t_{18} = -P_r(2m_5 A_1 m_1 A_{15})$, $t_{19} = -P_r(2m_5 A_1 m_{18} A_{16})$,
$t_{20} = -P_r(2m_5 A_1 m_9 L_{27})$, $t_{21} = -P_r(2m_5 A_1 m_{10} L_{28})$, $t_{22} = -P_r(2m_5 A_1 m_{11} L_{29})$,
$t_{23} = -P_r(2m_5 A_1 m_{12} L_{30})$, $t_{24} = -P_r(2m_5 A_2 m_{17} A_{15})$, $t_{25} = -P_r(2m_5 A_2 m_{18} A_{16})$,
$t_{26} = -P_r(2m_5 A_2 m_9 L_{27})$, $t_{27} = -P_r(2m_5 A_2 m_{10} L_{28})$, $t_{28} = -P_r(2m_5 A_2 m_{11} L_{29})$,
$t_{29} = -P_r(2m_5 A_2 m_{12} L_{30})$, $t_{30} = -P_r(2m_3 L_1 m_{17} A_{15})$, $t_{31} = -P_r(2m_3 L_1 m_{18} A_{16})$,
$t_{32} = -P_r(2m_3 L_1 m_9 L_{27})$, $t_{33} = -P_r(2m_3 L_1 m_{10} L_{28})$, $t_{34} = -P_r(2m_3 L_1 m_{11} L_{29})$,
$t_{35} = -P_r(2m_3 L_1 m_{12} L_{30})$, $t_{36} = -P_r(2m_4 L_2 m_{17} A_{15})$, $t_{37} = -P_r(2m_4 L_2 m_{18} A_{16})$,
$t_{38} = P_r(2m_4 L_2 m_9 L_{27})$, $t_{39} = P_r(2m_4 L_2 m_{10} L_{28})$, $t_{40} = P_r(2m_4 L_2 m_{11} L_{29})$,
$t_{41} = -P_r(2m_4 L_2 m_{12} L_{30})$, $t_{42} = -P_r(2r_2 L_3 m_{17} A_{15})$, $t_{43} = -P_r(2r_2 L_3 m_{18} A_{16})$,
$t_{44} = -P_r(2r_2 L_3 m_6 A_{23})$, $t_{45} = -P_r(2r_2 L_3 m_9 L_{27})$, $t_{46} = -P_r(2r_2 L_3 m_{10} L_{28})$,
$t_{47} = -P_r(2r_2 L_3 m_{11} A_{29})$, $t_{48} = -P_r(2r_2 L_{30} m_{12} L_3)$,

4.0. Results and Discussion

In this work,the effects of variable suction and mass transfer on unsteady free convective couette dissipating fluid flow are considered. The governing equations are solved for velocity, temperature, and concentration distribution and the various parameters on the flow have been considered and the results are presented graphically and discussed quantitatively. The velocity of the flow field varies to a great extent with the variation of the flow parameters.The main factors affecting the velocity of the flow field are suction parametersα_1 , α_2 ,Grashof number for heat and mass transferG_r , G_c; slip flow parameters h_1 , h_2 , radiation parameter F , permeability parameter K_p and Schmidt number S_c. The effects of these parameters on the velocity of the flow field are analyzed in Figures 1 to 15 with default fixed parameters as F=3 ,$G_r = 2$, $G_c = 2$, $P_r = 0.71$, $\omega = 0.1$, $t = 0.1$, $Ec = 0.02 S_c = 1.002$, $K_p = 1$, $A = 0.5$, $B = 0.7$, $h_1 = 0.1$,h_2 =0.5 , $\alpha_1 = 0.1$, $\alpha_2 = 0.7$. Figures 8 and 9 discuss the effect of suction parameters α_1 *and* α_2 ,respectively on the flow field. Analyzing the curves of both figures, it is observed that both parameters have accelerating effect on the velocity of the flow field at all points. Figures 1 and 2 respectively show the effect of Grashof number for heat transfer $G_r$and mass transfer G_c on the flow field. The effect of growing Grashof number for heat transfer and mass transfer is to accelerate the magnitude of velocity of the flow field at all points. The effects of slip flow parametersh_1 *and* h_2; on the velocity of the flow field are shown in Figures 3 and 6 respectively. Comparing the curves of both figures, it is observed that the parameter $h_1$enhance the velocity of the flow field in the range $0 \leq y < 0.5$ it converges at $y = 0.5$, and then diverges. The parameter h_2 tends to decreases the velocity profiles.

Figure 5 shows the effect of radiation parameter F on the flow field. A growing radiation parameter is found to accelerate the velocity of the flow field at all points. Figure 7, present the effect of permeability parameter K_p on the velocity field.The effect of increasing permeability parameter is to accelerate the velocity of the flow field at all points. Figure 4 shows the effect of Schmidt number S_c

on the velocity field. The effect of growing S_c decreases the velocity in the range $0 \leq y < 0.43$ It converges at point ,$y = 0.43$then diverges to have reverse effect on the velocity of the flow field. It is observed from figure 11 that as Ec increases the velocity increases also but it diverges at point $y = 0.5$ with a reverse effect on velocity field.In figure 12, Ec enhances the temperature field. Figure 10 shows the effect of magnetic parameter M on the velocity, as magnetic M increase the velocity decreases. Influence of schmidt parameter$S_c$on the concentration of the flow field is shown in figure 15,asschmidtS_c increases the concentration boundary layer reduces. Figure 13 present the effect of radiation parameter F the on temperature field.The effect of increasing radiation parameter is to accelerate the temperature of the flow field at all points. Figure 14 shows the effect of prandtl parameter P_r on the temperature field. The effect of growing P_r is to decrease the temperature at all points of the flow field. Table 1 shows the variation of Nusselt number Nn and skin friction at the wall with the variation of each parameter while others are fixed as F=3 ,$G_r = 2$, $G_c = 2$, $P_r = 0.71$, $\omega = 0.1$, $t = 0.1$, $Ec = 0.02 S_c = 1.002$, $K_p = 1$, $A = 0.5$, $B = 0.7$, $h_1 = 0.1$,h_2 =0.5 , $\alpha_1 = 0.1$, $\alpha_2 = 0.7$. An increase in magnetic parameter M and prandtl number make both Nu and skin friction to decrease. Also, an increase in Grashof numberG_r, enhance Nu and skin friction.Increase in slip flow condition h_1, suction parameter α_1 , and eckert number Ec make Nu to increases and skin friction to decreases. While increase in radiation parameter F, Grashofnumber G_c,suction parameterα_2, schmidt number S_c and permeability parameterK_p,decreases the Nusselt number and make skin friction to increase.

Table 1.Variation of Nu and skin friction of the walls with the parameter

	Nu	Skin friction		Nu	Skin friction
M=0.1	-1.3601	1.9677	h_2=0.5	-1.3601	1.9677
M=2	-1.4218	1.5931	h_2=1	-1.4421	3.1780
M=3	-1.4102	1.6795	$h_2 = 2$	-1.3483	0.0214
Ec=0.1	-1.4504	6.5112	$\alpha_1 = 0.1$	-1.3601	1.9677
Ec=0.01	-1.3601	1.9677	$\alpha_1 = 0.5$	-1.3406	1.0300
Ec=0.001	-1.3590	1.5134	$\alpha_1 = 1$	-1.3282	0.1011
F=2	-0.6345	-23.6524	$\alpha_2 = 0.7$	-1.3601	1.9677
F=3	-1.3601	-1.9677	$\alpha_2 = 1.7$	-1.4836	4.1736
F=4	-1.4639	1.9372	$\alpha_2 = 3$	-1.8274	6.9980
$G_r = 2$	-1.3601	1.9677	$S_c = 1.002$	-1.3601	1.9677
G_r=3	-1.3192	2.4499	$S_c = 1.02$	-1.3613	1.9718
$G_r = 4$	-1.2852	3.2078	$S_c = 1.2$	-1.3808	2.1312
$G_c = 2$	-1.3601	1.9677	P_r	-1.3601	1.9677
$G_c = 3$	-1.3986	2.4271	P_r	-1.4948	1.5897
$G_c = 4$	-1.4431	2.9474	P_r	-1.7781	1.4366
$h_1 = 0.1$	-1.3601	1.9677	$K_p = 1$	-1.3601	1.9677
$h_1 = 0.2$	-1.3530	1.5544	$K_p = 2$	-1.4870	1.7561
$h_1 = 0.5$	-1.3431	0.7012	$K_p = 3$	-1.6439	2.8155

5.0. Conclusions

We present below some of the important features of the flow field due to the variation of the flow parameters.

1. Suction parameters α_1 *and* α_2 are observed to have different effect on the velocity of the flow field .
2. The Grashof number for heat transfer $G_r$and Grashof number for mass transfer $G_c$is to accelerate the magnitude of velocity of the flow field at all points.
3. The slip flow parameters h_1 *and* $h_2$enhance the velocity of the flow field at all points in a different manner. The parameter h_1 tends to converge the velocity profiles at a point and there afterhas reverse effect on it. While the parameter h_2 tends to retard the velocity profiles at all points.

4. A growing radiation parameter F is found to accelerate the velocity of the flow field at all points.
5. The effect of increasing permeability parameter K_p is to accelerate the velocity of the flow field at all points.
6. The Prandtl number P_r retard the magnitude of temperature of the flow field at all points. While radiation parameter (F), accelerate it.
7. A growing Schmidt number reduces the concentration boundary layer thickness of the flow field at all points.
8. Eckert number (Ec) accelerate the velocity of the flow and also increases the magnitude of temperature of the flow field at all points.
9. Magnetic parameter (M) reduces the velocity of the flow field.

References

Al-Odat, M. Q., Al-Ghamdi A. (2012), Numerical investigation of Dufour and Soret effects on unsteady MHD natural convection flow past vertical plate embedded in non-Darcy porous medium, Appl. Math. Mech., 33,195-210.

Chamkha, A. J. (2003), MHD flow of a uniformly stretched vertical permeable surface in the presence of heat generation /absorption and chemical reaction, Int. Commun. Heat Mass Trans., 30, 413-422.

Chamkha, A. J. (2004), Unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption, Int. J. Engg. Science, 42, 217-230.Crpeau, J. C.,

Clarksean, R. (1997), Similarity solution of natural convection with internal heat generation, ASME Journal of Heat Transfer, 119, 183-185.

Das S. S.,Tripathy, U. K. and Das, J. K. (2010), Hydromagnetic convective flow past a vertical porous plate through a porous medium with suction and heat source, Int. J. Ener. Environ., 1(3), 467-478.

Das, S. S. (2009), Effect of suction and injection on MHD three dimensional couette flow and heat transfer through a porous medium, J. Naval Architecture Marine Engng., 6(1), 41-51.

Das, S.S., Mohanty, M., Panigrahi, S. K.,Padhy, R.K. and Sahu, M.(2012), Radioactive heat and mass transfer effects on natural convection couette flow through a porous medium in the slip flow regime., International Journal of Renewable Energy Technology Research, 1(1), 01 - 14

Hossain, M. A., Begum, R. A.(1984), Effect of mass transfer and free convection on the flow past a vertical plate, ASME Journal of Heat Transfer, 106, 664-668.

Ibrahim, F. S., Elaiw, A. M. and Bakr, A. A. (Bakr) (2008), Effect of the chemical reaction and radiation absorption on the unsteady MHD free convection flow past a semi-infinite vertical permeable moving plate with heat source and suction, Commun. Non- linear Sci. Numer.Simul., 13,1056-1066.

Kim, Y. J., (2000), Unsteady MHD convective heat transfer past a semi-infinite vertical porous moving plate with variable suction, Int. J. Engg. Sci., 38, 833-845.

Makinde O. D., Mango J. M., Theuri D. M., (2003), Unsteady free convection flow with suction on an accelerating porous plate, AMSE J. Mod. Meas. Cont., 72 (3), 39-46.

Ostrach S., (1952), An analysis of laminar free convective flow and heat transfer about a flat plate parallel to direction of the generating body force, NASA Technical Report 1111.

Singh A. K., (1983), Finite difference analysis of MHD free convective past an accelerated vertical porous plate, Astrophys. Space Sci., 248, 395-400.

Singh P.and Gupta, C. B.(2005), MHD free convective flow of viscous fluid through a porous medium bounded by an oscillating porous plate in slip flow regime with mass transfer, Ind. J. Theo. Phys., 53(2), 111-120.

Soundalgekar V. M., (1971), Viscous dissipation effects on unsteady free convective flow past an infinite vertical porous plate with constant suction, Int. J. heat Mass Transfer, 15, 1253-1261.

Sreekanth, S., A. S. Nagrajan, S. VenkataRamana. (2001), Transient MHD free convention flow of an incompressible viscous dissipative fluid, Ind. J. Pure Appl. Math., 32(7), 1051-1058.

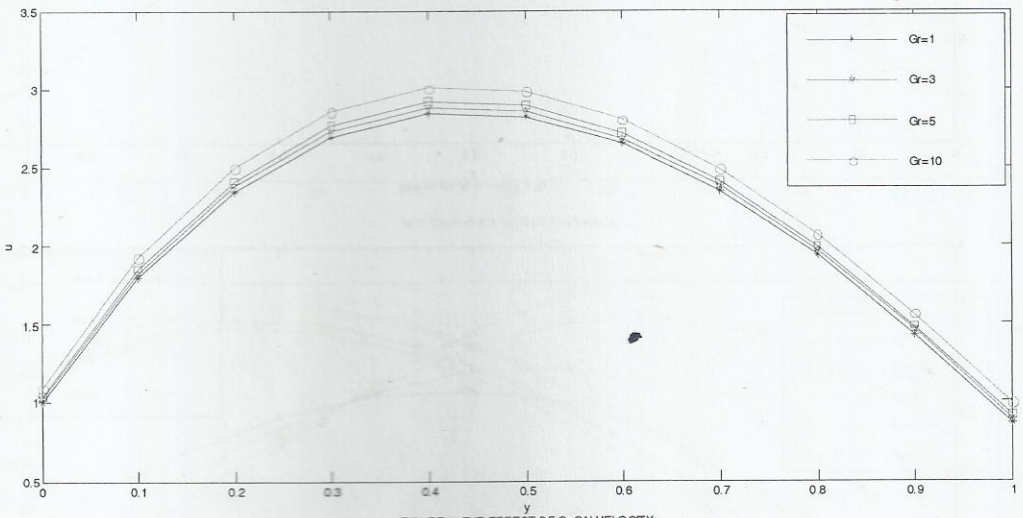


FIGURE 1: THE EFFECT OF Gr ON VELOCITY

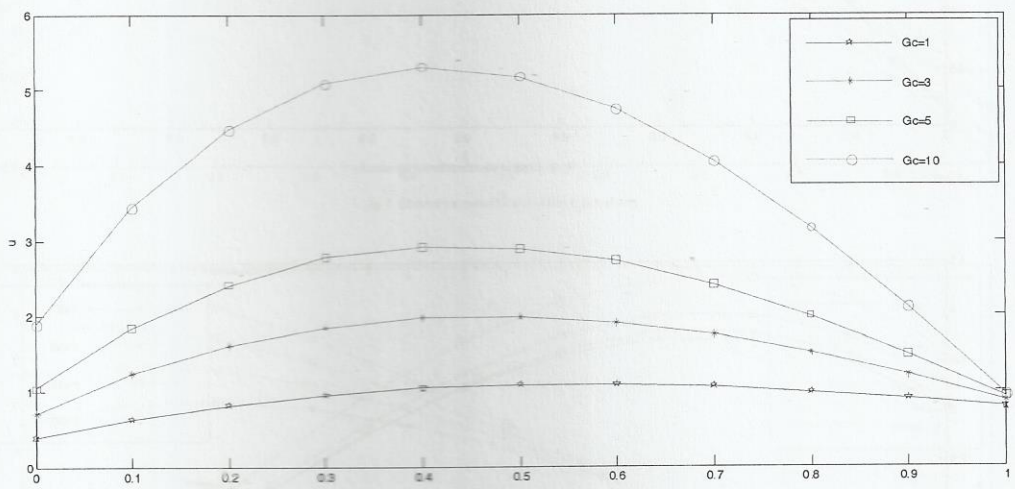


FIGURE 2: EFFECT OF Gc ON VELOCITY

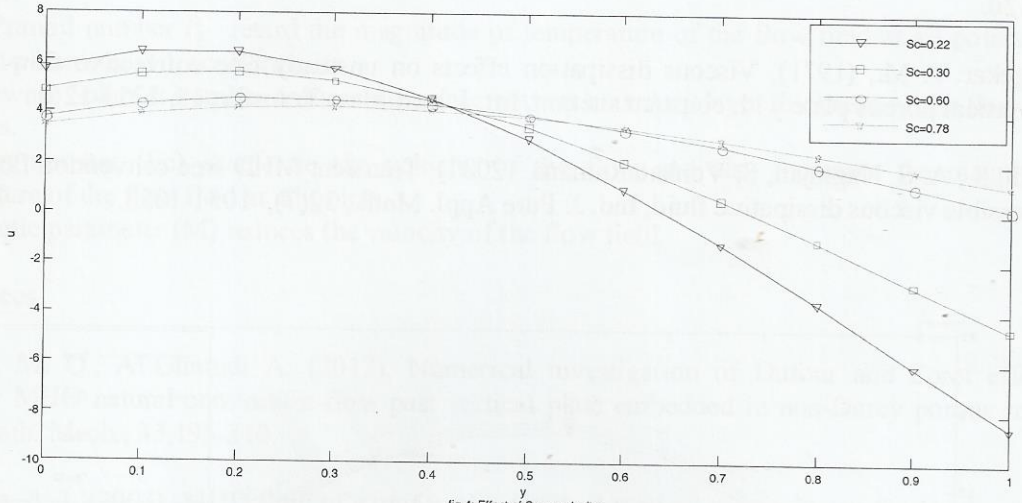


fig 4: Effect of Sc on velocity

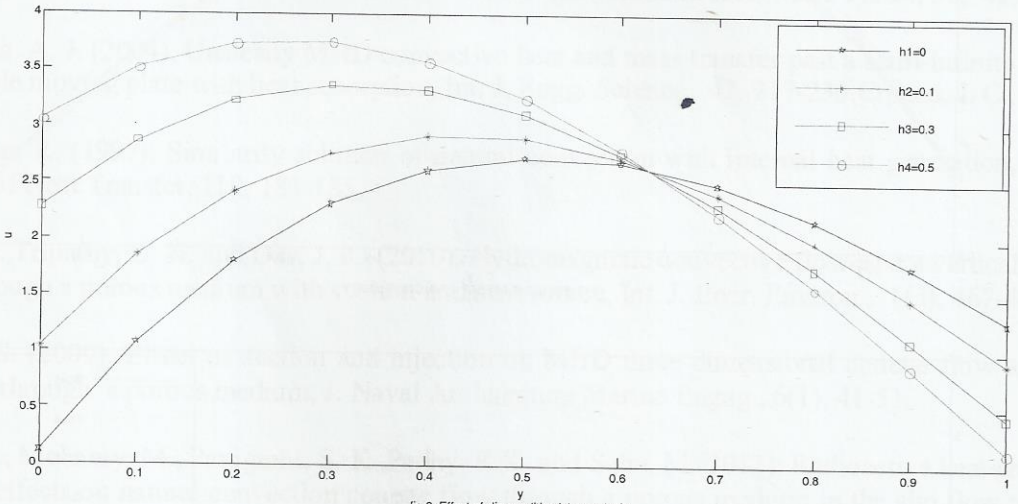


fig 3: Effect of slip condition h1 on velocity

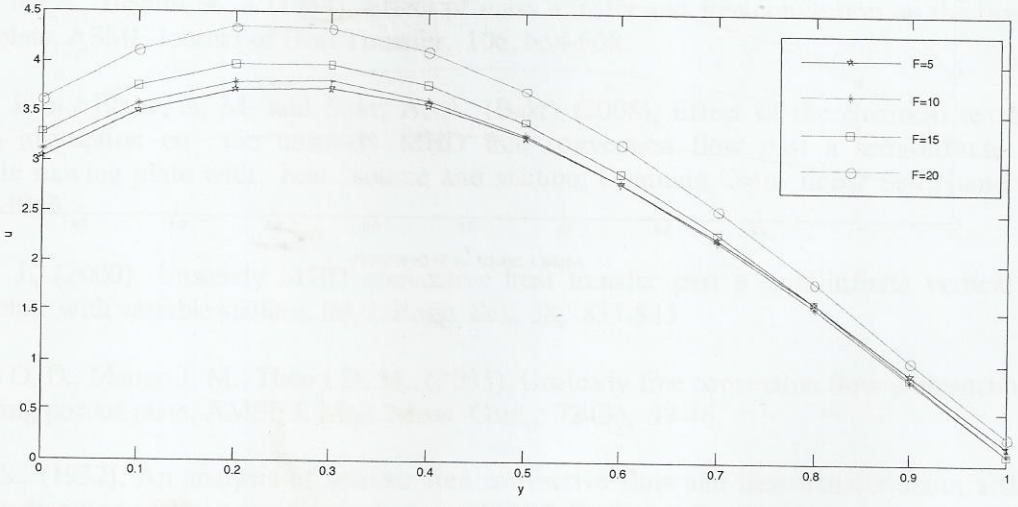


fig 5: Effect of radiation F on velocity

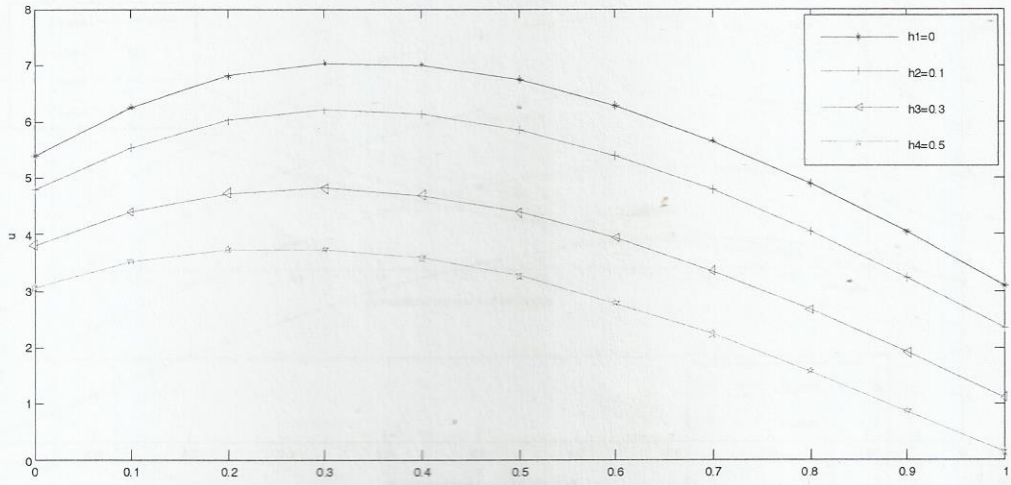


fig 6: Effect of slip condition h2 on velocity

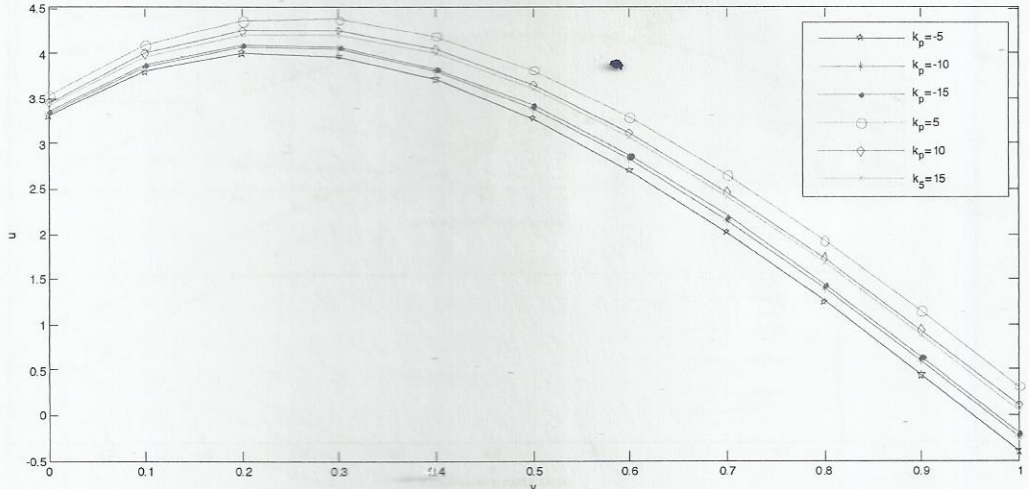


fig 7: Effect of permeability parameter k_p on velocity

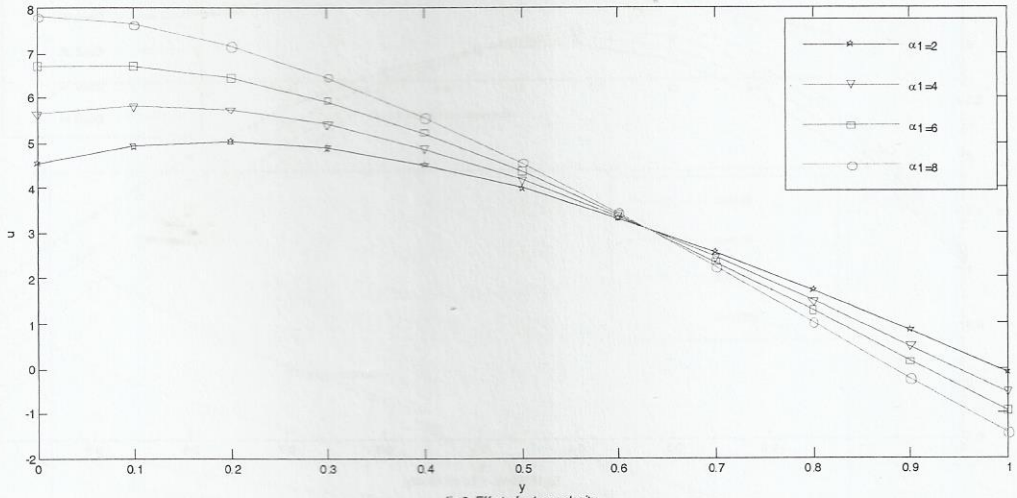


fig 8: Effect of α 1 on velocity

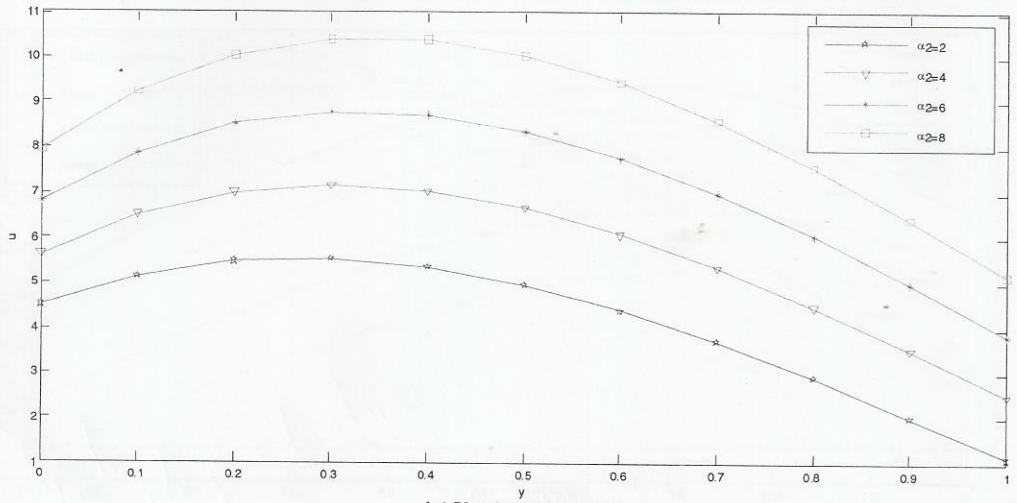


fig 9: Effect of suction α2 on velocity

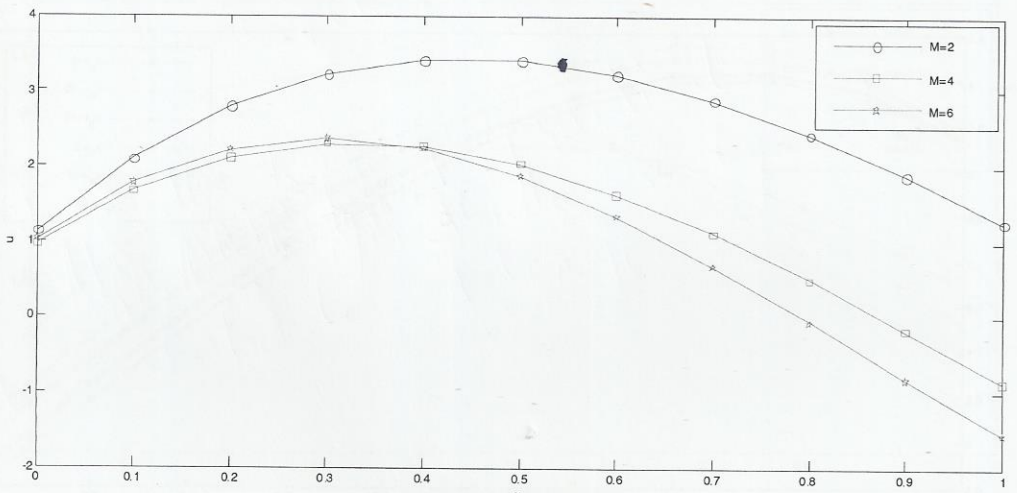


fig 10: Effect of M on velocity

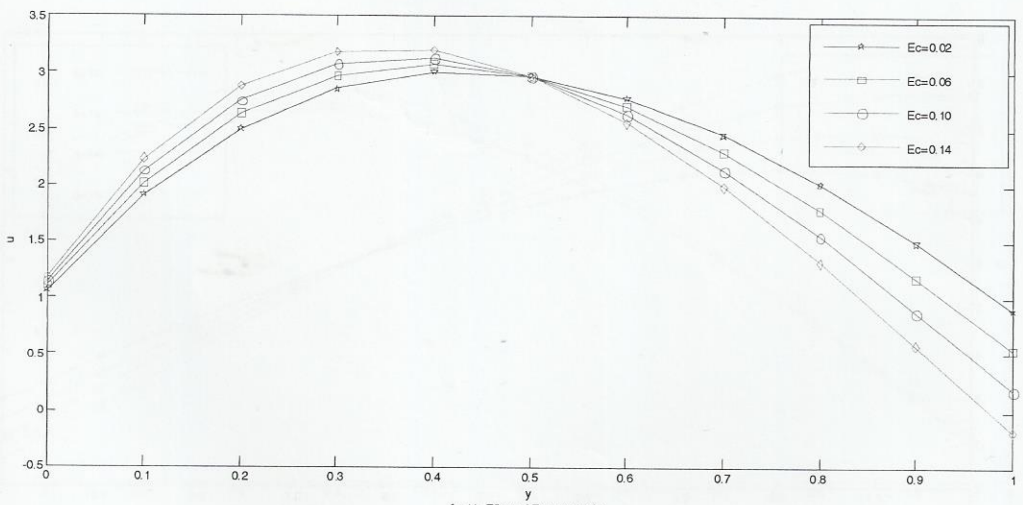


fig 11: Effect of Ec on velocity

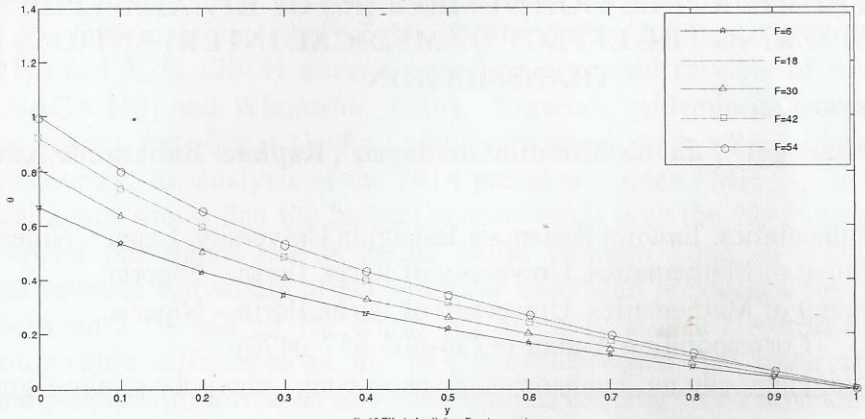


fig 12:Effect of radiation F on temperature

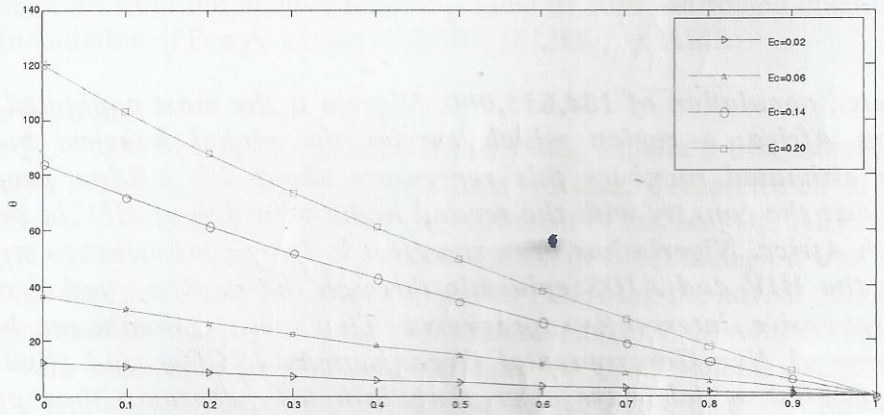


fig 13: Effect of Ec on temperature

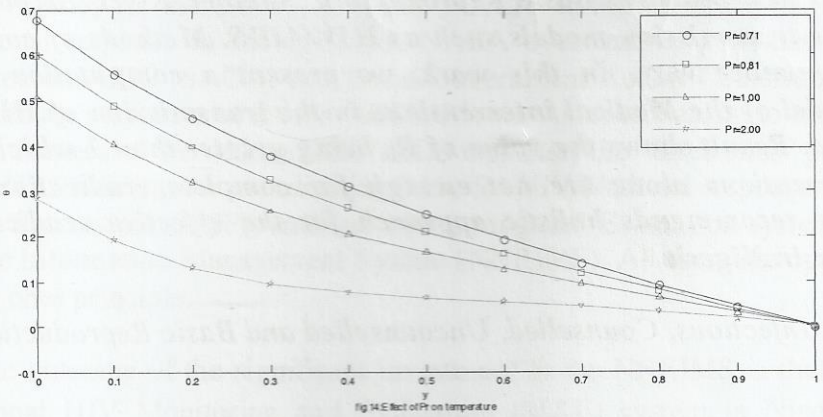


fig 14:Effect of Pr on temperature

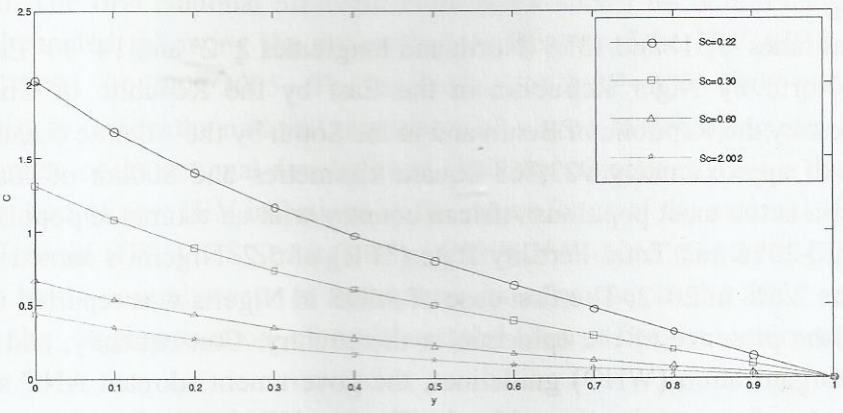


fig 15: Effect of Sc on concentration