



A Mathematical Model of the Tuberculosis Epidemic

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Received: 18 January 2020 / Accepted: 24 December 2020
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Abstract

Tuberculosis has continued to retain its title as “the captain among these men of death”. This is evident as it is the leading cause of death globally from a single infectious agent. TB as it is fondly called has become a major threat to the achievement of the sustainable development goals (SDG) and hence require inputs from different research disciplines. This work presents a mathematical model of tuberculosis. A compartmental model of seven classes was used in the model formulation comprising of the susceptible S , vaccinated V , exposed E , undiagnosed infectious I_1 , diagnosed infectious I_2 , treated T and recovered R . The stability analysis of the model was established as well as the condition for the model to undergo backward bifurcation. With the existence of backward bifurcation, keeping the basic reproduction number less than unity ($R_0 < 1$) is no more sufficient to keep TB out of the community. Hence, it is shown by the analysis that vaccination program, diagnosis and treatment helps to control the TB dynamics. In furtherance to that, it is shown that preference should be given to diagnosis over treatment as diagnosis precedes treatment. It is as well shown that at lower vaccination rate (0–20%), TB would still be endemic in the population. As such, high vaccination rate is required to send TB out of the community.

Keywords Tuberculosis · Vaccine · Diagnosis · Bifurcation · Contact rate

1 Introduction

According to the sustainable development goals (SDG) report 2019 of the United Nations (United Nations 2019), tuberculosis, HIV/AIDS and malaria are threats to achieving the third goal which is “Good Health and Well-Being”, with tuberculosis retaining its status as the leading cause of death from a single infectious agent (WHO 2018a). Tuberculosis simply referred to as TB is a bacterial infection caused by *Mycobacterium tuberculosis* (*Mtb*). TB which can either be pulmonary (affecting

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the lungs) or extra-pulmonary (affecting other body parts) has cough lasting for weeks, fatigue, night sweats and fever as major symptoms.

Since the early twentieth century, Bacille Calmette–Guérin (BCG) vaccine has been used as the prophylactic vaccine to prevent TB infection even before the availability of the therapeutic drugs (isoniazid and rifampicin) for TB treatment. However, the vaccine has not been potent enough to prevent TB infection and as such, requires the production of new vaccine.

The guide to the selection of candidate vaccine for TB was presented in (2019) Ayinla et al. (2019). It was established there that any candidate vaccine for TB should be able to give 50% reduction in the infectious power of *Mycobacterium tuberculosis*. Established also is the fact that if the TB vaccine is potent enough to give 100% immunity coverage against TB infection, then TB would soon be a thing of the past.

Researches have established awareness as a key element in TB incidence reduction (Moualeu et al. 2015; Okuonghae and Omosigho 2011; Okuonghae and Ikhimwin 2016). When there is a great awareness regarding the dangers that high contacts with TB patients portend, then the need for reduced contact rates between the susceptible and TB carriers shall be understood. Ayinla and Wan Ainun (2019); Ayinla et al. (2019); Castillo-Chavez and Feng (1997); Ullah et al. (2019) are among the authors that have established the contact rate as a promotional factor of TB. Except for South Africa and Philippines found on the list, countries that contribute to the two-thirds of the global TB cases in the year 2018 are relatively highly populated nations, which is an indication that contact rate plays significant role in the dynamics of TB. The countries with their percentages of incidences are; India (27%), China (9%), Indonesia (8%), Philippines (6%), Pakistan (6%), Nigeria (4%), Bangladesh (4%) and South Africa (3%) (WHO 2019). This is an indication that the African and Asian continents need to do more in mitigating the TB incidence as they have the most cases. This is as shown in Fig. 1 below. Also, attitudinal factor is one of the great promoters of TB incident. Bhunu et al. (2011) and Choi et al. (2015) show that smoking promotes TB. On the other hand, the authors in Huo and Zou (2016) established that treating TB patients at home without seeking proper medical attention at the clinic contributes to the persistence of TB in the society while (Xiang et al. 2019) show the need for early therapy if the fight against TB is actually meant.

Non compliance with the above established recommendations explain why the incidence of TB stands at 10 million in the year 2018 (WHO 2019) which is the same as that of 2017 (WHO 2018a). Although, the death toll drops by 100,000 from 1.3 million in 2017 (WHO 2018a) to 1.2 million in 2018 (WHO 2019) but yet, it is obvious that a lot needs to be done to reduce both the incident and death rates. For effective TB prevention and management in achieving the SDG, there is the need for multifaceted approaches to tackling TB.

This research focuses on the impact of TB diagnosis and vaccination in the presence of treatment. The research seeks to establish which approach reduces TB incidence in the face of contact control. There are four sections in the order; Sects. 1, 2, 3 and 4. This section which is Sect. 1 presents the introduction followed by the model formulation in Sect. 2. Section 3 presents the results and Sect. 4 gives the discussions and conclusion.

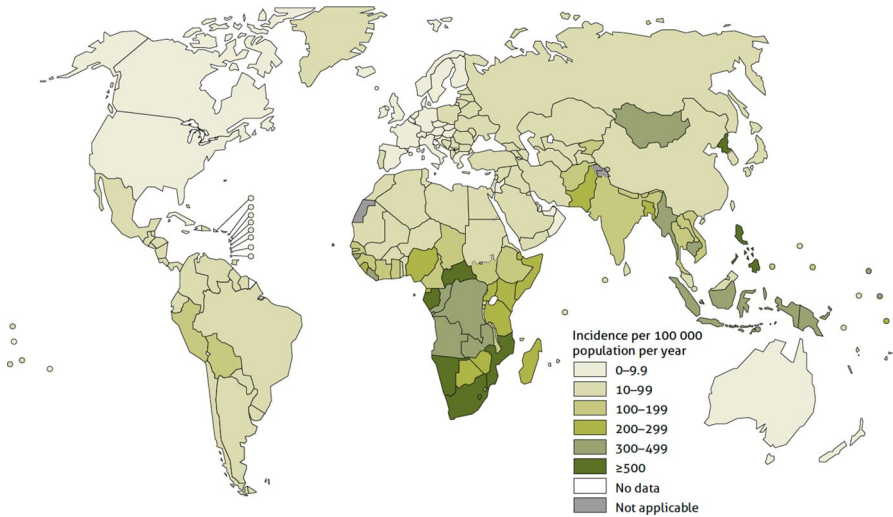


Fig. 1 Continental TB incidences WHO (2019)

2 Model Formulation

The total population at any time t , denoted by $N(t)$ is divided into 7 mutually exclusive classes. These classes are namely; susceptible ($S(t)$), vaccinated ($V(t)$), latently infected ($E(t)$), undiagnosed infectious ($I_1(t)$), diagnosed infectious ($I_2(t)$), treated individuals ($T(t)$) and the recovered individuals ($R(t)$). The latently infected individuals are unaware of their conditions, they can be aware of their condition by undergoing tuberculin skin test. These individuals can be in this state for 1 to 5 years for fast latent period (Colijn et al. 2007; Flynn and Chan 2001) or several years for slow latent period (Blower et al. 1995). There is the possibility of the treated getting reinfected while the recovered individuals are assumed to be fully fit.

The population grows at the recruitment rate Λ which can either be through child birth or immigration of uninfected individuals. After an effective contact with infectious persons, the susceptible individuals acquire TB at the rate

$$\lambda = \frac{\beta c(I_1 + I_2)}{N} \quad (1)$$

to either progress to the latently infected class or move directly to the undiagnosed infectious compartments at the rates f and $(1-f)$ respectively. In (1) above, β denotes the TB transmission rate while c denotes the effective contact rate between the susceptible and infectious beings, it is an effective contact between the susceptible and infectious that lead to the contraction of TB. Susceptible individuals are assumed to get vaccinated at the rate τ ($(1-\tau)$ represents unvaccinated fraction) and loose their immunity coverage to fall back into the susceptible class at the rate σ . This vaccine given wanes at the rate θ .

Exogenous reinfection has been established to be a major player in TB dynamics (Cohen et al. 2007; Feng et al. 2000; Gomes et al. 2004), hence, δ_1 denotes the rate at which exogenous reinfection occurs while endogenous reactivation occurs at the rate k . Infectious persons are diagnosed at the rate ϕ to proceed to I_2 compartment before being treated at the rate ω . These treated individuals are assumed to get reinfected at the rate δ_2 and fall back into the I_1 compartment or recover fully at the rate ψ and proceed to the compartment R . These reinfected individuals need to be rediagnosed to proceed to the I_2 compartment. Since death is inevitable, people die of TB infection at the rates ε_1 and ε_2 for I_1 and I_2 respectively or die naturally at the rate μ . All the parameter values are positive.

At any point in time, the total population $N(t)$ is

$$N(t) = S(t) + V(t) + E(t) + I_1(t) + I_2(t) + T(t) + R(t) \quad (2)$$

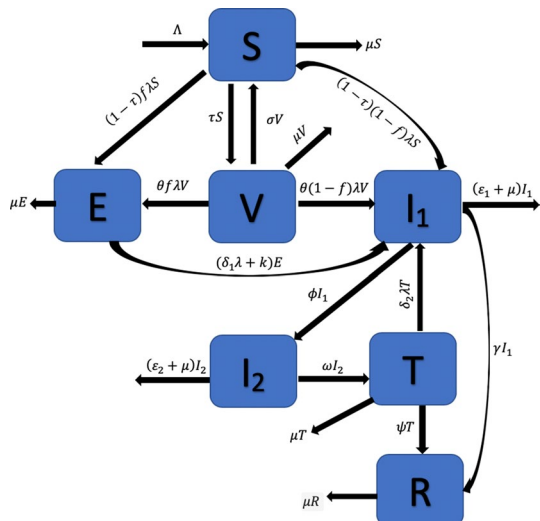
and the model is as presented in Eq. (3) and Fig. 2. Homogenous mixing of the population is assumed.

$$\left. \begin{aligned} S' &= \Lambda + \sigma V - [(1 - \tau)\lambda + \tau + \mu]S \\ V' &= \tau S - (\theta\lambda + \sigma + \mu)V \\ E' &= (1 - \tau)f\lambda S + \theta f\lambda V - (\delta_1\lambda + k + \mu)E \\ I_1' &= (1 - \tau)(1 - f)\lambda S + \theta(1 - f)\lambda V \\ &\quad + (\delta_1\lambda + k)E - (\gamma + \phi + \varepsilon_1 + \mu)I_1 + \delta_2\lambda T \\ I_2' &= \phi I_1 - (\omega + \varepsilon_2 + \mu)I_2 \\ T' &= \omega I_2 - (\delta_2\lambda + \psi + \mu)T \\ R' &= \gamma I_1 + \psi T - \mu R \end{aligned} \right\} \quad (3)$$

with the initial conditions:

$$S(0) > 0, \quad V(0) > 0, \quad E(0) > 0, \quad I_1(0) > 0, \quad I_2(0) > 0, \quad R(0) > 0.$$

Fig. 2 A TB vaccine model with two infectious stages



2.1 Basic Properties of the Model

2.1.1 Positivity and Boundedness of Solutions

For the formulated model (3) to be epidemiologically meaningful, it is necessary to show that all its state variables are non-negative for all time t , i.e., solutions of model (3) with positive initial conditions will remain positive for all time $t \geq 0$.

Theorem 2.1 *Suppose the initial conditions for the model (3) is $S(0) > 0, V(0) > 0, E(0) > 0, I_1(0) > 0, I_2(0) > 0, T(0) > 0$ and $R(0) > 0$, then the solutions $(S, V, E, I_1, I_2, T, R)$ of the model with positive initial conditions remain positive for all time $t > 0$.*

Proof Let $t_1 = \sup\{t > 0 : S(t), V(t), E(t), I_1(t), I_2(t), T(t), R(t) > 0\} > 0$, then

$$S''(t) = \Lambda + \sigma V - [(1 - \tau)\lambda + \tau + \mu]S(t) \geq \Lambda - [(1 - \tau)\lambda + \tau + \mu]S(t)$$

which can be written as

$$\frac{d}{dt} \left\{ S(t) \exp \left[(\tau + \mu)t + \int_0^t (1 - \tau)\lambda(u) du \right] \right\} \geq \Lambda \left\{ \exp \left[(\tau + \mu)t + \int_0^t (1 - \tau)\lambda(u) du \right] \right\}.$$

Thus,

$$\begin{aligned} S(t_1) \exp \left[(\tau + \mu)t_1 + \int_0^{t_1} (1 - \tau)\lambda(u) du \right] \\ - S(0) \geq \int_0^{t_1} \Lambda \exp \left[(\tau + \mu)\kappa + \int_0^\kappa (1 - \tau)(v) dv \right] d\kappa \\ \Rightarrow S(t_1) \geq S(0) \exp \left[-(\tau + \mu)t_1 - \int_0^{t_1} (1 - \tau)\lambda(u) du \right] \\ + \left\{ \exp \left[-(\tau + \mu)t_1 - \int_0^{t_1} (1 - \tau)\lambda(u) du \right] \right\} \\ \times \int_0^{t_1} \Lambda \exp \left[(\tau + \mu)\kappa + \int_0^\kappa (1 - \tau)(v) dv \right] d\kappa > 0. \end{aligned}$$

□

In similar manner, it can be shown that $V > 0, E > 0, I_1 > 0, I_2 > 0, T > 0$ and $R > 0$ for all time t . As such, all the solutions of model (3) are positive for non-negative initial conditions.

2.1.2 Invariant Region

Theorem 2.2 *The closed set*

$$\Omega = \left\{ (S, V, E, I_1, I_2, T, R) \in \mathbb{R}_+^7; \quad N \leq \frac{\Lambda}{\mu} \right\}$$

is positively invariant and attracts all positive solutions.

Proof Summing up the equations in model (3), then

$$\frac{dN}{dt} = \Lambda - \mu N - (\varepsilon_1 I_1 + \varepsilon_2 I_2). \quad (4)$$

Obviously, Eq. (4) above is bounded on the right by $\Lambda - \mu N$. With the aid of standard comparison theorem, it can be shown that

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}). \quad (5)$$

Particularly, $N(t) \leq \frac{\Lambda}{\mu}$ if $N(0) \leq \frac{\Lambda}{\mu}$. Thus, Ω is a positively invariant set. Hence according to Hethcote (2000), the model can be considered to be epidemiologically and mathematically well posed. Also, the solutions of the model (3) with its initial conditions remains in Ω for all $t > 0$ due to its positive invariance. $\square$

2.2 Stability of the Disease Free Equilibrium (DFE)

2.2.1 Local Asymptotic Stability of the DFE

Let $m_1 = 1 - \tau$, $m_2 = \tau + \mu$, $m_3 = \sigma + \mu$, $m_4 = k + \mu$, $m_5 = 1 - f$, $m_6 = \gamma + \phi + \varepsilon_1 + \mu$, $m_7 = \omega + \varepsilon_2 + \mu$, $m_8 = \psi + \mu$, $p_1 = m_1 m_3 + \theta \tau$ and $p_2 = m_2 m_3 - \sigma \tau$. Then equation (3) becomes

$$\left. \begin{aligned} S' &= \Lambda + \sigma V - m_1 \lambda S - m_2 S \\ V' &= \tau S - \theta \lambda V - m_3 V \\ E' &= m_1 f S + \theta f V - \delta_1 E \lambda - m_4 E \\ I_1' &= (m_1 m_5 S + \theta m_5 V + \delta_1 E + \delta_2 T) \lambda + k E - m_6 I_1 \\ I_2' &= \phi I_1 - m_7 I_2 \\ T' &= \omega I_2 - \delta_2 \lambda T - m_8 T \\ R' &= \gamma I_1 + \psi T - \mu R \end{aligned} \right\} (*)$$

At the DFE, there is no infection in the community, i.e., $E = 0$, $I_1 = 0$ and $I_2 = 0$. Setting the left hand side of (*) to zero at this point gives the DFE to be

$$\xi_0 = (S^*, V^*, E^*, I_1^*, I_2^*, T^*, R^*) = \left(\frac{m_3 \Lambda}{p_2}, \frac{\Lambda \tau}{p_2}, 0, 0, 0, 0, 0 \right),$$

adding S^* and V^* gives the population size $N^*(t) = \frac{\Lambda}{\mu}$ as established earlier.

With the aid of next generation matrix operator (Diekmann et al. 1990; van den Driessche and Watmough 2002), the local stability of the model (3) can be established. Following the notations used in van den Driessche and Watmough (2002), then $\mathcal{F}$ and $\mathcal{M}$ matrices for the new infection terms and the remaining transition terms are respectively given by

$$\mathcal{F} = \begin{bmatrix} (m_1 f S + \theta f V) \lambda \\ (m_1 m_5 S + \theta m_5 V + \delta_1 E + \delta_2 T) \lambda \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{M} = \begin{bmatrix} \delta_1 \lambda E + m_4 E \\ m_6 I_1 - k E \\ m_7 I_2 - \phi I_1 \\ \delta_2 \lambda T + m_8 T - \omega I_2 \end{bmatrix}.$$

$$\text{Hence, } F = \begin{bmatrix} 0 & \frac{\beta c f (m_1 S^* + \theta V^*)}{N^*} & \frac{\beta c f (m_1 S^* + \theta V^*)}{N^*} & 0 \\ 0 & \frac{\beta c m_5 (m_1 S^* + \theta V^*)}{N^*} & \frac{\beta c m_5 (m_1 S^* + \theta V^*)}{N^*} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Recall that } \frac{S^*}{N^*} = \frac{m_3 \mu}{m_2 m_3 - \sigma \tau} \text{ and } \frac{V^*}{N^*} = \frac{\tau \mu}{m_2 m_3 - \sigma \tau}.$$

$$\Rightarrow F = \begin{bmatrix} 0 & \frac{\beta c f \mu (m_1 m_3 + \theta \tau)}{m_2 m_3 - \sigma \tau} & \frac{\beta c f \mu (m_1 m_3 + \theta \tau)}{m_2 m_3 - \sigma \tau} & 0 \\ 0 & \frac{\beta c m_5 \mu (m_1 m_3 + \theta \tau)}{m_2 m_3 - \sigma \tau} & \frac{\beta c m_5 \mu (m_1 m_3 + \theta \tau)}{m_2 m_3 - \sigma \tau} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $p_1 = m_1 m_3 + \theta \tau$ and $p_2 = m_2 m_3 - \sigma \tau$, then

$$F = \begin{bmatrix} 0 & \frac{\beta c f p_1 \mu}{p_2} & \frac{\beta c f p_1 \mu}{p_2} & 0 \\ 0 & \frac{\beta c m_5 p_1 \mu}{p_2} & \frac{\beta c m_5 p_1 \mu}{p_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Also, from } \mathcal{M}, M = \begin{bmatrix} m_4 & 0 & 0 & 0 \\ -k & m_6 & 0 & 0 \\ 0 & -\phi & m_7 & 0 \\ 0 & 0 & -\omega & m_8 \end{bmatrix}.$$

Thus

$$R_0 = \rho(FM^{-1}) = \frac{\beta c \mu (m_1 m_3 + \theta \tau) (m_7 + \phi) (fk + m_4 m_5)}{m_4 m_6 m_7 p_2}, \quad (6)$$

$$\Rightarrow R_0 = \frac{\beta c \mu ((1 - \tau)(\sigma + \mu) + \theta \tau) (\omega + \phi + \varepsilon_2 + \mu) (fk + (1 - f)(k + \mu))}{(k + \mu) (\omega + \varepsilon_2 + \mu) (\gamma + \phi + \varepsilon_1 + \mu) ((\tau + \mu)(\sigma + \mu) - \sigma \tau)},$$

where ρ is the spectral radius, and R_0 is the basic reproduction number. By Theorem 2 of van den Driessche and Watmough (2002), the following Lemma 2.1 is presented.

Lemma 2.1 *The DFE of the model (3) is locally asymptotically stable (LAS) if $R_0 < 1$ and unstable otherwise.*

Proof The Jacobian matrix of the model (3) evaluated at the DFE is

$$J(\xi_0) = \begin{bmatrix} -m_2 & \sigma & 0 & \frac{-\beta c m_1 m_3 \mu}{p_2} & \frac{-\beta c m_1 m_3 \mu}{p_2} & 0 & 0 \\ \tau & -m_3 & 0 & \frac{-\beta c \theta \tau \mu}{p_2} & \frac{-\beta c \theta \tau \mu}{p_2} & 0 & 0 \\ 0 & 0 & -m_4 & \frac{-\beta c f p_1 \mu}{p_2} & \frac{-\beta c f p_1 \mu}{p_2} & 0 & 0 \\ 0 & 0 & k & \frac{\beta c m_5 p_1 \mu}{p_2} - m_6 & \frac{\beta c m_5 p_1 \mu}{p_2} & 0 & 0 \\ 0 & 0 & 0 & \phi & -m_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & -m_8 & 0 \\ 0 & 0 & 0 & \gamma & 0 & \psi & -\mu \end{bmatrix}$$

$$\text{Trace}[J(\xi_0)] = -\left(4\mu + \tau + \sigma + k + \gamma + \phi + \omega + \psi + \varepsilon_1 + \varepsilon_2 - \frac{\beta c m_5 p_1 \mu}{p_2}\right) < 0 \quad \text{for } \frac{\beta c m_5 p_1 \mu}{p_2} < (\gamma + \phi + \varepsilon_1 + \mu).$$

$\text{Det}[J(\xi_0)] = \mu[\beta p_1 (m_7 + \phi) (m_4 m_5 + fk) - p_2 m_4 m_6 m_7]$. This means that $\text{Det}[J(\xi_0)] > 0$ when

$$\frac{\beta c \mu ((1 - \tau)(\sigma + \mu) + \theta \tau) (\omega + \phi + \varepsilon_2 + \mu) (fk + (1 - f)(k + \mu))}{(k + \mu) (\omega + \varepsilon_2 + \mu) (\gamma + \phi + \varepsilon_1 + \mu) ((\tau + \mu)(\sigma + \mu) - \sigma \tau)} < 1.$$

$$\Rightarrow R_0 < 1 \quad \left(R_0 = \frac{\beta c \mu ((1 - \tau)(\sigma + \mu) + \theta \tau) (\omega + \phi + \varepsilon_2 + \mu) (fk + (1 - f)(k + \mu))}{(k + \mu) (\omega + \varepsilon_2 + \mu) (\gamma + \phi + \varepsilon_1 + \mu) ((\tau + \mu)(\sigma + \mu) - \sigma \tau)} \right)$$

The eigenvalues established from $\text{Det}[J(\xi_0) - \lambda] = 0$ all contain negative real parts whenever $R_0 < 1$. This means the DFE, ξ_0 is locally asymptotically stable. $\square$

The basic reproduction number (R_0) can simply be defined as the average number of secondary infections from a singly recorded case, in a completely susceptible environment with vaccination program in place. This implies that TB can be effectively eliminated in the community when $R_0 < 1$ with the condition that the initial sizes of the subpopulation are in the basis of attraction of the DFE (ξ_0) while maintaining the vaccination program (Okuonghae and Omosigho 2011; Okuonghae and Ikhimwin 2016; Sharomi and Gumel 2009). To ensure that this condition is met, it is necessary to show that the DFE is globally asymptotically stable (GAS) if $R_0 < 1$. This result is presented in Sect. 2.5.

2.3 Analysis of the Reproduction Number R_0 with Considerations for Vaccination, Diagnosis and Treatment

The aim of this section is to explore the impacts of vaccination, identification of infectious persons and subsequently treating the identified individuals on the dynamics of TB. This would guide on whether to vaccinate, treat and put up efforts to identify the TB patients or not.

Following Sharomi et al. (2008), then (7) is established from (6).

$$\begin{aligned}
 \lim_{\tau \rightarrow 1} R_0 &= \frac{\beta c \theta [fk + (k + \mu)(1 - f)] [\phi + \omega + \varepsilon_2 + \mu]}{(k + \mu)(1 + \sigma + \mu)(\omega + \varepsilon_2 + \mu)(\gamma + \phi + \varepsilon_1 + \mu)} > 0 \\
 \lim_{\phi \rightarrow 1} R_0 &= \frac{\beta c [fk + (k + \mu)(1 - f)] [1 + \omega + \varepsilon_2 + \mu] [(1 - \tau)(\sigma + \mu) + \theta \tau]}{(k + \mu)(\sigma + \tau + \mu)(\omega + \varepsilon_2 + \mu)(1 + \gamma + \varepsilon_1 + \mu)} > 0 \\
 \lim_{\omega \rightarrow 1} R_0 &= \frac{\beta c [fk + (k + \mu)(1 - f)] [1 + \phi + \varepsilon_2 + \mu] [(1 - \tau)(\sigma + \mu) + \theta \tau]}{(k + \mu)(\sigma + \tau + \mu)(1 + \varepsilon_2 + \mu)(\phi + \gamma + \varepsilon_1 + \mu)} > 0 \\
 \lim_{\tau \rightarrow 1} R_0 &= \frac{\beta c \theta [fk + (k + \mu)(1 - f)] [1 + \omega + \varepsilon_2 + \mu]}{(k + \mu)(1 + \sigma + \mu)(\omega + \varepsilon_2 + \mu)(1 + \gamma + \varepsilon_1 + \mu)} > 0 \quad (7) \\
 \lim_{\tau \rightarrow 1} R_0 &= \frac{\beta c \theta [fk + (k + \mu)(1 - f)] [1 + \phi + \varepsilon_2 + \mu]}{(k + \mu)(1 + \sigma + \mu)(1 + \varepsilon_2 + \mu)(\gamma + \phi + \varepsilon_1 + \mu)} > 0 \\
 \lim_{\omega \rightarrow 1} R_0 &= \lim_{\phi \rightarrow 1} R_0 = \frac{\beta c \theta (2 + \varepsilon_2 + \mu) [fk + (k + \mu)(1 - f)]}{(k + \mu)(1 + \sigma + \mu)(1 + \varepsilon_2 + \mu)(1 + \gamma + \varepsilon_1 + \mu)} > 0.
 \end{aligned}$$

Note that $\lim_{\tau \rightarrow 1} R_0 = \lim_{\phi \rightarrow 1} R_0$ means the converse is always true.

The implication of the above is that vaccination program done at the rate τ , TB patients' diagnosis (identification) done at the rate ϕ and treatment of TB patients done at the rate ω can lead to effective TB control if the right hand side of (7) is less than unity. This is shown with the contour plots below (Figs. 3, 4, 5) based on parameter values in Table 2. Figures 3 and 4 display the contour plots of R_0 as a function of diagnosis (ϕ) and vaccination (τ) and also as a function of treatment (ω) and vaccination (τ). It is observed from the plots that these measures reduce TB incident rate as the basic reproduction number (R_0) keeps on reducing at higher values of these measures. Figure 5 shows the contour plot of R_0 as a function of diagnosis (ϕ) and treatment (ω) rates. The plot shows that TB incidence would be on the rise when there are no enough diagnosis and treatment of the TB patients.

In furtherance to the above, the partial derivatives of R_0 with respect to these parameters τ , ϕ and ω are presented below to have a better understanding of the impact of these factors (parameters).

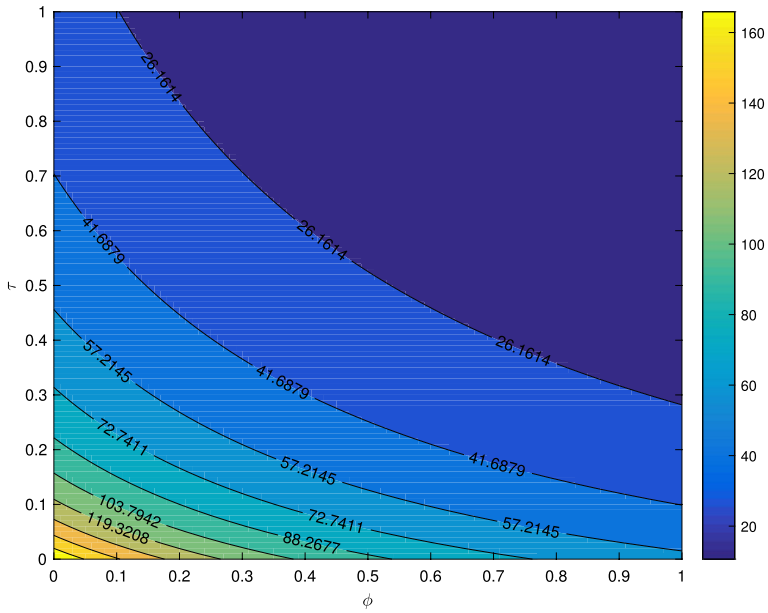


Fig. 3 A contour plot of R_0 as a function of ϕ and τ

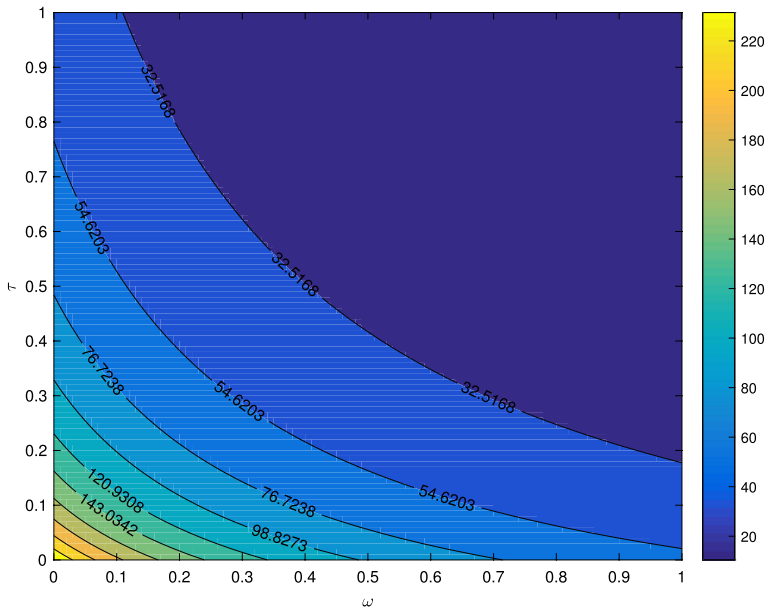


Fig. 4 A contour plot of R_0 as a function of ω and τ

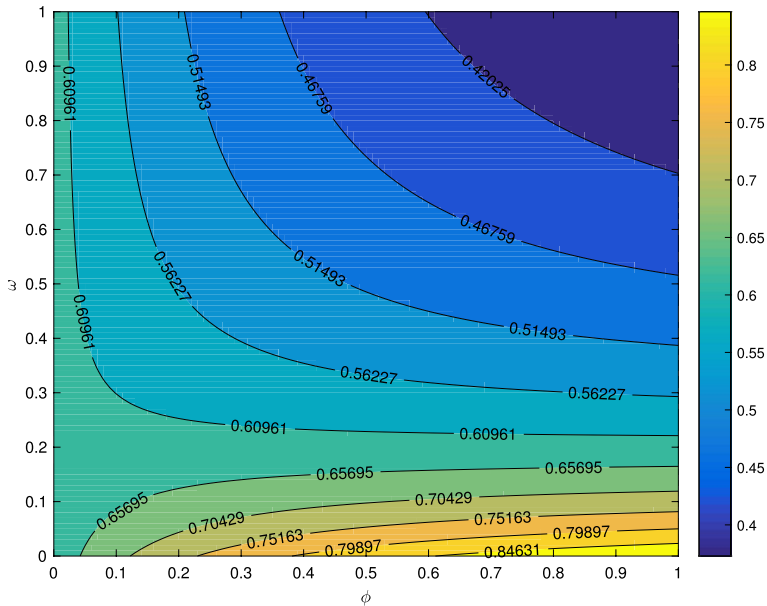


Fig. 5 A contour plot of R_0 as a function of ϕ and ω

$$\begin{aligned} \frac{\partial R_0}{\partial \tau} &= \frac{-\beta c(\sigma + \mu)[k + \mu(1 - f)](\phi + \omega + \varepsilon_2 + \mu)[(\sigma + \mu) + (1 - \theta)]}{(k + \mu)(\sigma + \tau + \mu)^2(\omega + \varepsilon_2 + \mu)(\gamma + \phi + \varepsilon_1 + \mu)} < 0 \\ \frac{\partial R_0}{\partial \phi} &= \frac{\beta c[k + \mu(1 - f)][(\sigma + \mu)(1 - \tau) + \theta\tau][\gamma + \varepsilon_1 - (\omega + \varepsilon_2)]}{(k + \mu)(\sigma + \tau + \mu)(\omega + \varepsilon_2 + \mu)(\gamma + \phi + \varepsilon_1 + \mu)^2} < 0 \\ \frac{\partial R_0}{\partial \omega} &= \frac{\beta c\phi[k + \mu(1 - f)][(\sigma + \mu)(1 - \tau) + \theta\tau]}{(k + \mu)(\sigma + \tau + \mu)(\omega + \varepsilon_2 + \mu)^2(\gamma + \phi + \varepsilon_1 + \mu)} < 0 \end{aligned} \quad (8)$$

The first inequality in (8) clearly shows the positive impact the vaccination program has on the control of TB (since $\frac{\partial R_0}{\partial \tau} < 0$). This means the program should be encouraged. The outcome is as well the same when the partial derivative of the basic reproduction number is done with respect to ω . The condition for $\frac{\partial R_0}{\partial \phi} < 0$ to hold is by letting $\frac{\gamma + \varepsilon_1}{\omega + \varepsilon_2} < 1$. This implies that TB case detection (diagnosis) contributes to the reduction in TB incidence. Since case detection precedes treatment, there can not be treatment without case detection (diagnosis). This gives an insight into why the incidence of TB seems non-subsiding (even with better treatment coverage), which is in agreement with the finding in Okuonghae and Korobeinikov (2007). It also explains the reason for treatment failures which have led to greater dangers in form of different TB strains like multi-drug resistant TB (MDR-TB), extensively drug resistant TB (XDR-TB) and totally drug resistant TB (TDR-TB). Thus, the following Lemma 2.2 is given.

Lemma 2.2 *The treatment of TB patients in the absence of case detection (diagnosis) may not necessarily curb TB infection.*

2.4 Bifurcation Analysis

As stated earlier that exogenous reinfection is a major player in the dynamics of TB, models with the consideration of exogenous reinfection are known to exhibit backward bifurcation (Benjamin and Denise 2004; Cohen et al. 2007; Feng et al. 2000; Gomes et al. 2004; Gumel 2012; Moualeu et al. 2016; Wangari and Stone 2018; Yang and Raimundo 2010). Hence, this section would focus on bifurcation analysis of the model. Backward bifurcation phenomenon is when the stable DFE coexists with a stable endemic equilibrium when the reproduction number is less than unity ($R_0 < 1$).

To start with, let $\xi_1 = (S^{**}, V^{**}, E^{**}, I_1^{**}, I_2^{**}, T^{**}, R^{**})$ be an arbitrary equilibrium point of model (3). Hence, from (1),

$$\lambda_e^{**} = \frac{\beta c(I_1^{**} + I_2^{**})}{N^{**}}. \quad (9)$$

Also, setting the right hand side of (3) to zero gives

$$\begin{aligned} S^{**} &= \frac{\Lambda J_2}{J_1 J_2 - \sigma \tau}, \quad V^{**} = \frac{\Lambda \tau}{J_1 J_2 - \sigma \tau}, \quad E^{**} = \frac{\Lambda f(m_1 J_2 + \theta \tau) \lambda^{**}}{J_3 (J_1 J_2 - \sigma \tau)}, \\ I_1^{**} &= \frac{\Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**}}{J_3 (J_1 J_2 - \sigma \tau) (J_4 m_6 m_7 - \delta_2 \omega \phi \lambda^{**})}, \\ I_2^{**} &= \frac{\phi \Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**}}{m_7 J_3 (J_1 J_2 - \sigma \tau) (J_4 m_6 m_7 - \delta_2 \omega \phi \lambda^{**})}, \\ T^{**} &= \frac{\omega \phi \Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**}}{J_4 m_7 J_3 (J_1 J_2 - \sigma \tau) (J_4 m_6 m_7 - \delta_2 \omega \phi \lambda^{**})}, \\ R^{**} &= \frac{(J_4 m_7 \gamma + \psi \omega \phi) \Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**}}{\mu J_4 m_7 J_3 (J_1 J_2 - \sigma \tau) (J_4 m_6 m_7 - \delta_2 \omega \phi \lambda^{**})}. \end{aligned} \quad (10)$$

From (3),

$$N^{**} = \frac{\Lambda - (\varepsilon_1 I_1^{**} + \varepsilon_2 I_2^{**})}{\mu}. \quad (11)$$

where $J_1 = m_1 \lambda^{**} + m_2$, $J_2 = \theta \lambda^{**} + m_3$, $J_3 = \delta_1 \lambda^{**}$ and $J_4 = \delta_2 \lambda^{**} + m_8$.

Subsequently, using I_1 and I_2 as given in (10) and as well as Eqs. (11) in (9) gives

$$\lambda_e^{**} = \frac{\beta c \mu (m_7 + \phi) (\Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**})}{J_3 (J_1 J_2 - \sigma \tau) (J_4 m_6 m_7 - \delta_2 \omega \phi \lambda^{**}) m_7 \Lambda \mu - (m_7 \varepsilon_1 + \phi \varepsilon_2) (\Lambda J_4 m_7 (m_1 J_2 + \theta \tau) (J_3 m_5 + \delta_1 f \lambda^{**} + f k) \lambda^{**})}. \quad (12)$$

After a tedious expansion and simplification of Eq. (12), the below quartic equation is given.

$$A_1 (\lambda_e^{**})^4 + A_2 (\lambda_e^{**})^3 + A_3 (\lambda_e^{**})^2 + A_4 \lambda_e^{**} + A_5 = 0, \quad (13)$$

where the coefficients $A_1 - A_5$ are as given below.

$$\begin{aligned}
A_1 &= L_7 f_1 - L_5 L_6, \\
A_2 &= L_7 f_2 - L_2 L_5 - \mu L_1 L_6, \\
A_3 &= L_7 f_3 - L_3 L_5 - \mu L_1 L_2, \\
A_4 &= L_7 f_4 - L_4 L_5 - \mu L_1 L_3, \\
A_5 &= \Lambda m_4 m_6 m_7 m_8 (m_2 m_3 - \sigma \tau) (1 - R_0).
\end{aligned} \tag{14}$$

$$f_1 = \delta_1 \delta_2 m_1 \theta (m_6 m_7 - \omega \phi),$$

$$f_2 = [m_1 m_4 \theta + \delta_1 (m_1 m_3 + m_2 \theta) (m_6 m_7 - \omega \phi) \delta_2 + \theta \delta_1 m_1 m_6 m_7 m_8],$$

$$f_3 = [\{m_4 (m_1 m_3 + m_2 \theta) + \delta_1 (m_2 m_3 - \sigma \tau)\} (m_6 m_7 - \omega \phi) \delta_2 + \{m_1 m_4 + \delta_1 (m_1 m_3 + m_2 \theta)\} m_6 m_7 m_8],$$

$$f_4 = [m_4 (m_2 m_3 - \sigma \tau) (m_6 m_7 - \omega \phi) \delta_2 + \{m_4 (m_1 m_3 + m_2 \theta) + \delta_1 (m_2 m_3 - \sigma \tau)\} m_6 m_7 m_8],$$

$$f_5 = m_1 \theta \delta_1 (f + m_5) \quad L_1 = \beta c \Lambda (m_7 + \phi), \quad L_2 = \delta_2 f_6 + m_8 f_5, \quad L_3 = \delta_2 f_7 + f_6 m_8, \\ L_4 = m_8 f_7, L_5 = \Lambda (\varepsilon_1 m_7 + \varepsilon_2 \phi), L_6 = \delta_2 f_5, L_7 = \Lambda.$$

By simplification of (14), $A_1 > 0$ is achieved. Also, $A_5 > 0$ (< 0) if $R_0 < 1$ (> 1). This implies that the number of positive real roots Eq. (13) may have depends on the signs of A_2, A_3 and A_4 . These signs can be analysed using the Decartes Rule of Signs on the function $f(x) = A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5$, given in equation (13) (such that $x = \lambda_e^{**}$). The possible roots of $f(x)$ are given in Table 1 below. Hence the following results (Theorem 2.3 and Lemma 2.3) could be deduced.

Table 1 Possible positive real roots of $f(x)$ for $R_0 < 1$ and $R_0 > 1$

Cases	A_1	A_2	A_3	A_4	A_5	R_0	No of times sign changed	No of positive real roots (endemic equilibrium point)
1	+	+	+	+	+	< 1	0	0
	+	+	+	+	-	> 1	1	1
2	+	+	+	-	+	< 1	2	0,2
	+	+	+	-	-	> 1	1	1
3	+	+	-	+	+	< 1	2	0,2
	+	+	-	+	-	> 1	3	1,3
4	+	-	+	+	+	< 1	2	0,2
	+	-	+	+	-	> 1	3	1,3
5	+	-	-	+	+	< 1	2	0,2
	+	-	-	+	-	> 1	3	1,3
6	+	-	-	-	+	< 1	2	0,2
	+	-	-	-	-	> 1	1	1
7	+	-	+	-	+	< 1	4	0,2,4
	+	-	+	-	-	> 1	3	1,3
8	+	+	-	-	+	< 1	2	0,2
	+	+	-	-	-	> 1	1	1

Theorem 2.3 *The tuberculosis model (3)*

1. *has a unique endemic equilibrium if $R_0 > 1$ and whenever cases 1, 2, 6 and 8 are satisfied;*
2. *could have more than one endemic equilibrium if $R_0 > 1$ and cases 3,4,5 and 7 are satisfied;*
3. *could have 2 or more endemic equilibria if $R_0 < 1$ and cases 2-8 are satisfied.*

The existence of multiple endemic equilibria when $R_0 < 1$ established using Theorem 2.3, Lemma 2.3 and Table 1 suggests the possibility of backward bifurcation (see (Zhang and Liu 2008, 2009)). As defined earlier, backward bifurcation phenomenon is when the stable DFE coexists with a stable endemic equilibrium when the reproduction number is less than unity ($R_0 < 1$). According to the authors in Castillo-Chavez and Song (2004), Haderler and van den Driessche (1997), Okuonghae and Omosigbo (2011), Okuonghae and Ikhimwin (2016), the backward bifurcation phenomenon could be triggered by exogenous re-infection. Figure 6 below shows the bifurcation diagram. The figure is plotted using the values as declared in Table 2 with $c = 8$. The blue curve indicates the DFE when $R_0 < 1$ while the red curve indicates the EEP when $R_0 > 1$. This shows that stable DFE coexists with stable EEP.

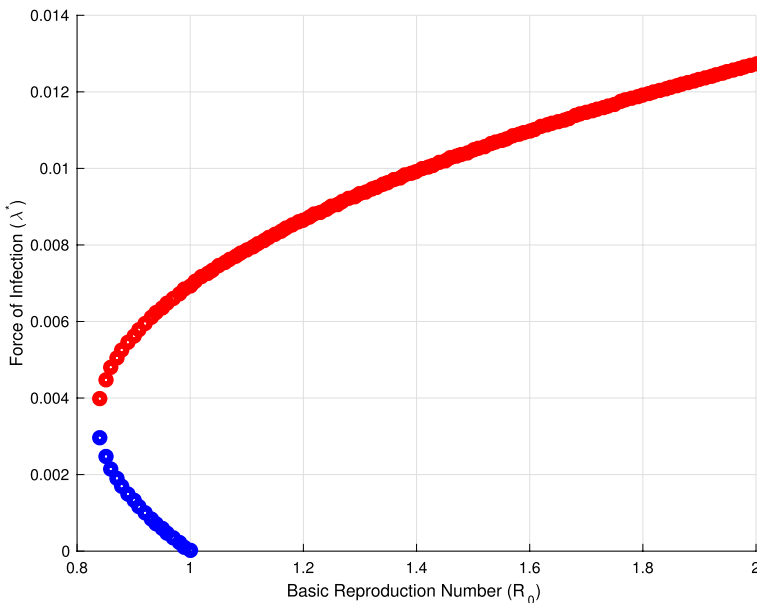


Fig. 6 Backward bifurcation curve for the model (3)

Lemma 2.3 *The tuberculosis model (3) has at least one positive endemic equilibrium whenever $R_0 > 1$ but could have zero, two or four positive endemic equilibrium whenever $R_0 < 1$.*

2.5 Global Stability of the DFE

To establish the global stability of the DFE of the model (3), the use of the global stability theorem discussed by Castillo-Chavez et al. (2002) is employed. This approach has been used by Agosto and Khan (2018); Bhunu et al. (2011); Goswami et al. (2018) and Srivastav and Ghosh (2016) among other researchers.

If model (3) can be written in the form:

$$\frac{dX}{dt} = F(X, Y) \quad \text{and} \quad \frac{dY}{dt} = G(X, Y) \quad \ni : \quad G(X, \mathbf{0}) = 0,$$

where the uninfected and infected compartments are respectively represented as

$X = (S, V, R)^T$ and $Y = (E, I_1, I_2, T)^T$, and also, the DFE, ξ_0 is simply expressed as

$$\xi_0 = (X_0^*, \mathbf{0})$$

where

$$X_0^* = \left(\frac{\Lambda(\sigma + \mu)}{(\sigma + \mu)(\tau + \mu) - \sigma\tau}, \frac{\Lambda\tau}{(\sigma + \mu)(\tau + \mu) - \sigma\tau}, 0 \right),$$

then the DFE is globally asymptotically stable (GAS) provided $R_0 < 1$ and as well satisfies the conditions H_1 and H_2 given below.

$H_1 : \frac{dX}{dt} = F(X_0, \mathbf{0}), X_0^*$ is globally asymptotically stable. is globally asymptotically stable.

$H_2 : G(X, Y) = AY - \hat{G}(X, Y), \hat{G}(X, Y) \geq 0$, where $A = D_Y G(X_0^*, \mathbf{0})$ is M-matrix (that is, all the non-diagonal elements of the matrix are non-negative).

To start with, let $\Gamma_1 = \{X \in \Gamma_1 : S \leq S^*, V \leq V^*\}$, where $X = \{S, V, E, I_1, I_2, T, R\}$. $S^* = \frac{\Lambda(\sigma + \mu)}{(\tau + \mu)(\sigma + \mu) - \sigma\tau}$ and $V^* = \frac{\Lambda\tau}{(\sigma + \mu)(\tau + \mu) - \sigma\tau}$, then the following Lemma 2.4 is established.

Lemma 2.4 *The region Γ_1 is positively invariant for model (3).*

Proof From the first equation of (3),

$$S' = \Lambda + \sigma V - [(1 - \tau)\lambda + \tau + \mu]S$$

then at the DFE,

$$S' \leq \Lambda + \sigma V - (\tau + \mu)S.$$

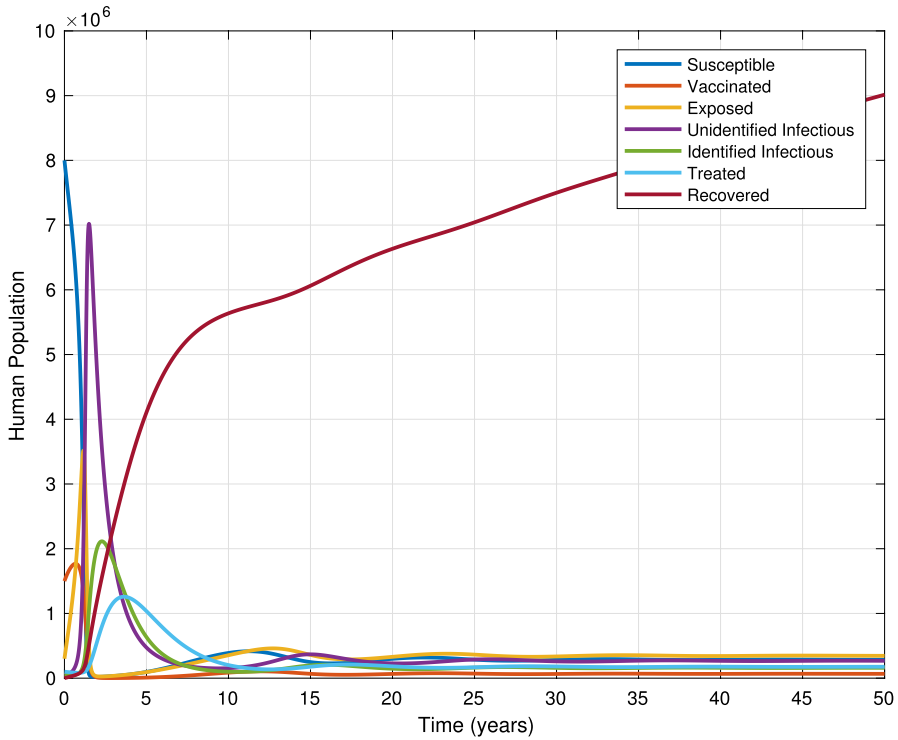


Fig. 7 TB dynamics with $c = 80$

Using the second equation in (3), then

$$\begin{aligned}
 S' &\leq \Lambda + \frac{\sigma(\tau S - V')}{\sigma + \mu} - (\tau + \mu)S \\
 &\Rightarrow S' + \left[\frac{(\sigma + \mu)(\tau + \mu) - \sigma\tau}{\sigma + \mu} \right] S \leq \Lambda - \frac{\sigma V'}{(\tau + \mu)} \\
 S(t) &\leq \frac{\Lambda(\sigma + \mu)}{(\tau + \mu)(\sigma + \mu) - \sigma\tau} - \frac{\sigma V(t)}{(\sigma + \mu)} e^{-\left[\frac{(\sigma + \mu)(\tau + \mu) - \sigma\tau}{\sigma + \mu} \right] t} \\
 &\quad + \left[S(0) + \frac{\sigma V(t)}{\sigma + \mu} - \frac{\Lambda(\sigma + \mu)}{(\sigma + \mu)(\tau + \mu) - \sigma\tau} \right] e^{-\left[\frac{(\sigma + \mu)(\tau + \mu) - \sigma\tau}{\sigma + \mu} \right] t}.
 \end{aligned}$$

As $t \rightarrow \infty$,

$$S(t) \leq \frac{\Lambda(\sigma + \mu)}{(\tau + \mu)(\sigma + \mu) - \sigma\tau}.$$

Hence, $S(t) \leq S^*$ for all $t \geq 0$. Following the same approach, $V \leq V^*$ for all $t \geq 0$. This shows that Γ_1 is positively invariant and attracts all solutions in $\mathbb{R}_+^7$ for the model (3).

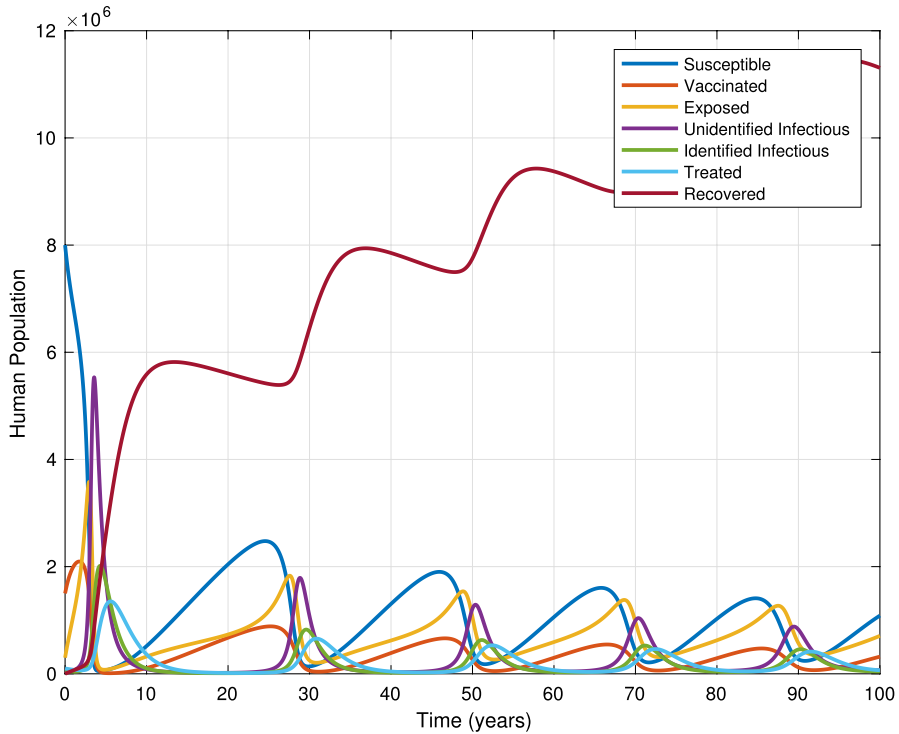


Fig. 8 TB dynamics with $c = 40$

Theorem 2.4 *The DFE, $\xi_0 = (X_0^*, \mathbf{0})$ of the nonlinear system (3) is globally asymptotically stable (GAS) provided it satisfies the conditions H_1 and H_2 above, as well as $R_0 < 1$.*

Proof The reduced system $\frac{dX}{dt} = F(X, \mathbf{0})$ is

$$\left. \begin{aligned} \frac{dS}{dt} &= +\sigma V - (\tau + \mu)S \\ \frac{dV}{dt} &= \tau S - (\sigma + \mu)V \\ \frac{dR}{dt} &= -\mu R. \end{aligned} \right\} (**)$$

If $X^* = (S^*, V^*, R^*) = \left(\frac{\Lambda(\sigma + \mu)}{(\tau + \mu)(\sigma + \mu) - \sigma\tau}, \frac{\Lambda\tau}{(\sigma + \mu)(\tau + \mu) - \sigma\tau}, 0 \right)$ is the equilibrium point of the reduced system (**), then the following results show that X^* is a globally stable equilibrium in Γ_1 .

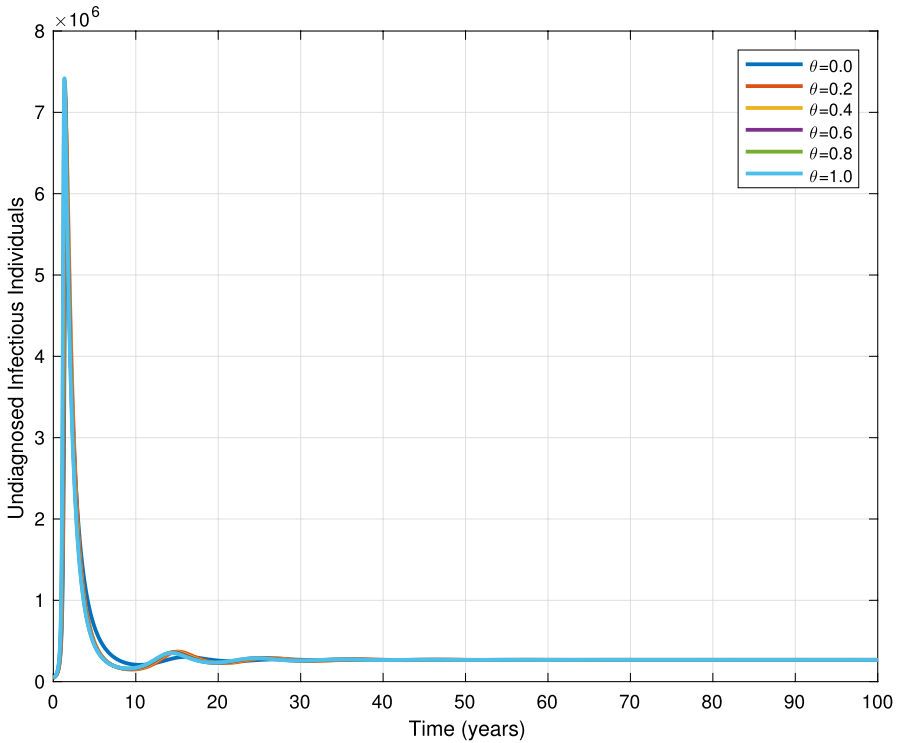


Fig. 9 Impact of vaccination (potency) with $c = 80$ (undiagnosed infectious compartment)

From (**), then

$$\left. \begin{aligned} S(t) &= \frac{\Lambda + \sigma V(t)}{\tau + \mu} - \left[S(0) + 2 \int_0^t \frac{e^{(\tau+\mu)w}}{\tau + \mu} Q dw - \frac{\Lambda + \sigma V(0)}{\tau + \mu} \right] e^{-(\tau+\mu)t} \\ V(t) &= \frac{\tau S(t)}{\sigma + \mu} - \left[V(0) + 2 \int_0^t \frac{e^{(\sigma+\mu)y}}{\sigma + \mu} N dy - \frac{\tau S(0)}{\sigma + \mu} \right] e^{-(\sigma+\mu)t} \\ R(t) &= [R(0) + \int_0^t e^{\mu z} dz] e^{-\mu t} \end{aligned} \right\} (***)$$

Taking the limit of $S(t)$, $V(t)$, $R(t)$ as $t \rightarrow \infty$, in (***) then $\lim_{t \rightarrow \infty} S(t) = \frac{\Lambda(\sigma+\mu)}{(\tau+\mu)(\sigma+\mu)-\sigma\tau}$, $\lim_{t \rightarrow \infty} V(t) = \frac{\Lambda\tau}{(\sigma+\mu)(\tau+\mu)-\sigma\tau}$ and $\lim_{t \rightarrow \infty} R(t) = 0$. This shows that convergence of the solutions of (*) is global in Γ_1 .

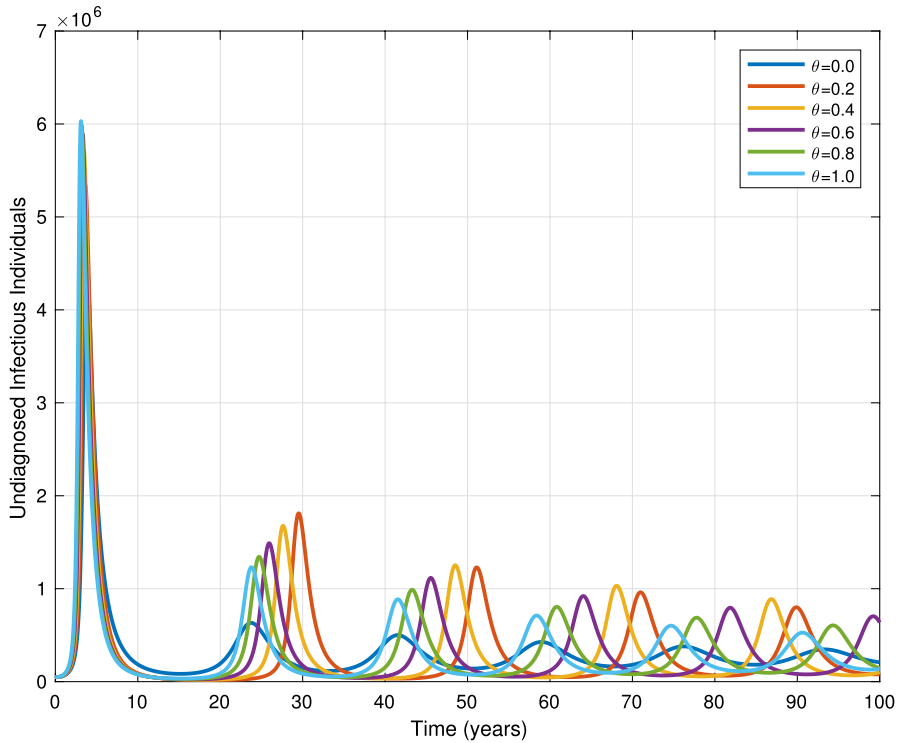


Fig. 10 Impact of vaccination (potency) with $c = 40$ (undiagnosed infectious compartment)

Also, the following are established from the model Eq. (3),

$$G(X, Y) = AY - \hat{G}(X, Y),$$

where

$$A = \begin{pmatrix} -(k + \mu) & \frac{\beta c}{N^*}((1 - \tau)fS^* + \theta fV^*) & \frac{\beta c}{N^*}((1 - \tau)fS^* + \theta fV^*) & 0 \\ k & A_{22} & \frac{\beta c}{N^*}((1 - \tau)(1 - f)S^* + \theta(1 - f)V^*) & 0 \\ 0 & \phi & -(\omega + \varepsilon_2 + \mu) & 0 \\ 0 & 0 & \omega & -(\psi + \mu) \end{pmatrix}, \quad (15)$$

such that

$$A_{22} = \frac{\beta c}{N^*}(1 - \tau)(1 - f)S^* + \frac{\beta c}{N^*}\theta(1 - f)V^* - (\gamma + \phi + \varepsilon_1 + \mu).$$

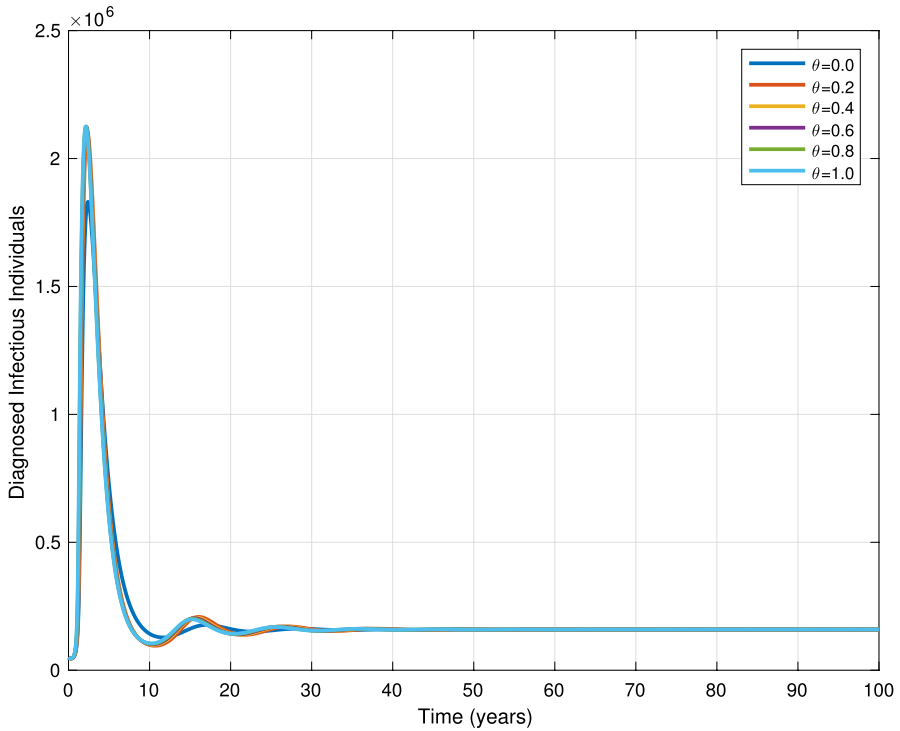


Fig. 11 Impact of vaccination (potency) with $c = 80$ (diagnosed infectious compartment)

It is clear from (15) that the non-diagonal elements are non-negative (because $\tau \leq 1$ and $f \leq 1$). Also, from H_2 , $\hat{G}(X, Y) = AY - G(X, Y)$

$$\Rightarrow \hat{G}(X, Y) = \begin{pmatrix} \hat{G}_1(X, Y) \\ \hat{G}_2(X, Y) \\ \hat{G}_3(X, Y) \\ \hat{G}_4(X, Y) \end{pmatrix} = \begin{pmatrix} f\beta c\lambda(1-\tau)\left(1 - \frac{SN^*}{S^*N}\right) + \theta f\beta\lambda c\left(1 - \frac{VN^*}{V^*N}\right) + \frac{\delta_1\beta c\lambda E}{N} \\ (1-\tau)(1-f)\beta c\lambda\left(1 - \frac{SN^*}{S^*N}\right) + \theta(1-f)\beta\lambda c\left(1 - \frac{VN^*}{V^*N}\right) \\ 0 \\ \frac{\delta_2\beta cT}{N} \end{pmatrix}, \quad (16)$$

such that $S^* = \frac{\Lambda(\sigma+\mu)}{(\sigma+\mu)(\tau+\mu)-\sigma\tau}$, $V^* = \frac{\Lambda\tau}{(\sigma+\mu)(\tau+\mu)-\sigma\tau}$ and $N^* = \frac{\Lambda}{\mu}$. From Γ_1 , we have $S \leq S^*$ and $V \leq V^*$. If the population is at equilibrium, then $\left(1 - \frac{SN^*}{S^*N}\right) > 0$ and $\left(1 - \frac{VN^*}{V^*N}\right) > 0$, $\Rightarrow \hat{G}(X, Y) \geq 0$. Hence, the DFE is globally asymptotically stable. $\square$

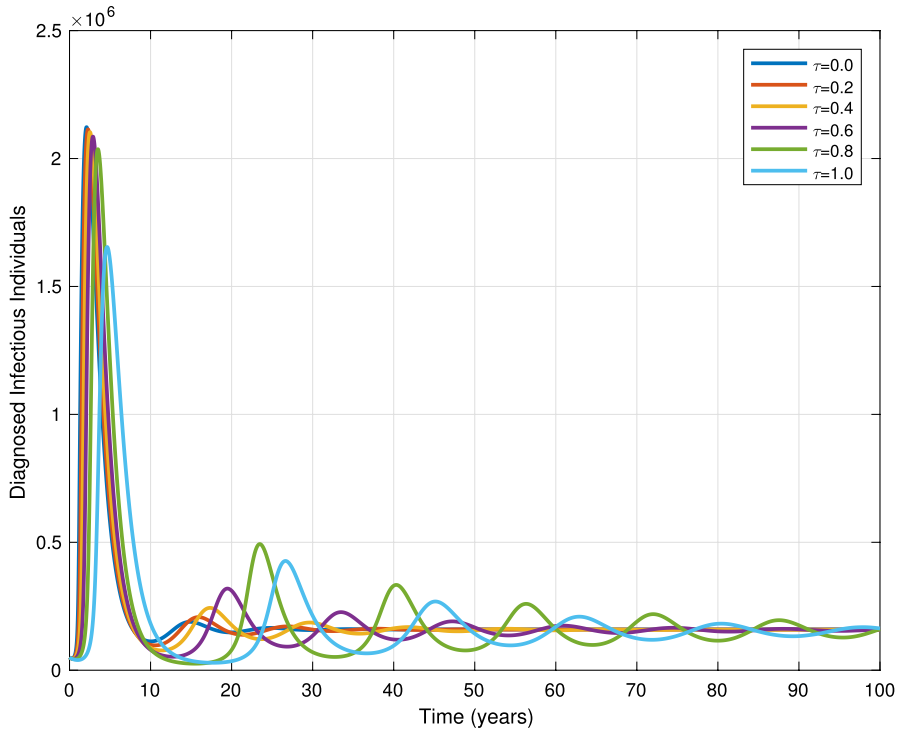


Fig. 12 Impact of vaccination (rate) with $c = 40$ (diagnosed infectious compartment)

3 Results

This section presents the results gotten when the model (3) is simulated numerically using *ODE45* package of MATLAB. The aim is to establish the impact of some key parameters (factors) considered in the model formulation. It should be noted that the simulation periods vary; some graphs are simulated for 50 years while some are for 100 years. This is done in order to understand the dynamics of TB infection over a long period of time. The parameter values are as declared in Table 2 and the initial conditions used for the simulations are $S = 8,000,000$, $V = 1,500,000$, $E = 300,000$, $I_1 = 50,000$, $I_2 = 50,000$, $T = 100,000$ and $R = 0$.

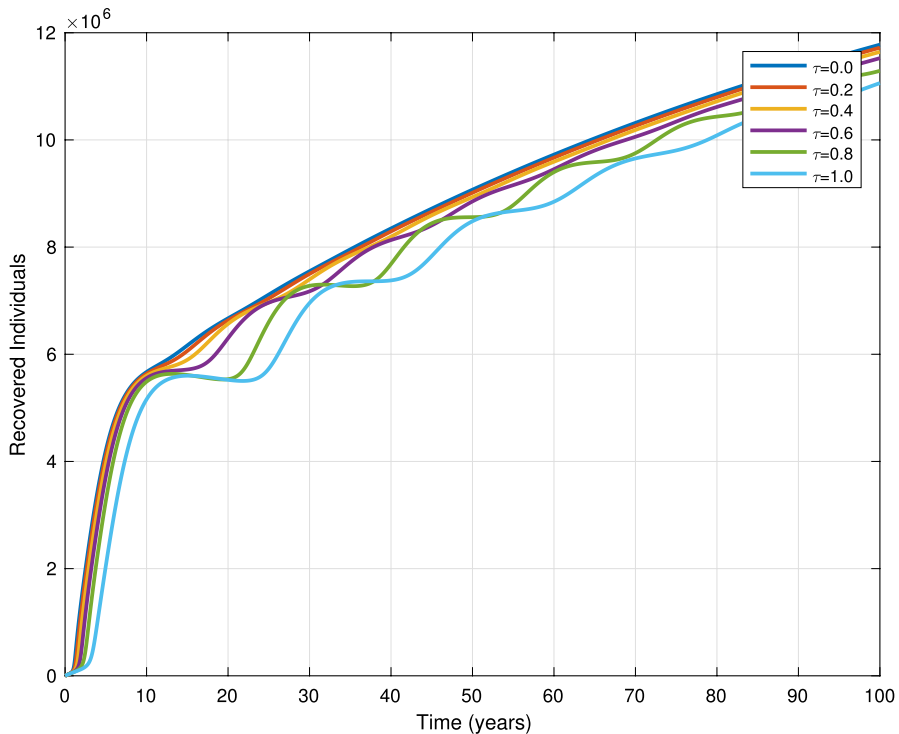


Fig. 13 Impact of vaccination (rate) with $c = 80$ (recovered compartment)

Figures 7 and 8 give the results for all the compartments as the impact of contact rate c on the disease dynamics was queried. Expectedly, reduction in the c value gives corresponding reduction in the infectious compartments (undiagnosed and diagnosed individuals) as shown in Fig. 7. However, it interestingly reveals that the infection would still persist in the population (although at lower rate). This is because the graph outcome is wavelike (i.e., the infectious population keeps on rising and falling without being steady).

Figures 9, 10, 11, 12, 13 and 14 display the relevance of vaccination in TB dynamics. The vaccination rate is varied from 0 (no vaccination program) to 1 (all susceptible individuals are vaccinated) to see its impact on the compartments I_1, I_2 and R . This variation in vaccination rate is done alongside the measurement of the relevance of the contact rate in mitigating the deadly impact of TB. Figures 9, 10, and 11 which display the results for infectious classes show that increment in the vaccination rate leads to corresponding decrement in the infectious classes. Further reductions in the population of the infectious individuals

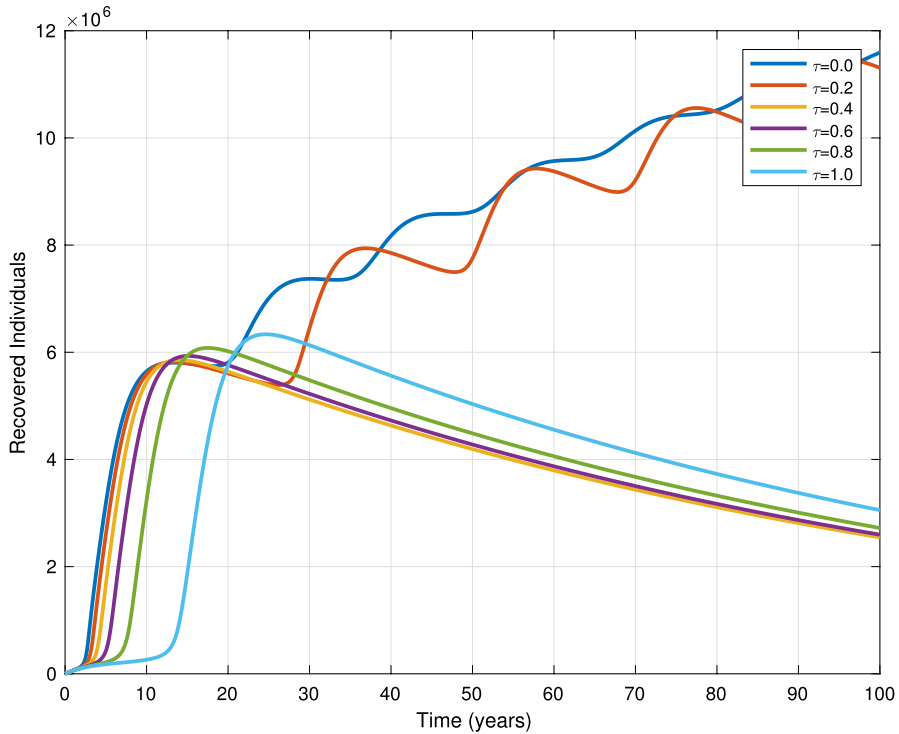


Fig. 14 Impact of vaccination (rate) with $c = 40$ (recovered compartment)

are recorded with the contact rate $c = 40$. Conversely, Fig. 13 shows that the recovered individuals' population keep on rising at different vaccination rates for the contact rate $c = 80$. This trend is as well maintained in Fig. 14 where the contact rate $c = 40$. However, Fig. 14 shows that at higher vaccination rates ($\tau = 0.6 - 1.0$), there are sharp drops in the population of the recovered individuals. This shows that lesser people get TB infection at higher vaccination rate. Subsequently, there would be less number of people getting recovered as there are little persons with the infection.

Lastly studied is the relevance of the case detection rate in the TB dynamics. It is through this detection that infectious individuals get diagnosed of TB. This was necessitated due to the established result in Sect. 2.3 that treatment may not necessarily reduce TB infection in the population, but case detection has big role to play.

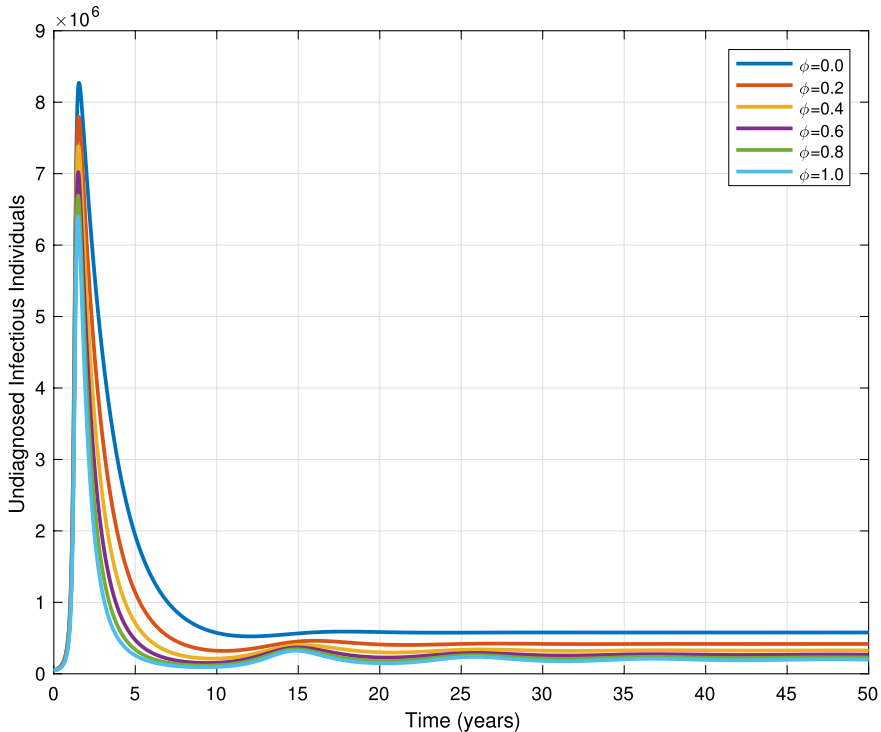


Fig. 15 Impact of TB diagnosis with $c = 80$ (undiagnosed infectious compartment)

Figures 14, 15, 16, and 17 display the impact of case detection rate on the infectious compartments. As expected, higher detection rates give corresponding decrement (increment) in the undiagnosed (diagnosed) populations. For the recovered class (Figs. 18 and 19), higher detection rate gives corresponding increment in the recovered class (at $c = 80$). This trend is replicated at $c = 40$. However, a sharp drop in the recovered group was observed at $\phi = 1$ (i.e., every infectious individual is diagnosed). This is due to the fact that diagnosing every TB patient means a reasonable proportion of the sick patients would be enrolled for treatment. Enrolment of this proportion for treatment implies there are less TB patients which translates to having less people with infection. When people with infection are less, then less number of people would get recovered (Fig. 20).

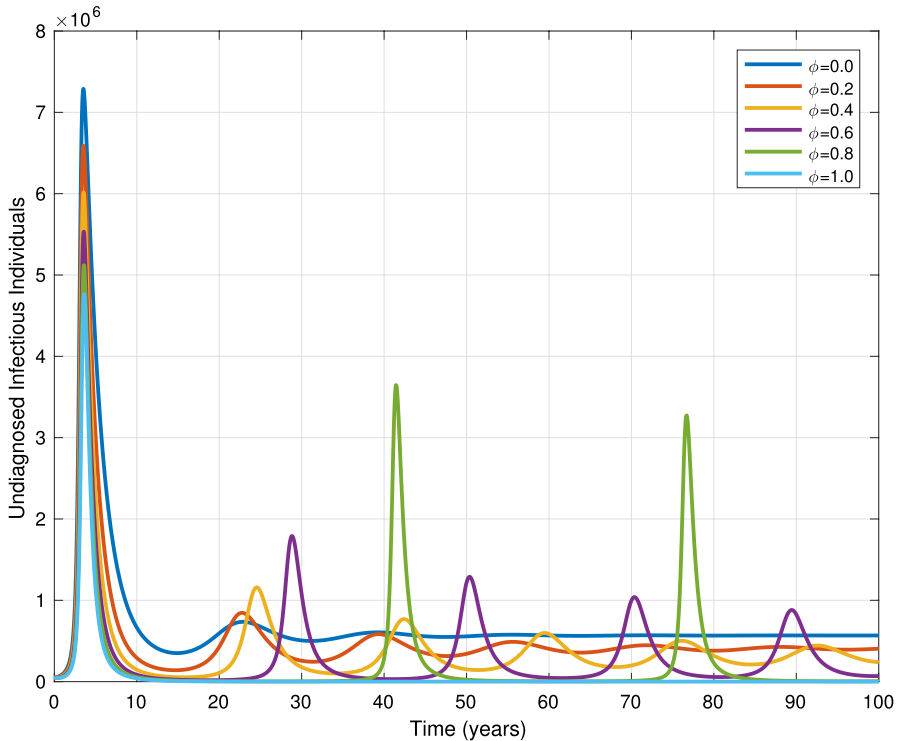


Fig. 16 Impact of TB diagnosis with $c = 40$ (undiagnosed infectious compartment)

4 Discussions and Conclusion

A mathematical model of tuberculosis (TB) with vaccination program was formulated and subsequently analyzed. The model is an extension of the model studied in Ayinla et al. (2019) where the major factor to be considered in the selection of TB vaccine candidate was presented. The extension here classifies the infectious individuals into two subgroups; diagnosed and undiagnosed subgroups.

The model is both locally and globally asymptotically stable whenever the basic reproduction number is less than unity ($R_0 < 1$). The possibility of the model undergoing backward bifurcation as a result of the exogenous reinfection characteristic of TB was established. This implies that a stable DFE coexists with a stable EEP. In

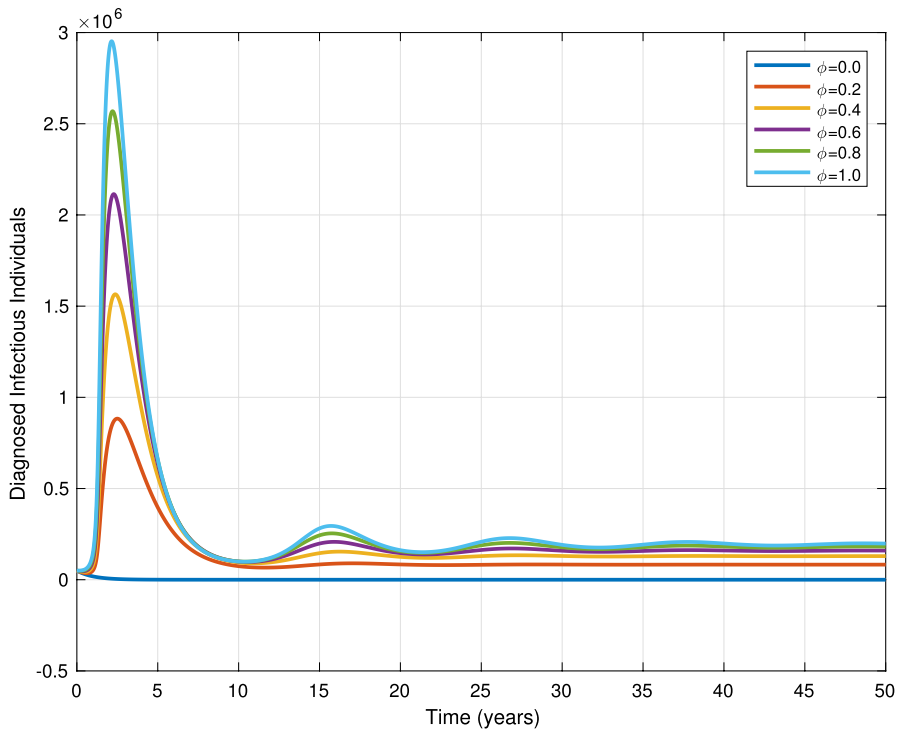


Fig. 17 Impact of TB diagnosis with $c = 80$ (diagnosed infectious compartment)

other words, keeping the basic reproduction number less than unity ($R_0 < 1$) does not guarantee keeping TB out of the population.

The basic reproduction number was studied qualitatively to determine the extent by which some of the parameters (factors) affect the TB dynamics. The factors considered are; vaccination program, diagnosis and treatment. These factors show their ability to help in managing TB epidemics. Furthermore, it was revealed that the treatment is dependent on the diagnosis, i.e., there can never be treatment without diagnosis. This shows that the major factors that actually contribute to mitigating TB spread are the vaccination program and diagnosis. For effective treatment, there must be high diagnosis of TB patients.

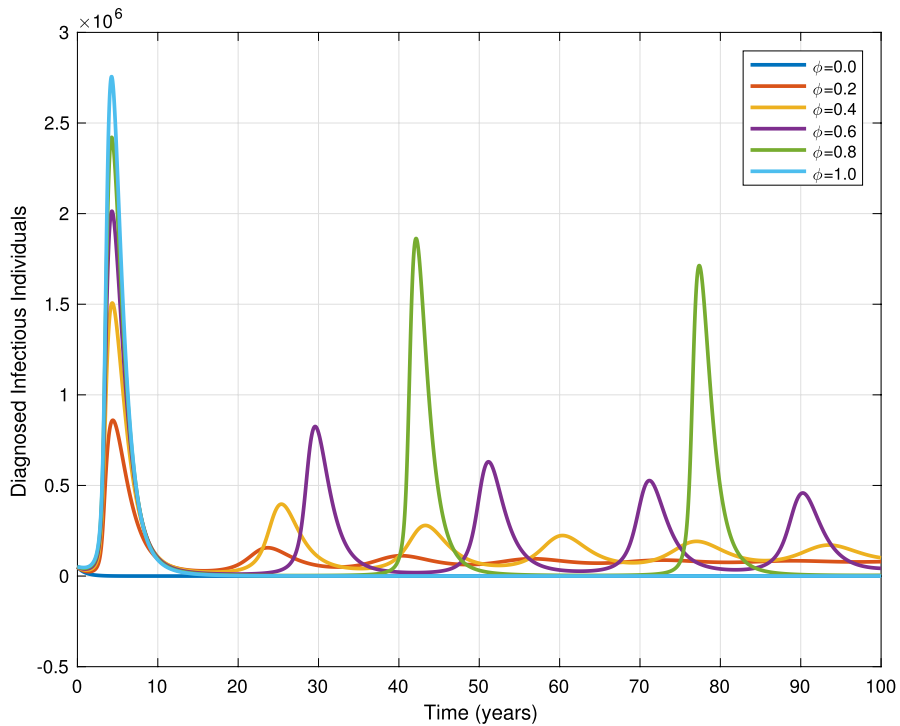


Fig. 18 Impact of TB diagnosis with $c = 40$ (diagnosed infectious compartment)

Table 2 Table of values

S/No	Parameter	Meaning	Value (yr^{-1})	Source
1	β	Probability of TB infection	0.35000	Bhunu et al. (2008)
2	Λ	Recruitment rate	300,000	Estimated
3	c	Contact rate	80.0000	Bhunu et al. (2008)
4	μ	Natural mortality rate	0.01000	Estimated
5	ε_1	TB induced death of the undiagnosed TB carrier	0.30000	Bhunu et al. (2008)
6	ε_2	TB induced death of the diagnosed TB carrier	0.30000	Estimated
7	k	Endogenous reactivation	0.00013	Bhunu et al. (2008)
8	δ_1	Exogenous reinfection	0.70000	Bhunu et al. (2008)
9	δ_2	Treatment relapse rate	0.02000	Estimated
10	τ	Vaccination rate	0.20000	Estimated
11	γ	Natural recovery rate	0.20000	Bhunu et al. (2008)
12	σ	Immunity loss rate	0.50000	Estimated
13	f	Progression rate to latent class	0.99000	Bhunu et al. (2008)
14	ϕ	Diagnosis rate	0.60000	Estimated
15	ω	Treatment rate of the diagnosed	0.70000	Estimated
16	θ	Rate at which vaccinated get infected	0.28500	WHO (2018b)
17	ψ	Recovery rate after treatment	0.60000	Estimated

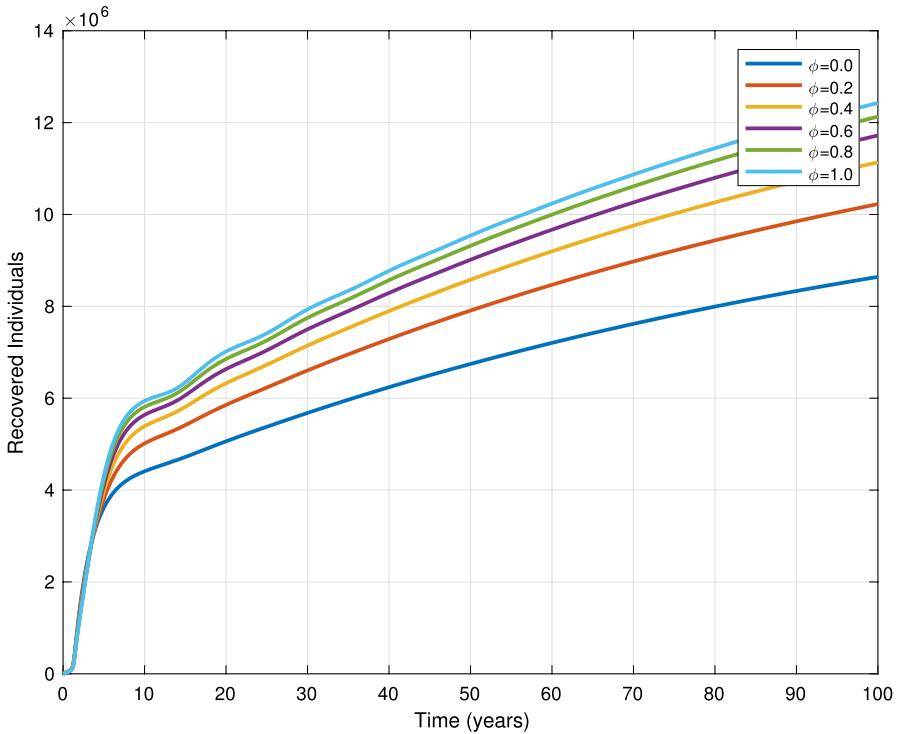


Fig. 19 Impact of TB diagnosis with $c = 80$ (recovered compartment)

The model was simulated numerically using some initial conditions to show the pictorial dynamics of the infection. Since the analysis done has established that vaccination and diagnosis are the most important factors in TB management, the impact of the contact rate alongside the vaccination and diagnosis rates were queried. Focus was given to the undiagnosed, diagnosed and recovered compartments as the infection dynamics revolve round them. In all groups (compartments), lower contact rate shows positive results in TB dynamics. It leads to the reduction in the number of infectious (undiagnosed and diagnosed) individuals and as well delay the TB disease outbreak. As for the recovered compartment, lower contact rate gives a sharp drop in the number of recovered individuals at higher vaccination and diagnosis rates. This is due to the fact that higher vaccination and diagnosis lead to fewer number of TB patients, which subsequently results to lesser number of individuals getting recovered. One of the interesting results given by the lower contact rate is that, at lower vaccination rates (0% and 20%), the TB epidemic would be endemic in the population. Increasing the vaccination rate changes the scenario and makes the epidemic manageable.

In conclusion, it has been shown from the model that vaccination and TB patients' diagnosis are important in TB management. The vaccination program should be done at reasonable rate, as 20% vaccination rate and below show that the TB infection would be endemic in the population, even at low contact rate. It is encouraged

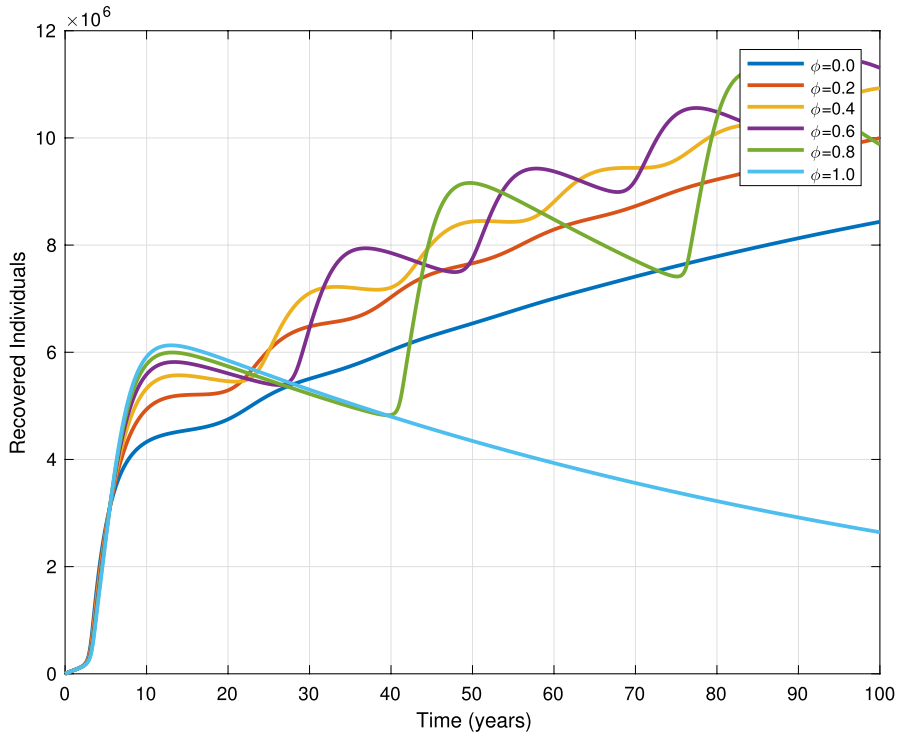


Fig. 20 Impact of TB diagnosis with $c = 40$ (recovered compartment)

that any individual with cough lasting for weeks and experiencing some of the TB symptoms as mentioned in the introduction should visit the clinic for proper diagnosis. Also, the government's policy regarding TB vaccine should be meant to cover a reasonable proportion of the population, as lesser vaccination proportion cannot send the TB epidemic out of the community.

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