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NEW INSIGHTS INTO BAZILEVIč MAPS

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Abstract. The well-known family of Bazilevič maps are studied via a class of Caratheodory-type analytic functions normalized by other than unity. The results are groundbreaking, especially that they concern this great class as a whole.

1. Backgound

This work is generally concerned with analytic functions normalized by $f(0) = 0$ and $f'(0) = 1$ and having series form $f(z) = z + a_2z^2 + \cdots$ and defined in the unit disk $E = \{z : |z| < 1\}$. Any subsequent mention of $f(z)$ or z refers to these definitions.

The study of such functions has a rich history beginning from around the turn of the last century. Among the landmark achievements of the field is the groundbreaking discovery by Bazilevič, in 1955, of functions defined by the integral

$$f(z) = \left\{ \frac{\alpha}{1+\beta^2} \int_0^z (p(t) - i\beta) t^{-\left(1+\frac{i\alpha\beta}{1+\beta^2}\right)} g(t)^{\frac{\alpha}{1+\beta^2}} \, dt \right\}^{\frac{1+i\beta}{\alpha}} \tag{1.1}$$

where $p(z) = 1 + c_1z + \cdots$ is a Caratheodory function and $g(z) = z + b_2z^2 + \cdots$ is starlike, that is, $\mathrm{Re}\ zg'(z)/g(z) > 0$. The parameters α and β are real numbers with $\alpha > 0$. Also all powers mean principal determinations only. We denote this class by $B(\alpha, \beta, g)$.

The all-important singular result of Bazilevič - that all such functions are univalent in the unit disk - have become the stairway upon which flights to many important results in the field are made. It is a great irony, however, that in spite of the important place taken by the discovery of Bazilevič, nothing more is yet known about those functions, as a whole, apart from univalence! It is our hope this work shall provide the desired direction and build bridges that would lead to successful investigation of the family as whole.

First, we shall redefine the concept upon which those functions were defined. Then we shall give a generalization of associated geometric conditions and then associate that with a modified class of Caratheodory functions, which are normalized by other than unity. The justification shall soon become evident.

To proceed, we say:

Definition. Let $\lambda = \eta + i\mu$ be a complex number with $\eta > 0$. Then we say an analytic function $h(z)$ belongs to the class P_λ if and only if

(a) $h(0) = \lambda/\eta = 1 + i\mu/\eta$, and

(b) $\mathrm{Re}\ h(z) > 0$.

Functions in P_λ have the form $h(z) = 1 + i\mu/\eta + c_1z + c_2z^2 + \cdots$ and the class of such functions include the well known class of Caratheodory functions as the case $\mathrm{Im}\ \lambda = \mu = 0$, which we denote by P as usual. It is easy to see that for any $p \in P$, $h(z)$ is in P_λ if and

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only if

$$h(z) = p(z) + i\frac{\mu}{\eta}.$$

Then it gladdens the heart to know that many of the large body of known inequalities for P remain unperturbed by the new normalization given that the real parts of both are positive and in fact the same. In particular, as we shall make use of, the well known Caratheodory inequality $|c_k| \leq 2,\ k = 1, 2, \cdots$ is also valid for $h \in P_\lambda$.

Next we say:

DEFINITION. With all parameters having their preceding definitions, a function $f(z)$ belongs to the class $B(\lambda, g)$ if and only if

$$\frac{z\left(f(z)^\lambda\right)'}{\eta z^{i\mu} g(z)^\eta} \in P_\lambda. \tag{1.2}$$

Remark 1.1. Functions in $B(\lambda, g)$ have the integral representation

$$f(z) = \left\{\eta \int_0^z t^{i\mu-1} g(t)^\eta h(t) dt\right\}^{\frac{1}{\lambda}} \tag{1.3}$$

for some $h \in P_\lambda$.

Choosing $\lambda = \alpha/(1 + i\beta)$ where $\alpha > 0$ and β are real numbers, and writing $h \in P_\lambda$ as $h(z) = p(z) + i\frac{\mu}{\eta} = p(z) - i\beta$, we have that great class, $B(\alpha, \beta, g)$, of Bazilevič functions (1.1).

Furthermore, if λ is purely real, that is $\mu = 0$, then we arrive at a generalization

$$\mathrm{Re}\frac{z\left(f(z)^\eta\right)'}{\eta g(z)^\eta} > 0$$

of many classes of functions which have had their start from setting $\beta = 0$ in (1.1). The literature in this direction is large. See [3, 5, 6] for example.

We denote the particular case $g(z) = z$ by $B(\lambda)$. If $\mu = 0$, this class is denoted by $B(\eta)$.

In the next section we give some preliminary lemmas. Section 3 contains the main results, each of which is followed by corollaries, especially for the class of Bazilevič functions, $B(\alpha, \beta, g)$.

2. Preliminary Lemmas

Let Ω denote the class of functions $\omega(z)$ which are analytic in the unit disk and satisfy $|\omega(z)| < 1$.

Lemma 2.1. [2] *Let $\omega \in \Omega$. Then*

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2}.$$

Strict inequality holds unless $\omega(z) = e^{i\sigma} z$ for some real number σ.

The well known relation between P and Ω is given by $p(z) = (1+\omega(z))/(1-\omega(z))$. Then for $h \in P_\lambda$ we say

$$h(z) = \frac{1 + \omega(z)}{1 - \omega(z)} + i\frac{\mu}{\eta}.$$

Then we find that $h'(z) = 2\omega'(z)/(1-\omega(z))^2$ so that by Lemma 1,

$$|zh'(z)| = \left|\frac{2z\omega'(z)}{(1-\omega(z))^2}\right| \leq \frac{2r}{1-r^2}\frac{1-|\omega(z)|^2}{|1-\omega(z)|^2}, |z| = r.$$

Thus we have

$$|zh'(z)| \leq \frac{2r}{1-r^2}\mathrm{Re}\, h(z).$$

This proves the following results.

Lemma 2.2. *Let $h \in P_\lambda$. Then*

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2}.$$

Corollary 2.3. *Let $h \in P_\lambda$. Then*

$$Re\frac{zh'(z)}{h(z)} \geq \frac{-2r}{1-r^2}.$$

Lemma 2.4. [4] *If $M(z)$ and $N(z)$ are regular in E, $M(0) = N(0)$, $N(z)$ maps E onto a many sheeted region which is starlike with respect to the origin and Re $M'(z)/N'(z) > 0$ in E, then Re $M(z)/N(z) > 0$ in E.*

Lemma 2.5. *If $g(z)$ is starlike, then so is the integral*

$$G(z)^\eta = \frac{\lambda + c}{z^{c+i\mu}}\int_0^z t^{c+i\mu-1}g(t)^\eta dt, \; c > 0$$

where all parameters have their preceding definitions.

Proof. From the integral $G(z)^\eta$, we have

$$z^{c+i\mu}(G(z)^\eta)' = (\lambda + c)\left[z^{c+i\mu+1}g(z)^\eta - (c+i\mu)\int_0^z t^{c+i\mu-1}g(t)^\eta dt\right].$$

Hence we have

$$\frac{zG'(z)}{G(z)} = \frac{\left[z^{c+i\mu}g(z)^\eta - (c+i\mu)\int_0^z t^{c+i\mu-1}g(t)^\eta dt\right]}{\eta\int_0^z t^{c+i\mu-1}g(t)^\eta dt} = \frac{M(z)}{N(z)}.$$

Then we have

$$\mathrm{Re}\;\frac{M'(z)}{N'(z)} = \mathrm{Re}\;\frac{zg'(z)}{g(z)} > 0$$

since $g(z)$ is starlike. Hence by the preceding lemma, we have

$$\mathrm{Re}\frac{M(z)}{N(z)} = \mathrm{Re}\frac{zG'(z)}{G(z)} > 0$$

which proves the lemma. □

3. Main Results

Theorem 3.1. *Let $f \in B(\lambda, g)$. Then*

$$Re\left\{1 + (\lambda - 1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right\} > 0 \tag{1.4}$$

in the disk $|z| < r_0(\eta)$ where $r_0(\eta)$ is given by

$$r_0(\eta) = \frac{1}{\eta}\left(\sqrt{1+\eta^2} - 1\right).$$

Proof. Since $f \in B(\lambda, g)$, then by definition there exists $h \in P_\lambda$ such that

$$\frac{\big(f(z)^\lambda\big)'}{\eta z^{i\mu-1}g(z)^\eta} = h(z).$$

Then by simple computation we find that

$$z(z(f(z)^\lambda)')' = \eta z^{i\mu}g(z)^\eta\left[zh'(z) + \left(i\mu + \eta\frac{zg'(z)}{g(z)}\right)h(z)\right]$$

so that

$$\frac{(z(f(z)^\lambda)')'}{(f(z)^\lambda)'} = i\mu + \eta\frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}.$$

Since g is any starlike functions, then by computation we find that the largest disk $|z| < r_0(\eta)$ is attained by choosing $g(z) = z$ so that, by Corollary 1,

$$\mathrm{Re}\,\frac{(z(f(z)^\lambda)')'}{(f(z)^\lambda)'} > \eta - \frac{2r}{1-r^2} \quad = \frac{\eta - 2r - \eta r^2}{1-r^2} > 0$$

provided $|z| < r_0(\eta)$ as defined. □

Observe that the disk $|z| < r_0(\eta)$ is independent of the imaginary part of the parameter λ. Thus the same result holds when $\mu = 0$. In particular we have the following corollaries.

Corollary 3.2. *If $f \in B(\alpha, \beta, g)$, then*

$$Re\left\{1 + \frac{\alpha - 1 - i\beta}{1 + i\beta}\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right\} > 0$$

in the disk $|z| < r_0(\alpha)$.

Corollary 3.3. *Let $f \in B(\lambda)$. Then $f(z)$ satisfies* (1.4) *in the disk $|z| < r_0(\eta)$.*

Furthermore, if $\mu = 0$ we have that

Corollary 3.4. *Let $f \in B(\eta)$. Then*

$$Re\left\{1 + (\eta - 1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right\} > 0$$

in the disk $|z| < r_0(\eta)$.

And if $\eta = 1$, then we have

Corollary 3.5. *The radius of convexity of functions of bounded turning is* $\sqrt{2} - 1$.

This result agrees with MacGregor's radius of convexity for functions whose derivatives have positive real parts [3].

Theorem 3.6. *Let $f \in B(\lambda, g)$. Then the set of all points $\log(\eta^{-1}z^{1-\lambda}(f(z)^\lambda)')$ is a convex set.*

Proof. Since $f \in B(\lambda, g)$, then by definition there exists $h \in P_\lambda$ such that

$$\frac{(f(z)^\lambda)'}{\eta z^{i\mu-1}g(z)^\eta} = h(z).$$

Then we have

$$\log(\eta^{-1}z^{1-\lambda}(f(z)^\lambda)') = \log h(z) + \eta\log\frac{g(z)}{z}.$$

Following similar argument as in the proof of Theorem 1 in [5], the proof is complete. □

For $B(\alpha, \beta, g)$, we have

Corollary 3.7. *If $f \in B(\alpha, \beta, g)$, then the set of all points*

$$\log\left(\frac{1+\beta^2}{1+i\beta} z^{\frac{1-\alpha+i\beta}{1+i\beta}} f(z)^{\frac{\alpha-1-i\beta}{1+i\beta}} f'(z)\right)$$

is a convex set.

Furthermore, if $\beta = 0$ we have the result of Singh [5] as follows:

Corollary 3.8. *If $f \in B(\alpha, g)$, then the set of all points $\log\left(z^{1-\alpha} f(z)^{\alpha-1} f'(z)\right)$ is a convex set.*

Theorem 3.9. *The class $B(\lambda, g)$ is closed under the integral*

$$F(z)^\lambda = \frac{\lambda + c}{z^c} \int_0^z t^{c-1} f(t)^\lambda dt, \ \lambda = \eta + i\mu, \ c > 0.$$

Proof. From the integral $F(z)^\lambda$, we have

$$z^{c+1}(F(z)^\lambda)' = (\lambda + c)\left[z^c f(z)^\lambda - c\int_0^z t^{c-1} f(t)^\lambda dt\right].$$

Since $f \in B(\lambda, g)$, then there exists a starlike function $g(z)$ such that (1.2) is satisfied for some $h \in P_\lambda$. Thus if we define $G(z)$ as

$$G(z)^\eta = \frac{\lambda + c}{z^{c+i\mu}} \int_0^z t^{c+i\mu-1} g(t)^\eta dt$$

then $G(z)$ is starlike using Lemma 4. Hence we have

$$\frac{z(F(z)^\lambda)'}{\eta z^{i\mu} G(z)^\eta} = \frac{\left[z^c f(z)^\lambda - c\int_0^z t^{c-1} f(t)^\lambda dt\right]}{\eta \int_0^z t^{c+i\mu-1} f(t)^\eta dt} = \frac{M(z)}{N(z)}$$

so that

$$\frac{M'(z)}{N'(z)} = \frac{z(f(z)^\lambda)'}{\eta z^{i\mu} g(z)^\eta}.$$

Hence by Lemma 3, we have

$$\mathrm{Re}\frac{M(z)}{N(z)} = \mathrm{Re}\frac{z(F(z)^\lambda)'}{\eta z^{i\mu} G(z)^\eta} > 0.$$

This concludes the proof. □

Corollary 3.10. *The class, $B(\alpha, \beta, g)$, of Bazilevič functions is closed under the integral*

$$F(z)^{\frac{\alpha}{1+i\beta}} = \frac{\alpha + c(1+i\beta)}{(1+i\beta)z^c} \int_0^z t^{c-1} f(t)^{\frac{\alpha}{1+i\beta}} dt, \ c > 0.$$

Theorem 3.11. *Let $f \in B(\lambda, g)$. Then*

$$|a_2| \le \frac{2\eta(1+|\lambda|)}{|\lambda(\lambda+1)|}.$$

The bound is sharp. Equality is attained by

$$f(z) = \left\{\int_0^z \frac{(\lambda + \bar{\lambda}t)t^{\lambda-1}}{(1-t)^{2\eta+1}} dt\right\}^{\frac{1}{\lambda}} = z + \frac{2\eta(1+|\lambda|)}{|\lambda(\lambda+1)|} z^2 + \cdots.$$

Proof. Since $f \in B(\lambda, g)$, then there exists a starlike function $g(z)$ and an $h \in P_\lambda$ such that

$$\frac{z\left(f(z)^\lambda\right)'}{\eta z^{i\mu} g(z)^\eta} = h(z).$$

Then we have

$$z\left(f(z)^\lambda\right)' = \eta z^{i\mu} g(z)^\eta h(z).$$

With $f(z)$, $g(z)$ and $h(z)$ having their defined series representations, by careful computation we have

$$\lambda z^\lambda + \lambda(\lambda+1)a_2 z^{\lambda+1} + \cdots = \lambda z^\lambda + \eta(\lambda b_2 + c_1)z^{\lambda+1} + \cdots.$$

Comparing coefficients we have

$$\lambda(\lambda+1)a_2 = \eta(\lambda b_2 + c_1)$$

from which the bound on a_2 follows using the inequality $|c_k| \le 2$ for $h \in P_\lambda$ and the well known $|b_2| \le 2$ for starlike functions.

Choosing $g(z) = z/(1-z)^2$ and $h(z) = (1+z)/(1-z) + i\mu/\eta$ in (1.3) we find that equality is attained by the extremal function given. □

Corollary 3.12. *Let $f \in B(\alpha, \beta, g)$. Then*

$$|a_2| \le \frac{2(1+\beta^2+\alpha\sqrt{1+\beta^2})}{\sqrt{(1+\beta^2)[(1+\alpha+\beta^2)^2+\alpha^2\beta^2]}}.$$

The inequality is sharp with equality attained by

$$f(z) = z + \frac{2(1+\beta^2+\alpha\sqrt{1+\beta^2})}{\sqrt{(1+\beta^2)[(1+\alpha+\beta^2)^2+\alpha^2\beta^2]}} z^2 + \cdots.$$

Furthermore, if $\beta = 0$, then we have $|a_2| \le 2$ and the equality is thus realized by the Koebe function $k(z) = z/(1-z)^2$.

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