SOME COMBINATORIAL RESULTS ON ALTERNATING SEMIGROUPS

BAKARE, G. N. & MAKANJUOLA, S. O.
Department of Mathematics, University of Ilorin, Nigeria
E-mail: bakaregattanaimat@gmail.com,
somakanjuola@unilorin.edu.ng
Phone No: +234-803-645-9002

Abstract
We constructed the elements of A_n^c alternating semigroup on n-objects. We also investigated the combinatorial properties of the idempotent and nilpotent elements.

Keywords: Semigroup, Symmetric Inverse semigroup, Alternating semigroup, Idempotent and Nilpotent Element.

Introduction

Let $X_n = \{1, 2, \ldots, n\}$. Then a (partial) transformation $\alpha : Dom\alpha \subseteq X_n \to Im\alpha$ is said to be full or total if $Dom\alpha = X_n$, otherwise is called Strictly Partial. It is denoted by P_n, when it is partial, T_n when it is full or total and C_n when it is one to one partial transformation semigroups. It is well known fact that if $S = C_n$ then $|S| = C_n = \sum_{r=0}^{n} \binom{n}{r}^2 r!$ Borwein [1989].

Alternating group is the group formed by even permutation on an n-objects denoted by A_n. A permutation is said to be even if it can be expressed as a product of an even number of transpositions.

The idea of even permutation can be extended to that of an-even chart.Theeven charts in C_n partial one to one forms the alternating semigroup on n-objects$A_n^c \subset C_n$.As expected $A_n^c \cap S_n = A_n$.

Even Chart: for $\alpha \in C_n$and $m \in d\alpha$, "α moves m" when $m\alpha \neq m$ and that "α *fixes* m" when $m\alpha = m$. For instance let N = {1, 2, ...} then the transposition $(i, j) \in S_n$ moves i and j while fixing each $k \in N - ij$

In other words, A chart is said to be even if it can be expressed as a product of an even number of transpositional.

Transposition: Transposition is a circuit of length two it formed by (i, j).

Transpositional: A transpositional is a chart either transposition (i, j) or semitransposition (i, j).

Idempotent: An element b in a transformation semigroup S is called an idempotent ($b^2 = b$) if and only if Im(b) = F(b)

Nilpotent: an element X in a semigroup with zero S, is a nilpotent if their exists a positive integer k such that $X^k = 0$.

Borwein (1989a) derived the formular for the number of semigroup$|S|$ in partial one-one transformation to be

$$C_n = \sum_{r=0}^{n} \binom{n}{r}^2 r!$$

$$C_0 = 1, C_1 = 2$$

Borwein (1989b) also derived the formular for the number of semigroup of the order decreasing partial one-one transformation to be the bell number

Where

$$B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} B_k, B_0 = 1 = B_1$$

Borwein (1989c) also derived the formular for the number of Idempotent $|E(S)|$of the order-decreasing partial one-one transformation to be 2^n for $n \geq 1$

Umar in (1992) derived the formular for the Nilpotent$|N(S)|$ of the order –decreasing partial one-one transformation to be bell number

B_n

Umar (2004) reported that the formular for the number of Idempotent$|E(S)|$ in partial one-one transformation to be 2^n for $n \geq 1$

Laradji (2007) also derived the formular for the number of nilpotent element for$|N(S)|$ in partial one-one transformation to be

$$|NC_n| = \sum_{r=0}^{n-1} \binom{n}{r} \binom{n-1}{r} r! = u_n$$

Where

$$U_n = (2n-1)U_{n-1} - (n-1)(n-2)U_{n-2} U_0 = 1, U_1 = 1$$

Garba (1994a) derived the formular for the number of semigroup of the order preserving partial one-one transformation to be

$$CO_n = \binom{2n}{n}$$

Garba (1994b) also derived the formular for the number of Idempotent $E(S)$ for the order preserving partial one-one transformation to be 2^n for $n \geq 1$.

Ganyuushkin in (2003a) derived the formular for the number of semigroup of the order preserving and order decreasing partial one-one transformation to be I_{n+1} where

$$I_n = \frac{1}{n} \binom{2n}{n-1}$$

Ganyuushkin in (2003b) also derived the formular for the number of Idempotent$|E(S)|$of the order preserving and order decreasing partial oneone transformation to be 2^n for $n \geq 1$

Stephen limscomb derived the formular for the number of alter=natingsemigroup in partial oneone transformation to be

$$|A_n^c| = \frac{n!}{2} + \frac{n^2}{2}(n-1)! + \sum_{r=0}^{n-2} \binom{n}{r}^2 r!$$

Identification of Elements of A_n^c InC_n

The List

The elements in C_1

For n=1, has 2 elements.

C_1	$\{1\}$
	$\binom{1}{1}\binom{1^+}{1}$

The elements in C_2

For n=2, has 7 elements.

C_2	$\{1, 2\}$
	$\binom{12^+}{12}\binom{12}{21}\binom{12^+}{1-}\binom{12}{-1}\binom{12}{2-}\binom{12^+}{-2}\binom{12^+}{--}$

The elements in C_3

For n=3, has 34 elements

C_3	$\{1, 2, 3\}$
	$\binom{123}{123^+}\binom{123}{132}\binom{123}{213}\binom{123^+}{231}\binom{123^+}{312}\binom{123}{321}\binom{123^+}{12-}\binom{123}{1-2}$
	$\binom{123^+}{-12}\binom{123}{13-}\binom{123^+}{1-3}\binom{123}{-13}\binom{123}{21-}\binom{123^+}{21-}\binom{123}{-21}\binom{123^+}{23-}$
	$\binom{123}{2-3}\binom{123^+}{-23}\binom{123^+}{31-}\binom{123}{3-1}\binom{123^+}{-31}\binom{123}{32-}\binom{123^+}{3-2}\binom{123}{-32}$
	$\binom{123^+}{1--}\binom{123^+}{-1-}\binom{123^+}{--1}\binom{123^+}{2--}\binom{123^+}{-2-}\binom{123^+}{--2}\binom{123^+}{3--}$
	$\binom{123^+}{-3-}\binom{123^+}{--3}\binom{123^+}{---}$

The elements of partial one-one transformation semigroup of C_4 and C_5 were also generated but because of the space we could not display it. The table below shown the results

The Elements of Partial one-one Transformation Semigroup and Alternating Semigroup

Table 1

N	C_n	A_n^c
1	2	1
2	7	4
3	34	22
4	209	149
5	1546	1186

From the table 1 above, the sequences 2, 7, 34, 209, 1546,... the general formular which was derived by [1] was

$$C_n = \sum_{r=0}^{n} \binom{n}{r}^2 r!$$

$n \geq 2$. moreover, for

$C_n = 2nC_{n-1} - (n-1)^2 C_{n-2}$ with $r_0 = 1$ and $r_1 = 2$

According to [12] the sequences 1, 4, 22, 149, 1186, ... given the formular

From the sequences 1, 4, 8, 16, 32,... and 1, 1, 13, 49, 501,... in table 2, we were able to derived the formular for the numbers of Idempotent and Nilpotent elements in Alternating semi groups as

$$|EA_n^c| = 2^n \text{ for } n \geq 2$$

and

$$|NA_n^c| = \begin{cases} \sum_{r=0}^{n-1} \binom{n}{r}\binom{n-1}{r} r! & \text{if n is odd} \\ \sum_{r=0}^{n-2} \binom{n}{r}\binom{n-1}{r} r! & \text{if n is even} \end{cases}$$

Discussion and Conclusion

In this paper, partial one-one transformation semigroups has been successfully used to identified the number of elements of alternating semigroup A_n^c and the formular for the number of elements of idempotent and nilpotent were derived. The proof for the formular obtained is still in progress. Based on this paper, we have strong believe that many things can be deduced if further studies could be done.The theory of semigroup has it's scope widened to embrace many aspects of theoretical computer sciences, such as: automata theory, coding theory,computational theory and formal languages as well as applications in the sciences. It's also assist people and (Computers) in sorting data and designing better networks.

Acknowledgement: we would like to gratefuly acknowledge the support of Dr. A. Umar, Department of Mathematics and Statistics Qaboos University OMAN.

References

Borwein, D., Rankin, S. & Renner, L. (1989). Enumeration of injective partial transformations. *Discrete Math.73, 291-296.*

Ganyushkin, E. & Mazorchuk, V. (2009). *Classical Finite Transformation Semi groups: An introduction.* London: Springer.

Ganyushkin, E. & Mazorchuk, V. (2004). Combinatorics of nilpotent in symmetric inverse semi groups. *Ann. Comb. 8 (2) 161-175.*

Garba, G. U. (1994). Nilpotents in semigroups of partial one-to-one order-preserving mappings. *Semigroup Forum, 48, 37-49.*

Howie, J. M. (1995). *Fundamentals of Semigroup Theory.* Oxford: Clarendon Press.

Laradji, A. & Umar, A. (2004). On the number of nilpotents in the partial symmetrics semigroup. *Comm. Algebra, 32, 3017-3023.*

Laradji, A. & Umar, A. (2007). Combinatorial results for the symmetric inverse semigroup. *Semigroup Forum, 75 221-236.*

Limpscomb, S. (1996). Symmetric Inverse Semigroups. Mathematical Surveys of the American mathematical Society, no 46 providence, R L.

Munn, W. D. (1957). *The characters of the symmetric inverse- Semigroup.* Proc. Cambridge Philos. Soc. 53, 13-18

Umar, A. (1993). *On the semigroup one-to-one order-decreasing finite transformations.* Proc. Roy. Soc. Edinburgh, 123A, 355-363.

Journal of Science, Technology, Mathematics and Education (JOSTMED), 9(3), August, 2013

JOSTMED, VOLUME 9(3), AUGUST, 2013

JOURNAL OF SCIENCE, TECHNOLOGY, MATHEMATICS AND EDUCATION (JOSTMED)

ARTICLES AND RESEARCH REPORTS ON EDUCATION