

CUBIC TRANSMUTED LINDLEY AS A BETTER DISTRIBUTION IN THE FAMILY OF RAYLEIGH DISTRIBUTIONS

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ABSTRACT: A new statistical model for non-normal data is proposed with some of its statistical properties as Cubic of the Transmuted Lindley distribution (CTLTD), based on a new family of life time distribution. The new model contains some of lifetime distributions as special cases such as exponentiated Lindley, transmuted Lindley and Lindley distributions. These include the density and hazard rate functions with their behavior, moments, and moment generating function, skewness and kurtosis measures. Maximum likelihood estimation of the parameters and their estimated are derived. An application of the model to a real data set is presented and compared with the fit attained by other well-known existing distributions.

KEYWORDS: Transmuted Lindley distribution; maximum likelihood method; transmutation map; hazard rate function, reliability function, parameter estimation

1. INTRODUCTION

The quality of the techniques or methods adopted in a statistical analysis relied heavily on the assumed probability model or distribution. As a result of this, considerable effort has been made in the development of large classes of standard probability distributions with relevant statistical methodologies. However, there are still remaining many important problems where the real data does not follow any of the classical or standard probability models. Since there is a clear need for extended forms of these distributions a significant progress has been made toward the generalization of some well known distributions and their successful application to problems in areas such as engineering, finance, economics and biomedical sciences, among others, see [1][4][7][9][14][16][21][22] and [23].

Numerous lifetime data used in statistical analysis depends on a particular statistical distribution. Knowledge of suitable distribution of real data will extremely ameliorates the efficiency and the power of the statistical tests involved with it. Therefore, several distributions are suggested for modeling lifetime data. However, there are still many life time data that does not follow any distribution and hence there is a need to extend some new distributions. Here we suggested a new distribution for fitting

lifetime data using one of the wellknown distribution function generation methods. The cubic transmuted Lindley distribution is defined for modeling waiting and survival time's data.

The quadratic transmuted family of distributions by [24], has opened doors to extend the existing probability models to capture the quadratic behavior of the data.

Properties and applications of Lindley distribution in reliability analysis were studied by [12], showing that this distribution may provide a better fit than the exponential distribution. [12] discussed Lindley distribution and its applications extensively and showed that the Lindley distribution fits better than the exponential distribution based on the waiting time at a bank for service, see also [3][9][11][13][18][19][20].

[12] argue that the Lindley distribution could be a better lifetime model than the exponential distribution through a numerical example. In addition, they show that the hazard function of the Lindley distribution does not exhibit a constant hazard rate, indicating the flexibility of the Lindley distribution over the exponential distribution.

2. CUBIC TRANSMUTED LINDLEY DISTRIBUTION (CTLTD)

2.1 Derivation and Development of CTLTD

A random variable X is said to have a cubic transmuted distribution if its cumulative distribution function (cdf) is given by (1)

$$f(x) = g(x)[(1 + \lambda) - 4\lambda G(x) + 3\lambda(G(x))^2] \quad (1)$$

$$|\lambda| \leq 1$$

And the density function is given by eq. (2)

$$F(x) = (1+\lambda)G(x) - 2\lambda(G(x))^2 + \lambda(G(x))^3$$

$$F(x) = G(x)[(1 + \lambda) - 2\lambda G(x) + \lambda(G(x))^2] \quad (2)$$

Where $g(x)$ and $G(x)$ are the cumulative distribution and density function of the parent distribution. The parent distribution can be any probability distribution from which we want to obtain its transmuted version.

Given that $g(x) = \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x}$ and $G(x) = 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta}$ are the pdf and cdf of Rayleigh distribution (parent distribution), then the Cubic transmuted Rayleigh can be obtained as follows:

$$\begin{aligned} f_{CTLD}(x; \theta, \lambda) &= \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} \left[1 + \lambda - 4\lambda \left(1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \right) + 3\lambda \left(1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \right)^2 \right] \\ &= \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} \left[1 + \lambda - 4\lambda + \frac{4\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + 3\lambda \left(1 - \frac{2e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \right) + \frac{e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \\ &= \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} \left[1 - 3\lambda + \frac{4\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + 3\lambda - \frac{6\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \frac{3\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \\ &= \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} \left[1 - \frac{2\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \frac{3\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \\ f_{CTLD}(x; \theta, \lambda) &= \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} - \frac{2\lambda\theta^2}{(1+\theta)^2}(1+x)(1+\theta+\theta x)e^{-2\theta x} + \frac{3\lambda\theta^2}{(1+\theta)^3}(1+x)(1+\theta+\theta x)^2e^{-3\theta x} \end{aligned} \quad (3)$$

Where eq. (3) above is the pdf of Cubic Transmuted Lindley Distribution (CTLD)

The Cumulative distribution function (cdf) of the same hybrid CTLD can be obtained using eq. (2).

From eq. (2), it follows that:

$$\begin{aligned} F_{CTLD}(x; \theta, \lambda) &= G(x)[(1+\lambda) - 2\lambda G(x) + \lambda(G(x))^2] \\ &= 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \left[1 + \lambda - 2\lambda \left(1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \right) + \lambda \left(1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \right)^2 \right] \\ &= 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \left[1 + \lambda - 2\lambda + \frac{2\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \lambda \left(1 - \frac{2e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \frac{e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right) \right] \\ &= 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \left[1 - \lambda + \frac{2\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \lambda - \frac{2\lambda e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \frac{\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \\ &= 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \left[1 + \frac{\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \\ &= 1 + \frac{\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} - \frac{\lambda e^{-3\theta x(1+\theta+\theta x)^3}}{(1+\theta)^3} \\ F_{CTLD}(x; \theta, \lambda) &= 1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \lambda \left(\frac{e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} - \frac{e^{-3\theta x(1+\theta+\theta x)^3}}{(1+\theta)^3} \right) \end{aligned} \quad (4)$$

The eq. (4) above represent the cdf of Cubic Transmuted Lindley distribution (CTLD)

2.2. Moment

$$E(x^r) = \int_0^\infty x^r f_{CTLD}(x; \theta, \lambda) dx \quad (5)$$

$$= \int_0^\infty x^r \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} - \frac{2\lambda\theta^2}{(1+\theta)^2}(1+x)(1+\theta+\theta x)e^{-2\theta x} + \frac{3\lambda\theta^2}{(1+\theta)^3}(1+x)(1+\theta+\theta x)^2e^{-3\theta x} dx \quad (16)$$

$$= \int_0^\infty x^r \frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} dx - \int_0^\infty x^r \frac{2\lambda\theta^2}{(1+\theta)^2}(1+x)(1+\theta+\theta x)e^{-2\theta x} dx + \int_0^\infty x^r \frac{3\lambda\theta^2}{(1+\theta)^3}(1+x)(1+\theta+\theta x)^2e^{-3\theta x} dx$$

$$E(x^r) = p - q + m \quad (6)$$

Where:

$$p = \frac{1}{\theta^r(1+\theta)} [\theta \Gamma(r+1) + \Gamma(r+2)] \quad (7)$$

$$q = \frac{2\lambda\theta^2}{(1+\theta)^2} \int_0^\infty x^r (1+x)(1+\theta+\theta x)e^{-2\theta x} dx \quad (8)$$

Let $y = 2\theta x$; $x = \frac{y}{2\theta}$ and $dx = \frac{dy}{2\theta}$,

the (8) becomes (9)

$$= \frac{2\lambda\theta^2}{(1+\theta)^2} \int_0^\infty x^r (1+x)(1+\theta+\theta x)e^{-y} \frac{dy}{2\theta} \quad (9)$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \int_0^\infty x^r (1+x)(1+\theta+\theta x)e^{-y} dy$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \int_0^\infty x^r + x^{r+1}(1+\theta+\theta x) e^{-y} dy$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \int_0^\infty (x^r + \theta x^r + 2\theta x^{r+1} + x^{r+1} + \theta x^{r+2}) e^{-y} dy$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \int_0^\infty \left(\left(\frac{y}{2\theta} \right)^r + \theta \left(\frac{y}{2\theta} \right)^r + 2\theta \left(\frac{y}{2\theta} \right)^{r+1} + \left(\frac{y}{2\theta} \right)^{r+1} + \theta \left(\frac{y}{2\theta} \right)^{r+2} \right) e^{-y} dy$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \int_0^\infty y^r e^{-y} dy + \frac{\theta}{(2\theta)^r} \int_0^\infty y^r e^{-y} dy + \frac{2\theta}{(2\theta)^{r+1}} \int_0^\infty y^{r+1} e^{-y} dy + \frac{1}{(2\theta)^{r+1}} \int_0^\infty y^{r+1} e^{-y} dy + \frac{\theta}{(2\theta)^{r+2}} \int_0^\infty y^{r+2} e^{-y} dy \right] =$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \int_0^\infty y^r e^{-y} dy + \frac{\theta}{(2\theta)^r} \int_0^\infty y^r e^{-y} dy + \frac{2\theta}{(2\theta)^{r+1}} \int_0^\infty y^{r+1} e^{-y} dy + \frac{1}{(2\theta)^{r+1}} \int_0^\infty y^{r+1} e^{-y} dy + \frac{\theta}{(2\theta)^{r+2}} \int_0^\infty y^{r+2} e^{-y} dy \right] =$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$= \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right]$$

$$\begin{aligned}
 &= \frac{\lambda\theta}{(1+\theta)^3} \int_0^\infty [x^r + 2\theta x^r + 2\theta x^{r+1} + \theta^2 x^r + 2\theta^2 x^{r+1} + \\
 &\quad \theta^2 x^{r+2} + x^{r+1} + 2\theta x^{r+1} + 2\theta x^{r+2} + \theta^2 x^{r+1} + \\
 &\quad 2\theta^2 x^{r+2} + \theta^2 x^{r+3}] e^{-w} dw \\
 &= \frac{\lambda\theta}{(1+\theta)^3} \int_0^\infty [x^r + 2\theta x^r + 4\theta x^{r+1} + \theta^2 x^r + 3\theta^2 x^{r+1} + \\
 &\quad 3\theta^2 x^{r+2} + x^{r+1} + 2\theta x^{r+2} + \theta^2 x^{r+3}] e^{-w} dw \\
 &= \frac{\lambda\theta}{(1+\theta)^3} \int_0^\infty [x^r (1 + 2\theta + \theta^2) + x^{r+1} (4\theta + 3\theta^2 + 1) + \\
 &\quad (3\theta^2 + 2\theta) x^{r+2} + \theta^2 x^{r+3}] e^{-w} dw \\
 &= \frac{\lambda\theta}{(1+\theta)^3} [(1 + 2\theta + \theta^2) \int_0^\infty x^r e^{-w} dw + (4\theta + 3\theta^2 + \\
 &\quad 1) \int_0^\infty x^{r+1} e^{-w} dw + (3\theta^2 + 2\theta) \int_0^\infty x^{r+2} e^{-w} dw + \\
 &\quad \theta^2 \int_0^\infty x^{r+3} e^{-w} dw] \\
 &= \frac{\lambda\theta}{(1+\theta)^3} \left[(1 + 2\theta + \theta^2) \int_0^\infty \left(\frac{w}{3\theta}\right)^r e^{-w} dw + \right. \\
 &\quad (4\theta + 3\theta^2 + 1) \int_0^\infty \left(\frac{w}{3\theta}\right)^{r+1} e^{-w} dw + \\
 &\quad \left. \theta^2 \int_0^\infty \left(\frac{w}{3\theta}\right)^{r+3} e^{-w} dw \right] \\
 &= \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)}{(3\theta)^r} \int_0^\infty w^r e^{-w} dw + \right. \\
 &\quad \frac{(4\theta+3\theta^2+1)}{(3\theta)^{r+1}} \int_0^\infty w^{r+1} e^{-w} dw + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \int_0^\infty w^{r+2} e^{-w} dw + \\
 &\quad \left. \frac{\theta^2}{(3\theta)^{r+3}} \int_0^\infty w^{r+3} e^{-w} dw \right] \\
 &= \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)}{(3\theta)^r} \Gamma(r+1) + \frac{(4\theta+3\theta^2+1)}{(3\theta)^{r+1}} \Gamma(r+2) + \right. \\
 &\quad \left. \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+3) + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+4) \right]
 \end{aligned}$$

$$E(x^r) = p - q + m \quad \text{From (6)}$$

$$\begin{aligned}
 E(x^r) &= \frac{1}{\theta^r(1+\theta)} [\theta \Gamma(r+1) + \Gamma(r+2)] - \\
 &\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \right. \\
 &\quad \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \\
 &\quad \left. \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)}{(3\theta)^r} \Gamma(r+1) + \right. \\
 &\quad \frac{(4\theta+3\theta^2+1)}{(3\theta)^{r+1}} \Gamma(r+2) + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+3) + \\
 &\quad \left. \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+4) \right]
 \end{aligned}$$

$$\begin{aligned}
 E(x^r) &= \left[\frac{1}{\theta^r(1+\theta)} [\theta \Gamma(r+1) + \Gamma(r+2)] - \right. \\
 &\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \right. \\
 &\quad \left. \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)}{(3\theta)^r} \Gamma(r+1) + \frac{(4\theta+3\theta^2+1)}{(3\theta)^{r+1}} \Gamma(r+2) + \right. \\
 &\quad \left. \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+3) + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+4) \right] \Bigg]
 \end{aligned}$$

$$\begin{aligned}
 E(x) &= \frac{1}{\theta(\theta+1)} [\theta \Gamma(2) + \Gamma(3)] - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{(1+\theta)\Gamma(2)}{2\theta} + \right. \\
 &\frac{(2\theta+1)\Gamma(3)}{(2\theta)^2} + \frac{\theta\Gamma(4)}{(2\theta)^3} + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)\Gamma(2)}{3\theta} + \right. \\
 &\quad \frac{(1+4\theta+3\theta^2)\Gamma(3)}{(3\theta)^2} + \frac{(2\theta+3\theta^2)\Gamma(4)}{(3\theta)^3} + \frac{\theta^2\Gamma(4)}{(3\theta)^4} \Bigg] \quad r = 1
 \end{aligned}$$

$$\begin{aligned}
 E(x^2) &= \frac{1}{\theta(\theta+1)} [\theta \Gamma(3) + \Gamma(4)] - \\
 &\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{(1+\theta)\Gamma(3)}{(2\theta)^2} + \frac{(2\theta+1)\Gamma(4)}{(2\theta)^3} + \frac{\theta\Gamma(5)}{(2\theta)^4} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)\Gamma(3)}{(3\theta)^2} + \frac{(1+4\theta+3\theta^2)\Gamma(4)}{(3\theta)^3} + \right. \\
 &\quad \left. \frac{(2\theta+3\theta^2)\Gamma(5)}{(3\theta)^4} + \frac{\theta^2\Gamma(6)}{(3\theta)^5} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2(1+\theta)}{(2\theta)^2} + \frac{6(2\theta+1)}{(2\theta)^3} + \frac{24\theta}{(2\theta)^4} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{2(1+2\theta+\theta^2)}{(3\theta)^2} + \frac{6(1+4\theta+3\theta^2)}{(3\theta)^3} + \frac{24(2\theta+3\theta^2)}{(3\theta)^4} + \right. \\
 &\quad \left. \frac{120\theta^2}{(3\theta)^5} \right]
 \end{aligned}$$

$$\begin{aligned}
 E(x^2) &= \frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \quad r = 2
 \end{aligned}$$

$$\begin{aligned}
 E(x^3) &= \frac{1}{\theta(\theta+1)} [\theta \Gamma(4) + \Gamma(5)] - \\
 &\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{(1+\theta)\Gamma(4)}{(2\theta)^3} + \frac{(2\theta+1)\Gamma(5)}{(2\theta)^4} + \frac{\theta\Gamma(6)}{(2\theta)^5} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)\Gamma(4)}{(3\theta)^3} + \frac{(1+4\theta+3\theta^2)\Gamma(5)}{(3\theta)^4} + \right. \\
 &\quad \left. \frac{(2\theta+3\theta^2)\Gamma(6)}{(3\theta)^5} + \frac{\theta^2\Gamma(7)}{(3\theta)^6} \right] \quad r = 3
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{6(1+\theta)}{(2\theta)^3} + \frac{24(2\theta+1)}{(2\theta)^4} + \frac{120\theta}{(2\theta)^5} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{6(1+2\theta+\theta^2)}{(3\theta)^3} + \frac{24(1+4\theta+3\theta^2)}{(3\theta)^4} + \frac{120(2\theta+3\theta^2)}{(3\theta)^5} + \right. \\
 &\quad \left. \frac{720\theta^2}{(3\theta)^6} \right] \\
 &= \frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right]
 \end{aligned}$$

$$\begin{aligned}
 E(x^4) &= \frac{1}{\theta(\theta+1)} [\theta \Gamma(5) + \Gamma(6)] - \\
 &\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{(1+\theta)\Gamma(5)}{(2\theta)^4} + \frac{(2\theta+1)\Gamma(6)}{(2\theta)^5} + \frac{\theta\Gamma(7)}{(2\theta)^6} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)\Gamma(5)}{(3\theta)^4} + \frac{(1+4\theta+3\theta^2)\Gamma(6)}{(3\theta)^5} + \right. \\
 &\quad \left. \frac{(2\theta+3\theta^2)\Gamma(7)}{(3\theta)^6} + \frac{\theta^2\Gamma(8)}{(3\theta)^7} \right] \quad r = 4 \\
 &= \frac{24(\theta+4)}{\theta^4(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{24(1+\theta)}{(2\theta)^4} + \frac{120(2\theta+1)}{(2\theta)^5} + \frac{720\theta}{(2\theta)^6} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{24(1+2\theta+\theta^2)}{(3\theta)^4} + \frac{120(1+4\theta+3\theta^2)}{(3\theta)^5} + \frac{720(2\theta+3\theta^2)}{(3\theta)^6} + \right. \\
 &\quad \left. \frac{5040\theta^2}{(3\theta)^7} \right]
 \end{aligned}$$

$$\begin{aligned}
 E(x^4) &= \frac{24(\theta+4)}{\theta^4(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{120\theta^2+72\theta+120}{8\theta^4} \right] + \\
 &\frac{\lambda}{(1+\theta)^3} \left[\frac{72\theta^3+5040\theta^2+1272\theta+1160}{243\theta^4} \right]
 \end{aligned}$$

2.3 Statistical properties

This section investigates the statistical properties of the CTLD distribution such as the moments, the moment generating function, skewness, kurtosis, reliability, hazard function, odd function cumulative hazard, mean time to failure and method of maximum likelihood.

3. MOMENT ABOUT THE MEAN

3.1 Variance

$$\begin{aligned}
 \mu_2 &= Var(x) = E(x^2) - (E(x))^2 \quad (12) \\
 &= \frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \\
 &\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] - \left(\frac{\theta+2}{\theta(\theta+1)} - \right.
 \end{aligned}$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right]^2$$

$$\mu_3 = E(x - \mu)^3$$

By applying Binomial Expansion

$$= \binom{3}{0} x^0 \cdot (-\mu)^{3-0} + \binom{3}{1} x^1 \cdot (-\mu)^{3-1} +$$

$$\binom{3}{2} x^2 \cdot (-\mu)^{3-2} + \binom{3}{3} x^3 \cdot (-\mu)^{3-3}$$

$$= E((-\mu)^3 + 3x(-\mu)^2 + 3x^2(-\mu) + x^3) = E(x^3 -$$

$$3\mu x^2 + 3x\mu^2 - \mu^3)$$

$$= Ex^3 - 3\mu Ex^2 + 3\mu^2 Ex - \mu^3$$

$$= \left[\frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \right.$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] - 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right)^2 +$$

$$3 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \right.$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^3 \Bigg]$$

$$= \left[\frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \right.$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] - 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right)^2 + 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^3 \Bigg]$$

$$\mu_3 = \left[\frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \right.$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] - 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right)^2 + 2 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^3 \Bigg]$$

$$\mu_4 = E(x - \mu)^4$$

$$= \binom{4}{0} x^0 \cdot (-\mu)^{4-0} + \binom{4}{1} x^1 \cdot (-\mu)^{4-1} +$$

$$\binom{4}{2} x^2 \cdot (-\mu)^{4-2} + \binom{4}{3} x^3 \cdot (-\mu)^{4-3} +$$

$$\binom{4}{4} x^4 \cdot (-\mu)^{4-4}$$

$$= E[(-\mu)^4 + 4x(-\mu)^3 + 6x^2(-\mu)^2 + 4x^3(-\mu) + x^4]$$

$$= Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 4\mu^3 Ex + \mu^4$$

$$= \frac{24(\theta+4)}{\theta^4(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{120\theta^2+72\theta+120}{8\theta^4} \right] +$$

$$\frac{\lambda}{(1+\theta)^3} \left[\frac{72\theta^3+504\theta^2+1272\theta+1160}{243\theta^4} \right] - 4 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{6(\theta+4)}{\theta^3(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right) - 4 \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^4 + \left(\frac{\theta+2}{\theta(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] +$$

$$\frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \left(\frac{6(\theta+4)}{\theta^3(\theta+1)} - \right.$$

$$\frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] +$$

$$\begin{aligned} & \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] + 6 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right) - 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \right. \\ & \left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \right. \\ & \left. \mu_4 = \left[\frac{24(\theta+4)}{\theta^4(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{120\theta^2+72\theta+120}{8\theta^4} \right] + \frac{\lambda}{(1+\theta)^3} \left[\frac{72\theta^3+5040\theta^2+1272\theta+1160}{243\theta^4} \right] - 4 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \right. \right. \\ & \left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right) \left(- \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] \right) + 6 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right) \left(- \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] \right)^2 \\ & \left. \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \right. \\ & \left. \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right) - 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^4 \left. \right] \\ & \gamma_1(x) = \frac{(\mu_3)^2}{(\mu_2)^3} \end{aligned} \quad (13)$$

3.2 Skewness

$$\gamma_1(x) = \frac{\left[\left[\frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] - \right. \right. \\ \left. \left. 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right) \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right)^2 \right. \right. \\ \left. \left. + 2 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^3 \right] \right] \\ \left[\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] - \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \right]^3 \right]$$

3.3 Kurtosis

$$\gamma_2(x) = \frac{\mu_4}{(\mu_2)^2} \quad (14)$$

$$\left[\left[\frac{24(\theta+4)}{\theta^4(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{120\theta^2+72\theta+120}{8\theta^4} \right] + \frac{\lambda}{(1+\theta)^3} \left[\frac{72\theta^3+5040\theta^2+1272\theta+1160}{243\theta^4} \right] - \right. \right. \\ \left. \left. 4 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right) \left(\frac{6(\theta+4)}{\theta^3(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{3\theta^2+15\theta+21}{4\theta^4} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+126\theta+184}{81\theta^4} \right] \right) + \right. \right. \\ \left. \left. 6 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \left(\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] \right) - \right. \right. \\ \left. \left. 3 \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^4 \right] \right] \\ \left[\frac{2(\theta+3)}{\theta^2(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{2\theta^2+8\theta+9}{4\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{18\theta^3+108\theta^2+162\theta+106}{81\theta^4} \right] - \left(\frac{\theta+2}{\theta(\theta+1)} - \frac{\lambda\theta}{(1+\theta)^2} \left[\frac{4\theta^2(1+\theta)+4\theta(2\theta+1)+6\theta}{8\theta^3} \right] + \frac{\lambda\theta}{(1+\theta)^3} \left[\frac{9\theta^3+36\theta^2+51\theta+26}{27\theta^2} \right] \right)^2 \right]^2 \right]$$

3.4 Reliability

$$\begin{aligned} R(t) &= 1 - F_{CTLD}(x; \theta, \lambda) = 1 - \left[1 - \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} + \right. \\ & \left. \lambda \left(\frac{e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} - \frac{e^{-3\theta x(1+\theta+\theta x)^3}}{(1+\theta)^3} \right) \right] \\ &= \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} - \lambda \left(\frac{e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} - \frac{e^{-3\theta x(1+\theta+\theta x)^3}}{(1+\theta)^3} \right) \\ &= \frac{e^{-\theta x(1+\theta+\theta x)}}{1+\theta} \left[1 - \frac{\lambda e^{-\theta x(1+\theta+\theta x)}}{(1+\theta)} + \frac{\lambda e^{-2\theta x(1+\theta+\theta x)^2}}{(1+\theta)^2} \right] \end{aligned}$$

3.5 Hazard Function

The other characteristic of interest of a random variable is the hazard function (HF). The new cubic transmuted Lindley distribution also known as instantaneous failure rate denoted by $H_{CTLD}(x)$, is an important quantity characterizing life phenomenon. It can be interpreted as the conditional probability of failure, given it has survived to the time t . The HF of the CTLD is defined by

$$H(x) = \frac{f_{CTLD}(x; \theta, \lambda)}{R(t)} \quad (15)$$

$$= \frac{\frac{\theta^2}{(\theta+1)}(x+1)e^{-\theta x} \left[1 - \frac{2\lambda e^{-\theta x}(1+\theta+\theta x)}{1+\theta} + \frac{3\lambda e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} \right]}{\frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} \left[1 - \frac{\lambda e^{-\theta x}(1+\theta+\theta x)}{(1+\theta)} + \frac{\lambda e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} \right]}$$

3.6 Odd Function

$$O(x) = \frac{F_{CTLD}(x; \theta, \lambda)}{R(t)} \quad (16)$$

$$= \frac{\left[1 - \frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} + \lambda \left(\frac{e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} - \frac{e^{-3\theta x}(1+\theta+\theta x)^3}{(1+\theta)^3} \right) \right]}{\frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} \left[1 - \frac{\lambda e^{-\theta x}(1+\theta+\theta x)}{(1+\theta)} + \frac{\lambda e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} \right]}$$

3.7 Cumulative Hazard

$$CH(x) = -\log(R(t)) \quad (17)$$

$$= -\log \left[\frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} \left[1 - \frac{\lambda e^{-\theta x}(1+\theta+\theta x)}{(1+\theta)} + \frac{\lambda e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} \right] \right]$$

3.8 Moment generating function

The r^{th} non-central moments or (moments about the origin) of the CTLD using is given by:

$$M_x(t) = \sum_{r=1}^{\infty} \frac{t^r}{r!} E(x^r) \quad (18)$$

$$= \sum_{r=1}^{\infty} \frac{t^r}{r!} \left[\left[\frac{1}{\theta^r(1+\theta)} [\theta \Gamma(r+1) + \Gamma(r+2)] - \frac{\lambda \theta}{(1+\theta)^2} \left[\frac{1}{(2\theta)^r} \Gamma(r+1) + \frac{\theta}{(2\theta)^r} \Gamma(r+1) + \frac{2\theta}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{1}{(2\theta)^{r+1}} \Gamma(r+2) + \frac{\theta}{(2\theta)^{r+2}} \Gamma(r+3) \right] + \frac{\lambda \theta}{(1+\theta)^3} \left[\frac{(1+2\theta+\theta^2)}{(3\theta)^r} \Gamma(r+1) + \frac{(4\theta+3\theta^2+1)}{(3\theta)^{r+1}} \Gamma(r+2) + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+3) + \frac{2\theta+3\theta^2}{(3\theta)^{r+2}} \Gamma(r+4) \right] \right] \right]$$

3.9 Mean time to Failure

$$MTTF = \int_0^{\infty} R(t) dt \quad (19)$$

$$= \int_0^{\infty} \frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta} - \lambda \left(\frac{e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} - \frac{e^{-3\theta x}(1+\theta+\theta x)^3}{(1+\theta)^3} \right) dx$$

$$= \frac{1}{(1+\theta)} \int_0^{\infty} e^{-\theta x}(1+\theta+\theta x) dx - \frac{\lambda}{(1+\theta)^2} \int_0^{\infty} e^{-2\theta x}(1+\theta+\theta x)^2 dx + \frac{\lambda}{(1+\theta)^3} \int_0^{\infty} e^{-3\theta x}(1+\theta+\theta x)^3 dx$$

$$= \frac{1}{(1+\theta)} j - \frac{\lambda}{(1+\theta)^2} k + \frac{\lambda}{(1+\theta)^3} m \quad (20)$$

From (20),

$$j = \int_0^{\infty} e^{-\theta x}(1+\theta+\theta x) dx = \int_0^{\infty} e^{-\theta x} dx + \theta \int_0^{\infty} e^{-\theta x} dx + \theta \int_0^{\infty} x e^{-\theta x} dx$$

$$= \left[\frac{e^{-\theta x}}{-\theta} + \theta \left(\frac{e^{-\theta x}}{-\theta} \right) \right] + \theta (uv - \int_0^{\infty} v du), \text{ let } u = x, du = dx \text{ and } v = \frac{e^{-\theta x}}{-\theta}$$

$$= \left[\frac{e^{-\theta x}}{-\theta} + \theta \left(\frac{e^{-\theta x}}{-\theta} \right) \right] + \left(\frac{x e^{-\theta x}}{-\theta} - \int_0^{\infty} \frac{e^{-\theta x}}{-\theta} \right) = \left[\frac{e^{-\theta x}}{-\theta} + \left(\frac{x e^{-\theta x}}{-\theta} + \frac{1}{\theta} \left(\frac{e^{-\theta x}}{-\theta} \right) \right) \right]$$

$$= \left[\frac{e^{-\theta x}}{-\theta} + \left(\frac{e^{-\theta x}}{-\theta} \right) \right] + \left(\frac{x e^{-\theta x}}{-\theta} - \frac{e^{-\theta x}}{\theta^2} \right) = 0 - \left[\frac{1}{-\theta} - 1 - \frac{1}{\theta^2} \right] = \frac{1}{\theta} + 1 + \frac{1}{\theta^2}$$

$$= \frac{1}{\theta} + 1 + \frac{1}{\theta^2} = \frac{\theta^2 + \theta + 1}{\theta^2} \quad (21)$$

$$k = \int_0^{\infty} e^{-2\theta x} (1+\theta+\theta x)^2 dx$$

$$= \int_0^{\infty} e^{-2\theta x} (1+2\theta+2\theta x+\theta^2+2\theta^2 x+\theta^2 x^2) dx$$

$$= \int_0^{\infty} e^{-2\theta x} dx + 2\theta \int_0^{\infty} e^{-2\theta x} dx + 2\theta \int_0^{\infty} x e^{-2\theta x} dx + \theta^2 \int_0^{\infty} e^{-2\theta x} dx + 2\theta^2 \int_0^{\infty} x e^{-2\theta x} dx + \theta^2 \int_0^{\infty} x^2 e^{-2\theta x} dx$$

$$= \left(\frac{e^{-2\theta x}}{-2\theta} \right) + 2\theta \left(\frac{e^{-2\theta x}}{-2\theta} \right) + \theta^2 \left(\frac{e^{-2\theta x}}{-2\theta} \right) + 2\theta \left(\frac{x e^{-2\theta x}}{-2\theta} - \int_0^{\infty} \frac{e^{-2\theta x}}{-2\theta} dx \right) + 2\theta^2 \left(\frac{x e^{-2\theta x}}{-2\theta} - \int_0^{\infty} \frac{e^{-2\theta x}}{-2\theta} dx \right) + \theta^2 \left(\frac{x^2 e^{-2\theta x}}{-2\theta} + \frac{1}{\theta} \int_0^{\infty} x e^{-2\theta x} dx \right)$$

$$= \left[\frac{e^{-2\theta x}}{-2\theta} - e^{-2\theta x} - \frac{\theta e^{-2\theta x}}{2} \right] + 2\theta \left(\frac{x e^{-2\theta x}}{-2\theta} - \frac{e^{-2\theta x}}{4\theta^2} \right) + 2\theta^2 \left(\frac{x e^{-2\theta x}}{-2\theta} - \frac{e^{-2\theta x}}{4\theta^2} \right) + \theta^2 \left(\frac{x^2 e^{-2\theta x}}{-2\theta} + \frac{1}{\theta} \left(\frac{x e^{-2\theta x}}{-2\theta} - \int_0^{\infty} \frac{e^{-2\theta x}}{-2\theta} dx \right) \right)$$

$$= \left[\frac{e^{-2\theta x}}{-2\theta} - e^{-2\theta x} - \frac{\theta e^{-2\theta x}}{2} \right] + 2\theta \left(\frac{x e^{-2\theta x}}{-2\theta} - \frac{e^{-2\theta x}}{4\theta^2} \right) + 2\theta^2 \left(\frac{x e^{-2\theta x}}{-2\theta} - \frac{e^{-2\theta x}}{4\theta^2} \right) + \theta^2 \left(\frac{x^2 e^{-2\theta x}}{-2\theta} + \frac{1}{\theta} \left(\frac{x e^{-2\theta x}}{-2\theta} - \frac{e^{-2\theta x}}{4\theta^2} \right) \right)$$

$$\begin{aligned}
 &= 0 - \left\{ \left[\frac{1}{-2\theta} - 1 - \frac{\theta}{2} \right] + 2\theta \left(-\frac{1}{4\theta^2} \right) + 2\theta^2 \left(-\frac{1}{4\theta^2} \right) + \right. \\
 &\theta^2 \left(0 + \frac{1}{\theta} \left(-\frac{1}{4\theta^2} \right) \right) \Big\} \\
 &= - \left\{ \left[\frac{1}{-2\theta} - 1 - \frac{\theta}{2} \right] - \frac{1}{2\theta} - \frac{1}{2} + \theta^2 \left(-\frac{1}{4\theta^3} \right) \right\} = \\
 &- \left\{ \left[\frac{1}{-2\theta} - 1 - \frac{\theta}{2} \right] - \frac{1}{2\theta} - \frac{1}{2} - \frac{1}{4\theta} \right\} \\
 &k = \frac{(2\theta^2 + 6\theta + 5)}{4\theta} \quad (22) \\
 &m = \int_0^\infty e^{-3\theta x} (1 + \theta + \theta x)^3 dx \\
 &= \int_0^\infty e^{-3\theta x} [1 + 3\theta + 3\theta^2 + \theta^3 + 3\theta x + 6\theta^2 x + \\
 &3\theta^2 x^2 + 3\theta^3 x + 4\theta^3 x^2] dx \\
 &= \int_0^\infty e^{-3\theta x} dx + 3\theta \int_0^\infty e^{-3\theta x} dx + 3\theta^2 \int_0^\infty e^{-3\theta x} dx + \\
 &\theta^3 \int_0^\infty e^{-3\theta x} dx + 3\theta \int_0^\infty x e^{-3\theta x} dx + \\
 &6\theta^2 \int_0^\infty x e^{-3\theta x} dx + 3\theta^3 \int_0^\infty x e^{-3\theta x} dx + \\
 &3\theta^2 \int_0^\infty x^2 e^{-3\theta x} dx + 4\theta^3 \int_0^\infty x^2 e^{-3\theta x} dx \\
 &= (1 + 3\theta + 3\theta^2 + \theta^3) \int_0^\infty e^{-3\theta x} dx + (3\theta + 6\theta^2 + \\
 &3\theta^3) \int_0^\infty x e^{-3\theta x} dx + (3\theta^2 + 4\theta^3) \int_0^\infty x^2 e^{-3\theta x} dx \\
 &\text{Let } u = x, du = dx; v = \frac{e^{-3\theta x}}{-3\theta} \text{ and } u = x^2; du = \\
 &2x dx, \text{ then} \\
 &= (1 + 3\theta + 3\theta^2 + \theta^3) \left(\frac{e^{-3\theta x}}{-3\theta} \right) + (3\theta + 6\theta^2 + \\
 &3\theta^3) \left(\frac{x e^{-3\theta x}}{-3\theta} + \frac{1}{3\theta} \int_0^\infty e^{-3\theta x} dx \right) + (3\theta^2 + \\
 &4\theta^3) \left(\frac{2x e^{-3\theta x}}{-3\theta} + \frac{2}{3\theta} \left(\frac{x e^{-3\theta x}}{-3\theta} + \frac{1}{3\theta} \int_0^\infty e^{-3\theta x} dx \right) \right) \\
 &= (1 + 3\theta + 3\theta^2 + \theta^3) \left(\frac{e^{-3\theta x}}{-3\theta} \right) + (3\theta + 6\theta^2 + \\
 &3\theta^3) \left(\frac{x e^{-3\theta x}}{-3\theta} + \frac{1}{3\theta} \left(\frac{e^{-3\theta x}}{-3\theta} \right) \right) + (3\theta^2 + 4\theta^3) \left(\frac{2x e^{-3\theta x}}{-3\theta} + \right. \\
 &\left. \frac{2}{3\theta} \left(\frac{x e^{-3\theta x}}{-3\theta} + \frac{1}{3\theta} \left(\frac{e^{-3\theta x}}{-3\theta} \right) \right) \right) \\
 &= 0 - \left\{ \frac{(1+3\theta+3\theta^2+\theta^3)}{-3\theta} + (3\theta + 6\theta^2 + 3\theta^3) \left(-\frac{1}{9\theta^2} \right) + \right. \\
 &(3\theta^2 + 4\theta^3) \left(\frac{2}{3\theta} \left(-\frac{1}{9\theta^2} \right) \right) \Big\} \\
 &= - \left\{ \frac{(1+3\theta+3\theta^2+\theta^3)}{-3\theta} - \frac{(3\theta+6\theta^2+3\theta^3)}{9\theta^2} - \frac{(6\theta^2+8\theta^3)}{27\theta^3} \right\} = \\
 &\left\{ \frac{(1+3\theta+3\theta^2+\theta^3)}{3\theta} + \frac{(3\theta+6\theta^2+3\theta^3)}{9\theta^2} + \frac{(6\theta^2+8\theta^3)}{27\theta^3} \right\} \\
 &m = \frac{(1+3\theta+3\theta^2+\theta^3)}{3\theta} + \frac{(3\theta+6\theta^2+3\theta^3)}{9\theta^2} + \frac{(6\theta^2+8\theta^3)}{27\theta^3} = \\
 &\frac{9\theta^5+36\theta^4+53\theta^3+24\theta^2}{27\theta^3} \quad (23)
 \end{aligned}$$

From eq. (20),

$$MTTF = \frac{1}{(1+\theta)} j - \frac{\lambda}{(1+\theta)^2} k + \frac{\lambda}{(1+\theta)^3},$$

we can substitute for j, k and m to obtain eq. (24)

$$\begin{aligned}
 &= \frac{1}{(1+\theta)} \left(\frac{\theta^2 + \theta + 1}{\theta^2} \right) - \frac{\lambda}{(1+\theta)^2} \left(\frac{2\theta^2 + 6\theta + 5}{4\theta} \right) + \\
 &\frac{\lambda}{(1+\theta)^3} \left(\frac{9\theta^5 + 36\theta^4 + 53\theta^3 + 24\theta^2}{27\theta^3} \right) \\
 &MTTF = \frac{1}{\theta(1+\theta)} \left[\frac{\theta^2 + \theta + 1}{\theta} - \frac{\lambda(2\theta^2 + 6\theta + 5)}{4(1+\theta)} + \right. \\
 &\left. \frac{\lambda(9\theta^5 + 36\theta^4 + 53\theta^3 + 24\theta^2)}{27\theta^2(1+\theta)^2} \right] \quad (24)
 \end{aligned}$$

The eq.(60) above is the expected time to system failure.

Method of Maximum Likelihood

The maximum likelihood estimators (MLEs) for the parameters of the Cubic transmuted Lindley distribution CTLD (λ, θ) is discussed in this section. Consider the random sample $x_1, x_2, \dots, x_n$ of size n from CTLD (λ, θ) with probability density function, then the likelihood function can be expressed as follows:

$$L_{f_{CTLD}}(x; \theta, \lambda) = \prod_{i=1}^n \frac{\theta^2}{(\theta+1)} (x+1) e^{-\theta x} \left[1 - \frac{2\lambda e^{-\theta x}(1+\theta+\theta x)}{1+\theta} + \frac{3\lambda e^{-2\theta x}(1+\theta+\theta x)^2}{(1+\theta)^2} \right] \quad (25)$$

$$\ln L_{f_{CTLD}}(x; \theta, \lambda) = 2n \ln \theta - n \ln(\theta+1) \prod_{i=1}^n x_i - \theta \sum_{i=1}^n x_i + \sum_{i=1}^n \ln \left[1 - 2\lambda(1+\theta)^{-1} e^{-\theta x_i} (1+\theta + \theta x_i) + 3\lambda(1+\theta)^{-2} e^{-2\theta x_i} (1+\theta + \theta x_i)^2 \right]$$

$$\begin{aligned}
 \frac{\partial \ln L_{f_{CTLD}}(x; \theta, \lambda)}{\partial \lambda} &= \sum_{i=1}^n \left[\frac{2(1+\theta)^{-1} e^{-\theta x_i} (1+\theta+\theta x_i) + 3(1+\theta)^{-2} e^{-2\theta x_i} (1+\theta+\theta x_i)^2}{1 - 2\lambda(1+\theta)^{-1} e^{-\theta x_i} (1+\theta+\theta x_i) + 3\lambda(1+\theta)^{-2} e^{-2\theta x_i} (1+\theta+\theta x_i)^2} \right] \\
 \frac{\partial \ln L_{f_{CTLD}}(x; \theta, \lambda)}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{(\theta+1)} \prod_{i=1}^n x_i - \sum_{i=1}^n x_i + \sum_{i=1}^n \left[\frac{2\lambda(1+\theta)^{-1} \theta e^{-\theta x_i} + 24\theta(1+\theta)^{-2} e^{-2\theta x_i}}{(1 - 2\lambda(1+\theta)^{-1} e^{-\theta x_i} (1+\theta+\theta x_i) + 3\lambda(1+\theta)^{-2} e^{-2\theta x_i} (1+\theta+\theta x_i)^2)} \right]
 \end{aligned}$$

The method of maximum likelihood is displayed above and the parameter of the model can be obtained by using the numerical method such as Newton Raphson method and the fisher information matrix can be obtained using the same approach.

4. PLOT OF CUBIC TRANSMUTED LINDLEY DISTRIBUTION (CTLD)

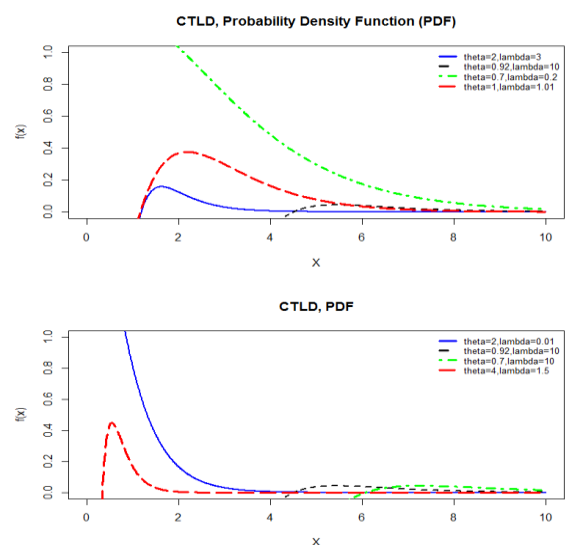


Fig 1: PDF of CTLD at Different Parameter Values

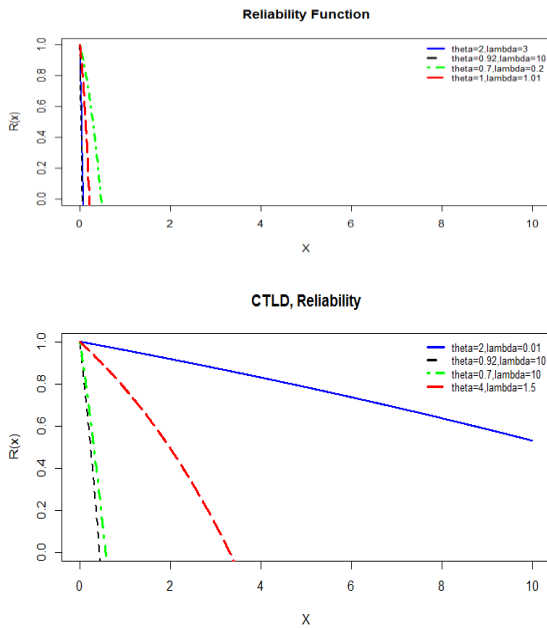


Fig 2: Reliability of CTLD at Different Parameter Values

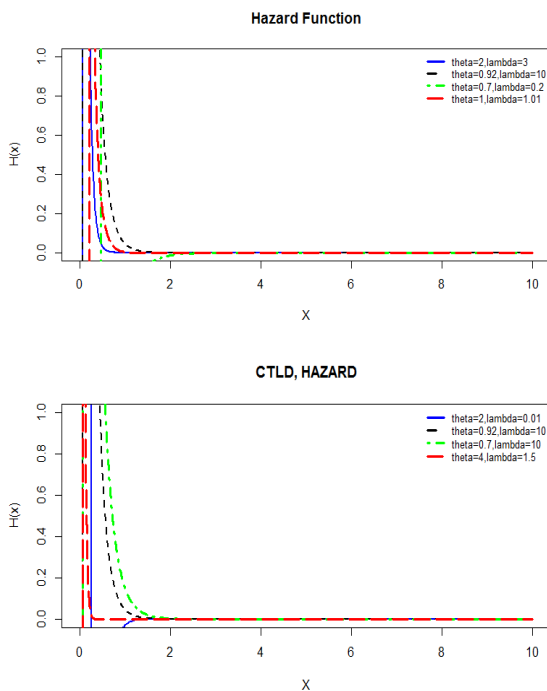


Fig 3: Hazard of CTLD at Different Parameter Values

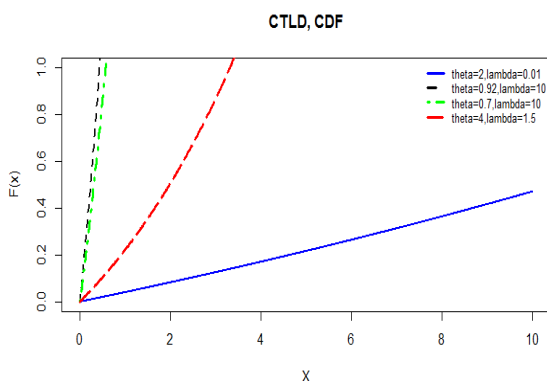


Fig 4: The CDF of CTLD at Different Parameter Values

5. APPLICATION TO LIFETIME DATA

5.1 Accessing Normality

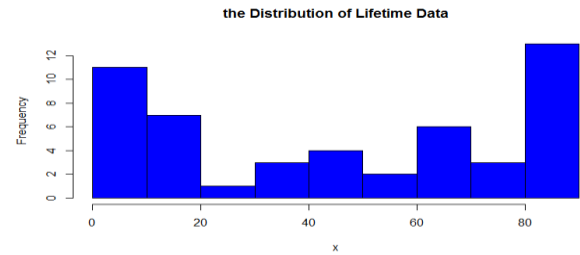


Fig 5: The Histogram that shows the distribution of the lifetime data

Since the distribution of the data is heavily tailed and U-shaped, then CTLD can be used to capture non-normal data. That is, CTLD will be reasonably good in modeling the data with heavily tailed distribution or non-normal process.

5.2. Fitting the CTLD to the Data and Comparing with Other Distributions

Table 1: the estimation of parameter of the hybrid distributions and other parent distribution

Cubic Transmuted Lindley Distribution (CTLD) $l_{CTLD} = -422.8583$				
Parameter	Estimate	Std. Error	t.value	Pr(>t)
$\hat{\theta}$	0.044079	0.004439	9.931	<2e-16 ***
$\hat{\lambda}$	0.008066	0.006069	6.329	0.04*
Lindley Distribution ($l_{LD} = -414.9851$)				
$\hat{\theta}$	0.04288	0.00429	9.996	<2e-16 ***
Cubic Transmuted Weibull Distribution (CTWD) $(l_{CTWD} = -240.9888)$				
$\hat{\alpha}$	0.02581	0.01521	5.697	0.019
$\hat{\beta}$	0.95999	0.13790	6.961	3.37e-12 ***
$\hat{\lambda}$	0.05925	0.36991	0.160	0.8728
Weibull Distribution ($l_{WD} = -441.0018$)				
$\hat{\alpha}$	0.9489	0.1167	8.128	4.35e-16 ***
$\hat{\beta}$	44.8738	4.2524	10.553	< 2e-16 ***

From the table 1 above, we can see that the parameters of the CTLD and weibull distribution are stable and efficient as compared to CTWD. This is because those parameters are significant (p-values < ($\alpha = 0.05$)) as compared to that of CTWD in which not all its parameters are statistical significant.

5.3. Test for Significance of transmuted parameter in the Model

Since the p-value (0.04*) < ($\alpha = 0.05$), we have statistical reason to conclude that the transmuted parameter is significant in the hybrid model and thus it contributes to the flexibility of the modeling in modeling non-normal process.

The goodness-of-fit statistics Wald and likelihood ratio test are also presented below. These statistics can be used to verify which distribution provides the best fit to the data. In general, the smaller the values of Wald and likelihood ratio test, the better the fit.

5.4. WALD Test

Hypothesis

$H_0: \hat{\lambda} = 0, \alpha = 0.05$

$H_1: \hat{\lambda} \neq 0$

Decision Rule:

Accept H_0 if $w < \chi_q^2$, otherwise do not accept q is the number of parameters in the model or the number of rows of the variance-covariance matrix χ_q^2 , at $q = 2$, is 5.991

Decision

Since the $w (100.5542) > \chi_q^2$, then we have statistical reason not to accept H_0 and conclude that the shape parameter transmuted parameter contributed significantly to the modeling efficiency of the hybrid CTLD model. The parameters have increased the flexibility of CTLD over Lindley Distribution in modeling non-normal data.

5.5. Likelihood Ratio Test

Hypothesis

$H_0:$

CTLD is is not better than LD and and CTLD, $\alpha = 0.05$

$H_1:$ *CTLD is is better*

$$T_1 = 2(l_{LD} - l_{CTLD}) = 2(-414.9851 + 422.8583) = 2(7.8732) = 15.7464$$

$$T_2 = 2(l_{CTWD} - l_{CTLD}) = 2(-240.9888 + 422.8583) = 2(181.8695) = 363.739$$

Decision Rule:

We accept H_0 , if $T < \chi^2_{(\alpha)}$. We fail to accept if Otherwise

Decision:

Since $T_1 (15.7464)$ and $T_2 (363.739) > \chi^2_{(\alpha)} (5.991)$, we have utmost statistical ground to conclude that the hybrid Cubic Transmuted Lindely Distribution (CTLD) is far better than Lindley Distribution (LD) and Cubic Transmuted Weibull Distribution (CTLD) in modeling lifetime data.

Hypothesis:

$H_0:$

CTLD is not a good fit as compared to Weibull, $\alpha = 0.05$

$H_1:$ *CTLD is better than Weibull*

$$T_3 = 2(l_{WD} - l_{CTLD}) = 2(-441.0018 + 422.8583) = 2(-18.1435) = -36.287$$

Decision

Since $T_3 (-36.287) < \chi^2_{(\alpha)} (5.991)$, we have statistical reasons to accept H_0 and conclude that CTLD is not a good fit as compared to Weibull when fitting the two models to lifetime data.

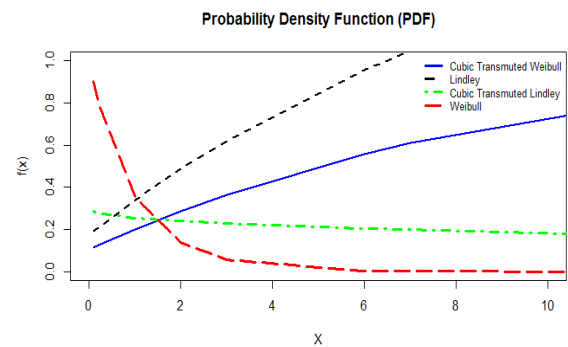


Fig 6: The PDF of the Different Fitted Distribution

From the figure 6 above, we can conclude that Weibull distribution perfectly model the lifetime data and Cubic Transmuted Lindley also approximately model the process reasonably well. The Lindley and Cubic Transmuted Weibull did not model the data reasonably well, thus they are not a good fit for this process. It is on this not that we can picture the performance of Cubic Transmuted Lindley Distribution and Weibull Distribution over Lindley and Cubic Transmuted Weibull Distribution in modeling lifetime data (Reliability Analysis)

5.6. Mean Time to System Failure

Expected time to failure is 515.3959 hrs. That is, the average time that the cooling system is expected to operate before failure is 515.3959 hrs. From $f(x) = 7.334249e-04$, we can say that about 8 failures will be expected within 10,000hrs

CONCLUSIONS

Mixing parent distributions, we obtain the hybrid distribution version with increased number of parameters which is believed to give the newly compounded distribution more flexibility, consistency, stability, sufficiency, uniqueness and wider applicability as compare to its parent distribution. From analysis, CTLD provides good fit as compared to Lindley and CTWD in modeling lifetime data. According to CTLD, the expected time system is expected to operate before failure is 515.3959hrs.

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