

**ALGEBRAIC AND COMBINATORIAL RESULTS OF ORDER-PRESERVING FULL
CONTRACTION TRANSFORMATION SEMIGROUP**

BY

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Certification

This is to certify that the research work reported in this thesis was carried out by **Ibrahim-Garba, Risqot** with Matriculation Number **01/55EB048** in the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, for the award of the degree of Doctor of Philosophy (Ph.D.) in Mathematics.

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Dedication

This thesis is dedicated to Almighty Allah who gave me strength, wisdom and understanding to carry out this research work successfully.

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Abstract

Let X_n be a finite set, CT_n the semigroup of full contraction transformation and OCT_n the subsemigroup of CT_n of all order-preserving full contraction transformation semigroup. Several works have been done on algebraic properties of semigroups and results were obtained among these are generating set, the structure of starred Green's relations in OCT_n , the local and global U-depth of singular self and strictly partial one-one mappings but the combinatorics properties of OCT_n has not be considered. Therefore, this study focuses on combinatorics properties of OCT_n using the classes of the starred Green's relations and other algebraic properties such as the local U-depth, status of OCT_n which as not be investigated were examined. The structure of Green's relations of OCT_n were also examined, which extended some results in the literature. The aim of this study is to develop the algebraic and combinatorics properties of order-preserving full contraction transformation semigroup and objectives are to: (i) determine the local and global U-depth of α , where U is the generating set; (ii) obtain the status of OCT_n using the global U-depth; (iii) examine the number of D^* , L^* , R^* and H^* classes of height r ; (iv) determine the total number of D^* , L^* , R^* and H^* classes of OCT_n ; (v) investigate the number of elements in each D^* , L^* , R^* and H^* classes within D^* ; and (vi) characterize Green's relations of OCT_n . The following procedures were used to obtained the results of the study: the elements of the semigroup OCT_n were arranged based on their height, within each height by their images sets and their kernel sets; from the table obtained triangular array and sequences were formed; the patterns of the arrangement were studied; formulas were deduced in each case through the combinatorial principles. The gap software were used to confirmed the total number of elements. Also, the minimum length of factorisation that gives α were obtained from the known generating set, for all $\alpha \in OCT_n$. The findings of the study were:

- for each $\alpha \in OCT_n$ the local U-depth of α is equal to its defect and global U-depth is $n-1$. ;
- the status of OCT_n satisfies the property $Stat(OCT_n) \leq 2(n-1)^2$;
- the number of D^* , L^* , R^* and H^* classes of height r are : 1 , $n-r+1$;
 $\binom{n-1}{r-1}$; and $(n-r+1)\binom{n-1}{r-1}$ respectively.
- the total number of D^* , L^* , R^* and H^* classes in OCT_n are: n ; $\frac{n(n+1)}{2}$;
 2^{n-1} and $(n+1)2^{n-2}$ respectively.
- the number of elements in each L^* , R^* and H^* within D^* -class are: $\binom{n-1}{r-1}$;
 $n-r+1$; and $(n-r+1)\binom{n-1}{r-1}$ respectively.

- the equivalence classes of Green's relations were also characterized based on their image set and kernel classes.

In conclusion, some algebraic and combinatorics properties results of subsemigroup OCTn were obtained with relevant examples. This research work and its findings are expected to be beneficial in the areas such as: computational theory, automata theory and formal languages. It can assist in sorting data and designing better network and also, new in.

NOTATIONS	MEANING
$Im(\alpha)$	the image set or range of the mapping α
$Dom(\alpha)$	the domain of α
$ker(\alpha)$	kernel of α
$height(\alpha)$	the order of image set or range of the mapping α
T_n	semigroup of full transformation on X_n
P_n	semigroup of partial transformation on X_n
I_n	semigroup of partial one-one transformation on X_n
CT_n	semigroup of full contraction transformation on X_n

CP_n	semigroup of partial contraction transformation on X_n
OCT_n	semigroup of order-preserving full contraction transformation on X_n
$ S $	order of semigroup
S^1	semigroup with identity
CI_n	semigroup of partial one-one contraction transformation on X_n
SI_n	semigroup of all strictly partial one-one transformation on X_n
$Sing_n$	semigroup of all singular full transformation on X_n
L^*	starred Green's relation defined by

	equality of principal left ideals
R^*	starred Green's relation defined by equality of principal right ideals
J^*	starred Green's relation defined by equality of principal ideals
D^*	starred Green's relation defined by $L^* \cup R^*$
H^*	starred Green's relation defined by $L^* \cap R^*$
L	Green's relation defined by equality of principal left ideals
R	Green's relation defined by equality of principal right ideals
J	Green's

	relation defined by equality of principal ideals
D	Green's relation defined by $L \cup R$
H	Green's relation defined by $L \cap R$

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1 GENERAL INTRODUCTION

1 Background of the Study

The algebraic theory of semigroup has been well studied during the second half of the twentieth century. Many ideas in study of semigroup were directly motivated by their analogues in the ring theory. However, certain important and interesting concepts such as the concepts of Green's relation, idempotent rank, nilpotent rank, product of idempotent and so on developed independent, just as the set of all permutation of a finite set proves to be an important source of examples in group theory, the set of all mappings of a finite set in to itself provide us with a corresponding objects in the semigroup theory. The importance of mappings in semigroup theory can be judged by the fact that every semigroup is isomorphic to a semigroup of mappings in this sense it turns out that the understanding of semigroup of mappings in semigroup theories is of paramount significance. Transformation semigroups are one of the most fundamental mathematical objects. They occur in theoretical computer science, where properties of language depend on algebraic properties of various transformation semigroups related to them.

Algebraic and Combinatorial properties of transformation semigroup have been studied over a long period and interesting results have emerged for example Ganyushkin and Mazorchuk (2009); Garba (1990,1994a,b,c); Umar (1996). One of the reasons is due to the fact that every finite inverse semigroup can be embedded in a symmetric inverse semigroup. On studying a new class of semigroup it is mostly of interest to know the type of semigroup in question for example, whether the semigroup is regular, inverse (see Clifford and Preston, 1961,1967). Another interest is in the characterisation of the five classical Green's relations, mostly when the semigroup is regular. For non-regular semigroup, generalisation of Green's relations like starred analogues are investigated (Fountain 1979,1982; Maclister 1976). Regular semigroups were introduced by Green (1951) and was the paper in which popular Green's relations were introduced. It was his study of regular semigroups which led Green to define relations. Let S be a semigroup $a, b \in S$. If a and b generate the same left principal ideal, that is $S^1a = S^1b$, then we say that a and b are L -equivalent and write $aL b$ or $(a,b) \in L$. If a and b generate the same right principal ideal that is $aS^1 = bS^1$, then we say that a and b are R -equivalent and write $aR b$ or $(a,b) \in R$. If a and b generate the same principal ideal that is S^1aS^1 , then we say that a and b are J equivalent and write $aJ b$ or $(a,b) \in J$, Let $H = L \cap R$, $D = L \cup R$, then H, D are equivalences on S too. It is well known that $J = D$ in any finite semigroup. These five equivalences are usually called Green's equivalence relations. They were first introduced by Green(1951) and have a great importance in the study of the algebraic structure of semigroups. Two functions are L -related if and only if they have the same image, R -related if and only if for x and y in the set X_n ; such functions are in the same table row(kernel). Similarly, two functions are D -related if and only if their image sets have the same size.

2 Statement of the Problems

In the symmetric inverse semigroup of X_n Garba and Imam (2015) showed that the semigroup SI_n , of all strictly partial one-one maps on X_n , is generated by quasi-idempotent elements (non-idempotent elements satisfying $\alpha^4 = \alpha^2$) and give the least upper bound for the minimum length factorisation of each $\alpha \in SI_n$ into a product of quasi-idempotent. Also, in semigroup of full contraction mappings much work have not been done. For example Kehinde (2012) looked into the isometry injective contraction mappings, Zhao and Yang (2012) researched in Algebraic Structure of Partial compression (contraction) mappings. Adeshola (2013) and Muhammedjamil (2015) studied the algebraic nature of the following three subsemigroups of full contraction mapping .i.e. OCT_n , $ORCT_n$ and ODT_n and the order of their idempotent elements. The research work is therefore focuses on some algebraic properties such as Depth, Status and Structure of Green's relations of semigroup OCT_n , also, combinatorics nature of the five equivalence classes in Green's relations.

3 Aim and Objectives of the study

The aim of the study is to determine some algebraic and combinatorics properties of order-preserving full contraction transformation semigroup and the objectives are to:

- find local and global U-depth of α , where U is the generating set;
- obtain the status of OCT_n using the global U-depth in OCT_n ;
- calculate the number of D^* , L^* , R^* and H^* classes of height r ;
- determine the total number of D^* , L^* , R^* and H^* classes of OCT_n ;
- find the number of elements in each L^* , R^* and H^* classes within D^* class;

and

- characterize Green's relations on OCT_n using a better and easier method than

the one used by Adeshola (2013).

4 Significance of the Study

The theory of semigroup embrace many aspects of theoretical computer sciences. This research work and its findings are expected to be beneficial in the areas such as :computational theory, formal languages and automata theory. For example, Colcombet (2011) and Lata (2011) used Green's relations for proving classical results related to automata theory. It can assist in sorting data and designing and also, the integer sequences obtained in the work will be useful for other researchers in relevant fields.

5 Methodology of the Study

Generally, we investigated the semigroup of order preserving full contraction transformation of X_n . The following procedures were used to obtained the results of the study: the elements of the semigroup OCT_n were arranged based on their height, within each height by their images sets and their kernel sets; from the table obtained triangular array and sequences were formed; the patterns of the arrangement were studied; formulas were deduced in each case through the combinatorial principles. The gap software were used to confirmed the total number of elements. Also, the minimum length of factorisation that gives α were obtained from the known generating set, for all $\alpha \in OCT_n$.

6 Thesis Outline

The thesis is categorized into five chapters. Chapter 1 comprises of the background to the study, general introduction, statement of the problem, aim and objectives of the study, significant of the study, research methodology used, outlines of the chapters and definitions of relevant terms. Chapter 2 deals with literature review, which includes brief introductions and some existing results so far related to the work. Chapter 3 contains brief introduction on generating set and main results obtained for the depth and status of semigroup OCT_n . Chapter 4 consists of introduction on equivalence classes, combinatorial results of starred Green's relations on semigroup OCT_n , number of elements in each equivalence classes within D^* -class as well as the characterisation of Green's relations of OCT_n . Lastly, chapter 5 contains summary and the conclusion on results, contribution to knowledge and recommendation for further study.

7 Definition of Relevant Terms

All the basic definitions needed to understand the results of the thesis were discussed in this section and mostly taken from Howie (1995), Clifford and Preston (1961,1967), Gauchushkin and Mazorchuk (2009) and Lallement (1979).

1.7.1 Semigroups

A non empty set S together with an associative binary operation $*$ is called semigroup. That is, for all $x, y \in S$ $x*(y*c) = (x*y)*c$. If there exists an element $1 \forall x \in S$, $1x = x = x1$, we say that S is a semigroup with identity or monoid. If S has no identity, then we adjoin one to make a monoid. This is written as follows,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

where the element 1 is with the property that $1x = x = x1$ and $1.1 = 1 \forall x \in S$. An element $x \in S$ is called left identity of S , if $\forall y \in S : x.y = y$. similarly, x is called right identity of S , $\forall y \in S : y.x = y$. If x is both left and right identity of S , then S is called an identity of S .

Also, if a semigroup S contains an element 0 that satisfies $x0 = 0x = 0 \forall x \in S$ then 0 is called a zero element and S is called a semigroup with zero. If semigroup has no zero element, then we adjoin zero to it and defining S^0 as follow:

$$S^0 = \begin{cases} S & \text{if } S \text{ has zero element} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

where 0 is with the property that $0x = x0 = 0$ and $0.0 = 0 \forall x \in S$

Also, An element $x \in S$ is called left zero of S , if $\forall y \in S : x.y = 0$. Similarly, x is called right zero of S , $\forall y \in S : y.x = 0$, and it is trivial that a semigroup can have atmost one such element.

A semigroup S is called left cancellative if

$$\forall x, y, z \in S, zx = zy \Rightarrow x = y.$$

Similarly, S is right cancellative if

$$\forall x, y, z \in S, xz = yz \Rightarrow x = y$$

and cancellative if it is both left and right cancellative.

If semigroup has the property that $\forall x, y \in S$.

$$xy = yx$$

we say that S is a commutative semigroup.

A great importance in semigroup theory is that of inverse of an element. We say that $y \in S$ is an inverse element of $x \in S$, if

$$x = xyx \text{ and } y = yxy$$

An element x may have more than one inverse if it exists. Indeed, in a rectangular band (Howie, 1995), every element is an inverse of every other element. If S is a monoid and $x \in S$ such that there exists an element y with $xy = 1$, then y is right inverse for x and x is called right invertible.

Similarly, suppose there exists an element z such that $zx = 1$, then z is left inverse for x and x is called left invertible. If x is both left and right invertible then x is invertible.

Remark

- A semigroup S has atmost one identity, that is, the identity of monoid is unique.

• Let S be a monoid, and let $x \in S$. If y and z are right and left inverse of x respectively, then $y = z$.

1.7.2 Subsemigroup

A subsemigroup of a semigroup S is a non-empty subset of S which is itself a semigroup. That is, a non-empty subset T of a semigroup S is called a subsemigroup if it is closed with respect to multiplication. i.e.

$$\forall x, y \in T, xy \in T$$

The associativity condition that hold in S certainly holds in T and T itself semigroup.

1.7.3 Generating set of a semigroup

A non-empty subset U of semigroup S is said to be a generating set for S , written as $\langle U \rangle = S$, if every elements of S is expressible as a product of elements in U . if S is finite, local U -depth of $s \in S$ is the smallest integer k such that $s = u_1 u_2 u_3 \dots u_k$ for some $u_1 u_2 \dots u_k \in U$ that is, the minimum length factorisation of each element of semigroup. The global U -depth is the maximum possible value of local U -depth.

$OCT_n = CT_n \cap O_n$ be the subsemigroups of $T_n \setminus S_n$ consisting of all order-preserving maps, all contraction maps and all order -preserving maps contraction maps respectively.

On the set X_n , the equivalence $Ker(\alpha) = \{(x, y): x\alpha = y\alpha\}$ partitions X_n into $Ker(\alpha)$ -classes called blocks of α . A block of α is called stationary if it contains the image of its members under α . Let A be a subset of X_n and let $\{A_1, A_2, \dots, A_r\}$ be partition of X_n . then A is called convex if for all $x, y \in X_n$ ($x, y \in A$ and $x \leq y \Rightarrow z \in A$).

The status of semigroup S is defined as the least value of the product of the order of generating set $|U|$ and the global U -depth $\Delta(U)$.

That is, $Stat(S) = \min\{|U| \Delta(U): \langle U \rangle = S\}$

The defect of α is define as non image point of α i.e $n - |Im(\alpha)|$.

$Rank(OCT_n)$ is the smallest number of generator of semigroup OCT_n

1.7.4 Ideal

A non-empty subset A of S is called a left ideal if $SA \subseteq A$, a right ideal if $AS \subseteq A$ and a two sided ideal if it is both a left and right ideal. Every right, left or two sided ideal is a semigroup, but converse is not true.

Principal Ideal

If a is an element of a semigroup S , the smallest left ideal containing a is $S_a \cup a$. Which may be written as $S^1 a$ and shall be called the principal left ideal generated by a . Similarly, $a \cup Sa = aS^1$ is the principal right ideal generated by a . And $S^1 a S^1 a = Sa \cup aSa$ is the principal ideal generated by a .

1.7.5. Transformation Semigroups

Following the pattern of Ganyushkin and Mazorchuk (2009), let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set and n is a positive integer. Transformation of X is an array of the form:

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \alpha(x_1) & \alpha(x_2) & \cdots & \alpha(x_n) \end{pmatrix}$$

where all $\alpha(x_i) \in X$ and are called the value of the transformation of α . The fact that α is transformation of X is usually written as $\alpha: X \rightarrow X$. The composition of this transformation is also a transformation and it form a semigroup called transformation semigroup.

Partial Transformation Semigroup

Apart from transformation of X , we also have another transformation called a partial transformation. A transformation of the form:

$$\beta: A \rightarrow X$$

is called partial transformation of X if A is a subset of X . The set of all partial transformation of X is denoted by P_X and P_n if the transformation is on X_n . Also, the composition of partial transformation will be partial transformation and set P_n is a semigroup under composition of mapping called the partial transformation semigroup.

Full Transformation Semigroup

A transformation where $\text{dom}(\alpha) = X$, is called full or total transformation. The set of full transformation of X is denoted by T_X and T_n if $X_n = \{1, 2, \dots, n\}$. This set of transformation forms a semigroup under composition of mappings called the full transformation semigroup. The full transformation has special use in the theory of semigroup, since a common way of representing an algebraic object is to find an embedding into the existing object. Then, according to Cayley's theorem: every finite group of order n is isomorphic to a permutation group on n elements, thus it can be embedded into the full symmetric group of $[n]$.

For each map $\alpha \in T_n$, we denote the image set $\{x\alpha : x \in X_n\}$ by $\text{im}(\alpha)$ and define the height $h(\alpha)$, defect $\text{def}(\alpha)$ and fix $f(\alpha)$ by $h(\alpha) = |\text{im}(\alpha)|$, $\text{def}(\alpha) = |X_n \setminus \text{im}(\alpha)|$ and $f(\alpha) = |\{x\alpha = x : x \in X_n\}|$ respectively.

Symmetric Inverse Semigroup

Let $X_n = \{1, 2, \dots, n\}$ be a finite set and $\alpha: \text{dom}(\alpha) \subseteq X_n \rightarrow \text{im}(\alpha) \subseteq X_n$ be a partial one-one transformation on X_n . The set of all partial one-one transformations on X_n denoted by I_n become a semigroup under the operation $\circ$ of composition of relations. If $\alpha, \beta \in I_n$, then $(\alpha, \beta) \in \alpha \circ \beta$ if and only if there exists $z \in X$ such that $(x, y) \in \alpha$ and $(z, y) \in \beta$. Thus, $z = x\alpha$ and $y = z\beta$, and so $y = (x\alpha)\beta$. It follows that, for all $x_1, x_2 \in \text{dom}(\alpha \circ \beta)$,

$$x_1(\alpha \circ \beta) = x_2(\alpha \circ \beta) \Rightarrow (x_1\alpha)\beta = (x_2\alpha)\beta \Rightarrow x_1\alpha = x_2\alpha \Rightarrow x_1 = x_2.$$

Notice that $x \in \text{dom}(\alpha\beta)$ if and only if there exists z and y such that $(x, z) \in \alpha$ and $(z, y) \in \beta$. Thus, $z \in \text{im}(\alpha) \cap \text{dom}(\beta)$, so

$$\begin{aligned} \text{dom}(\alpha\beta) &= (\text{im}\alpha \cap \text{dom}\beta)\alpha^{-1} \\ \text{im}(\alpha\beta) &= (\text{im}\alpha \cap \text{dom}\beta)\beta \end{aligned}$$

8 Green's Relation

The relations L, R, H, D and J on the semigroup S were introduced by Green (1951) and called the Green's relations.

An equivalence L on S is defined by the rule that $\alpha L \beta$ if and only if α and β have the same image set, that is $im(\alpha) = im(\beta)$. Also, the equivalence R by the rule that $\alpha R \beta$ if and only if they have the same kernel, that is $ker(\alpha) = ker(\beta)$. Equally, $\alpha D \beta$ if and only if the order of their image are equal, $|im(\alpha)| = |im(\beta)|$. The intersection of two equivalence is again an equivalence (see Howie, 1995), The intersection of L and R is denoted as H , and the join L and R that is, $L \vee R$ is denoted by D . Also, if S is a periodic semigroup then, $D = J$.

9 Structure of D

Each D -class in a semigroup S is a union of L -class and R and their intersection of is either empty or an H -class. Hence, it is convenient to visualize D -class as what Clifford and Preston (1961) called an egg box, in which row represents an R -class, each column represents L -class and each cell represents H -class. Note that, it is possible for the eggbox to contain a single row or single column of cells. even to contain only one cell. Similarly, it may be an infinite eggbox.

Structure of D -class

[illegible]

10 Starred Green's Relations

In a semigroup S the relation L^* is defined by the $(a, b) \in L^*$ if and only if a, b are related by the Green's relation L in some over semigroup of S . The relation R^* is defined the same way. These relations also have the following characterisations, (see Fountain, 1979).

$$L^*(S) = \{(a, b) \mid (\forall x, y \in S') ax = ay \Leftrightarrow bx = by\} \quad (1)$$

and

$$R^*(S) = \{(a, b) \mid (\forall x, y \in S') xa = ya \Leftrightarrow xb = yb\} \quad (2)$$

The join of the relations L^* and R^* is denoted by D^* and their intersection by H^* .

The starred relations play an important role in the theory of abundant semigroups analogues to that of Green's relation in the theory regular semigroups.

A Semigroup is said to be abundant if each L^* -class and R^* of S contains an idempotent.

The L^* -class containing an element a is denoted by L_a^* and corresponding notations are used for the remaining starred relations.

2 LITERATURE REVIEW

Higgins et. al. (2003) first described the nature of the mappings with the property that $|x\alpha - y\alpha| \leq |x - y|$ for all x and y to be called a contraction mapping. Umar (2012), stated many subsemigroups of contraction of the partial symmetric semigroup and symmetric inverse semigroup.

The sets of all contraction maps and of all order-preserving contraction maps in T_n are respectively denoted by CT_n and OCT_n respectively and constitute subsemigroups of T_n . Subsemigroups of order- preserving (i.e maps α satisfying $x \leq y \Rightarrow x\alpha \leq y\alpha$ for all $x \in X_n$) and of order-decreasing (i.e. maps $\alpha : x\alpha \leq x$ for all $x \in X_n$) elements of T_n have been studied by Howie (1971) and Umar (1992). Adeshola (2013) studied the algebraic and combinatorial problems CT_n and OCT_n by considering their orders and regularity. Garba and Imam (2015) found local and global depth in both SI_n , the semigroup consisting all strictly partial one-one and $sing_n$, the semigroup of all singular self maps of X_n respectively. Due to the fact that semigroup of contraction mapping is a new class of transformation semigroup only few researchers have worked on semigroup of contraction mapping; amongst them are Zhao and Yang (2012) considered regularity and Green's relations in semigroup OCP_n ; Kehinde (2012) presented the order of the set of idempotent of each semigroup of contraction mappings and isometries of finite chain of CI_n (injective contraction mapping), DP_n (partial

isometries), DDP_n (order-decreasing partial isometries) and DP_n (order-preserving and order-decreasing partial isometries); Adeshola (2013), Mohammed Jamil (2015) investigated the structure of Green's relations and their starred analogue in semigroup CT_n and OCT_n and also computed the rank (defined to be the smallest size of a generating set) of semigroup CI_n , CT_n and OCT_n . The algebraic properties in the research work uses the results of Adeshola (2013), Garba and Imam (2015) and Mohammedjamil (2015) to further the study on OCT_n . In addition the work of Kehinde (2012) was applied to get the combinatorial results of OCT_n .

Here are some of their results,

1 Algebraic properties

Adeshola (2013) studied the condition for which the semigroup OCT_n is said to be regular by the following observation : Let

$$O_n = \{\alpha \in T_n : (\forall x, y \in X_n) x \leq y \Rightarrow x(\alpha) \leq y(\alpha)\}$$

$$CT_n = \{\alpha \in T_n : (\forall x, y \in X_n) |x - y| \geq |x(\alpha) - y(\alpha)|\}$$

be the subsemigroup of T_n consisting of all order-preserving full transformations of X_n and full contraction respectively. Also, let

$OCT_n = \{\alpha \in O_n : (\forall x, y \in \text{dom } \alpha) x \leq y \Rightarrow x(\alpha) \leq y(\alpha)\}$ be the subsemigroup of O_n consisting of all order-preserving full contraction.

Theorem 2.1 [Adeshola, (2013)]: Let O_n and CT_n be as defined as above and let $OCT_n = O_n \cap CT_n$,

- If $h(\alpha) \leq 2$, then α is regular
- If $h(\alpha) > 2$, then α is regular if and only if $\min A_p - a_p = d$ and $A_i = a_i + d$

Theorem 2.2 [Adeshola, (2013)]: Let $(\alpha, \beta) \in OCT_n$ be regular elements with $\text{im}(\alpha) = \text{im}(\beta)$ ($|\text{im}(\alpha)| \geq 2$). Then $\alpha \ker \beta$. Then Green's relation on OCT_n by Adeshola (2013) is given below:

Theorem 2.3 [Adeshola, (2013)]: Let $\alpha, \beta \in OCT_n$.

- If $h(\alpha) \leq 2$ then $\alpha, \beta \in L$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$
- If $h(\alpha) > 2$ then $\alpha, \beta \in L$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$ and $\alpha \ker \beta$

Theorem 2.4 [Adeshola, (2013)]: Let $\alpha, \beta \in OCT_n$.

- If $h(\alpha) \leq 2$ then $\alpha, \beta \in R$ if and only if $\ker(\alpha) = \ker(\beta)$
- If $h(\alpha) > 2$ then $\alpha, \beta \in R$ if and only if $\ker(\alpha) = \ker(\beta)$ and $\alpha \text{im } \beta$

Theorem 2.5 [Adeshola, (2013)]: Let $\alpha, \beta \in OCT_n$

- If $h(\alpha) \leq 2$ then $\alpha, \beta \in D$ if and only if $|\text{im}(\alpha)| = |\text{im}(\beta)|$
- If $h(\alpha) > 2$ then $\alpha, \beta \in D$ if and only if $|\text{im}(\alpha)| = |\text{im}(\beta)|$ and $\alpha \ker \beta$ and $\alpha \text{im } \beta$.

Theorem 2.6 [Garba and Imam (2015)]: The $(QE_1 \cap J_{n-1})$ -depth of $\alpha \in SI_n$ is

$$K(\alpha) = \begin{cases} g(\alpha) & \text{if } v(\alpha) \leq c(\alpha) \\ g'(\alpha) & \text{if } v(\alpha) > c(\alpha) \end{cases}$$

where $g(\alpha) = s(\alpha) + c(\alpha)$, $g'(\alpha) = s(\alpha) + v(\alpha)$

Theorem 2.7 [Garba and Imam (2015)]: For each $n \geq 2$, the global

$(QE_1 \cap J_{n-1})$ -depth of SI_n is $[\frac{3}{2}(n-1)]$.

Theorem 2.8 [Garba and Imam (2015)]: For $n \geq 3$, each $\alpha \in E \setminus E_1$ is expressible as a product of $g(\alpha)$ 3-paths in $sing_n$. Where E is the set of all idempotents in $sing_n$ and E_1 is the set of all idempotents of defect 1 in E .

Recently, Mohammedjamil (2015) characterised the Green's relations and starred Green's relations in CT_n , CI_n and also starred Green's relation in OCT_n . Here are some of his results:

Theorem 2.9 [Mohammedjamil, (2015)]: Let $\alpha, \beta \in CI_n$

- $(\alpha, \beta) \in L(CI_n)$ if and only if $im(\alpha) = im(\beta) = (a_i, \dots, a_r)$ and the mapping $\theta: a_i \alpha^{-1} \rightarrow a_i \beta^{-1}$ is isometry.
- $(\alpha, \beta) \in R(CI_n)$ if and only if $dom(\alpha) = dom(\beta) = (b_i, \dots, b_r)$ and the mapping $\theta: b_i \alpha \rightarrow b_i \beta$ is isometry.
- $(\alpha, \beta) \in H(CI_n)$ if and only if $im(\alpha) = im(\beta) = (a_i, \dots, a_r)$ and $dom(\alpha) = dom(\beta) = (b_i, \dots, b_r)$ and the mapping $\theta: a_i \alpha^{-1} \rightarrow a_i \beta^{-1}$ and $\theta: b_i \alpha \rightarrow b_i \beta$ are isometries.

Theorem 2.10 [Mohammedjamil, (2015)]: Let $\alpha, \beta \in CT_n$, then

- $(\alpha, \beta) \in L(CT_n)$ if and only if $im(\alpha) = im(\beta)$ and both $ker(\alpha)$ and $ker(\beta)$ are convex partitions of X_n ;
- $(\alpha, \beta) \in R(CT_n)$ if and only if $ker(\alpha) = ker(\beta)$;
- $(\alpha, \beta) \in H(CT_n)$ if and only if $im(\alpha) = im(\beta)$ and $ker(\alpha) = ker(\beta)$ are convex partitions of X_n ; and
- $(\alpha, \beta) \in D(CT_n)$ if and only if $|im(\alpha)| = |im(\beta)|$ and both $ker(\alpha)$ and $ker(\beta)$ are convex partitions of X_n .

Theorem 2.11 [Mohammedjamil, (2015)]: Let $\alpha, \beta \in CT_n$, then

- $(\alpha, \beta) \in L^*(CT_n)$ if and only if $im(\alpha) = im(\beta)$;
- $(\alpha, \beta) \in R^*(CT_n)$ if and only if $ker(\alpha) = ker(\beta)$;
- $(\alpha, \beta) \in H^*(CT_n)$ if and only if $\alpha = \beta$ and $ker(\alpha) = ker(\beta)$ are convex partitions of X_n ; and
- $(\alpha, \beta) \in D^*(CT_n)$ if and only if $|im(\alpha)| = |im(\beta)|$.

Theorem 2.12 [Mohammedjamil, (2015)]: Let $\alpha, \beta \in OCT_n$, then

- $(\alpha, \beta) \in L^*(OCT_n)$ if and only if $im(\alpha) = im(\beta)$;
- $(\alpha, \beta) \in R^*(OCT_n)$ if and only if $ker(\alpha) = ker(\beta)$;
- $(\alpha, \beta) \in H^*(OCT_n)$ if and only if $\alpha = \beta$ and $ker(\alpha) = ker(\beta)$ are convex

partitions of X_n ; and

- $(\alpha, \beta) \in D^*(OCT_n)$ if and only if $|im(\alpha)| = |im(\beta)|$.

He also computed the rank of the semigroup OCT_n and its order by partitioned elements into classes $K_1, K_2, \dots, K_{n-1}$, where for each $1 \leq r \leq n-1$, $K_r = \{\alpha \in OCT_n \mid |im(\alpha)| = r\}$

Lemma 2.1 [Mohammedjamil, (2015)]: $rank(OCT_n) \leq n-1$.

Now, it follows that any generating set for OCT_n must pick from each of the possible kernel classes and also from each of the possible image sets in K_{n-1} . Thus,

$rank(OCT_n) \geq n-1$.

This together with lemma 2.2. give the following theorem.

Theorem 2.13 [Mohammedjamil, (2015)]: For any $n \geq 3$, $rank(OCT_n) = n-1$

Corollary 2.1 [Mohammedjamil, (2015)]: $|K_{n-1}| = 2(n-1)$.

2 Combinatorial Properties

Kehinde (2012) came up with some results on combinatorial properties of the semigroup of injective partial contraction mappings and isometries of finite chain that is CI_n and some of its subsemigroup.

Theorem 2.14 [Kehinde(2012)]: Let $S = DP_n$. Then, $(\alpha, \beta) \in D$ if and only if there exist an isometry $\theta: Dom(\alpha) \rightarrow Dom(\beta)$.

• The order of the set of idempotent element of CI_n , DP_n , DDP_n and $ODDP_n$ are the same and equal to 2^n

• The number of order-preserving partial isometries (of an n-chain) of height p height of $\alpha = |im(\alpha)| (F(n; p))$ of

$$DP_n = \begin{cases} \frac{2(2n-p+1)}{p+1} \binom{n}{p+1} & p > 1 \\ n^2 & p = 1 \\ 1 & p = 0 \end{cases}$$

$$ODP_n = \begin{cases} \frac{2n-p+1}{p+1} \binom{n}{p+1} & p > 1 \\ n^2 & p = 1 \\ 1 & p = 0 \end{cases}$$

$$ODDP_n = \begin{cases} \binom{n+1}{p+1} & p > 1 \\ \binom{n+1}{2} & p = 1 \end{cases}$$

- The number of order-preserving partial isometries (of an n-chain)

For DP_n

$$F(n, m) = \binom{n}{m} m > 1$$

For ODP_n

$$F(n, m) = \begin{cases} \binom{n}{m} & m > 1 \\ 1 & m = 0 \end{cases}$$

For DDP_n

$$F(n, m) = \begin{cases} \binom{n}{m} & m > 1 \\ 2^{n+2} - 2 & m = 1, \text{ if } n \text{ is even} \\ 3 \cdot 2^{n-1} - 2 & m = 1, \text{ if } n \text{ is odd} \end{cases}$$

and

For $ODDP_n$

$$F(n, m) = \begin{cases} \binom{n}{m} & m > 0 \\ a_{n-1} & m = 0 \end{cases}$$

- The number of order-preserving partial isometries (of an n-chain)

For DP_n

$$F(n, m) = \begin{cases} \frac{13 \cdot 2^n - (3n^2 + 9n + 10)}{3} & \text{if } n \text{ is even} \\ \frac{25 \cdot 2^{n-1} - (3n^2 + 9n + 10)}{3} & \text{if } n \text{ is odd} \end{cases}$$

and

For ODP_n

$$F(n, m) = 2^{n+1} - (2n+1)$$

- The order of

$$DP_n = 3 \cdot 2^{n+1} - (n+1)^2 - 1$$

$$ODP_n = 3 \cdot 2^n - 2(n+1);$$

$$DDP_n = 3a_{n-1} - 2a(n-2) - 2^{\frac{n}{2}} + n + 1(n > 1), a_0 = 1, a_1 = 2$$

and

$$DP_n = 2^{n+1} - (n+1)$$

3 DEPTH AND STATUS OF ORDER-PRESERVING FULL CONTRACTION TRANSFORMATION SEMIGROUP

The work is motivated by Mohammedjamil (2015) on which some results on rank (generating set) were obtained. Here, generating set of OCT_n were used to find the local and global depths of OCT_n as well as the status in semigroup OCT_n

Let $S = OCT_n$, $n \geq 3$ and let U be the generating set, any arbitrary $\alpha \in OCT_n$ is expressed as $\alpha = \beta\gamma$ where $\beta, \gamma \in U$. According to Mohammedjamil (2015) the generating set of semigroup (OCT_n) is expressed as

$$U = \left\{ \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i & . & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 2 & 3 & \cdots & i+1 & \cdots & n \end{pmatrix} \right\}$$

$$1 \leq i \leq n. \text{ Now, let } \beta = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n-1 \end{pmatrix} \equiv \beta_{i,i+1}$$

$$\text{and } \gamma = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 2 & 3 & \cdots & . & i+1 & \cdots & n \end{pmatrix} \equiv \gamma_{i,i+1}, \forall \beta, \gamma \in U.$$

and from definition of generating set, any arbitrary $\alpha \in OCT_n$ is expressed as product of β and γ i.e. $\alpha = \beta\gamma \equiv \beta_{i,i+1} \cdot \gamma_{i,i+1}$

Semigroup of Full Contraction Mappings:

For CT_n on $X_n = \{1, 2, \dots, n\}$

OCT_1 on $X_1 = \{1\} = 1$ element

$|im(\alpha)| = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^*$

For CT_2 on $X_2 = \{1,2\} = 4$ elements
 $|im(\alpha)| = 2$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$
1 2	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}^{\hat{a}}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$|im(\alpha)| = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$
1,2	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{\hat{a}}$	$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{\hat{a}}$

For CT_3 on $X_3 = \{1,2,3\} = 17$ elements
 $|im(\alpha)| = 3$

$ker(\alpha)/im(\alpha)$	$\{1,2,3\}$
1 2 3	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$$|im(\alpha)|=2$$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$	$\{2,3\}$
$1 2,3$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$
$1,3 2$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$
$1,2 3$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$

$$|im(\alpha)|=1$$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$
$1,2,3$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}^*$

For CT_4 on $X_4 = \{1,2,3,4\} = 68$ elements

$$|im(\alpha)|=4$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4\}$
$1 2 3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}^*$

$$|im(\alpha)|=3$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3\}$	$\{2,3,4\}$
$1 2 3,4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 2 \end{pmatrix}$
$1 2,4 3$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 3 \end{pmatrix}$
$1 2,3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 2 \end{pmatrix}$
$1,3 2 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 2 \end{pmatrix}$
$1,2 3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 2 \end{pmatrix}$

$|im(\alpha)| = 2$

[illegible]

1,2,3 4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 3 \end{pmatrix}$
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$$|im(\alpha)|=1$$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1,2,3,4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}^*$

*indicates mappings that is order-preserving full contraction

The following table record the first six cases of $|OCT_n|$.

Table 1: The first six cases of $|OCT_n|$

n/r	1	2	3	4	5	6	
	1						
	2	1					
	3	4	1				
	4	9	6	1			
	5	16	18	8	1		
	6	25	40	30	10	1	

1 Results on Depth and Status of Order-preserving Full Contraction Transformation Semigroup

Theorem 3.1: Let $S = OCT_n$, for each $\alpha \in OCT_n$, the local U-depth of α is equal to $def(\alpha)$.

Proof: Let $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_i & \cdots & A_r \\ a_1 & a_2 & \cdots & a_{i+1} & \cdots & a_r \end{pmatrix}$ where $A_1 < A_2 < \dots < A_s$ are non singleton blocks in α and $a_i, a_j, \dots, a_s$ are the order of non single blocks respectively, that is $|A_i| = a_i, |A_j| = a_j, \dots, |A_s| = a_s$.

Now, if $\alpha \in OCT_n$, there must exist a block A_i for which $|A_i| \geq 2$. Three cases were considered to prove this theorem.

Case 1: If $a_1 = 1$, then α is expressed as a product of β that is,

$$\alpha = \beta_{i,i+1}^{a_i-1} \beta_{j,j+1}^{a_j-1} \cdots \beta_{s,s+1}^{a_s-1}$$

Case 2: If $a_1 = 2$, in this case α can be expressed as a product of β and γ as follow

$$\alpha = \begin{cases} \beta_{i,i+1} \gamma_{i,i+1} \beta_{i+1,i+2}^{a_i-3} \beta_{j+1,j+2}^{a_j-1} \cdots \beta_{s+1,s+2}^{a_s-1}, & \text{if } |A_i| > 2 \\ \beta_{i,i+1} \gamma_{\min[A_j]-1, \min[A_j]} \beta_{j+1,j+2}^{a_j-2} \cdots \beta_{s+1,s+2}^{a_s-2}, & \text{if } |A_j| \& |A_s| > 2 \\ & \text{and } s+1, s+2 \in A_s \\ \beta_{i,i+1} \gamma_{\min[A_j]-1, \min[A_j]} \beta_{j+1,j+2}^{a_j-1} \cdots \beta_{s+1,s+2}^{a_s-1}, & \text{if } |A_s| \geq 2 \\ & \text{and if at least one of } s+1, s+2 \text{ is not in } A_s \end{cases}$$

Case 3: If $a_1 \geq 3$, in this case α can be expressed as product of β and γ i.e.

$$\alpha = \begin{cases} \gamma_{i,i+1} \gamma_{i+2,i+3} \beta_{j+2,j+3}^{a_j-1} \cdots \beta_{s+2,s+3}^{a_s-1}, & \text{if } |A_i| \leq 3 \\ \gamma_{i,i+1} \gamma_{i+1,i+2} \beta_{i+2,i+3} \beta_{j+2,j+3}^{a_j-1} \cdots \beta_{s+2,s+3}^{a_s-1}, & \text{if } |A_s| > 3 \end{cases}$$

Example 1 Case 1: Let $\alpha \in OCT_8$

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 & 6 & \{7,8\} \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

Here, $def(\alpha) = 3$, the local U-depth of α is 3, these are $\beta_{23}\beta_{23}\beta_{56}$

Example 2 Case 2: Let $\alpha \in OCT_n$

$$\alpha = \begin{pmatrix} 1 & 2 & \{3,4\} & 5 & \{6,7,8\} \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \{3,4\} & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \{6,7\} & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$def(\alpha) = 3$, the local U-depth of α is 3, these are $\beta_{34}\gamma_{56}\beta_{67}$

Example 3 Case 3: Let $\alpha \in OCT_n$ Let

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 \\ 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \{2,3\} & 4 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & \{3,4\} & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

The $def(\alpha) = 2$, the local U-depth of α is 2, these are $\gamma_{23}\gamma_{34}$

Example 4 Case 3 Let

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 & 6 & \{7,8\} \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$def(\alpha) = 3$ and the local U-depth of α is 3, these are $\gamma_{23}\gamma_{34}\beta_{78}$

Example 5 Case 3: Let $\alpha \in OCT_n$ Let

$$\alpha = \begin{pmatrix} \{1,2,3\} & 4 & 5 & \{6,7\} & \{8,9\} & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

$\text{def}(\alpha) = 4$ and the local U-depth of α is 4, these are $\gamma_{23}\gamma_{34}\gamma_{67}\beta_{89}$

Remark

We observe that as the value of a_1 increases, the more the factor γ appears.

Lemma 3.1: $\max\{\text{def}(\alpha) : \alpha \in OCT_n\} = n-1$

Proof: For each map $\alpha \in OCT_n$, since the local U-depth of α is equal to its defect, that is the smallest positive integer k for which $\alpha = u_1, u_2, \dots, u_k$ where $u_1, u_2, \dots, u_k \in U$. Maximizing $\text{def}(\alpha)$ by making $h(\alpha)$ as large as possible and since $\text{def}(\alpha) = n - |h(\alpha)|$, clearly, the maximum value of $h(\alpha)$ is $n-1$. Thus, $\max\{\text{def}(\alpha) : \alpha \in OCT_n\} = n-1$. Hence, for all $n \geq 3$, the global U-depth of OCT_n deduced easily.

Theorem 3.2: For each $n \geq 3$, the global U-depth of OCT_n is $n-1$

Proof: The proof follows from lemma 3.1.

Theorem 3.3: Let $S = OCT_n$, $n \geq 3$ Then $\text{Stat}(OCT_n) \leq 2(n-1)^2$.

Proof: Let U be the generating set, $|U|$ be the order of U and $\Delta(U)$ be the global U-depth of OCT_n . Then, $|U|$ is equal to $2(n-1)$ by corollary 2.1 and from theorem 3.2. above $\Delta(U)$ equal to $n-1$. Since the status of semigroup is the least value of the product of order of generating set and global U-depth, that is $\text{Stat}(S) = \min\{|U| \Delta(U) : \langle U \rangle = S\}$ Hence, $\text{Stat}(OCT_n) \leq 2(n-1)^2$.

4 SOME COMBINATORICS PROPERTIES OF ORDER-PRESERVING FULL CONTRACTION TRANSFORMATION SEMIGROUP BY THEIR EQUIVALENCE CLASSES

This chapter investigate cardinalities of starred Green's relations and characterization of Green's relations on OCT_n .

On a semigroup S the relation L^* is defined by the rule that $(a, b) \in L^*$ if and only if a and b are related by the Green's relation L in some bigger semigroup of S (Fountain (1979)). The relation R^* is defined dually, these relations also have the following characterizations (see Fountain (1979)).

$$L^*(S) = (a, b) : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by \text{ and}$$

$R^*(S) = (a, b) : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb$. The join of the L^* and R^* is denoted by D^* and their intersection by H^* .

Recently, Mohammedjamil (2015) characterized starred Green's relations in semigroup OCT_n which we used in the research work to determine the cardinalities of equivalence classes in semigroup OCT_n .

Some of the elements of OCT_n were arranged based on this characterization as follow:

Let $\alpha \in OCT_n$

Table 1:For $n = 1, r = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Table 2:For $n = 2, r = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$
12	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

For $n = 2, r = 2$

Table 3:For $n = 2, r = 2$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$
1 2	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

Table 4:For $n = 3, r = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$
123	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

Table 5:For $n = 3, r = 2$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$	$\{2,3\}$
12 3	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$

Table 6:For $n = 3, r = 3$

$ker(\alpha)/im(\alpha)$	$\{1,2,3\}$
$1 2 3$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Table 7:For $n = 4, r = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1234	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$

Table 8:For $n = 4, r = 2$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$	$\{2,3\}$	$\{3,4\}$
$1 234$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 4 \end{pmatrix}$
$12 34$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}$
$123 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}$

For $n = 4 \quad r = 3$

Table 9:For $n = 4, r = 3$

$ker(\alpha)/im(\alpha)$	$\{1,2,3\}$	$\{2,3,4\}$
$1 2 34$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 \end{pmatrix}$
$12 3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}$

1 23 4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix}$
--------	--	--

Table 10:For $n = 4, r = 4$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4\}$
1 2 3 4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

From these elements listed above the following tables were formed which show the cardinalities of starred Green’s relations of height r :

Table 11: D^* Classes of OCT_n

n/r	1	2	3	4	5	6	
	1						
	1	1					
	1	1	1				
	1	1	1	1			
	1	1	1	1	1		
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	1	1	1	1	

Table 13: L^* -Classes of OCT_n

n/r	1	2	3	4	5	6	
	1						
	2	1					
	3	2	1				
	4	3	2	1			
	5	4	3	2	1		
	6	5	4	3	2	1	
	7	6	5	4	3	2	

Table 14: R^* - Classes of OCT_n

n/r	1	2	3	4	5	6	
	1						
	1	1					
	1	2	1				
	1	3	3	1			
	1	4	6	4	1		
	1	5	10	10	5	1	
	1	6	15	20	15	6	

Table 15: H^* Classes of OCT_n

n/r	1	2	3	4	5	6	
	1						
	2	1					
	3	4	1				
	4	9	6	1			
	5	16	18	8	1		
	6	25	40	30	10	1	
	7	36	75	80	45	12	

1 Combinatorial Results of Starred Green's Relations OCT_n

Theorem 4.1: Let $S = OCT_n$, the number of

- D^* -class of height r is 1;
- L^* -classes of height r is $n-r+1$;
- R^* -classes of height r is $\binom{n-1}{r-1}$; and
- H^* -classes of height r is $(n-r+1)\binom{n-1}{r-1}$.

Proof:

• From the characterisation of Mohammedjamil (2015) that if $|im(\alpha)| = |im(\beta)|$ then $(\alpha, \beta) \in D^*(OCT_n)$. Hence, D^* -class of height r is equal to 1.

• Since for each $\alpha \in OCT_n$, $im(\alpha)$ is convex subset of X_n , by characterisation of L^* -relation in OCT_n , the number of L^* -class of height r corresponds to the number of ways of

selecting r consecutively terms out of n . This is given by $n-r+1$.

- From the order-preservedness of OCT_n each block of the $\ker(\alpha)$, $\alpha \in OCT_n$ must be convex in X_n . Thus, by characterisation of R -relation, the number of R -classes of height r in OCT_n is equal to the number of partitions of X_n into r convex classes. This is not difficult to see that, this number equal to the number of ways of inserting $r-1$ identical strokes into $n-1$ places since inserting a stroke in space is the same as selecting that place. Hence, the number of R -classes of height r in OCT_n is the number of ways of selecting $r-1$ places out of $n-1$ places, that is, $\binom{n-1}{r-1}$

- By characterisation of the H -relation in OCT_n , the number of H -classes of height r is equal to the product of number of the columns and number of the rows of height r . That is, $(n-r+1)\binom{n-1}{r-1}$

. From the of theorem 4.1 above, we obtained the following corollary

Corollary 4.1: Let $S = OCT_n$, the total number of

- D^* -classes in OCT_n is n ;
- L^* -classes in OCT_n is $\frac{n(n+1)}{2}$;
- R^* -classes in OCT_n is 2^{n-1} ; and
- H^* -classes in OCT_n is $(n+1)2^{n-2}$

Proof From the theorem 4.1,

- Since for each $r = 1, 2, \dots, n$ there is an $\alpha \in OCT_n$ with $|im \alpha| = r$, we see that there is n D^* -class.

- The total number of L^* -classes in OCT_n is

$$\sum_{r=1}^n (n-r+1)$$

$$= \sum_{r=1}^n n - \sum_{r=1}^n r + \sum_{r=1}^n 1$$

$$= n^2 - \frac{n(n+1)}{2} + n = \frac{2n^2 - n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

- The total number of R -classes in OCT_n is equal to

$$\sum_{r=1}^n \binom{n-1}{r-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} = 2^{n-1}$$

• Since H -classes are trivial in OCT_n , then the total number of H -classes in OCT_n is equal to the order of OCT_n , That is, $|OCT_n|$, and this has been found by Adeshola (2013) to be $(n+1)2^{n-2}$. Hence,

$$\sum_{r=1}^n \binom{n-1}{r-1} (n-r+1) = |OCT_n| = (n+1)2^{n-2}$$

Example 4.1: Let $\alpha \in OCT_4$ and $|im \alpha| = r$, For $r = 1$

n/r	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1234	$\begin{pmatrix} \{1234\} \\ 1 \end{pmatrix}$	$\begin{pmatrix} \{1234\} \\ 2 \end{pmatrix}$	$\begin{pmatrix} \{1234\} \\ 3 \end{pmatrix}$	$\begin{pmatrix} \{1234\} \\ 4 \end{pmatrix}$

For $r = 2$

n/r	$\{12\}$	$\{23\}$	$\{34\}$
1 234	$\begin{pmatrix} 1 & \{234\} \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & \{234\} \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & \{234\} \\ 3 & 4 \end{pmatrix}$
12 34	$\begin{pmatrix} \{12\} & \{34\} \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} \{12\} & \{34\} \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} \{12\} & \{34\} \\ 3 & 4 \end{pmatrix}$
123 4	$\begin{pmatrix} \{123\} & 4 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} \{123\} & 4 \\ 2 & 3 \end{pmatrix}$	$\begin{pmatrix} \{123\} & 4 \\ 3 & 4 \end{pmatrix}$

For $r = 3$

n/r	$\{123\}$	$\{234\}$
1 2 34	$\begin{pmatrix} 1 & 2 & \{34\} \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & \{34\} \\ 2 & 3 & 4 \end{pmatrix}$
1 23 4	$\begin{pmatrix} 1 & \{23\} & 4 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & \{23\} & 4 \\ 2 & 3 & 4 \end{pmatrix}$

12 3 4	$\left(\begin{array}{ccc} \{12\} & 3 & 4 \\ 1 & 2 & 3 \end{array} \right)$	$\left(\begin{array}{ccc} \{12\} & 3 & 4 \\ 2 & 3 & 4 \end{array} \right)$
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For $r = 4$

n/r	$\{1234\}$
1234	$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array} \right)$

number of L^* -classes in OCT_4 is 10

number of R^* -classes in OCT_4 is 8

number of D^* -classes in OCT_4 is 4

number of H^* -classes in OCT_4 is 20

2 Number of elements in each equivalence classes within D^* -class

In this section the number of elements in each equivalence classes within D^* -class were investigated

Table 6: The number of elements in L^* - Class within D^* -class

n/r	1	2	3	4	5	6	
	1						
	1	1					
	1	2	1				
	1	3	3	1			
	1	4	6	4	1		
	1	5	10	10	5	1	
	1	6	15	20	15	6	

Table 7: The number of elements in R^* -class within D^* -class

n/r	1	2	3	4	5	6	7
1	1						
2	2	1					
3	3	2	1				
4	4	3	2	1			
5	5	4	3	2	1		
6	6	5	4	3	2	1	
7	7	6	5	4	3	2	1

Theorem 4.2: Let D^* be a D^* -class of height r in OCT_n . Then

- The number of elements in each L^* -class within D^* -class is $\binom{n-1}{r-1}$;
- The number of elements in each R^* -class within D^* -class is $n-r+1$; and
- The number of elements in each H^* -class within D^* -class is 1

Proof: The proof of (1) and (2) follows from the proof of theorem 4.1

(3) Since H^* -class is the intersection of L^* -class and R^* -class and each cell contained only one element. Hence, the number of element in each H^* -class is equal to 1 **Example**

4.2: From example 4.1 above we obtained the following:

For $n=4, r=1$ we have

The number of elements in each L^* -class is 1

The number of elements in each R^* -class is 4

The number of elements in each L^* -class is 1

For $n=4, r=2$ we have

The number of elements in each L^* -class is 3

The number of elements in each R^* -class is 3

The number of elements in each H^* -class is 1

For $n=4, r=3$ we have

The number of elements in each L^* -class is 3

The number of elements in each R^* -class is 2

The number of elements in each H^* -class is 1

For $n=4, r=4$ we have

The number of elements in each L^* -class is 1

The number of elements in each R^* -class is 1

The number of elements in each H^* -class is 1

Remark It was observed that table 5 and table 7 have the same entries, this shows that the order of R^* in OCT_n is equal to the number of elements in L^* -class within D^* -class.

3 Green's Relations of order-preserving full contraction transformation semigroup

Theorem 4.3 Let $S = OCT_n$ and $\alpha, \beta \in S$, then

- $(\alpha, \beta) \in L(OCT_n)$ if and only if either $\alpha = \beta$ or $im(\alpha) = im(\beta)$ and $ker(\alpha), ker(\beta)$ are convex partitions of X_n ;
- $(\alpha, \beta) \in R(OCT_n)$ if and only if $ker(\alpha) = ker(\beta)$;
- $(\alpha, \beta) \in D(OCT_n)$ if and only if either $ker(\alpha) = ker(\beta)$ or $|im(\alpha)| = |im(\beta)|$ and both $ker(\alpha), ker(\beta)$ are convex partitions of X_n ; and
- $(\alpha, \beta) \in H(OCT_n)$ if and only if $\alpha = \beta$.

Proof

• Suppose that $(\alpha, \beta) \in L(OCT_n)$, then $\delta\beta = \alpha$ and $\gamma\alpha = \beta$ for some $\delta, \gamma \in OCT_n$. This clearly implies that $im(\alpha) = im(\beta)$. Therefore, $im(\gamma)$ and $im(\delta)$ must be transversal of $ker(\alpha)$ and $ker(\beta)$ respectively. But since $\delta, \gamma \in OCT_n$. It follow from [Mohammedjamil, (2015)], that $im(\gamma)$ and $im(\delta)$ are convex subsets of X_n . Thus, $ker(\alpha)$ and $ker(\beta)$ are convex partitions of X_n .

Conversely, suppose that $im(\alpha) = im(\beta) = \{c_1, c_2, \dots, c_r\}$ and that $ker(\beta)$ are convex partitions of X_n . Let $\{a_1, a_2, \dots, a_r\}$ and $\{b_1, b_2, \dots, b_r\}$ be convex transversal of $ker(\alpha)$ and $ker(\beta)$ respectively, arranged in away that $a_i \in c_i\alpha^{-1}$ and $b_i \in c_i\beta^{-1}$ for each $1 \leq i \leq r$. define maps δ and γ by $ker(\delta) = ker(\alpha), ker(\gamma) = ker(\beta), (c_i\alpha^{-1})\delta = b_i$ and $(c_i\beta^{-1})\gamma = a_i$ for each $1 \leq i \leq r$. Then, $\delta, \gamma \in OCT_n$ and $\delta\beta = \alpha$ and $\gamma\alpha = \beta$. So that $(\alpha, \beta) \in L(OCT_n)$

• Suppose that $(\alpha, \beta) \in R(OCT_n)$, then $\beta\delta = \alpha$ and $\alpha\gamma = \beta$ for some $\delta, \gamma \in OCT_n$. From this it follows that $ker(\alpha) = ker(\beta)$. Conversely, suppose that $ker(\alpha) = ker(\beta) = \{c_1, c_2, \dots, c_r\}$. Then, since $\alpha, \beta \in OCT_n$, without loss of generality we write

$$\alpha = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ i & i+1 & \dots & i+r-1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ j & j+1 & \dots & j+r-1 \end{pmatrix} \text{ for some } i, j \in X_n,$$

then the maps

$$\delta = \begin{pmatrix} \{1, 2, \dots, j\} & j+1 & \dots & j+r-2 & \{j+r-1, j+r, \dots, n\} \\ i & i+1 & \dots & i+r-2 & i+r-1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} \{1, 2, \dots, i\} & i+1 & \dots & i+r-2 & \{i+r-1, i+r, \dots, n\} \\ j & j+1 & \dots & j+r-2 & j+r-1 \end{pmatrix}$$

are in OCT_n and satisfy $\beta\delta = \alpha$ and $\alpha\gamma = \beta$, So that $(\alpha, \beta) \in R(OCT_n)$

• Suppose that $\alpha \neq \beta$ and $(\alpha, \beta) \in D(OCT_n)$, then $(\alpha\gamma) \in L(OCT_n)$ and $(\gamma, \beta) \in R(OCT_n)$, for some $\gamma \in OCT_n$ from the theorem $im(\alpha) = im(\gamma)$, $ker(\gamma) = ker(\beta)$ and that $ker(\gamma), ker(\alpha)$ are convex partitions of X_n . This implies that $|im(\alpha)| = |im(\beta)|$ and

$ker(\alpha), ker(\beta)$ are convex partitions of X_n . Conversely, suppose that $|im(\alpha)|=|im(\beta)|$, and both $ker(\alpha), ker(\beta)$ are convex partitions of X_n . Then we can choose $\gamma \in OCT_n$ such that $ker(\alpha)=ker(\beta)$ and $im(\gamma)=im(\alpha)$. Hence $(\alpha\gamma) \in L(OCT_n)$ and $(\gamma, \beta) \in R(OCT_n)$.

5 GENERAL SUMMARY AND CONCLUSION

1 Summary

The nature of contraction mapping were first described by Higgins et.al. (2003) and some algebraic and combinatorials properties of order-preserving full contraction mapping such as Green's relations, rank and characterization of starred Green's relations were studied by some researchers among them are Adeshola (2013), Mohammedjamil (2015) and many more while the Depth and Status of OCT_n as well as the cardinalities of Equivalence relations and their elements have not been studied . In the thesis, the Depth and Status of semigroup OCT_n were studied .

The results obtained from this study were summarized below:

- the local $U - depth$ of α is equal to its defect and global U-depth of α is $n-1$
- Obtained the status of OCT_n to be $Stat(OCT_n) \leq 2(n-1)^2$;
- The table below shows the cardinalities of Equivalence classes of starred Green's relations of height r .

Table 9: The results on Equivalence classes of starred Green's relations of height r

$ D^* $	1
$ L^* $	$n-r+1$
$ R^* $	$\binom{n-1}{r-1}$
$ H^* $	$(n-r+1)\binom{n-1}{r-1}$

- The following table shows the total number of each equivalence classes in OCT_n

Table 10: Results on total number of each equivalence classes in OCT_n

$\sum D^* $	n
--------------	-----

$\sum L^* $	$\frac{n(n+1)}{2}$
$\sum R^* $	2^{n-1}
$\sum H^* $	$(n+1)2^{n-2}$

• The table below also shows the order of elements in each equivalence classes within D^* class.

Table 11: Results on order of elements in each equivalence classes within D^* class.

$ L_a^* $	$\binom{n-1}{r-1}$
$ R_a^* $	$n-r+1$
$ H_a^* $	$(n-r+1)\binom{n-1}{r-1}$

- The Green's relations of OCT_n were also characterized based on their kernel class and image sets

2 Conclusion

In this thesis, we have shown that in the semigroup OCT_n , the minimum length factorisation of each $\alpha \in OCT_n$ is equal to the defect of α and the Status of semigroup OCT_n were obtained using the global U -depth to be $\text{Stat}(OCT_n) \leq 2(n-1)^2$. Also, the cardinalities of equivalence Starred Green's relation and their elements in order-preserving full contraction transformation semigroup OCT_n were shown with relevant examples.

3 Contribution to Knowledge

In this thesis, formulas were deduced for the number of sequences as seen above. The fomulas will be useful for other researchers in this field and other related field. Its also has applications in computational theory: formal languages, coding theory and automata theory as well as applications in other sciences.

4 Recommendation

It is hereby recommended for other researchers in this field to determine the cardinalities of Green's relations and their elements in semigroup OCT_n based on the characterisation given in this work and other semigroup such as CT_n , OCI_n , CI_n and so on. The theory of semigroup can also assist in sorting data and designing better networks.

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APPENDIX Semigroup of Full Contraction Mappings:

For CT_n on $X_n = \{1, 2, \dots, n\}$

OCT_1 on $X_1 = \{1\} = 1$ element

$|im(\alpha)| = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^*$

For CT_2 on $X_2 = \{1, 2\} = 4$ elements

$|im(\alpha)| = 2$

$ker(\alpha)/im(\alpha)$	$\{1, 2\}$
1 2	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}^{\hat{a}}$ $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$|im(\alpha)| = 1$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$
1,2	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{\hat{a}}$	$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}^{\hat{a}}$

For CT_3 on $X_3 = \{1, 2, 3\} = 17$ elements

$|im(\alpha)| = 3$

$ker(\alpha)/im(\alpha)$	$\{1, 2, 3\}$
1 2 3	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$$|im(\alpha)|=2$$

$ker(\alpha)/im(\alpha)$	$\{1,2\}$	$\{2,3\}$
$1 2,3$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}$
$1,3 2$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$
$1,2 3$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$

$$|im(\alpha)|=1$$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$
$1,2,3$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}^*$

For CT_4 on $X_4 = \{1,2,3,4\} = 68$ elements
 $|im(\alpha)|=4$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4\}$
$1 2 3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}^*$

$$|im(\alpha)|=3$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3\}$	$\{2,3,4\}$
$1 2 3,4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 2 \end{pmatrix}$
$1 2,4 3$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 3 \end{pmatrix}$
$1 2,3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 2 \end{pmatrix}$
$1,3 2 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 2 \end{pmatrix}$
$1,2 3 4$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}^*$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 2 \end{pmatrix}$

$$|im(\alpha)| = 2$$

[illegible]

1,2,3 4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \end{pmatrix}^*$
	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 3 \end{pmatrix}$

$$|im(\alpha)|=1$$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1,2,3,4	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}^*$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}^*$

For CT_5 on $X_5 = \{1,2,3,4,5\} = 259$ elements

$$|im(\alpha)|=5$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4,5\}$
1 2 3 4 5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

$$|im(\alpha)|=4$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4\}$	$\{2,3,4,5\}$
1 2 3 4,5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 5 \end{pmatrix}$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 2 \end{pmatrix}$

1 2 3,5 4	$\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 3 & 2 & 3 & 4 & 5 & 4 \end{array} \right)$ $\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 2 & 5 & 4 & 3 & 2 & 3 \end{array} \right)$	
1 2 3,4 5	$\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 2 & 3 & 4 & 4 & 5 \end{array} \right)$ $\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 2 & 1 & 5 & 4 & 3 & 3 & 2 \end{array} \right)$	
1 2,3 4 5	$\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 & 2 & 3 & 3 & 4 & 5 \end{array} \right)$ $\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 3 & 2 & 1 & 5 & 4 & 4 & 3 & 2 \end{array} \right)$	
1,2 3 4 5	$\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 2 & 2 & 3 & 4 & 5 \end{array} \right)$ $\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 3 & 2 & 1 & 5 & 5 & 4 & 3 & 2 \end{array} \right)$	
1,3 2 4 5	$\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 & 1 & 3 & 2 & 3 & 4 & 5 \end{array} \right)$ $\left(\begin{array}{cccc cccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 & 4 & 5 & 4 & 3 & 2 \end{array} \right)$	

$$|im(\alpha)| = 3$$

[illegible]

[illegible]

[illegible]

$$|im(\alpha)| = 1$$

$ker(\alpha)/im(\alpha)$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
1,2,3,4,5	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 5 & 5 \end{pmatrix}$

For CT_6 on $X_6 = \{1,2,3,4,5,6\} = 950$ elements

$$|im(\alpha)| = 6$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4,5,6\}$
1 2 3 4 5 6	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix}$

$$|im(\alpha)| = 5$$

$ker(\alpha)/im(\alpha)$	$\{1,2,3,4,5\}$	$\{2,3,4,5,6\}$
$1 2 3 4 5,6$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix}$
$1 2 3 4,6 5$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix}$
$1 2 3 4,5 6$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix}$
$1 2 3,4 5 6$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 4 & 4 \end{pmatrix}$

1 2,3 4 5 6	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 4 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 5 & 4 \end{pmatrix}$
1,2 3 4 5 6	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 4 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 6 & 5 & 4 \end{pmatrix}$
1,3 2 4 5 6	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 4 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 5 & 4 \end{pmatrix}$