

EXPONENTIALLY-FITTED NUMERICAL INTEGRATION
SCHEMES FOR FIRST AND SECOND ORDER DIFFERENTIAL
EQUATIONS

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B.Sc., M.Sc. (UNILORIN)

APRIL, 2021

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BY

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BEING A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS,
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APRIL, 2021

Certification

This is to certify that the research work reported in this thesis was done by FANIYI, Olugbade Ezekiel with Matriculation number 05/55EB054 in the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the award of a Doctor of Philosophy

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Dedication To the

angels of my life.

Declaration

I, Olugbade Ezekiel FANIYI, a Ph.D. student in the Department of Mathematics, University of Ilorin, Ilorin, hereby declare that this thesis entitled “ Exponentiaaly-Fitted Numerical Integration Scheme for First and Second Order Differential Equations”, submitted by me is based on my actual and original work. Any materials obtained from other sources or work done by any other persons or institutions have been duly acknowledged. In addition, the research has been approved by the University of Ilorin Ethical Review Committee.

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Abbreviations and Acronyms

ODE- Ordinary Differential Equation

PDE-Partial Differential Equation

IVP-Initial Value Problem

BVP-Boundary Value Problem

IBVP-Initial-Boundary Value Problem

ENIS-Exponentially-fitted Numerical Integration Scheme

HENIS-Hybrid Exponentially-fitted Numerical Integration Scheme

PCM-Predictor-Corrector Method

SSM-Self-Starting Mode DE-Differential Equation

LMM-Linear Multistep Method

MOL-Method of Lines

Errk4-Error for ENIS for $k=4$

Errk5-Error for ENIS for $k=5$

RKM-Runge-Kutta Method yapp-

y approximate solution yex- y

exact solution

CPU-Central Processing Unit

BVM-Boundary Value Method

BUM-Block Unification Method

BLOCK2-2-step block method

BLOCK10- 10-step block method

BLOCK2SIMP-simplest 2-step block method

BLOCK10SIMP-10-step block method

Abstract

Mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called Differential Equations (DEs). A DE in which the unknown function is a function of two or more independent variables is called a Partial Differential Equation (PDE). Those in which the unknown function is a function of only one independent variable are called Ordinary Differential Equations (ODEs). This work concerns the study of numerical solutions of both types. Since most DEs do not have closed form solution, the development of numerical techniques becomes necessary. Methods used for solving Initial Value Problems (IVPs) of general ODEs have been reported to have setbacks as regards to handling certain attributes like singularity of stiffness and oscillatory nature of problems. Furthermore, implementation via predictor-corrector mode in Linear Multistep Methods (LMMs) have been found to be exhausting in terms of the number of function evaluations per step. K-step exponentially-fitted block methods circumvent these challenges associated with

LMMs, for solving IVPs, Boundary Value Problems (BVPs) and Initial Boundary Value Problems (IBVPs) of special nature; hence the motivation of this work.

This study is therefore aimed at developing a class of efficient numerical integration schemes, for direct solution of first order IVPs, second order BVPs and IBVPs. To this end, the objectives were to: (i) construct trial solution with power series fitted with exponential function as basis function; (ii) develop new classes of k-step continuous

exponentially-fitted block methods; (iii) develop continuous 2-step hybrid exponentially-fitted block method; (iv) analyze the methods for zero stability, consistency and convergence; and (v) compare the accuracy of the proposed schemes with those of existing methods using some existing problems. The first order IVP of the form

$$y' = f(x, y), \quad a < x < b \quad (1)$$

and the second order BVPs of the form

$$y'' = f(x, y, y'), \quad a < x < b \quad (2)$$

with appropriate boundary conditions were considered. The trial solution of the form

$$p(x) = \sum_{i=0}^k a_i x^i + a_{k+1} e^{\omega x} \quad (3)$$

$$\tilde{y} = y(x)$$

was assumed where the a_i 's are unknown constants and ω is the frequency. In order to determine the unknown constants, interpolation and collocation methods were employed. The resulting values were later substituted back into (3), which after some algebraic manipulations yielded the desired continuous exponentially fitted block method.

The findings of this study were;

- (i) new classes of k-step exponentially fitted block methods for direct solution of first order IVPs and second order BVPs were derived;
- (ii) new 2-step Continuous hybrid exponentially fitted block method for solving second order IBVPs were developed;
- (iii) the methods are zero stable, consistent and convergent;

- (iv) the methods have small global error and less computation time compared to existing methods; and
- (v) the methods are self-starting and admit easy change of step-sizes and function evaluation at off grid points.

The study concluded that an efficient class of continuous and discrete numerical integration schemes of implicit form for first and second order problems were developed and successfully implemented. These schemes are therefore recommended for solution of first and second order IVPs, second order BVPs and IBVPs.

Chapter 1

GENERAL INTRODUCTION

1.1 Background of the study

Differential Equations (DEs) are mathematical models developed to help understand physical phenomena or processes which often time arise in natural sciences, engineering, medicine, economics and so on. Such physical phenomena include: motions of the planet in a gravitational field like the Kepler problem, the simple pendulum, electrical circuits and chemical kinetic problems. The models often yield equations that contain some derivatives of an unknown function of one or several variables. A Differential Equation (DE) which has function of only one independent variable is called an Ordinary Differential Equation (ODE), while that in which the unknown function is a function of two or more independent variables is called a Partial Differential Equation (PDE).

In most cases, DEs often do not have analytic solutions. This is due to the fact that most real life problems are modeled into highly nonlinear equations and such equations lack methods for obtaining exact solutions. Therefore the need for numerical methods to obtain approximate solutions is necessary. The objective of numerical methods is to solve complex numerical problems using only the simple operations of arithmetic to develop and evaluate methods for computing numerical results from given data. The methods of computation are called algorithms. An algorithm is a finite sequence of rules for performing computations on a computer

such that at each instant the rules determine exactly what the computer has to do next. Numerical methods tend to emphasize the implementation of the algorithms.

Some of the numerical techniques include: one-step (or single-step) methods and multistep methods especially the Linear Multistep Methods (LMMs). One-step methods such as Euler method, Runge-Kutta method, Taylor series, etc., give approximation to the true solution $y(x)$ at x_{n+1} once y_n , y_{n+1} and h , the step size are known. LMMs such as Adam-Bashforth method, Adam-Moulton method and Numerov method, used recurrence relations that express the function value $y(x_{n+1})$ in terms of the function value $y(x)$ and the derivative value $y^0(x)$ at $x = x_{n+1}$ and at previous nodal points. In this work, a first order Initial Value Problem (IVP) of the form

$$\left. \begin{aligned} y^0 &= f(x, y), \quad a < x < b, \\ y(a) &= \alpha_0 \end{aligned} \right\} \quad (1.1)$$

where α_0 is an arbitrary finite real constants, $x \in (a, b)$, $f \in C[a, b]$, is a continuous function defined on the interval (a, b) and a second order Boundary Value Problem

(BVP) of the form

$$\left. \begin{aligned} y^{00} &= f(x, y, y^0), \quad a < x < b, \\ y(a) &= \alpha_0, \quad y(b) = \beta_0 \end{aligned} \right\} \quad (1.2)$$

where α_0, β_0 are an arbitrary finite real constants, $x \in (a, b)$, $f \in C^2[a, b]$, is a continuous function defined on the interval (a, b) were considered. The boundary conditions in (1.2) are the Dirichlet type.

where α_0, β_0 are an arbitrary finite real constants, $x \in (a, b)$, $f \in C^2[a, b]$, is a continuous function defined on the interval (a, b) were considered. The boundary conditions in (1.2) are the Dirichlet type.

Any of the following boundary conditions were also considered: Robin or mixed boundary conditions $y^0(a) = \alpha_0, y(b) = \beta_0$ or $y(a) = \alpha_0, y^0(b) = \beta_0$, the Neumann Boundary condition $y^0(a) = \alpha_0, y^0(b) = \beta_0$.

The equations above originated from models in Mathematical Biology triggered by a random walk of organisms, such as bacterial in the presence of a chemical resulting in interesting spatial patterns (Tyson, Lubkin and Murray, 1999; Tyson, Stern and LeVeque, 2000). On the other hand Two-point BVPs of the form (1.1) and (1.2) occur in a large number of applications. A short list of examples includes but not limited to;

- Material science and structural analysis.
- Microphysics, as exemplified by the (stationary) Schrödinger equation

$$-\frac{\hbar^2}{2m}\phi'' + V(x)\phi = E\phi$$

where ϕ is the wave function, $V(x)$ is a “potential well”, m is the particle mass and E is the energy of a particle in the state ϕ .

- Eigenmode analysis, such as in $-u'' = \lambda u$, which may describe buckling of structures and eigenfrequency oscillation modes of materials such as a bridge or a musical instrument.

The last two examples are Sturm-Liouville problems, which in many cases are directly related to Fourier analysis. Examples arise in the mathematical modeling of deflection of cantilever beams under concentrated load (Na, 1979), deformation of beams and plate deflection theory (Glabisz, 2004), obstacle problems (Siraj-ul-Islam, Noor, Tirmizi and Khan 2006), Troesch’s problem relating to the confinement of a plasma column by radiation pressure, (Robert and Shipman, 1972; Wiebel, 1958)), temperature distribution of the radiation fin of trapezoidal profile (Na, 1979; Keller and Holdrege, 1970), and a number of other engineering applications. We note that the solutions of the non-stiff or singular ODEs may become unbounded in finite time and such a

phenomenon is often called blow-up, while the finite time is called the blow-up time (Cho, 2013; Groisman, 2006; Guo and Yang, 2015; Takayasu, Matsue, Sasaki, Tanaka, Mizuguchi and Oishi, 2017). It is also reported in Takayasu et al. (2017) that blow-up solutions describe the combustion of solid fuels via the nonlinear heat equation.

1.2 Aim and Objectives

The aim of this study is to develop continuous implicit k-step methods for the direct solution of initial value problems of general first order ODEs, BVPs and IBVPs of second order DEs. The objectives of the study were to:

1. construct exponential fitting which used as basis function of the trial solution;
2. develop continuous implicit k-step methods by collocating and interpolating at both the step and off-step points;
3. employ the Method of Lines (MOLs) technique to transform second order PDEs to system of second order ODEs;
4. analyse the methods developed for their order, consistency, zero stability and convergence; and
5. implement the methods using a written code to test the performance of the new methods for accuracy and efficiency.

1.3 Statement of the Problem

Methods used for solving higher order IVPs of general ODEs by reducing them to system of first order have been reported to have setbacks such as computational cost as regards to many subroutines and computer time inability of the method to handle certain attributes like singularity of the problems, oscillatory nature and stiffness of the solution (Vigo- Aguiar and Ramos, 2006). Furthermore, implementation via predictorcorrector mode in linear multistep methods have been found to be very exhausting in terms of the number of function evaluations per step. K-step exponentially fitted block methods tend to circumvent these challenges associated with linear multistep methods for solving IVPs and BVPs of special ODEs such as heat equation and wave equation.

1.4 Justification of the Study

This research is aimed at addressing the set-backs associated with the existing methods as stated above, by developing an exponentially fitted method that captures the properties of these existing methods. This method is less expensive, in terms of the number of functions evaluation per step; highly efficient in terms of accuracy and error term; flexible in change of step size; and easy to program resulting in economy of computer

time.

1.5 Scope of the Study or Limitations

The research is limited to the following: (i) only continuously differentiable functions in the interval of integration were considered;

(ii) the basis function considered in this work is the power series polynomial fitted with exponential function in view of its smoothness;

(iii) for development of methods for solution of first order IVPs, the research work adopted only continuous linear multistep methods where the step number $k = 1(1)5$;

(iv) for development of methods for solution of second order BVPs, the research work adopted only continuous linear multistep methods where the step number $k = 2(1)4$;

(v) for development of methods for solution of second order IBVPs, the research work adopted only continuous hybrid linear multistep methods where the step number $k = 2$; (vi) only implicit block methods were adopted in this research work.

1.6 Definition of Terms

Definition 1.6.1. *The sequence of points $\{x_n\}$ in the interval (a,b) defined by $a = x_0 <$*

$x_1 < x_2 < \dots < x_n < x_{n+1} \dots x_N = b$ Such that $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, \dots, N - 1$

where h_i is the step size.

Definition 1.6.2. (One-Step Methods) *One-step numerical integration scheme for Initial Value Problems (IVPs) are described as $y_{n+1} = y_n + h\phi_n$ where $h\phi_n = \phi(x_n, y_n, h)$, h is the step size which is adopted in the interval $[x_n, x_{n+1}]$. The method can both be formulated in explicit and implicit forms.*

Examples of one- step methods are the backward and forward Euler method, midpoint method etc

Definition 1.6.3. *Linear Multistep Methods (LMMs) require two or more proceeding values to be able to calculate y_{n+1} unlike the one-step methods that require only a single value y_n to compute the next approximation y_{n+1} .*

Definition 1.6.4. *The general k – step LMM for the μ^{th} order IVP*

$$\left. \begin{aligned} y^{(\mu)} &= f(x, y, y_0, y_0', \dots, y^{(\mu-1)}), \\ y^{(k)}(x_0) &= \rho_k, k = 0(1)\mu - 1 \end{aligned} \right\} \quad (1.3)$$

is defined as

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^{\mu} \beta_j f_{n+j} \quad (1.4)$$

where $\alpha_0, \dots, \alpha_k$ and $\beta_0, \dots, \beta_k$ are real constants and $y_{n+j} = y(x_{n+i})$, $f_{n+i} = f(x_{n+1}, y_{n+1}, y'_{n+j}, \dots, y^{(\mu-1)}_{n+j})$, μ is the order of the differential equation. (Lambert, 1973) . This can also be written symbolically as

$$\rho(r)y_n - h^\mu \sigma(r)f_n = 0 \quad (1.5)$$

where $\rho(r)$ and $\sigma(r)$ respectively, are the first and second characteristic polynomials of the LMM and are defined as

$$\rho(r) = \sum_{j=0}^k \alpha_j r^j, \quad \sigma(r) = \sum_{j=0}^{\mu} \beta_j r^j \quad (1.6)$$

The coefficients $\alpha_0, \dots, \alpha_k$ and $\beta_0, \dots, \beta_k$ are real constants and $y_{n+j} = y(x_{n+j})$ and

$$f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, \dots, y^{(\mu-1)}_{n+j})$$

(Lambert, 1991)

Remark 1.6.1. 1. It is assumed that $\alpha_k \neq 0$ and that α_0 and β_0 are both not equal to zero so as to avoid degenerate cases.

2. If $\beta_k = 0$ then y_{n+k} is obtained explicitly from previous values of y_{n+j} and f_{n+j} , and the k -step method is said to be explicit. However, if $\beta_k \neq 0$ then y_{n+k} appears not only on the left-hand side of (1.4) but also on the right within f_{n+k} ; due to this implicit dependence on y_{n+k} , the method is then called implicit.

3. The k -step LMM (1.4) is called linear because it involves only linear combinations of the y_{n+j} and f_{n+j} .

Definition 1.6.5. Order of LMM: The LMM (1.4) is said to be of order p if $C_0 = C_1 = C_2 = \dots = C_p = 0$. $C_{p+\mu} \neq 0$ is the error constant, and $C_{p+\mu} h^{p+\mu} y^{(p+\mu)}(x_n)$ is the principal Local Truncation Error (LTE) at the point x_n . The C 's are given by

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_k$$

$$C_1 = (\alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_k) - (\beta_0 + \beta_1 + \dots + \beta_k)$$

$$C_q = \frac{1}{q!} (\alpha_1 + 2^q \alpha_2 + \dots + k^q \alpha_k) - \frac{1}{(q-3)!} (\beta_1 + 2^q \beta_2 + \dots + k^q \beta_k)$$

According to Lambert (1973) the LTE associated with (1.4) is given by the difference operator

$$L[y(x) : h] = \sum_{j=0}^k [\alpha_j y(x + jh) - h^\mu \beta_j f(x + jh)]$$

where y is an arbitrary function, continuously differentiable on $[a, b]$. Expanding the last equation in Taylor's series about the point x , the expression is obtained as:

$$L[y(x) : h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_{p+3} h^{p+3} y^{(p+3)}(x)$$

Definition 1.6.6. A LMM is said to be zero stable if all roots of the characteristic poly-

nomial

k

$$\rho(r) = \sum_{j=0}^k \alpha_j r^j \quad (1.7)$$

$j=0$

satisfies $|r_i| \leq 1$ and every root such that $|r_i| = 1$ has multiplicity not exceeding the order of the differential equation.

Definition 1.6.7. An implicit method for the IVP (1.4) is said to be consistent if it satisfies the following conditions

1. the order $p \geq 1$
2. $\sum_{j=0}^k \alpha_j = 0$
3. $\rho(1) = \rho^0(1) = 0, \quad \rho^\mu(1) = \mu! \sigma(1)$

1.6.1 Zero-stability

Definition 1.6.8. The first characteristic polynomial of a k -step linear multistep method is the degree- k polynomial

$$\rho(z) = \sum_{j=0}^k \alpha_j z^j \quad (1.8)$$

Definition 1.6.9. A polynomial is said to satisfy the root condition if all its roots lie within or on the unit circle, with those on the boundary being simple. In other words, all roots satisfy $|z| \leq 1$ and any that satisfy $|z| = 1$ are simple. A polynomial satisfies the strict root condition if all its roots lie inside the unit circle; that is, $|r| < 1$, (Amodio and Mazzia, 1995).¹

¹ We say that λ is a simple root of $\rho(z)$ if $\lambda - z$ is a factor of $\rho(z)$.

1.6.2 Hybrid Methods

The LMMs have the advantage of being more efficient in terms of accuracy and weak stability properties for a given number of functions evaluated per step, but have the disadvantages of requiring starting values and special procedure for changing step size when compared to the Runge-kutta method. This technique was used independently by Gragg and Stetter (1964). The method was first named Hybrid by Gear (1965), and the hybrid method and Runge-Kutta methods share the property of utilizing data at off-step points and the flexibility of changing step length.

Definition 1.6.10. A k -step hybrid formula for the IVP (1.6.1) is defined as

$$\sum_{j=0}^k \alpha_j y_{n+j} = h^\mu \sum_{j=0}^k \beta_j f_{n+j} + h \beta_{-v} f_{n+v} \quad (1.9)$$

where $\alpha_k = 1$, α_0, β_0 are both non-zero and $y_{n+k} = y(x_{n+kh})$, $f_{n+v} = f(x_{n+vh}, y_{n+v})$

(Lambert, (1973))

1.6.3 Block Methods

The block method has a traditional advantage over single step method of being selfstarting and permitting easy change of step length (Lambert 1973). The methods are less expensive in terms of number of functions evaluated for a given order when compared with the Runge-Kutta method.

Definition 1.6.11. Let Y_m and F_m be defined by $Y_m = (y_n, y_{n+1}, \dots, y_{n+r-1})^T$, $F_m = (f_n, f_{n+1}, \dots, f_{n+r-1})^T$. The general k -block is a matrix of finite difference equation of the form

$$Y_m = \sum_{j=0}^k A_j y_{m-j} + h \sum_{j=0}^k B_j F_{m-j} \quad (1.10)$$

where A_j, B_j are properly chosen square matrices (Chu and Hamilton, 1987)

1.7 Organization of the Thesis

The thesis is organized as follows:

Chapter 1 is concerned with the introductory aspects comprising of the background, objectives, statement of problem, justification, definition and organization of the thesis. Chapter 2 deals with review of related literature.

Chapter 3 focuses on the methodology which includes the development and analysis of different schemes for IVPs, BVPs and IBVPs.

Chapter 4 concerns the application of the derived schemes, analysis and results of the schemes.

Chapter 5 concludes the work with some remarks.

Chapter 2

LITERATURE REVIEW

An Ordinary Differential Equation (ODE) is an equation with one or more functions of independent variable and its derivatives. On the other hand, Partial differential equations are differential equations having more than one independent variable. When physical phenomena are modeled as non-linear equations, they are generally approximated by linear differential equations for an easier solution. The few nonlinear ODEs that can be solved explicitly are generally solved by transforming the equations into equivalent linear ODEs. Some ODEs may be solved explicitly in terms of known

functions and integrals. Real life phenomena comprise of several variables and PDEs are the only suitable models for such situations. In most cases, solutions to both ODEs and PDEs are difficult to find since most are nonlinear in nature, hence numerical solutions are highly needed. In the section that follows, research done so far on numerical approximations of solution of differential equations in general are discussed.

2.1 The Review

In applications, most equations in the form (1.1) and (1.2) do not possess exact solutions, hence numerical methods are imperative for solving (1.1) and (1.2). Several numerical methods based on Runge-Kutta methods, linear multistep methods, Boundary value methods, and block methods have been extensively discussed in the literature for solving non-stiff and stiff IVPs (Akinfenwa and Jator, 2013; Brugnano and Trigiante, 1998; Jator, 2010; Cash, 1981; Hairer and Wanner, 1976; Lambert, 1991; Onumanyi, Awoyemi, Jator, and Sirisena, 1994; Onumanyi, Sirisena, and Jator 1999). Most of these methods do not perform well when applied to solve singular problems of the form (1.1) since they are based on polynomial approximation (Hairer and Wanner, 1996; Lambert, 1991). In particular, methods constructed using polynomial basis functions perform poorly in the proximity of a singular point when applied to singular IVPs. Hence, rational interpolants have been proposed for solving singular IVPs and were first introduced by Lambert and Shaw (1965). Other techniques based on rational functions have been proposed to cope especially with singular IVPs (Fatunla, 1982; Hasan, Ahamed, Alam, and Hossain, 2013; Ikhile, 2004; Odekunle, Oye, Adey, and Ademiluyi, 2004; Van Niekerk, 1988, Jator and Coleman, 2017). Other notable methods can be found in Van Niekerk (1988), Ramos and Vigo-Aguiar (2008) and Garwood and Jator (2016).

Numerical methods of solution to second-order BVPs of the form (1.2) are numerous. The details about the related numerical methods are found in the literature. (Na, 1979; Glabisz, 2004; Siraj-ul-Islam *et al.*, 2006; Tatari and Dehgan, 2006; Al-Said, 2001; Lakestani and Dehgan, 2006; Tirmizi and Twizell, 2002; Katti and Baboo, 1996 ; Siraj-ul-Islam *et al.*, 2006). Other techniques include the Walsh wavelets and the semiorthogonal B-spline wavelets, which were used in Glabisz (2004) and Lakestani and Dehgan (2006), to construct the numerical algorithms for the solution of second-order BVPs with Dirichlet and Neumann boundary conditions. Na (1979) obtained the numerical solution of second, third- and fourth-order BVPs by converting them into initial value problems and applying a class of methods such as nonlinear shooting, method of reduced physical parameters, method of invariant imbedding among others. The present approach can be applied to both BVPs and IVPs with a slight modification, but without the transformation of BVPs into IVPs or vice versa. Siraj-ul-Islam *et al.* (2010) , used collocation method with the Haar wavelets for the solution of second-order boundary value problems, Islam and Shirin (2011) solved numerically a class of second order boundary value problems using Bernoulli Polynomials, Adeyeye and Omar (2016), used maximal order block method for solving second order BVPs, Majid *et al.* (2013) applied a method called block method to solve second order linear dirichlet and neumann boundary value problems. Other methods are found in Akinfenwa and Jator, (2013); Brugnano and Trigiante, (1998); Jator, (2010); Cash, (1981); Hairer and Wanner, (1996); Lambert, (1991) and Onumanyi *et al.*, (1994, 1999).

Jator and Agyingi (2004), developed a block hybrid k -step backward differentiation formula for solving large stiff systems including large system resulting from the semidiscretization parabolic partial differential equations (PDEs) with zeros of the second degree chebyshev polynomial as basis functions. Awari (2017) proposed a class of generalized two-step block hybrid numerov method for solving general second order ordinary differential equation without predictor. Both the main and the

additional methods were derived via interpolation and collocation procedure. Muhammad and Zurni (2017) used an implicit one-step block hybrid third derivative method for the direct solution of the initial value problems of second-order ordinary differential equations. Ngwane and Jator (2014) applied a block method based on hybrid second derivative algorithm to parabolic partial differential equations. The existing methods of block hybrid have not examined exponential functions as basis function, which in this study compared favourably in terms of errors with the existing ones.

Linear Multistep Methods (LMM) are one of the classical approaches. If the interval of integration is discretized with a uniform partition with step-size $h = (x_N - x_0)/N$, then a k -step LMM with coefficients α_j 's and β_j 's replaces the equation in (1.1) with the following difference equation

$$\sum_{j=0}^k \alpha_j y_{n+j} = h^2 \sum_{j=0}^k \beta_j f_{n+j} \quad (2.1)$$

where $y_n = y(x_n)$, $f_n = f(x_n, y_n)$, with $x_n = x_0 + nh$, for all $n = 0, 1, \dots, N$.

The main idea on which such schemes rely is that of completing the discrete problem generated by a LMM with a set of boundary conditions instead of just initial ones as classically done. This approach was introduced in the nineties for the definition of schemes for solving first order ODEs and the principal reference is Brugnano and Trigiante (1998). Their linear stability properties have been studied in detail in several papers where it was proved rigorously that they are able to overcome the second Dahlquist barrier (Brugnano and Trigiante, 1996).

2.2 Exponential Fitting

The basic idea behind Exponential Fitting (EF) is to derive numerical methods that are better suited for oscillatory problems. The exponentially fitted methods are always based on their non-fitted counterparts. To make a clear distinction between, for example, an exponentially fitted Trapezoidal rule and the Trapezoidal rule, referred to the latter as the classical Trapezoidal rule. It is remarked here that exponential fitting is also applicable to numerical differentiation, integration and interpolation. In this study, however, we only consider the numerical solution of ordinary differential equations (Ixaru and Vanden, 2004).

A classical method performs best when the solution is a polynomial or can aptly be represented as one locally. A k-step Adams-Bashforth method can even yield a polynomial solution of degree k without errors. In exponential fitting terminology, a few of the highest-order monomials are replaced by exponentials. The approach considered in Ixaru and Paternoster (1999) was that the fitting space of the form

$$\{1, x, x^2, \dots, x^k, e^{\pm wx}, xe^{\pm wx}, \dots, x^p e^{\pm wx}\}$$

was employed. This approach was also employed by Ixaru and Paternoster (1999), Vanden Berge *et al.*, (1990), De Meyer, *et al.* (1990) and Simos (1991-1992). Regardless of the form of the fitting space, it is usually imposed that the parameter value(s) are either real or imaginary.

The exponential fitting space of the form

$$\{1, x, x^2, \dots, x^k, e^{wx}\} \quad (2.2)$$

is employed in the course of this work.

2.3 Existence and Uniqueness

In this section, the existence and uniqueness of first and second order ordinary differential equations are discussed.

2.3.1 Existence and Uniqueness of First Order ODEs

The general first-order ODE of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad x \in [a, b] \quad (2.3)$$

The following theorem guarantees the existence of at least one solution of (2.3).

Theorem 2.3.1 (Existence). *Suppose that $f(x, y(x))$ is a continuous function defined in some region*

$$R = \{(x, y(x)) : x_0 - \delta < x < x_0 + \delta, \quad y_0 - \epsilon < y < y_0 + \epsilon\}$$

containing the point (x_0, y_0) . Then there exists a number $\delta_1 < \delta$ such that a solution y

$y = f(x)$ of (2.3) is defined for $x_0 - \delta_1 < x < x_0 + \delta_1$.

Theorem 2.3.2 (Uniqueness). *Suppose that both $f(x, y(x))$ and $\frac{\partial f(x, y(x))}{\partial y}$ are continuous functions defined on the region*

$$R = \{(x, y(x)) : x_0 - \delta < x < x_0 + \delta, \quad y_0 - \epsilon < y < y_0 + \epsilon\}$$

Then there exists a number $\delta_2 < \delta_1$ such that the solution $y = f(x)$ of (2.3) whose existence was guaranteed by Theorem 2.3.1, has a unique solution for $x_0 - \delta_2 < x < x_0 + \delta_2$.

Brugnano and Trigiante, (1998).

2.3.2 Existence and Uniqueness of Second Order ODEs

Consider the n th-order ordinary differential equation

$$y^{(n)} = f(x, y, y^0, \dots, y^{(n-1)}) \quad x \in (a, b) \quad (2.4)$$

$$\begin{aligned} g_i(y^{(i)}(a), y^{(i+1)}(a), \dots, y^{(n-1)}(a)) &= 0, \quad i = 0, 1, \dots, n \\ h(y(a), y^0(a), \dots, y^{(n-1)}(a); y(b), y^0(b), \dots, y^{(n-1)}(b)) & \end{aligned} \quad (2.5)$$

where $f(x, y_0, y_1, \dots, y_{n-1})$ is continuous on $[a, b] \times \mathbb{R}^n$, $g_i(y^{(i)}(a), y^{(i+1)}(a), \dots, y^{(n-1)}(a))$, $i = 0, 1, \dots, n$ are continuous on $\mathbb{R}^{n-1}$ and $h(y_0, y_1, \dots, y_{n-1}; z_0, z_1, \dots, z_{n-1})$ is con-

tinuous on $\mathbb{R}^{2n}$. Considering the following conditions:

H_1 : $f(x, y_0, y_1, \dots, y_{n-1})$ is continuous on $[a, b] \times \mathbb{R}^n$;

H_2 : $(x, y_0, y_1, \dots, y_{n-1}), (x, y_0^-, y_1^-, \dots, y_{n-1}^-) \in [a, b] \times \mathbb{R}^n$, if $y_i \leq y_i^-$ for $i = 0, 1, \dots, n$, then $f(x, y_0, y_1, \dots, y_{n-1}) \leq f(x, y_0^-, y_1^-, \dots, y_{n-1}^-)$;

H_3 : for any $(x, y_0, y_1, \dots, y_{n-1}), (x, y_0^-, y_1^-, \dots, y_{n-1}^-) \in [a, b] \times \mathbb{R}^n$, $|f(x, y_0, y_1, \dots, y_{n-1}) - f(x, y_0^-, y_1^-, \dots, y_{n-1}^-)| \leq \sum_{i=0}^{n-1} L_i |y_i - y_i^-|$, where L_i are nonnegative constants.; H_3' : for any $(x, y_0, y_1, \dots, y_{n-1}), (x, y_0^-, y_1^-, \dots, y_{n-1}^-) \in [a, b] \times \mathbb{R}^n$, $|f(x, y_0, y_1, \dots, y_{n-1}) - f(x, y_0^-, y_1^-, \dots, y_{n-1}^-)| \leq L_i |y_i - y_i^-|$, where L_i are nonnegative constants.;

H_4 : $g_i(y^{(i)}(a), y^{(i+1)}(a), \dots, y^{(n-1)}(a))$, $i = 0, 1, \dots, n$ are continuous on $\mathbb{R}^{n-1}$ and $h(y_0, y_1, \dots, y_{n-1}; z_0, z_1, \dots, z_{n-1})$ is continuous differentiable on $\mathbb{R}^{2n}$;

H_5 : $\frac{\partial g_i}{\partial y_i} \geq \delta > 0$ $i = 0, 1, \dots, n$ on $\mathbb{R}^{n-i}$, $\frac{\partial g_i}{\partial y_j} \leq 0$, $i = 0, 1, \dots, n-1$, $j = i+1, i+2, \dots, n-1$ on $\mathbb{R}^{n-i}$;

H_6 : $\frac{\partial h}{\partial y_i} \geq 0$ $i = 0, 1, \dots, n$ on $\mathbb{R}^{2n}$;

H_7 : $\frac{\partial h}{\partial z_i} \geq 0$ $i = 0, 1, \dots, n$, $\sum_{i=0}^{n-1} \frac{\partial h}{\partial z_i} \geq \delta > 0$ on $\mathbb{R}^{2n}$;

Theorem 2.3.3. Assume that H_1, H_2, H_3, H_4, H_5 hold. Suppose also that there exist $\varphi_1, \varphi_2 \in \Gamma$ which are defined on $[0, 1]$ such that

$$h(\phi_1(0), \phi_1'(0), \dots, \phi_1^{(n-1)}(0); \phi_1(1), \phi_1'(1), \dots, \phi_1^{(n-1)}(1)) \geq 0$$

and

$$h(\phi_2(0), \phi_2'(0), \dots, \phi_2^{(n-1)}(0); \phi_2(1), \phi_2'(1), \dots, \phi_2^{(n-1)}(2)) \leq 0$$

Then problem (2.4) and (2.5) has at least one solution.

Theorem 2.3.4. Assume that $H_1, H_2, H_3', H_4, H_5, H_6, H_7$ hold, then problem (2.4) and (2.5) has at most one solution.

For proof of Theorems 2.3.3 and 2.3.4 see Pei and Chang, (2013).

Chapter 3

METHODOLOGY

3.1 Development of ENIS for First Order IVPs

We consider the k -step block methods ($k > 0$) for the problem of the form

$$\begin{aligned} & \text{[?]} \\ y' &= f(x, y), \quad a < x < b, \quad \text{[?]} \\ y(a) &= \alpha_0 \quad \text{[?]} \end{aligned} \tag{3.1}$$

For the k -step methods, $k \in \mathbb{N}$, we consider the grid points given by $x_n, x_{n+1} = x_n + h, \dots, x_{n+k} = x_n + kh$, for solving the problem in (3.1) on the interval $[x_n, x_{n+k+1}]$.

We consider the approximation of its solution $y(x)$ by a polynomial $p(x)$ given by

$$y(x) \simeq p(x) = \sum_{i=0}^k a_i x^i + a_{k+1} e^{wx} \quad (3.2)$$

which yields

$$y^0(x) - p^0(x) = \sum_{i=1}^k a_i x^{i-1} + w a_{k+1} e^{wx} = f(x, y) \quad (3.3)$$

with the $a_i \in \mathbb{R}$ real unknown parameters to be determined. Imposing appropriate interpolation conditions to $p(x)$ at x_n and collocation condition $p^0(x)$ at the points $x_{n+1}, \dots, x_{n+k+1}$, leads to the system of $k+2$ equations.

$$p(x_n) = y_n, p^0(x_n + ih) = f_{n+i}; i = 1(1)(k+1)$$

for the determination of a_i 's, $i = 0(1)(k+1)$. The collocation conditions give the system

$$\begin{aligned} a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \dots + a_k x_n^k + a_{k+1} e^{w x_n} - y_n &= 0 \\ a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + \dots + k a_k x_{n+1}^{k-1} + w a_{k+1} e^{w x_{n+1}} - f_{n+1} &= 0 \\ a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + \dots + k a_k x_{n+2}^{k-1} + w a_{k+1} e^{w x_{n+2}} - f_{n+2} &= 0 \end{aligned}$$

...

$$\begin{aligned} a_1 + 2a_2 x_{n+k} + 3a_3 x_{n+k}^2 + \dots + k a_k x_{n+k}^{k-1} + w a_{k+1} e^{w x_{n+k}} - f_{n+k} &= 0 \\ a_1 + 2a_2 x_{n+k+1} + 3a_3 x_{n+k+1}^2 + \dots + k a_k x_{n+k+1}^{k-1} + w a_{k+1} e^{w x_{n+k+1}} - f_{n+k+1} &= 0 \end{aligned}$$

The equivalent matrix equation is $\mathbf{A}_k \underline{x} = \underline{b}$ where:

$$\begin{aligned}
\mathbf{A}_k &= \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^k & e^{wx_n} \\ 0 & 1 & 2x_{n+1} & \cdots & (k)x_{n+1}^{k-1} & w e^{wx_{n+1}} \\ & & x & \cdots & (k)x^{k-1} & w e^{wx_{n+2}} \end{pmatrix} \quad ; \quad x = \begin{pmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{pmatrix} ; \quad b = \begin{pmatrix} f_{n+2} \\ f_{n+1} \\ f_n \end{pmatrix} \\
&\quad - \begin{pmatrix} a_0 \\ a_1 \\ a \end{pmatrix} - \begin{pmatrix} y_n \\ f_{n+1} \end{pmatrix} \quad 0 \quad 1 \quad 2 \\
&\quad - \begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} - \begin{pmatrix} f_{n+k} \\ f_{n+k+1} \end{pmatrix} \\
\underline{e} &= (1, x, x^2, \dots, x^k, e^{wx})^T
\end{aligned}$$

A

Hence, we state the following theorem without proof.

Theorem 3.1.1. *Biala (2016).*

Let (3.2) and (3.3) be satisfied, then the k -step continuous linear multistep method is equivalent to the equation

$$y(x) = \underline{b}^T (\mathbf{A}_k^{-1})^T \underline{e} \quad (3.4)$$

where $\underline{b}$, $\mathbf{A}_k$ and $\underline{e}$ are as defined above.

Applying the above theorem, the ENIS derived in (3.4) is of the form

$$k+1$$

$$y_{n+k} = y_n + \sum_{i=1}^k \beta(w, h) f_{n+i} \quad (3.5)$$

In the next section, we present the ENISs for $k = 1(1)6$

3.1.1 Specification of the ENIS

Case $k=1$

The trial solution is given as

$$y(x) \simeq p(x) = \sum_{i=0}^1 a_i x^i + a_2 e^{wx} = a_0 + a_1 x + a_2 e^{wx} \quad (3.6)$$

which yields

$$y'(x) \simeq p'(x) = \sum_{i=1}^1 i a_i x^{i-1} + w a_2 e^{wx} = a_1 + w a_2 e^{wx} \equiv f(x, p(x)) \quad (3.7)$$

Interpolating (3.6) at the point x_0 and collocating (3.7) at the points x_1 and x_2 yields the matrix $\mathbf{A}_{k1}\underline{x} = \underline{b}$, where;

$$\mathbf{A}_{11} = \begin{pmatrix} 0 & 1 & w e^{wx_1} \\ 0 & 1 & w e^{wx_2} \\ 1 & x_0 & e^{wx_0} \end{pmatrix}; \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}; \underline{b} = \begin{pmatrix} y_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \quad \mathbf{A}(3.8)$$

Applying the above theorem and after some algebraic simplification, the following 2step ENIS are derived;

$$\left. \begin{aligned} y_{n+1} &= y_n + \frac{e^{-w} (1 - e^w + e^{2w} w) h f_{n+1}}{(-1 + e^w) w} - \frac{e^{-w} (1 - e^w + e^w w) h f_{2+n}}{(-1 + e^w) w} \\ y_{n+2} &= y_n + \frac{e^{-w} (1 - e^{2w} + 2e^{2w} w) h f_{n+1}}{(-1 + e^w) w} - \frac{e^{-w} (1 - e^{2w} + 2e^w w) h f_{n+2}}{(-1 + e^w) w} \end{aligned} \right\} \quad (3.9)$$

Case $k = 2$

The trial solution is given as

$$y(x) \approx p(x) = \sum_{i=0}^2 a_i x^i + a_3 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 e^{wx} \quad (3.10)$$

which yields

$$y^{(0)}(x) \approx p^{(0)}(x) = \sum_{i=1}^2 i a_i x^{i-1} + w a_3 e^{wx} = a_1 + a_2 x + w a_3 e^{wx} = f(x, y) \quad (3.11)$$

Interpolating (3.10) at the point x_0 and collocating (3.11) at the points x_1, x_2 and x_3 yields the matrix $\mathbf{A}_{k2} \underline{x} = \underline{b}$, where

$$\mathbf{A}_{22} = \begin{pmatrix} 1 & x_0 & x_0^2 & e^{wx_0} \\ 0 & 1 & 2x_1 & w e^{wx_1} \\ 0 & 1 & 2x_2 & w e^{wx_2} \\ 0 & 1 & 2x_3 & w e^{wx_3} \end{pmatrix}; \quad \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}; \quad \underline{b} = \begin{pmatrix} y_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \quad (3.12)$$

Applying the above theorem and after some algebraic simplification, the following 3step ENISs are derived;

$$\left. \begin{aligned}
y_{n+1} &= - \frac{e^{-w} (2 - 2e^w + 5e^{2w}w - 3e^{3w}w) hf_{n+1}}{2(-1+e^w)^2 w} + \frac{e^{-w} (4 - 4e^w + 5e^w w - e^{3w}w) hf_{n+2}}{2(-1+e^w)^2 w} \\
&\quad - \frac{e^{-w} (2 - 2e^w + 3e^w w - e^{2w}w) hf_{n+3}}{2(-1+e^w)^2 w} + y_n \\
y_{n+2} &= - \frac{e^{-w} (2 - 2e^{2w} + 8e^{2w}w - 4e^{3w}w) hf_{n+1}}{2(-1+e^w)^2 w} + \frac{e^{-w} (4 - 4e^{2w} + 8e^w w) hf_{n+2}}{2(-1+e^w)^2 w} \\
&\quad - \frac{e^{-w} (2 - 2e^{2w} + 4e^w w) hf_{n+3}}{2(-1+e^w)^2 w} + y_n \\
y_{n+3} &= - \frac{e^{-w} (2 - 2e^{3w} + 9e^{2w}w - 3e^{3w}w) hf_{n+1}}{2(-1+e^w)^2 w} + \frac{e^{-w} (4 - 4e^{3w} + 9e^w w + 3e^{3w}w) hf_{n+2}}{2(-1+e^w)^2 w} \\
&\quad - \frac{e^{-w} (2 - 2e^{3w} + 3e^w w + 3e^{2w}w) hf_{n+3}}{2(-1+e^w)^2 w} + y_n
\end{aligned} \right\} \quad (3.13)$$

Case $k = 3$

The trial solution is given as

$$y(x) \simeq p(x) = \sum_{i=0}^3 a_i x^i + a_4 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 e^{wx} \quad (3.14)$$

which yields

$$y'(x) \simeq p'(x) = \sum_{i=1}^3 i a_i x^{i-1} + w a_4 e^{wx} = a_1 + a_2 x + a_3 x^2 + w a_4 e^{wx} \quad (3.15)$$

Interpolating (3.14) at the point x_0 and collocating (3.15) at the points x_1, x_2, x_3 and x_4 yields the matrix $\mathbf{A}_{k3}\underline{x} = \underline{b}$, where

$$\mathbf{A}_{33} = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & e^{wx_0} \\ 0 & 1 & 2x_1 & 3x_1^2 & w e^{wx_1} \\ 0 & 1 & 2x_2 & 3x_2^2 & w e^{wx_2} \\ 0 & 1 & 2x_3 & 3x_3^2 & w e^{wx_3} \\ 0 & 1 & 2x_4 & 3x_4^2 & w e^{wx_4} \end{pmatrix}; \quad \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}; \quad \underline{b} = \begin{pmatrix} y_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad (3.16)$$

Applying the above theorem and after some algebraic simplification, the following 4step ENISs are derived:

$$\begin{aligned}
y_{n+1} &= \frac{e^{-w} (6 - 32e^{3w}w - 6 \cosh(w) + 38e^{3w}w \cosh(w) - 6 \sinh(w) - 15e^{3w}w \sinh(w)) h f_{n+1}}{6(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (36 - 36e^w + 53e^w w - 33e^{3w}w + 16e^{4w}w) h f_{n+2}}{12(-1 + e^w)^3 w} + \\
&\quad \frac{e^{-w} (36 - 36e^w + 64e^w w - 33e^{2w}w + 5e^{4w}w) h f_{n+3}}{12(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (6 - 6e^w - 8e^{2w}w + 14e^{2w}w \cosh(w) - 9e^{2w}w \sinh(w)) h f_{n+4}}{6(-1 + e^w)^3 w} + y_n \\
y_{n+2} &= \frac{e^{-w} (6 - 40e^{3w}w + 52e^{3w}w \cosh(w) - 6 \cosh(2w) - 24e^{3w}hw \sinh(w) - 6 \sinh(2w)) h f_{n+1}}{6(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (36 - 36e^{2w} + 76e^w w - 12e^{3w}w + 8e^{4w}w) h f_{n+2}}{12(-1 + e^w)^3 w} + \\
&\quad \frac{e^{-w} (36 - 36e^{2w} + 80e^w hw - 12e^{2w}w + 4e^{4w}w) h f_{n+3}}{12(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (6 - 6e^{2w} - 4e^{2w}w + 16e^{2w}w \cosh(w) - 12e^{2w}w \sinh(w)) h f_{n+4}}{6(-1 + e^w)^3 w} + y_n
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
y_{n+3} &= \frac{e^{-w} (6 - 36e^{3w}w + 54e^{3w}w \cosh(w) - 6 \cosh(3w) - 27e^{3w}w \sinh(w) - 6 \sinh(3w)) h f_{n+1}}{6(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (36 - 36e^{3w} + 81e^w w + 27e^{3w}w) h f_{n+2}}{12(-1 + e^w)^3 w} + \\
&\quad \frac{e^{-w} (36 - 36e^{3w} + 72e^w w + 27e^{2w}w + 9e^{4w}w) h f_{n+3}}{12(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (6 - 6e^{3w} + 18e^{2w}w \cosh(w) - 9e^{2w}w \sinh(w)) h f_{n+4}}{6(-1 + e^w)^3 w} + y_n \\
y_{n+4} &= \frac{e^{-w} (6 - 32e^{3w}w + 56e^{3w}w \cosh(w) - 6 \cosh(4w) - 24e^{3w}w \sinh(w) - 6 \sinh(4w)) h f_{n+1}}{6(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (36 - 36e^{4w} + 80e^w w + 48e^{3w}w + 16e^{4w}w) h f_{n+2}}{12(-1 + e^w)^3 w} + \\
&\quad \frac{e^{-w} (36 - 36e^{4w} + 64e^w w + 48e^{2w}w + 32e^{4w}w) h f_{n+3}}{12(-1 + e^w)^3 w} \\
&\quad - \frac{e^{-w} (6 - 6e^{4w} - 8e^{2w}w + 32e^{2w}w \cosh(w)) h f_{n+4}}{6(-1 + e^w)^3 w} + y_n
\end{aligned} \tag{3.18}$$

Case $k = 4$

The trial solution is given as

$$y(x) = \sum_{i=0}^4 a_i x^i + a_5 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 e^{wx} \tag{3.19}$$

$$i=0$$

which yields

$$y^0(x) - p^0(x) = \sum_{i=1}^4 a_i x^{i-1} + w a_5 e^{wx} = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + w a_5 e^{wx} = f(x, y) \quad (3.20)$$

Interpolating (3.19) at the point x_0 and collocating (3.20) at the points x_1, x_2, x_3, x_4 and x_5 yields the matrix $\mathbf{A}_{k4}\underline{x} = \underline{b}$, where

$$\mathbf{A}_{44} = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & e^{wx_0} \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & w e^{wx_1} \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & w e^{wx_2} \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & w e^{wx_3} \\ 0 & 1 & 2x_4 & 3x_4^2 & 4x_4^3 & w e^{wx_4} \\ 0 & 1 & 2x_5 & 3x_5^2 & 4x_5^3 & w e^{wx_5} \end{pmatrix}; \quad \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}; \quad \underline{b} = \begin{pmatrix} y_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \quad (3.21)$$

Applying the above theorem and after some algebraic simplification, the following 5step ENISs are derived

$$\begin{aligned}
y_{n+1} = & - \frac{e^{-w} (24 - 24e^w + 161e^{2w}w - 293e^{3w}w + 211e^{4w}w - 55e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^w + 161e^w w - 206e^{3w}w + 200e^{4w}w - 59e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (-144 + 144e^w - 293e^w w + 206e^{2w}w - 94e^{4w}w + 37e^{5w}w) hf_{n+3}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^w + 211e^w w - 200e^{2w}w + 94e^{3w}w - 9e^{5w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^w + 55e^w w - 59e^{2w}w + 37e^{3w}w - 9e^{4w}w) hf_{n+5}}{24(-1+e^w)^4 w} + y_n \\
y_{n+2} = & - \frac{e^{-w} (24 - 24e^{2w} + 216e^{2w}w - 352e^{3w}w + 248e^{4w}w - 64e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{2w} + 216e^w w - 112e^{3w}w + 128e^{4w}w - 40e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} + \\
& \frac{e^{-w} (-144 + 144e^{2w} - 352e^w w + 112e^{2w}w - 80e^{4w}w + 32e^{5w}w) hf_{n+3}}{24(-1+e^{hw})^4 w} \\
& + \frac{e^{-w} (96 - 96e^{2w} + 248e^w w - 128e^{2w}w + 80e^{3w}w - 8e^{5w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^{2w} + 64e^w w - 40e^{2w}w + 32e^{3w}w - 8e^{4w}w) hf_{n+5}}{24(-1+e^w)^4 w} + y_n \\
y_{n+3} = & - \frac{e^{-w} (24 - 24e^{3w} + 225e^{2w}w - 333e^{3w}w + 243e^{4w}w - 63e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{3w} + 225e^w w + 18e^{3w}w + 72e^{4w}w - 27e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} + \\
& \frac{e^{-w} (-144 + 144e^{3w} - 333e^w w - 18e^{2w}w - 126e^{4w}w + 45e^{5w}w) hf_{n+3}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{3w} + 243e^w w - 72e^{2w}w + 126e^{3w}w - 9e^{5w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^{3w} + 63e^w w - 27e^{2w}w + 45e^{3w}w - 9e^{4w}w) hf_{5+n}}{24(-1+e^w)^4 w} + y_n \\
y_{n+4} = & - \frac{e^{-w} (24 - 24e^{4w} + 224e^{2w}w - 320e^{3w}w + 256e^{4w}w - 64e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{4w} + 224e^w w + 64e^{3w}w + 128e^{4w}w - 32e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} + \\
& \frac{e^{-w} (-144 + 144e^{4w} - 320e^w w - 64e^{2w}w - 256e^{4w}w + 64e^{5w}w) hf_{n+3}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{4w} + 256e^w w - 128e^{2w}w + 256e^{3w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^{4w} + 64e^w w - 32e^{2w}w + 64e^{3w}w) hf_{n+5}}{24(-1+e^w)^4 w} + y_n
\end{aligned}
\tag{3.22}$$

$$\begin{aligned}
y_{n+3} = & - \frac{e^{-w} (24 - 24e^{3w} + 225e^{2w}w - 333e^{3w}w + 243e^{4w}w - 63e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{3w} + 225e^w w + 18e^{3w}w + 72e^{4w}w - 27e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} + \\
& \frac{e^{-w} (-144 + 144e^{3w} - 333e^w w - 18e^{2w}w - 126e^{4w}w + 45e^{5w}w) hf_{n+3}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{3w} + 243e^w w - 72e^{2w}w + 126e^{3w}w - 9e^{5w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^{3w} + 63e^w w - 27e^{2w}w + 45e^{3w}w - 9e^{4w}w) hf_{5+n}}{24(-1+e^w)^4 w} + y_n \\
y_{n+4} = & - \frac{e^{-w} (24 - 24e^{4w} + 224e^{2w}w - 320e^{3w}w + 256e^{4w}w - 64e^{5w}w) hf_{n+1}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{4w} + 224e^w w + 64e^{3w}w + 128e^{4w}w - 32e^{5w}w) hf_{n+2}}{24(-1+e^w)^4 w} + \\
& \frac{e^{-w} (-144 + 144e^{4w} - 320e^w w - 64e^{2w}w - 256e^{4w}w + 64e^{5w}w) hf_{n+3}}{24(-1+e^w)^4 w} \\
& + \frac{e^{-w} (96 - 96e^{4w} + 256e^w w - 128e^{2w}w + 256e^{3w}w) hf_{n+4}}{24(-1+e^w)^4 w} - \\
& \frac{e^{-w} (24 - 24e^{4w} + 64e^w w - 32e^{2w}w + 64e^{3w}w) hf_{n+5}}{24(-1+e^w)^4 w} + y_n
\end{aligned}
\tag{3.23}$$

$$y_{n+5} = - \left. \begin{aligned} & \frac{e^{-w} (24 - 24e^{5w} + 225e^{2w}w - 325e^{3w}w + 275e^{4w}w - 55e^{5w}w) hf_{n+1}}{24(-1 + e^w)^4 w} \\ & + \frac{e^{-w} (96 - 96e^{5w} + 225e^w w + 50e^{3w}w + 200e^{4w}w + 5e^{5w}w) hf_{n+2}}{24(-1 + e^w)^4 w} + \\ & \frac{e^{-w} (-144 + 144e^{5w} - 325e^w w - 50e^{2w}w - 350e^{4w}w + 5e^{5w}w) hf_{n+3}}{24(-1 + e^w)^4 w} \\ & + \frac{e^{-w} (96 - 96e^{5w} + 275e^w w - 200e^{2w}w + 350e^{3w}w + 55e^{5w}w) hf_{n+4}}{24(-1 + e^w)^4 w} - \\ & \frac{e^{-w} (24 - 24e^{5w} + 55e^w w + 5e^{2w}w + 5e^{3w}w + 55e^{4w}w) hf_{n+5}}{24(-1 + e^w)^4 w} + y_n \end{aligned} \right\} \quad (3.24)$$

Case $k = 5$

The trial solution is given as

$$y(x) = \sum_{i=0}^5 a_i x^i + a_6 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 e^{wx} \quad (3.25)$$

which yields

$$y^0(x) = \sum_{i=1}^5 a_i x^{i-1} + w a_6 e^{wx} = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + w a_6 e^{wx} = f(x, y)$$

(3.26) Interpolating (3.25) at the point x_0 and collocating (3.26) at the points $x_1,$

x_2, x_3, x_4, x_5 and x_6 yields the matrix $\mathbf{A}_{k5X} = \underline{b}$, where

$$\mathbf{A}_{55} = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & e^{wx_0} \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & w e^{wx_1} \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 & w e^{wx_2} \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 & w e^{wx_3} \\ 0 & 1 & 2x_4 & 3x_4^2 & 4x_4^3 & 5x_4^4 & w e^{wx_4} \\ 0 & 1 & 2x_5 & 3x_5^2 & 4x_5^3 & 5x_5^4 & w e^{wx_5} \\ 0 & 1 & 2x_6 & 3x_6^2 & 4x_6^3 & 5x_6^4 & w e^{wx_6} \end{pmatrix}; \quad \underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}; \quad \underline{b} = \begin{pmatrix} y_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} \quad (3.27)$$

Applying the above theorem and after some algebraic simplification, the following 6step ENISs are derived

$$\begin{aligned}
y_{n+1} = & \left\{ \begin{aligned} & \frac{e^{-w}(720-720e^w+6731e^{2w}w-16394e^{3w}w+17736e^{4w}w-9254e^{5w}w+1901e^{6w}w)h_{f_{n+1}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^w-6731e^w w+14660e^{3w}w-21370e^{4w}w+12615e^{5w}w-2774e^{6w}hw)h_{f_{n+2}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^w+8197e^w w-7330e^{2w}w+6710e^{4w}w-5285e^{5w}w+1308e^{6w}w)h_{f_{n+3}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^w-8868e^w w+10685e^{2w}w-6710e^{3w}w+1930e^{5w}w-637e^{6w}w)h_{f_{n+4}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^w+9254e^w w-12615e^{2w}w+10570e^{3w}w-3860e^{4w}w+251e^{6w}w)h_{f_{n+5}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-720+720e^w-1901e^w w+2774e^{2w}w-2616e^{3w}w+1274e^{4w}w-251e^{5w}w)h_{f_{n+6}}}{720(-1+e^w)^5w} + y_n \end{aligned} \right\} \\
y_{n+2} = & \left\{ \begin{aligned} & \frac{e^{-w}(720-720e^{2w}+8632e^{2w}w-19168e^{3w}w+20352e^{4w}w-10528e^{5w}w+2152e^{6w}w)h_{f_{n+1}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^{2w}-8632e^w w+9520e^{3w}w-15440e^{4w}w+9480e^{5w}w-2128e^{6w}w)h_{f_{n+2}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^{2w}+9584e^w w-4760e^{2w}w+5920e^{4w}w-4720e^{5w}w+1176e^{6w}w)h_{f_{n+3}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^{2w}-10176e^w w+7720e^{2w}w-5920e^{3w}w+1760e^{5w}w-584e^{6w}hw)h_{f_{n+4}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^{2w}+10528e^w w-9480e^{2w}w+9440e^{3w}w-3520e^{4w}w+232e^{6w}hw)h_{f_{n+5}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-720+720e^{2w}-2152e^w w+2128e^{2w}w-2352e^{3w}w+1168e^{4w}w-232e^{5w}w)h_{f_{n+6}}}{720(-1+e^w)^5w} + y_n \end{aligned} \right\} \\
y_{n+3} = & \left\{ \begin{aligned} & \frac{e^{-w}(720-720e^{3w}+8883e^{2w}w-18522e^{3w}w+20088e^{4w}w-10422e^{5w}w+2133e^{6w}w)h_{f_{n+1}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^{3w}-8883e^w w+3780e^{3w}w-11610e^{4w}hw+7695e^{5w}w-1782e^{6w}w)h_{f_{n+2}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^{3w}+9261e^w w-1890e^{2w}w+7830e^{4w}w-5805e^{5w}w+1404e^{6w}hw)h_{f_{n+3}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(-3600+3600e^{3w}-10044e^w w+5805e^{2w}w-7830e^{3w}w+1890e^{5w}w-621e^{6w}w)h_{f_{n+4}}}{360(-1+e^w)^5w} + \\ & \frac{e^{-w}(3600-3600e^{3w}+10422e^w w-7695e^{2w}w+11610e^{3w}w-3780e^{4w}w+243e^{6w}w)h_{f_{n+5}}}{720(-1+e^w)^5w} + \\ & \frac{e^{-w}(-720+720e^{3w}-2133e^w w+1782e^{2w}w-2808e^{3w}w+1242e^{4w}w-243e^{5w}w)h_{f_{n+6}}}{720(-1+e^w)^5w} + y_n \end{aligned} \right\}
\end{aligned}
\tag{3.28}$$

$$\begin{aligned}
y_{n+4} &= \overbrace{\begin{aligned} &e^{-w} \frac{(720 - 720e^{4w} + 8864e^{2w}w - 18176e^{3w}w + 20544e^{4w}w - 10496e^{5w}w + 2144e^{6w}hw)hf_{n+1}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{4w} - 8864e^{2w}w + 2240e^{3w}w - 14080e^{4w}w + 8160e^{5w}w - 1856e^{6w}w)hf_{n+2}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{4w} + 9088e^{2w}w - 1120e^{3w}w + 11840e^{4w}w - 7040e^{5w}w + 1632e^{6w}w)hf_{n+3}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{4w} - 10272e^{2w}w + 7040e^{3w}w - 11840e^{4w}w + 1120e^{5w}w - 448e^{6w}w)hf_{n+4}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{4w} + 10496e^{2w}w - 8160e^{3w}w + 14080e^{4w}w - 2240e^{5w}w + 224e^{6w}w)hf_{n+5}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-720 + 720e^{4w} - 2144e^{2w}w + 1856e^{3w}w - 3264e^{4w}w + 896e^{5w}w - 224e^{6w}w)hf_{n+6}}{720(-1+e^w)^5w} + y_n \end{aligned}} \\
y_{n+5} &= \overbrace{\begin{aligned} &e^{-w} \frac{(720 - 720e^{5w} + 8875e^{2w}w - 18250e^{3w}w + 21000e^{4w}w - 10150e^{5w}w + 2125e^{6w}w)hf_{n+1}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{5w} - 8875e^{2w}w + 2500e^{3w}w - 16250e^{4w}w + 6375e^{5w}w - 1750e^{6w}w)hf_{n+2}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{5w} + 9125e^{2w}w - 1250e^{3w}w + 13750e^{4w}w - 5125e^{5w}w + 1500e^{6w}hw)hf_{n+3}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{5w} - 10500e^{2w}w + 8125e^{3w}w - 13750e^{4w}w - 1750e^{5w}w - 125e^{6w}w)hf_{n+4}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{5w} + 10150e^{2w}w - 6375e^{3w}w + 10250e^{4w}w + 3500e^{5w}w + 475e^{6w}w)hf_{n+5}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-720 + 720e^{5w} - 2125e^{2w}w + 1750e^{3w}w - 3000e^{4w}w + 250e^{5w}w - 475e^{6w}w)hf_{n+6}}{720(-1+e^w)^5w} + y_n \end{aligned}} \\
y_{n+6} &= \overbrace{\begin{aligned} &e^{-w} \frac{(720 - 720e^{6w} + 8856e^{2w}w - 18144e^{3w}w + 20736e^{4w}w - 9504e^{5w}w + 2376e^{6w}w)hf_{n+1}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{6w} - 8856e^{2w}w + 2160e^{3w}w - 15120e^{4w}w + 3240e^{5w}w - 3024e^{6w}hw)hf_{n+2}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{6w} + 9072e^{2w}w - 1080e^{3w}w + 12960e^{4w}w - 2160e^{5w}w + 2808e^{6w}w)hf_{n+3}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(-3600 + 3600e^{6w} - 10368e^{2w}w + 7560e^{3w}w - 12960e^{4w}w - 4320e^{5w}w - 1512e^{6w}w)hf_{n+4}}{360(-1+e^w)^5w} + \\ &e^{-w} \frac{(3600 - 3600e^{6w} + 9504e^{2w}w - 3240e^{3w}w + 4320e^{4w}w + 8640e^{5w}w + 2376e^{6w}w)hf_{n+5}}{720(-1+e^w)^5w} + \\ &e^{-w} \frac{(-720 + 720e^{6w} - 2376e^{2w}w + 3024e^{3w}w - 5616e^{4w}w + 3024e^{5w}w - 2376e^{6w}w)hf_{n+6}}{720(-1+e^w)^5w} + y_n \end{aligned}}
\end{aligned}$$

3.2 Development of Non-Hybrid ENIS for Second Order IVPs and BVPs

For values of $k \geq 2$, we consider the k -step block methods for the problem of the form

$$y^{(0)} = f(x, y, y^{(0)}), \quad a < x < b, \quad (3.29)$$

with the Dirichlet, Neumann or mixed boundary condition as stated in the previous section. For the k -step methods, $k \in \mathbb{N}$, we consider the grid points given by $x_n, x_{n+1} = x_n + h, \dots, x_{n+k} = x_n + kh$. For solving the problem in (3.29) on the interval $[x_n, x_{n+k}]$, we

consider the approximation of its solution $y(x)$ by a polynomial $p(x)$ whose degree depends solely on the number of interpolation/collocation points

$$y(x) \approx p(x) = \sum_{i=0}^{m-1} a_i x^i + a_m e^{wx} \quad (3.30)$$

which yields

$$y''(x) \simeq p''(x) = \sum_{i=2}^{m-1} i(i-1) a_i x^{i-2} + w^2 a_m e^{wx} = f(x, y, y') \quad (3.31)$$

where $m = r + s - 1$, r is the number of interpolation points, s is the number of collocation points, $a_i \in \mathbb{R}$ real unknown parameters to be determined. Imposing appropriate interpolation conditions to $p(x)$ at x_n and collocation condition to $p^{(0)}(x)$ at the points $x_n, \dots, x_{n+k}$, leads to the system of $k + 3$ equations.

$p(x_{n+i}) = y_{n+i}; i = 0(1)r, p^{(0)}(x_n + ih) = f_{n+i}; i = 0(1)s$ for the determination of a_i 's, $i = 0(1)(k + 2)$. The collocation conditions give the system

$$\begin{aligned} a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \dots + a_{m-1} x_n^{m-1} + a_m e^{wx_n} - y_n &= 0 \\ a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \dots + a_{m-1} x_{n+1}^{m-1} + a_m e^{wx_{n+1}} - y_{n+1} &= 0 \\ \dots \\ a_0 + a_1 x_{n+r} + a_2 x_{n+r}^2 + a_3 x_{n+r}^3 + \dots + a_{m-1} x_{n+r}^{m-1} + a_m e^{wx_{n+r}} - y_{n+r} &= 0 \\ 2a_2 + 3a_3 x_n + 6a_4 x_n^2 + \dots + (m-1)a_{m-1} x_n^{m-2} + w^2 a_m e^{wx_n} - f_n &= 0 \\ 2a_2 + 3a_3 x_{n+1} + 6a_4 x_{n+1}^2 + \dots + (m-1)a_{m-1} x_{n+1}^{m-2} + w^2 a_m e^{wx_{n+1}} - f_{n+1} &= 0 \\ \dots \\ 2a_2 + 3a_3 x_{n+s} + 6a_4 x_{n+s}^2 + \dots + (m-1)a_{m-1} x_{n+s}^{m-2} + w^2 a_m e^{wx_{n+s}} - f_{n+s} &= 0 \end{aligned}$$

The corresponding matrix equation $\mathbf{A}_k \underline{x} = \underline{b}$ is

$$\begin{aligned}
 & \begin{bmatrix}
 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{m-1} & e^{wx_n} \\
 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & \cdots & x_{n+1}^{m-1} & e^{wx_{n+1}} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{bmatrix}
 + \begin{bmatrix}
 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & \cdots & x_{n+r}^{m-1} & e^{wx_{n+r}} \\
 1 & x_{n+r+1} & x_{n+r+1}^2 & x_{n+r+1}^3 & \cdots & x_{n+r+1}^{m-1} & e^{wx_{n+r+1}} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{bmatrix}
 + \cdots + \begin{bmatrix}
 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & \cdots & x_{n+r}^{m-1} & e^{wx_{n+r}} \\
 1 & x_{n+r+1} & x_{n+r+1}^2 & x_{n+r+1}^3 & \cdots & x_{n+r+1}^{m-1} & e^{wx_{n+r+1}} \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{bmatrix}
 \\
 & \mathbf{A} \underline{x} = \underline{b}
 \end{aligned}$$

$$\underline{x} = (a_0, a_1, a_2, \dots, a_{m-2}, a_{m-1}, a_m)^T$$

$$\underline{b} = (y_n, y_{n+1}, \dots, y_{n+r}, f_n, f_{n+1}, \dots, f_{n+s})^T$$

$$\underline{e} = (1, x, x^2, \dots, x^{m-1}, e^{wx})^T$$

The parameters r and s which are the number of interpolation points and the number of collocation points satisfy $0 \leq r \leq k$ and $0 \leq s \leq k + 1$ respectively and k is the step number of the method.

The above system is solved using Mathematica 8.0 to obtain $\underline{x}$. Then, substituting the values of a 's obtains the entries in the vector $\underline{x}$ into (3.30), and after some algebraic simplifications, the following discrete k -step linear multistep method is obtained

$$p(x) = \sum_{i=0}^{r-1} \alpha_i y_{n+i} + h^2 \sum_{i=0}^{s-1} \beta_i(w;h) f_{n+i} \quad (3.32)$$

where α is a constant and β is a functions of h, w . In the next section, the formulae for $k = 2(1)5$ are presented.

3.2.1 Specification of the Method

Case $k=2, r=2, s=3$

For the step number $k = 2$, the number of interpolation points $r = 2$, the number of collocation points $s = 3$. Therefore $m = 4$ The trial solution is given as

$$y(x) \simeq p(x) = \sum_{i=0}^3 a_i x^i + a_4 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 e^{wx} \quad (3.33)$$

whose second derivative is given as

$$y''(x) \simeq p''(x) = \sum_{i=2}^3 i(i-1)a_i x^{i-2} + w^2 a_4 e^{wx} = 2a_2 + 6a_3 x + w^2 a_4 e^{wx} = (x, y, y') \quad (3.34)$$

Interpolating (3.33) at the point x_n, x_{n+1} and collocating (3.34) at the points x_n, x_{n+1}, x_{n+2} , yields the matrix $\mathbf{A}_{k2} \mathbf{X} = \mathbf{b}$, where

$$\mathbf{A}_{k2} = \begin{pmatrix} 0 & 0 & 2 & 6x_n & w^2 e^{wx_n} \\ 1 & x_n & x_n^2 & x_n^3 & e^{wx_n} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & e^{wx_{n+1}} \end{pmatrix} \begin{pmatrix} a_2 \\ a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f_n \\ y_n \\ y_{n+1} \end{pmatrix} - \begin{pmatrix} x_{n+1} & w^2 e^{wx_{n+1}} \\ x_{n+2} & w^2 e^{wx_{n+2}} \end{pmatrix} \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} \quad (3.35)$$

Solving (3.35) to obtain $a_i, j = 0(1)4$, and after some algebraic simplification, the following 2-step ENISs are derived

$$\left. \begin{aligned}
 y_{n+2} &= -y_n + 2y_{n+1} + \frac{1}{\xi_2} (1 - 2e^w + e^{2w} - e^w w^2) h^2 f_n \\
 &\quad - (2 - 4e^w + 2e^{2w} - w^2 - e^{2w} w^2) h^2 f_{n+1} \\
 &\quad + (1 - 2e^w + e^{2w} - e^w w^2) h^2 f_{n+2}) \\
 y'_n &= -\frac{y_n - y_{n+1}}{h} - \frac{1}{6\xi_2 h} (-6 + 6e^w - 6w - 5e^w w^2 + 2e^{2w} w^2) h f_n \\
 &\quad + (12 - 12e^w + 12w + 5w^2 + e^{2w} w^2) h f_{n+1} \\
 &\quad - (6 - 6e^w + 6w + 2w^2 + e^w w^2) h f_{n+2}) \\
 y'_{n+1} &= -\frac{y_n - y_{n+1}}{h} + \frac{1}{6\xi_2 h} (6 - 6e^w + 6e^w w - 4e^w w^2 + e^{2w} w^2) h f_n \\
 &\quad - (12 - 12e^w + 12e^w w - 4w^2 - 2e^{2w} w^2) h f_{n+1} \\
 &\quad + (6 - 6e^w + 6e^w w - w^2 - 2e^w w^2) h f_{n+2}) \\
 y'_{n+2} &= -\frac{y_n - y_{n+1}}{h} - \frac{1}{6\xi_2 h} (-6 + 6e^w - 6e^{2w} w + 7e^w w^2 + 2e^{2w} w^2) h f_n \\
 &\quad + (12 - 12e^w + 12e^{2w} w - 7w^2 - 11e^{2w} w^2) h f_{n+1} \\
 &\quad - (6 - 6e^w + 6e^{2w} w + 2w^2 - 11e^w w^2) h f_{n+2})
 \end{aligned} \right\} \quad (3.36)$$

where $\xi_2 = (-1 + e^w)^2 w^2$.

Case $k=3, r=2, s=4$

For the step number $k = 3$, the number of interpolation points $r = 2$, the number of collocation points $s = 4$. Therefore $m = 5$ The trial solution is given as

$$y(x) \approx p(x) = \sum_{i=0}^4 a_i x^i + a_5 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 e^{wx} \quad (3.37)$$

whose second derivative is given as

$$y''(x) \approx p''(x) = \sum_{i=2}^4 i(i-1) a_i x^{i-2} + w^2 a_5 e^{wx} = 2a_2 + 6a_3 x + 12a_4 x^2 + w^2 a_5 e^{wx} \quad (3.38)$$

Interpolating (3.37) at the point x_n, x_{n+1} and collocating (3.38) at the points $x_n, x_{n+1}, x_{n+2}, x_{n+3}$,

yields the matrix $\mathbf{A}_{k3} \mathbf{x} = \mathbf{b}$, where

$$\begin{pmatrix} e^{wx_n} \\ e^{wx_{n+1}} \\ w^2 e^{wx_n} \end{pmatrix} - \begin{pmatrix} a_0 \\ a_1 \\ a \end{pmatrix} - \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} \\
\begin{pmatrix} x_{n+1} & 12x_{n+1} & w e^{n+1} \\ x_{n+2} & 12x_{n+2}^2 & w^2 e^{wx_{n+2}} \\ x_{n+3} & 12x_{n+3}^2 & w^2 e^{wx_{n+3}} \end{pmatrix} - \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} - \begin{pmatrix} n+1 \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \quad (3.39)$$

$$\begin{matrix} 1 & x_n & x_{2n} & x_{3n} & x_{4n} \\ 1 & x_{n+1} & x_{2n+1} & x_{3n+1} & x_{4n+1} \end{matrix}$$

$$k_3 = \begin{vmatrix} 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \end{vmatrix}_n \quad ; x = \begin{vmatrix} 2 \\ 2 \end{vmatrix}_{wx} \quad ; b = \begin{vmatrix} f_n \\ f \end{vmatrix} \quad x$$

$$12x_{2n}$$

$$\mathbf{A} \begin{pmatrix} 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

Solving for $\underline{x}$ to obtain $a_i, i = 0(1)5$, and after some algebraic simplification, the following 2-step ENIS are derived

$$\begin{aligned}
y_{n+2} &= -y_n + 2y_{n+1} + \frac{1}{12\xi_3} \left(-12 + 24e^w - 12e^{2w} + 13e^w w^2 - 2e^{2w} w^2 + e^{3w} w^2 \right) h^2 f_n \\
&\quad + 36 - 72e^w + 36e^{2w} - 13w^2 - 33e^{2w} w^2 + 10e^{3w} w^2 \Big) h^2 f_{n+1} \\
&\quad - 36 - 72e^w + 36e^{2w} - 2w^2 - 33e^w w^2 - e^{3w} w^2 \Big) h^2 f_{n+2} \\
&\quad + 12 - 24e^w + 12e^{2w} - w^2 - 10e^w w^2 - e^{2w} w^2 \Big) h^2 f_{n+3} \Big) \\
y_{n+3} &= -2y_n + 3y_{n+1} + \frac{1}{4\xi_3} \left(-8 + 12e^w - 4e^{3w} + 9e^w w^2 + 2e^{2w} w^2 + e^{3w} w^2 \right) h^2 f_n \\
&\quad + 24 - 36e^w + 12e^{3w} - 9w^2 - 33e^{2w} w^2 + 6e^{3w} w^2 \Big) h^2 f_{n+1} \\
&\quad - 24 - 36e^w + 12e^{3w} + 2w^2 - 33e^w w^2 - 5e^{3w} w^2 \Big) h^2 f_{n+2} \\
&\quad + 8 - 12e^w 4e^{3w} - w^2 - 6e^w w^2 - 5e^{2w} w^2 \Big) h^2 f_{n+3} \Big) \\
hy'_n &= -\frac{y_n - y_{n+1}}{h} - \frac{1}{24\xi_3 h} \left(24 - 24e^w + 24w + 27e^w w^2 - 22e^{2w} w^2 + 7e^{3w} w^2 \right) h f_n \\
&\quad - 72 - 72e^w + 72w + 27w^2 + 15e^{2w} w^2 - 6e^{3w} w^2 \Big) h f_{n+1} \\
&\quad + 72 - 72e^w + 72w + 22w^2 + 15e^w w^2 - e^{3w} w^2 \Big) h f_{n+2} \\
&\quad - 24 - 24e^w + 24w + 7w^2 + 6e^w w^2 - e^{2w} w^2 \Big) h f_{n+3} \Big) \\
&\hspace{15em} (3.40) \\
y'_{n+1} &= -\frac{y_n - y_{n+1}}{h} + \frac{1}{24\xi_3 h} \left(-24 + 24e^w - 24e^w w + 19e^w w^2 - 10e^{2w} w^2 + 3e^{3w} w^2 \right) h f_n \\
&\quad + 72 - 72e^w + 72e^w w - 19w^2 - 27e^{2w} w^2 + 10e^{3w} w^2 \Big) h f_{n+1} \\
&\quad - 72 - 72e^w + 72e^w w - 10w^2 - 27e^w w^2 + e^{3w} w^2 \Big) h f_{n+2} \\
&\quad + 24 - 24e^w + 24e^w w - 3w^2 - 10e^w w^2 + e^{2w} w^2 \Big) h f_{n+3} \Big) \\
y'_{n+2} &= -\frac{y_n - y_{n+1}}{h} + \frac{1}{24\xi_3 h} \left(-24 + 24e^w - 24e^{2w} w + 29e^w w^2 + 6e^{2w} w^2 + e^{3w} w^2 \right) h f_n \\
&\quad + 72 - 72e^w + 72e^{2w} w - 29w^2 - 105e^{2w} w^2 + 26e^{3w} w^2 \Big) f_{n+1} \\
&\quad - 72 - 72e^w + 72e^{2w} w + 6w^2 - 105e^w w^2 - 9e^{3w} w^2 \Big) h f_{n+2} \\
&\quad + 24 - 24e^w + 24e^{2w} w - w^2 - 26e^w w^2 - 9e^{2w} w^2 \Big) h f_{n+3} \Big) \\
y'_{n+3} &= -\frac{y_n - y_{n+1}}{h} + \frac{1}{24\xi_3 h} \left(-24 + 24e^w - 24e^{3w} w + 27e^w w^2 + 22e^{2w} w^2 + 11e^{3w} w^2 \right) h f_n \\
&\quad + 72 - 72e^w + 72e^{3w} w - 27w^2 - 147e^{2w} w^2 - 6e^{3w} w^2 \Big) h f_{n+1} \\
&\quad - 72 - 72e^w + 72e^{3w} w + 22w^2 - 147e^w w^2 - 55e^{3w} w^2 \Big) h f_{n+2} \\
&\quad + 24 - 24e^w + 24e^{3w} w - 11w^2 + 6e^w w^2 - 55e^{2w} w^2 \Big) h f_{n+3} \Big) \\
&\hspace{15em} (3.41)
\end{aligned}$$

where $\xi_3 = (-1 + e^w)^3 w^2$.

Case $k=4, r=2, s=5$

For the step number $k = 4$, the number of interpolation points $r = 2$, the number of collocation points $s = 5$. Therefore $m = 6$ The trial solution is given as

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$$y(x) \cdot p(x) = \sum_{i=0}^5 a_i x^i + a_6 e^{wx} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 e^{wx} \quad (3.42)$$

$$i=0$$

whose second derivative is given as

$$y^{(5)}(x) = p_5(x) = \sum_{i=2}^5 a_i x^{i-2} + w_2 a_6 e^{wx} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + w_2 a_6 e^{wx} \quad (3.43)$$

Interpolating (3.42) at the point x_n, x_{n+1} and collocating (3.43) at the points $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$,

yields the matrix $\mathbf{A}_{k_4} \underline{x} = \underline{b}$, where;

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_{3n} & x_{4n} & x_{5n} & e^{wx_n} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{3n+1} & x_{4n+1} & x_{5n+1} & e^{wx_{n+1}} \\ 0 & 0 & 2 & 6x_n & 12x_{2n} & 20x_{2n} & w_2 e^{wx_n} \end{pmatrix} \quad k_3 = \begin{vmatrix} \end{vmatrix}$$

$$\mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 6x_{n+1} & 12x_{2n+1} & 20x_{2n+1} & w_2 e^{wx_{n+1}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{2n+2} & 20x_{2n+2} & w_2 e^{wx_{n+2}} \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{2n+3} & 20x_{2n+3} & w_2 e^{wx_{n+3}} \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{2n+4} & 20x_{2n+4} & w_2 e^{wx_{n+4}} \end{pmatrix} \underline{x} = \underline{b} \quad (3.44)$$

$$\underline{x} = (a_0, a_1, a_2, a_3, a_4, a_5, a_6)^T; \underline{b} = (y_n, y_{n+1}, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4})^T$$

Solving for $\underline{x}$ to obtain $a_i, i = 0(1)6$, and after some algebraic simplification, the following 2-step ENIS are derived

$$\begin{aligned}
y_{n+2} = & -y_n + 2y_{n+1} + \frac{1}{12\xi_4} \left. \begin{aligned}
& (12 - 24e^w + 12e^{2w} - 14e^w w^2 + 5e^{2w} w^2 - 4e^{3w} w^2 + e^{4w} w^2) h^2 f_n \\
& - (48 - 96e^w + 48e^{2w} - 14w^2 - 64e^{2w} w^2 + 40e^{3w} w^2 - 10e^{4w} w^2) h^2 f_{n+1} \\
& + (72 - 144e^w + 72e^{2w} - 5w^2 - 64e^w w^2 - 4e^{3w} w^2 + e^{4w} w^2) h^2 f_{n+2} \\
& - (48 - 96e^w + 48e^{2w} - 4w^2 - 40e^w w^2 - 4e^{2w} w^2) h^2 f_{n+3} \\
& + (12 - 24e^w + 12e^{2w} - w^2 - 10e^w w^2 - e^{2w} w^2) h^2 f_{n+4})
\end{aligned} \right\} \\
y_{n+3} = & -2y_n + 3y_{n+1} + \frac{1}{12\xi_4} \left. \begin{aligned}
& (24 - 36e^w + 12e^{3w} - 29e^w w^2 - 9e^{3w} w^2 + 2e^{4w} w^2) h^2 f_n \\
& - (96 - 144e^w + 48e^{3w} - 29w^2 - 174e^{2w} w^2 + 80e^{3w} w^2 - 21e^{4w} w^2) h^2 f_{n+1} \\
& + (144 - 216e^w + 72e^{3w} - 174e^w w^2 - 54e^{3w} w^2 + 12e^{4w} w^2) h^2 f_{n+2} \\
& - (96 - 144e^w + 48e^{3w} - 9w^2 - 80e^w w^2 - 54e^{2w} w^2 - e^{4w} w^2) h^2 f_{n+3} \\
& + (24 - 36e^w + 12e^{3w} - 2w^2 - 21e^w w^2 - 12e^{2w} w^2 - e^{3w} w^2) h^2 f_{n+4})
\end{aligned} \right\} \quad (3.45)
\end{aligned}$$

$$\begin{aligned}
y_{n+4} = & -3y_n + 4y_{n+1} + \frac{1}{6\xi_4} \left. \begin{aligned}
& (18 - 24e^w + 6e^{4w} - 22e^w w^2 - 3e^{2w} w^2 - 12e^{3w} w^2 + e^{4w} w^2) h^2 f_n \\
& - (72 - 96e^w + 24e^{4w} - 22w^2 - 144e^{2w} w^2 + 40e^{3w} w^2 - 18e^{4w} w^2) h^2 f_{n+1} \\
& + (108 - 144e^w + 36e^{4w} + 3w^2 - 144e^w w^2 - 84e^{3w} w^2 + 9e^{4w} w^2) h^2 f_{n+2} \\
& - (72 - 96e^w + 24e^{4w} - 12w^2 - 40e^w w^2 - 84e^{2w} w^2 - 8e^{4w} w^2) h^2 f_{n+3} \\
& + (18 - 24e^w + 6e^{4w} - w^2 - 18e^w w^2 - 9e^{2w} w^2 - 8e^{3w} w^2) h^2 f_{n+4})
\end{aligned} \right\} \\
y'_n = & -\frac{y_n - y_{n+1}}{h} - \frac{1}{360\xi_4 h} \left. \begin{aligned}
& (360 - 360e^w + 360w + 502e^w w^2 - 621e^{2w} w^2 + 396e^{3w} w^2 - 97e^{4w} w^2) h f_n \\
& - (1440 + 1440e^w - 1440w - 502w^2 - 528e^{2w} w^2 + 424e^{3w} w^2 - 114e^{4w} w^2) h f_{n+1} \\
& + (2160 + 2160e^w - 2160w - 621w^2 - 528e^w w^2 + 108e^{3w} w^2 - 39e^{4w} w^2) h f_{n+2} \\
& + (1440 - 1440e^w + 1440w + 396w^2 + 424e^w w^2 - 108e^{2w} w^2 + 8e^{4w} w^2) h f_{n+3} \\
& - (360 + 360e^w - 360w - 97w^2 - 114e^w w^2 + 39e^{2w} w^2 - 8e^{3w} w^2) h f_{n+4})
\end{aligned} \right\} \quad (3.46)
\end{aligned}$$

$$\begin{aligned}
y'_{n+1} = & -\frac{y_n - y_{n+1}}{h} + \frac{1}{360\xi_4 h} \left\{ \begin{aligned} & 360 - 360e^w + 360e^w w - 323e^w w^2 + 264e^{2w} w^2 - 159e^{3w} w^2 + 38e^{4w} w^2) h f_n \\ & - 1440 - 1440e^w + 1440e^w w - 323w^2 - 882e^{2w} w^2 + 656e^{3w} w^2 - 171e^{4w} w^2) h f_{n+1} \\ & + 2160 - 2160e^w + 2160e^w w - 264w^2 - 882e^w w^2 + 102e^{3w} w^2 - 36e^{4w} w^2) h f_{n+2} \\ & - 1440 - 1440e^w + 1440e^w w - 159w^2 - 656e^w w^2 + 102e^{2w} w^2 - 7e^{4w} w^2) h f_{n+3} \\ & + 360 - 360e^w + 360e^w w - 38w^2 - 171e^w w^2 + 36e^{2w} w^2 - 7e^{3w} w^2) h f_{n+4} \end{aligned} \right\} \\
y'_{n+2} = & -\frac{y_n - y_{n+1}}{h} + \frac{1}{360\xi_4 h} \left\{ \begin{aligned} & 360 - 360e^w + 360e^{2w} w - 458e^w w^2 - 21e^{2w} w^2 - 84e^{3w} w^2 + 23e^{4w} w^2) h f_n \\ & - 1440 - 1440e^w + 1440e^{2w} w - 458w^2 - 2832e^{2w} w^2 + 1496e^{3w} w^2 - 366e^{4w} w^2) h f_{n+1} \\ & + 2160 - 2160e^w + 2160e^{2w} w + 21w^2 - 2832e^w w^2 - 588e^{3w} w^2 + 159e^{4w} w^2) h f_{n+2} \\ & - 1440 - 1440e^w + 1440e^{2w} w - 84w^2 - 1496e^w w^2 - 588e^{2w} w^2 + 8e^{4w} w^2) h f_{n+3} \\ & + 360 - 360e^w + 360e^{2w} w - 23w^2 - 366e^w w^2 - 159e^{2w} w^2 + 8e^{3w} w^2) h f_{n+4} \end{aligned} \right\} \quad (3.47) \\
y'_{n+3} = & -\frac{y_n - y_{n+1}}{h} + \frac{1}{360\xi_4 h} \left\{ \begin{aligned} & 360 - 360e^w + 360e^{3w} w - 443e^w w^2 - 216e^{2w} w^2 - 279e^{3w} w^2 + 38e^{4w} w^2) h f_n \\ & - 1440 + 1440e^w - 1440e^{3w} w + 443w^2 + 3522e^{2w} w^2 - 656e^{3w} w^2 + 291e^{4w} w^2) h f_{n+1} \\ & + 2160 - 2160e^w + 2160e^{3w} w + 216w^2 - 3522e^w w^2 - 2538e^{3w} w^2 + 444e^{4w} w^2) h f_{n+2} \\ & - 1440 - 1440e^w + 1440e^{3w} w - 279w^2 - 656e^w w^2 - 2538e^{2w} w^2 + 127e^{4w} w^2) h f_{n+3} \\ & + 360 - 360e^w + 360e^{3w} w - 38w^2 - 291e^w w^2 - 444e^{2w} w^2 - 127e^{3w} w^2) h f_{n+4} \end{aligned} \right\} \\
y'_{n+4} = & -\frac{y_n - y_{n+1}}{h} - \frac{1}{360\xi_4 h} \left\{ \begin{aligned} & 360 - 360e^w + 360e^{4w} w - 458e^w w^2 - 141e^{2w} w^2 - 564e^{3w} w^2 - 97e^{4w} w^2) f_n \\ & + 1440 - 1440e^w + 1440e^{4w} w - 458w^2 - 3312e^{2w} w^2 - 424e^{3w} w^2 - 846e^{4w} w^2) h f_{n+1} \\ & - 2160 - 2160e^w + 2160e^{4w} w + 141w^2 - 3312e^{hw} w^2 - 3948e^{3w} w^2 - 441e^{4w} w^2) h f_{n+2} \\ & + 1440 - 1440e^w + 1440e^{4w} w - 564w^2 + 424e^w w^2 - 3948e^{2w} w^2 - 952e^{4w} w^2) h f_{n+3} \\ & - 360 - 360e^w + 360e^{4w} w + 97w^2 - 846e^w w^2 + 441e^{2w} w^2 - 952e^{3w} w^2) h f_{n+4} \end{aligned} \right\} \quad (3.48)
\end{aligned}$$

where $\xi_4 = (-1 + e^w)^4 w^2$.

3.3 Development of Hybrid ENIS (HENIS) for Second Order IBVPs

For the hybrid 2-step methods, we consider the grid points given by $x_{n+j} = x_n + jh$, $j = 0(1/2)2$. For solving the problem in (3.1) on the interval $[x_n, x_{n+2}]$. We consider the

approximation of its solution $y(x)$ by a polynomial $p(x)$ whose degree depends solely on the number of interpolation/collocation points

5

$$y(x) \approx p(x) = \sum_{i=0}^5 a_i x^i + a_6 e^{wx} \quad (3.49)$$

$i=0$

which yields

5

$$y^{(0)}(x) \approx p^{(0)}(x) = \sum_{i=2}^5 a_i (i-1) a_i x^{i-2} + w^2 a_6 e^{wx} = f(x, y, y^{(0)}) \quad (3.50)$$

$i=2$

where $a_i \in \mathbb{R}$ real unknown parameters to be determined. Interpolating (3.49) at the points $x = x_n$ and $x = x_{n+1}$ and collocating (3.50) at the points $x = x_{n+i}$, $i = 0(1/2)2$. The two off-step points are $x = x_{n+1/2}$ and $x = x_{n+3/2}$. Hence the following system of

equation

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & e^{wx_n} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & e^{wx_{n+1}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & w^2 e^{wx_n} \\ 0 & 0 & 2 & 6x_{n+1/2} & 12x_{n+1/2}^2 & 20x_{n+1/2}^3 & w^2 e^{wx_{n+1/2}} \\ & & & x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & w^2 e^{wx_{n+1}} \\ & & & x_{n+3/2} & 12x_{n+3/2}^2 & 20x_{n+3/2}^3 & w^2 e^{wx_{n+3/2}} \\ & & & x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & w^2 e^{wx_{n+2}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1/2} \\ f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{pmatrix} \quad \begin{matrix} 0 & 0 & 2 & 6 \end{matrix}$$

The above system is solved using Mathematica 8.0 to obtain the values of a_i 's. Then, substituting the values of a_i 's obtained into (3.49), and after some algebraic simplifications, the following discrete 2-step linear multistep method of the form is obtained

$$y(x) = \sum_{i=0}^1 \alpha_i y_i + \sum_{i=0}^3 \beta_i f_i + \sum_{i=0}^1 \hat{\beta}_i f_{\frac{2i+1}{2}} \quad (3.51)$$

where β and $\hat{\beta}$ are functions of x and w . By evaluating 3.51 and its first derivative at the points $x = x_{n+1/2}, x_{n+3/2}$ and $x = x_{n+i/2}, i = 0(\frac{1}{2})2$, respectively, the following 2- step discrete implicit methods are obtained;

$$\begin{aligned}
y_{n+\frac{1}{2}} = & \frac{1}{2} (y_n + y_{n+1}) - \frac{\left(48 - 96e^{\frac{w}{2}} + 48e^w - 14e^{\frac{w}{2}}w^2 + 5e^w w^2 - 4e^{\frac{3w}{2}}w^2 + e^{2w}w^2\right) h^2 f_n}{96 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(-96 + 192e^{\frac{w}{2}} - 96e^w + 7h^2w^2 + 32e^w w^2 - 20e^{\frac{3w}{2}}w^2 + 5e^{2w}w^2\right) h^2 f_{n+\frac{1}{2}}}{48 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(288 - 576e^{\frac{w}{2}} + 288e^w - 5w^2 - 64e^{\frac{w}{2}}w^2 - 4e^{\frac{3w}{2}}w^2 + e^{2w}w^2\right) h^2 f_{n+1}}{96 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{48 + 96e^{\frac{w}{2}} - 48e^w + w^2 + 10e^{\frac{w}{2}}w^2 + e^w w^2}{24 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} h^2 f_{n+\frac{3}{2}} \\
& - \frac{48 + 96e^{\frac{w}{2}} - 48e^w + w^2 + 10e^{\frac{w}{2}}w^2 + e^w w^2}{96 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} h^2 f_{2+n} \\
y_{n+\frac{3}{2}} = & \frac{1}{2} (-y_n + 3y_{1+n}) + \frac{\left(48 - 144e^w + 96e^{\frac{3w}{2}} - 16e^{\frac{w}{2}}w^2 - 15e^w w^2 - 6e^{\frac{3w}{2}}w^2 + e^{2w}w^2\right) h^2 f_n}{96 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(-48 + 144e^w - 96e^{\frac{3w}{2}} + 4w^2 + 39e^w w^2 - 10e^{\frac{3w}{2}}w^2 + 3e^{2w}w^2\right) h^2 f_{\frac{1}{2}+n}}{24 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(96 - 288e^w + 192e^{\frac{3w}{2}} + 5w^2 - 52e^{\frac{w}{2}}w^2 - 32e^{\frac{3w}{2}}w^2 + 7e^{2w}w^2\right) h^2 f_{1+n}}{32 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(-96 + 288e^w - 192e^{\frac{3w}{2}} + 3w^2 + 20e^{\frac{w}{2}}w^2 + 48e^w w^2 + e^{2w}w^2\right) h^2 f_{\frac{3}{2}+n}}{48 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{1 + 2e^{\frac{w}{2}}}{96 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \left(-48 + 96e^{\frac{w}{2}} - 48e^w + w^2 + 10e^{\frac{w}{2}}w^2 + e^w w^2\right) h^2 f_{2+n}
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
y_{n+2} = & -y_n + 2y_{1+n} + \frac{\left(3 - 6e^w + 3e^{2w} - e^{\frac{w}{2}}w^2 - e^w w^2 - e^{\frac{3w}{2}}w^2\right) h^2 f_n}{3 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{(-12 + 24e^w - 12e^{2w} + w^2 + 10e^w w^2 + e^{2w} w^2) h^2 f_{\frac{1}{2}+n}}{3 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(18 - 36e^w + 18e^{2w} + w^2 - 10e^{\frac{w}{2}}w^2 - 10e^{\frac{3w}{2}}w^2 + e^{2w} w^2\right) h^2 f_{1+n}}{3 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{(-12 + 24e^w - 12e^{2w} + w^2 + 10e^w w^2 + e^{2w} w^2) h^2 f_{\frac{3}{2}+n}}{3 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(3 - 6e^w + 3e^{2w} - e^{\frac{w}{2}}w^2 - e^w w^2 - e^{\frac{3w}{2}}w^2\right) h^2 f_{2+n}}{3 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
y'_n = & -\frac{y_n - y_{1+n}}{h} - \frac{\left(-90 + 90e^w - 90w - 89e^{\frac{w}{2}}w^2 + 87e^w w^2 - 57e^{\frac{3w}{2}}w^2 + 14e^{2w} w^2\right) h f_n}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(360 - 360e^w + 360w + 89w^2 + 186e^w w^2 - 128e^{\frac{3w}{2}}w^2 + 33e^{2w} w^2\right) h f_{\frac{1}{2}+n}}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(180 - 180e^w + 180w + 29w^2 + 62e^{\frac{w}{2}}w^2 - 2e^{\frac{3w}{2}}w^2 + e^{2w} w^2\right) h f_{1+n}}{30 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{360 - 360e^w + 360w + 57w^2 + 128e^{\frac{w}{2}}w^2 - 6e^w w^2 + e^{2w} w^2}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} h f_{\frac{3}{2}+n} \\
& + \frac{\left(90 - 90e^w + 90w + 14w^2 + 33e^{\frac{w}{2}}w^2 - 3e^w w^2 + e^{\frac{3w}{2}}w^2\right) h f_{2+n}}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2}
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
y'_{n+\frac{1}{2}} = & -\frac{y_n - y_{1+n}}{h} + \frac{\left(720 - 720e^w + 720e^{\frac{w}{2}}w - 113e^{\frac{w}{2}}w^2 + 189e^w w^2 - 99e^{\frac{3w}{2}}w^2 + 23e^{2w}w^2\right)hf_n}{720 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(-2880 + 2880e^w - 2880e^{\frac{w}{2}}w + 113w^2 - 78e^w w^2 - 56e^{\frac{3w}{2}}w^2 + 21e^{2w}h^2w^2\right)hf_{\frac{1}{2}+n}}{720 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(-1440 + 1440e^w - 1440e^{\frac{w}{2}}w + 63w^2 - 26e^{\frac{w}{2}}w^2 - 54e^{\frac{3w}{2}}w^2 + 17e^{2w}w^2\right)hf_{1+n}}{240 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(2880 + 2880e^w - 2880e^{\frac{w}{2}}w + 99w^2 + 56e^{\frac{w}{2}}w^2 - 162e^w w^2 + 7e^{2w}w^2\right)hf_{\frac{3}{2}+n}}{720 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(-720 + 720e^w - 720e^{\frac{w}{2}}w + 23w^2 + 21e^{\frac{w}{2}}w^2 - 51e^w w^2 + 7e^{\frac{3w}{2}}w^2\right)hf_{2+n}}{720 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
y'_{n+1} = & -\frac{y_n - y_{1+n}}{h} + \frac{\left(90 - 90e^w + 90e^w w - 31e^{\frac{w}{2}}w^2 - 12e^w w^2 - 3e^{\frac{3w}{2}}w^2 + e^{2w}w^2\right)hf_n}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(-360 + 360e^w - 360e^w w + 31w^2 + 234e^w w^2 - 112e^{\frac{3w}{2}}w^2 + 27e^{2w}w^2\right)hf_{\frac{1}{2}+n}}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(90 - 90e^w + 90e^w w + 2w^2 - 39e^{\frac{w}{2}}w^2 - 11e^{\frac{3w}{2}}w^2 + 3e^{2w}w^2\right)hf_{1+n}}{15 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& - \frac{\left(360 - 360e^w + 360e^w w - 3w^2 - 112e^{\frac{w}{2}}w^2 - 66e^w w^2 + e^{2w}w^2\right)hf_{\frac{3}{2}+n}}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2} \\
& + \frac{\left(90 - 90e^w + 90e^w w - w^2 - 27e^{\frac{w}{2}}w^2 - 18e^w w^2 + e^{\frac{3w}{2}}w^2\right)hf_{2+n}}{90 \left(-1 + e^{\frac{w}{2}}\right)^4 w^2}
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
y'_n + \frac{3}{2} = & -\frac{y_n - y_{1+n}}{h} + \frac{\left(720 - 720e^w + 720e^{\frac{3w}{2}} - 233e^{\frac{w}{2}}w^2 - 291e^w w^2 - 219e^{\frac{3w}{2}}w^2 + 23e^{2w}w^2\right)hf_n}{720 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& + \frac{\left(-2880 + 2880e^w - 2880e^{\frac{3w}{2}}w + 233w^2 + 2562e^w w^2 - 56e^{\frac{3w}{2}}w^2 + 141e^{2w}w^2\right)hf_{\frac{1}{2}+n}}{720 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& + \frac{\left(1440 - 1440e^w + 1440e^{\frac{3w}{2}}w + 97w^2 - 854e^{\frac{w}{2}}w^2 - 826e^{\frac{3w}{2}}w^2 + 143e^{2w}w^2\right)hf_{1+n}}{240 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& + \frac{\left(-2880 + 2880e^w - 2880e^{\frac{3w}{2}}w + 219w^2 + 56e^{\frac{w}{2}}w^2 + 2478e^w w^2 + 127e^{2w}w^2\right)hf_{\frac{3}{2}+n}}{720 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& - \frac{\left(720 + 720e^w - 720e^{\frac{3w}{2}}w + 23w^2 + 141e^{\frac{w}{2}}w^2 + 429e^w w^2 + 127e^{\frac{3w}{2}}w^2\right)hf_{2+n}}{720 - 1 + e^{\frac{w}{2}})^4 w^2} \\
y'_{n+2} = & -\frac{y_n - y_{1+n}}{h} - \frac{\left(-90 + 90e^w - 90e^{2w}w + 31e^{\frac{w}{2}}w^2 + 27e^w w^2 + 63e^{\frac{3w}{2}}w^2 + 14e^{2w}w^2\right)hf_n}{90 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& + \frac{\left(-360 + 360e^w - 360e^{2w}w + 31w^2 + 294e^w w^2 + 128e^{\frac{3w}{2}}w^2 + 87e^{2w}w^2\right)hf_{\frac{1}{2}+n}}{90 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& - \frac{\left(180 + 180e^w - 180e^{2w}w - 9w^2 + 98e^{\frac{w}{2}}w^2 + 162e^{\frac{3w}{2}}w^2 + 19e^{2w}w^2\right)hf_{1+n}}{30 - 1 + e^{\frac{w}{2}})^4 w^2} \\
& + \frac{360 + 360e^w - 360e^{2w}w + 63w^2 - 128e^{\frac{w}{2}}w^2 + 486e^w w^2 + 119e^{2w}w^2}{90 - 1 + e^{\frac{w}{2}})^4 w^2} hf_{\frac{3}{2}+n} \\
& + \frac{\left(90 - 90e^w + 90e^{2w}w + 14w^2 - 87e^{\frac{w}{2}}w^2 + 57e^w w^2 - 119e^{\frac{3w}{2}}w^2\right)hf_{2+n}}{90 - 1 + e^{\frac{w}{2}})^4 w^2}
\end{aligned} \tag{3.55}$$

for $n = 0, 2, \dots, N - 2$.

Remark 3.3.1. *Only hybrid 2-step continuous implicit method was derived in this work. The motivation is to derive method with less number of functions so as to reduce the number of function evaluations in the implementation process. For non-hybrid, the number of occurrences of functions is $k + 1$. For the hybrid method, the number of occurrences of functions is $2k + 1$. The more the functions, the number of function evaluations and this increases computation time.*

3.4 Analysis of the Methods

3.4.1 Local Truncation Error and Order of the first order block method

Let us consider the linear difference operators associated with the formulas in (3.5), which could be written as

$$L_k[y(x); h] \equiv y(x + nh) - y(x) - \sum_{i=1}^{k+1} \beta_i(w; h) y^{(i)}(x + ih) \quad (3.56)$$

for $i = 1(1)k$. The local truncation error of each of the formulae in (3.9), (3.13), (3.18), (3.24), (3.28) is the amount by which the exact solution of the ODEs fails to satisfy the corresponding difference equation. Thus, if we consider the exact solution $y(x)$ in (3.56), after expanding in Taylor series around x we get that each of the local truncation errors is of the form

$$L[y(x); h] = C_0 y(x) + C_1 h y^{(1)}(x) + C_2 h^2 y^{(2)}(x) + \dots + C_q h^q y^{(q)}(x) + O(h^{(q+1)}). \quad (3.57)$$

where the C_i are constants. If the first $p + 1$ constants vanish, that is,

$$C_0 = C_1 = C_2 = \dots = C_p = 0, \text{ and } C_{p+1} \neq 0$$

then, it is

$$L[y(x); h] = C_{p+1}h^{p+1}y^{(p+1)}(x) + O(h^{p+2}) \quad (3.58)$$

and p is called the order of the formula and C_{p+1} is error constant. In this the case for all the formulae, the method is said to be consistent of order p , (see Lambert, 1973). Expanding each of the methods in (3.9), (3.13), (3.18), (3.24), (3.28), the LTEs are as shown below respectively:

$$\begin{aligned} C_{p+1}^{k=1} &= \begin{pmatrix} \frac{5}{12} & wy''[x] - y^{(3)}[x] & h^3 + O[h]^4 \\ \frac{1}{3} & wy''[x] - y^{(3)}[x] & h^3 + O[h]^4 \end{pmatrix} \\ C_{p+1}^{k=2} &= \begin{pmatrix} -\frac{3}{8} & y^{(3)}[x] - y^{(4)}[x] & h^4 + O[h]^5 \\ \frac{1}{3} & -y^{(3)}[x] + y^{(4)}[x] & h^4 + O[h]^5 \\ -\frac{3}{8} & y^{(3)}[x] - y^{(4)}[x] & h^4 + O[h]^5 \end{pmatrix} \\ C_{p+1}^{k=3} &= \begin{pmatrix} \frac{251}{720} & wy^{(4)}[x] - y^{(5)}[x] & h^5 + O[h]^6 \\ \frac{29}{90} & wy^{(4)}[x] - y^{(5)}[x] & h^5 + O[h]^6 \\ \frac{27}{80} & wy^{(4)}[x] - y^{(5)}[x] & h^5 + O[h]^6 \\ \frac{14}{45} & wy^{(4)}[x] - y^{(5)}[x] & h^5 + O[h]^6 \end{pmatrix} \\ C_{p+1}^{k=4} &= \begin{pmatrix} -\frac{95}{288} & wy^{(5)}[x] - y^{(6)}[x] & h^6 + O[h]^7 \\ -\frac{14}{45} & wy^{(5)}[x] - y^{(6)}[x] & h^6 + O[h]^7 \\ -\frac{51}{160} & wy^{(5)}[x] - y^{(6)}[x] & h^6 + O[h]^7 \\ -\frac{14}{45} & wy^{(5)}[x] - y^{(6)}[x] & h^6 + O[h]^7 \\ -\frac{95}{288} & wy^{(5)}[x] - y^{(6)}[x] & h^6 + O[h]^7 \end{pmatrix} \\ C_{p+1}^{k=5} &= \begin{pmatrix} \frac{19087}{60480} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \\ \frac{1139}{3780} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \\ \frac{137}{448} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \\ \frac{286}{945} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \\ \frac{3715}{12096} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \\ \frac{41}{140} & wy^{(6)}[x] - y^{(7)}[x] & h^7 + O[h]^8 \end{pmatrix} \end{aligned}$$

Conclusively, the order p for k -step method is $p = k + 1$.

3.4.2 Zero stability

A Linear Multistep Method is said to be zero-stable if its first characteristic polynomial $\rho(z)$ satisfies the root condition (see Definitions 1.6.6, 1.6.8 and 1.6.9).

The methods in (3.9) to (3.28) can each be put in a block form

$$A_1 Y_{r+n} = A_0 Y_r + h B_1 F_{r+n} \quad (3.59)$$

The zero-stability is concerned with the stability of the difference system (3.59) as $h \rightarrow 0$. Thus, as $h \rightarrow 0$, (3.59) becomes

$$A_0^k Y_{r+n} = A_1^k Y_r \quad (3.60)$$

where

$Y_{r+n} = (y_{n+1}, y_{n+2}, \dots, y_{n+k+1})^T$, $Y_r = (y_{n-1}, y_{n-2}, \dots, y_{n-k-1})^T$, A_0^k is a $(k+1) \times (k+1)$ identity matrix and A_1^k is a $(k+1) \times (k+1)$ matrix with its $(k+1)$ column entries are 1. k is the number of steps. Hence we sought for the the

characteristic polynomial

$$\rho(z) = |zA_0^k - A_1^k| = 0 \quad (3.61)$$

such that the roots $|z| \leq 1$. For $k = 1$;

$$|zA_0^1 - A_1^1| = \left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} z & -1 \\ 0 & z-1 \end{pmatrix} \right| = z(z-1) = 0$$

Hence the roots are (0,1).

For $k = 2$;

$$|zA_0^2 - A_1^2| = \left| \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} z & 0 & -1 \\ 0 & z & -1 \\ 0 & 0 & z-1 \end{pmatrix} \right| = z^2(z-1) = 0$$

Hence the roots are (0,0,1).

For $k = 3$;

$$|zA_0^3 - A_1^3| = \left| \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} z & 0 & 0 & -1 \\ 0 & z & 0 & -1 \\ 0 & 0 & z & -1 \\ 0 & 0 & 0 & z-1 \end{pmatrix} \right| = z^3(z-1) = 0$$

Hence the roots are (0,0,0,1).

For $k = 4$;

$$|zA_0^4 - A_1^4| = \left| \begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ & z & 0 & 0 & 0 \\ & & z & 0 & 0 \\ & & & z & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^4(z-1) = 0$$

Hence the roots are $(0,0,0,0,1)$.

For $k = 5$;

$$\begin{aligned}
 |zA_0^5 - A_1^5| &= \left| \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} z & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & z-1 \end{pmatrix} \right| = z^5(z-1) = 0
 \end{aligned}$$

Hence the roots are $(0,0,0,0,0,1)$.

Consequently, the methods are all zero-stable, since the roots of the characteristic polynomial are all zero except one, whose modulus is one (Dahlquist 1959; Lambert 1973).

For convergence, we state the following theorem.

3.4.3 Convergence

Theorem 3.4.1. *Henrici (1962) . A linear multistep method is said to be convergent if it is consistent (with order $p \geq 1$) and it is zero-stable.*

By the above analysis, the method has order $p = k+1 \geq 1$, and is zero-stable. Thus, by the above theorem, the method is convergent.

3.4.4 Local Truncation Error and Order of the second order block method

The Linear difference operators associated with the formulas in (3.32), is defined by the formulae

$$\mathcal{L}_k[y(x); h] \equiv y(x + nh) - \left[\sum_{i=0}^{r-1} \alpha_i(x + ih) - h^2 \sum_{i=1}^{s-1} \beta_i(w; h) f(x + ih) \right] \quad (3.62)$$

The local truncation error of each of the formulae in (3.36), (3.41), (3.45), (3.46), (3.47) and (3.48) is the amount by which the exact solution of the ODEs fails to satisfy the corresponding difference operator. Thus, expanding (3.62), in Taylor series around x the local truncation errors takes the form

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_q h^q y^{(q)}(x) + O(h^{(q+1)}). \quad (3.63)$$

where the $C_i (i \geq 0)$ are constants. Suppose the first $p + 1$ constants vanish, that is,

$$C_0 = C_1 = C_2 = \dots = C_{p+1} = 0, \text{ and } C_{p+2} \neq 0$$

then

$$L[y(x); h] = C_{p+2} h^{p+2} y^{(p+2)}(x) + O(h^{p+3}) \quad (3.64)$$

and p is called the order of the formulae and C_{p+2} is known as the Local Truncation Error (LTE) or the principal error constant. In this the case for all the formulae, the method is said to be consistent of order p , Lambert (1973). Expanding each of the methods in (3.36), (3.41), (3.45)-(3.47), the LTEs are as shown below;

$$C_{p+2}^{k=2} = \begin{pmatrix} \tau_1 = -\frac{1}{240} & -w^2 y^{(4)}[x] + y^{(6)}[x] & h^6 + O[h]^7 \\ \tau_2 = \frac{1}{45} & -w y^{(4)}[x] + y^{(5)}[x] & h^4 + O[h]^5 \\ \tau_3 = \frac{7}{360} & w y^{(4)}[x] - y^{(5)}[x] & h^4 + O[h]^5 \\ \tau_4 = \frac{1}{45} & -w y^{(4)}[x] + y^{(5)}[x] & h^4 + O[h]^5 \end{pmatrix} \quad (3.65)$$

$$C_{p+2}^{k=3} = \begin{pmatrix} \tau_1 = \frac{1}{240} & -w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \\ \tau_2 = \frac{1}{80} & -w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \\ \tau_3 = \frac{7}{480} & w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \\ \tau_4 = \frac{-17}{1440} & w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \\ \tau_5 = \frac{1}{288} & w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \\ \tau_6 = \frac{-11}{480} & w y^{(5)}[x] - y^{(6)}[x] & h^5 + O[h]^6 \end{pmatrix} \quad (3.66)$$

$$C_{p+2}^{k=4} = \begin{pmatrix} \tau_1 = \frac{1}{240} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_2 = \frac{1}{120} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_3 = \frac{1}{120} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_4 = -\frac{107}{10080} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_5 = \frac{41}{5040} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_6 = \frac{1}{2016} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_7 = -\frac{41}{5040} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \\ \tau_8 = \frac{107}{10080} & w y^{(6)}[x] - y^{(7)}[x] & h^6 + O[h]^7 \end{pmatrix} \quad (3.67)$$

Conclusively, the order p for k -step method is $p = k$.

3.4.5 Local Truncation Error and Order of the 2-step hybrid method

The Linear difference operators associated with the formulas in (3.4), is defined by the formulae

$$\begin{aligned} \mathcal{L}_k[y(x); h] \equiv & y^{(j)}(x + nh) - \left[\sum_{i=0}^1 \alpha_i(x + vh) - \sum_{i=0}^3 \beta_i(w; h) f(x + ih) \right. \\ & \left. - \sum_{i=0}^1 \hat{\beta}_{\frac{2i+1}{2}}(w; h) f(x + h \frac{2i+1}{2}) \right], \quad j = 0, 1, \quad v = 0(1/2)2. \end{aligned} \quad (3.68)$$

By expanding (3.68) in Taylor series around x the local truncation errors takes the form

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_q h^q y^{(q)}(x) + O(h^{(q+1)}). \quad (3.69)$$

where the C_i are constants. Suppose the first $p + 1$ constants vanish, that is,

$$C_0 = C_1 = C_2 = \dots = C_{p+1} = 0, \text{ and } C_{p+2} \neq 0$$

then

$$L[y(x); h] = C_{p+2} h^{p+2} y^{(p+2)}(x) + O(h^{p+3}) \quad (3.70)$$

and p is called the order of the formulae and C_{p+2} is known as the Local Truncation

Error (LTE) or the principal error constant. In this the case for all the formulae, the method is said to be consistent of order p , (Lambert, 1973). Expanding each of the methods in (3.52)-(3.55), the LTEs are as shown below;

$$C_{p+2}^{k=2} = \begin{pmatrix} \tau_1 = -\frac{1}{61440} - w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_2 = \frac{1}{61440} w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_3 = \frac{1}{120960} - w y^{(6)}[x] - y^{(7)}[x] h^7 + O[h]^8 \\ \tau_4 = \frac{1}{5040} - w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_5 = \frac{61}{645120} w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_6 = \frac{1}{40320} - w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_7 = \frac{61}{645120} w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \\ \tau_8 = \frac{1}{5040} - w y^{(6)}[x] + y^{(7)}[x] h^7 + O[h]^8 \end{pmatrix} \quad (3.71)$$

Conclusively, the order p for the 2-step hybrid method is $p = 5$.

3.4.6 Convergence Analysis

Convergence of the ENIS

In this section, the convergence of the ENIS is discussed. Each method of step number k can be written in block form

$$AY - h^2 BF(Y) + C + \tau(h) = 0 \quad (3.72)$$

where A is a $2N \times 2N$ matrix defined by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Where the entries of A are $N \times N$ matrices and similarly, B is a $2N \times 2N$ matrix defined by

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

The matrices A_{ij} and B_{ij} are given below.

For $k=2$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \end{bmatrix}$$

$$B_{11} = \begin{pmatrix} \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \beta_{\frac{1}{11}} & \beta_{\frac{1}{12}} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \beta_{\frac{1}{21}} & \beta_{\frac{1}{22}} \end{pmatrix}$$

$$\beta_{\frac{1}{11}} = -\frac{1}{6\xi_2 h}(12 - 12e^{hw} + 12hw + 5h^2w^2 + e^{2hw}h^2w^2)$$

$$\beta_{\frac{1}{12}} = \frac{1}{6\xi_2 h}(6 - 6e^{hw} + 6hw + 2h^2w^2 + e^{hw}h^2w^2)$$

$$\beta_{\frac{1}{21}} = -\frac{1}{\xi_2}(2 - 4e^{hw} + 2e^{2hw} - h^2w^2 - e^{2hw}h^2w^2)$$

$$\beta_{\frac{1}{22}} = \frac{1}{\xi_2}(1 - 2e^{hw} + e^{2hw} - e^{hw}h^2w^2)$$

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$A_{22} = I$ where I is an $N \times N$ identity matrix and $A_{12} = 0$

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$$\begin{array}{cccccccccccc}
 \beta_{112} & \beta_{122} & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 \beta_{212} & \beta_{222} & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 0 & \beta_{112} & \beta_{121} & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 0 & \beta_{212} & \beta_{221} & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 0 & 0 & \beta_{112} & \beta_{122} & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 0 & 0 & \beta_{212} & \beta_{222} & 0 & 0 & 0 & \dots & \dots & 0 & 0 & ? \\
 0 & 0 & 0 & \beta_{112} & \beta_{122} & 0 & 0 & 0 & 0 & 0 & 0 & ? \\
 0 & 0 & 0 & \beta_{212} & \beta_{222} & 0 & 0 & 0 & 0 & 0 & 0 & ? \\
 0 & 0 & 0 & 0 & \beta_{112} & \beta_{122} & 0 & 0 & 0 & 0 & 0 & ? \\
 0 & 0 & 0 & 0 & \beta_{212} & \beta_{222} & 0 & 0 & 0 & 0 & 0 & ? \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & ? \\
 0 & 0 & 0 & \dots & \dots & 0 & 0 & \beta_{112} & \beta_{122} & \beta_{212} & \beta_{222} & ? \\
 0 & 0 & 0 & & & 0 & 0 & \beta_{112} & \beta_{122} & \beta_{212} & \beta_{222} & ?
 \end{array}$$

$$\begin{array}{ccc}
\begin{array}{c}
\beta_{11} \\
\beta_{12} \dots \\
\beta_{21} \\
\beta_{22} \\
0 \\
\beta_{31} \\
\beta_{32} \\
0
\end{array}
&
&
\begin{array}{c}
\beta_{122} \\
\beta_{222}
\end{array}
\end{array}$$

$B_{12} = B_{22} = 0$ where

$$\begin{aligned}
\beta_{11} &= -\frac{1}{6\xi_2 h} (12 - 12e^{hw} + 12e^{hw}hw - 4h^2w^2 - 2e^{2hw}h^2w^2) \\
\beta_{12} &= \frac{1}{6\xi_2 h} (6 - 6e^{hw} + 6e^{hw}hw - h^2w^2 - 2e^{hw}h^2w^2) \\
\beta_{21} &= -\frac{1}{6\xi_2 h} (12 - 12e^{hw} + 12e^{2hw}hw - 7h^2w^2 - 11e^{2hw}h^2w^2) \\
\beta_{22} &= \frac{1}{6\xi_2 h} (6 - 6e^{hw} + 6e^{2hw}hw + 2h^2w^2 - 11e^{hw}h^2w^2)
\end{aligned}$$

$$C = (-1, \eta_1, 1/h, \eta_2, 1/h, \eta_3, -1/h, \eta_4, 0, 0, \dots, 0)^T$$

$$\eta_1 = \frac{1}{\xi_2} (1 - 2e^{hw} + e^{2hw} - e^{hw}h^2w^2)$$

$$\eta_2 = -\frac{1}{6\xi_2 h} (-6 + 6e^{hw} - 6hw - 5e^{hw}h^2w^2 + 2e^{2hw}h^2w^2)$$

$$\eta_3 = \frac{1}{6\xi_2 h} (6 - 6e^{hw} + 6e^{hw}hw - 4e^{hw}h^2w^2 + e^{2hw}h^2w^2)$$

$$\eta_4 = -\frac{1}{6\xi_2 h} (-6 + 6e^{hw} - 6e^{2hw}hw + 7e^{hw}h^2w^2 + 2e^{2hw}h^2w^2)$$

(3.73)

$$\tau(h) = (\tau_1, \dots, \tau_N, 0, 0, \dots, 0, h\tau'_1, \dots, h\tau'_N, 0, 0, \dots, 0)^T$$

τ_n are all the entries of C_{p+2}^k in (3.65)

$$Y = (y(x_1), \dots, y(x_N), hy^0(x_1), \dots, hy^0(x_N), 0, 0, \dots, 0)^T$$

[illegible]

$$F = (f(x_1), \dots, f(x_N), hf^0(x_1), \dots, hf^0(x_N), 0, 0, \dots, 0)^T$$

For $k=3$

[illegible]

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	$?$	$?$	
	0		
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	$?$	$?$	
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			0

$?$
$?$

	$?$
$\dots$	$?$
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	$?$
0	$?$
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	$?$
0	$?$
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$A_{22} = I$ where I is an $N \times N$ identity matrix and $A_{12} = 0$

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$$B_{11} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{11} & \beta_{12} & \beta_{13} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{21} & \beta_{22} & \beta_{23} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \beta_{11} & \beta_{12} & \beta_{13} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \beta_{21} & \beta_{22} & \beta_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_{21} & \beta_{22} & \beta_{23} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}$$

$$\beta_{11} = \frac{1}{12\xi_3}(36 - 72e^{hw} + 36e^{2hw} - 13h^2w^2 - 33e^{2hw}h^2w^2 + 10e^{3hw}h^2w^2)$$

$$\beta_{12} = \frac{1}{12\xi_3}(36 - 72e^{hw} + 36e^{2hw} - 2h^2w^2 - 33e^{hw}h^2w^2 - e^{3hw}h^2w^2)$$

$$\beta_{13} = \frac{1}{12\xi_3}(12 - 24e^{hw} + 12e^{2hw} - h^2w^2 - 10e^{hw}h^2w^2 - e^{2hw}h^2w^2)$$

$$\beta_{21} = \frac{1}{4\xi_3}(24 - 36e^{hw} + 12e^{3hw} - 9h^2w^2 - 33e^{2hw}h^2w^2 + 6e^{3hw}h^2w^2)$$

$$\beta_{22} = \frac{1}{4\xi_3}(24 - 36e^{hw} + 12e^{3hw} + 2h^2w^2 - 33e^{hw}h^2w^2 - 5e^{3hw}h^2w^2)$$

$$\beta_{23} = \frac{1}{4\xi_3}(8 - 12e^{hw}4e^{3hw} - h^2w^2 - 6e^{hw}h^2w^2 - 5e^{2hw}h^2w^2)$$

$$\beta_{31} = \frac{1}{24\xi_3h}(72 - 72e^{hw} + 72hw + 27h^2w^2 + 15e^{2hw}h^2w^2 - 6e^{3hw}h^2w^2)$$

$$\beta_{32} = -\frac{1}{24\xi_3h}(72 - 72e^{hw} + 72hw + 22h^2w^2 + 15e^{hw}h^2w^2 - e^{3hw}h^2w^2)$$

$$\beta_{33} = \frac{1}{24\xi_3h}(24 - 24e^{hw} + 24hw + 7h^2w^2 + 6e^{hw}h^2w^2 - e^{2hw}h^2w^2)$$

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$$\begin{aligned}
\beta_{11}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{hw}hw - 19h^2w^2 - 27e^{2hw}h^2w^2 + 10e^{3hw}h^2w^2) \\
\beta_{12}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{hw}hw - 10h^2w^2 - 27e^{hw}h^2w^2 + e^{3hw}h^2w^2) \\
\beta_{13}^2 &= \frac{1}{24\xi_3 h} (24 - 24e^{hw} + 24e^{hw}hw - 3h^2w^2 - 10e^{hw}h^2w^2 + e^{2hw}h^2w^2) \\
\beta_{21}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{2hw}hw - 29h^2w^2 - 105e^{2hw}h^2w^2 + 26e^{3hw}h^2w^2) \\
\beta_{22}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{2hw}hw + 6h^2w^2 - 105e^{hw}h^2w^2 - 9e^{3hw}h^2w^2) \\
\beta_{23}^2 &= \frac{1}{24\xi_3 h} (24 - 24e^{hw} + 24e^{2hw}hw - h^2w^2 - 26e^{hw}h^2w^2 - 9e^{2hw}h^2w^2) \\
\beta_{31}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{3hw}hw - 27h^2w^2 - 147e^{2hw}h^2w^2 - 6e^{3hw}h^2w^2) \\
\beta_{32}^2 &= \frac{1}{24\xi_3 h} (72 - 72e^{hw} + 72e^{3hw}hw + 22h^2w^2 - 147e^{hw}h^2w^2 - 55e^{3hw}h^2w^2) \\
\beta_{33}^2 &= \frac{1}{24\xi_3 h} (24 - 24e^{hw} + 24e^{3hw}hw - 11h^2w^2 + 6e^{hw}h^2w^2 - 55e^{2hw}h^2w^2)
\end{aligned}$$

$$\begin{aligned}
C &= [-1/h, \eta_1, -1, \eta_2, -2, \eta_3, -1/h, \eta_4 - 1/h, \eta_5 - 1/h, \eta_6, 0, 0, \dots, 0]^T \\
\eta_1 &= -\frac{1}{24\xi_3 h} (24 - 24e^{hw} + 24hw + 27e^{hw}h^2w^2 - 22e^{2hw}h^2w^2 + 7e^{3hw}h^2w^2) \\
\eta_2 &= \frac{1}{12\xi_3} (-12 + 24e^{hw} - 12e^{2hw} + 13e^{hw}h^2w^2 - 2e^{2hw}h^2w^2 + e^{3hw}h^2w^2) \\
\eta_3 &= \frac{1}{4\xi_3} (-8 + 12e^{hw} - 4e^{3hw} + 9e^{hw}h^2w^2 + 2e^{2hw}h^2w^2 + e^{3hw}h^2w^2) \\
\eta_4 &= \frac{1}{24\xi_3 h} (-24 + 24e^{hw} - 24e^{hw}hw + 19e^{hw}h^2w^2 - 10e^{2hw}h^2w^2 + 3e^{3hw}h^2w^2) \\
\eta_5 &= \frac{1}{24\xi_3 h} (-24 + 24e^{hw} - 24e^{2hw}hw + 29e^{hw}h^2w^2 + 6e^{2hw}h^2w^2 + e^{3hw}h^2w^2) \\
\eta_6 &= \frac{1}{24\xi_3 h} (-24 + 24e^{hw} - 24e^{3hw}hw + 27e^{hw}h^2w^2 + 22e^{2hw}h^2w^2 + 11e^{3hw}h^2w^2)
\end{aligned} \tag{3.74}$$

$$\tau(h) = (\tau_1, \dots, \tau_N, 0, 0, \dots, 0, h\tau'_1, \dots, h\tau'_N, 0, 0, \dots, 0)^T$$

τ_n are all the entries of C_{p+2}^k in (3.66).

$$Y = (y(x_1), \dots, y(x_N), hy^0(x_1), \dots, hy^0(x_N), 0, 0, \dots, 0)^T$$

$$F = (f(x_1), \dots, f(x_N), hf^0(x_1), \dots, hf^0(x_N), 0, 0, \dots, 0)^T$$

For $k=4$

[illegible]

0

0 0
0
0
0
1

$$\begin{pmatrix} -1 & ? \\ ? & ? \\ -1 & ? \end{pmatrix}$$

$A_{22} = I$ where I is an $N \times N$ identity matrix and $A_{12} = 0$

$$\begin{pmatrix} ? & 0 & 0 & 0 & 0 & \dots & \dots & 0 & ? \\ \beta_{111} & \beta_{121} & \beta_{131} & \beta_{141} & 0 & 0 & 0 & 0 & \dots & \dots & 0 & ? \\ ? & \beta_{211} & \beta_{221} & \beta_{231} & \beta_{241} & 0 & 0 & 0 & 0 & \dots & \dots & 0 & ? \\ ? & & \beta_{321} & \beta_{331} & \beta_{341} & 0 & 0 & 0 & 0 & \dots & & 0 & ? \\ ? & \beta_{311} & \beta_{421} & \beta_{431} & \beta_{441} & \beta_{121} & \beta_{131} & \beta_{141} & 0 & \dots & & 0 & ? \\ ? & & 0 & 0 & \beta_{111} & \beta_{221} & \beta_{231} & \beta_{241} & 0 & \dots & & 0 & ? \\ ? & \beta_{411} & 0 & 0 & \beta_{211} & \beta_{321} & \beta_{331} & \beta_{341} & 0 & \beta_{131} & & 0 & ? \\ ? & ? & 0 & 0 & \beta_{311} & \beta_{421} & \beta_{431} & \beta_{441} & \dots & \beta_{331} & & 0 & ? \\ 0 & & 0 & 0 & \beta_{411} & \dots & \dots & \dots & \dots & & & 0 & ? \\ ? & & \dots & \dots & & & & & & & & ? & ? \\ ? & ? & & & & & & & & & & ? & ? \\ 0 & & & & & & & & & & & ? & ? \\ ? & & & & & & & & & & & ? & ? \end{pmatrix}$$

$$\begin{aligned}
\beta_{11}^1 &= -\frac{1}{360\xi_4 h}(1440 + 1440e^{hw} - 1440hw - 502h^2w^2 - 528e^{2hw}h^2w^2 + 424e^{3hw}h^2w^2 - 114e^{4hw}h^2w^2) \\
\beta_{12}^1 &= -\frac{1}{360\xi_4 h}(2160 + 2160e^{hw} - 2160hw - 621h^2w^2 - 528e^{hw}h^2w^2 + 108e^{3hw}h^2w^2 - 39e^{4hw}h^2w^2) \\
\beta_{13}^1 &= -\frac{1}{360\xi_4 h}(1440 - 1440e^{hw} + 1440hw + 396h^2w^2 + 424e^{hw}h^2w^2 - 108e^{2hw}h^2w^2 + 8e^{4hw}h^2w^2) \\
\beta_{14}^1 &= -\frac{1}{360\xi_4 h}(360 + 360e^{hw} - 360hw - 97h^2w^2 - 114e^{hw}h^2w^2 + 39e^{2hw}h^2w^2 - 8e^{3hw}h^2w^2) \\
\beta_{21}^1 &= \frac{1}{12\xi_4}(48 - 96e^{hw} + 48e^{2hw} - 14h^2w^2 - 64e^{2hw}h^2w^2 + 40e^{3hw}h^2w^2 - 10e^{4hw}h^2w^2) \\
\beta_{22}^1 &= \frac{1}{12\xi_4}(72 - 144e^{hw} + 72e^{2hw} - 5h^2w^2 - 64e^{hw}h^2w^2 - 4e^{3hw}h^2w^2 + e^{4hw}h^2w^2) \\
\beta_{23}^1 &= -\frac{1}{12\xi_4}(48 - 96e^{hw} + 48e^{2hw} - 4h^2w^2 - 40e^{hw}h^2w^2 - 4e^{2hw}h^2w^2) \\
\beta_{24}^1 &= \frac{1}{12\xi_4}(12 - 24e^{hw} + 12e^{2hw} - h^2w^2 - 10e^{hw}h^2w^2 - e^{2hw}h^2w^2) \\
\beta_{31}^1 &= \frac{1}{12\xi_4}(96 - 144e^{hw} + 48e^{3hw} - 29h^2w^2 - 174e^{2hw}h^2w^2 + 80e^{3hw}h^2w^2 - 21e^{4hw}h^2w^2) \\
\beta_{32}^1 &= \frac{1}{12\xi_4}(144 - 216e^{hw} + 72e^{3hw} - 174e^{hw}h^2w^2 - 54e^{3hw}h^2w^2 + 12e^{4hw}h^2w^2) \\
\beta_{33}^1 &= \frac{1}{12\xi_4}(96 - 144e^{hw} + 48e^{3hw} - 9h^2w^2 - 80e^{hw}h^2w^2 - 54e^{2hw}h^2w^2 - e^{4hw}h^2w^2) \\
\beta_{34}^1 &= \frac{1}{12\xi_4}(24 - 36e^{hw} + 12e^{3hw} - 2h^2w^2 - 21e^{hw}h^2w^2 - 12e^{2hw}h^2w^2 - e^{3hw}h^2w^2) \\
\beta_{41}^1 &= \frac{1}{6\xi_4}(72 - 96e^{hw} + 24e^{4hw} - 22h^2w^2 - 144e^{2hw}h^2w^2 + 40e^{3hw}h^2w^2 - 18e^{4hw}h^2w^2) \\
\beta_{42}^1 &= \frac{1}{6\xi_4}(108 - 144e^{hw} + 36e^{4hw} + 3h^2w^2 - 144e^{hw}h^2w^2 - 84e^{3hw}h^2w^2 + 9e^{4hw}h^2w^2) \\
\beta_{43}^1 &= \frac{1}{6\xi_4}(72 - 96e^{hw} + 24e^{4hw} - 12h^2w^2 - 40e^{hw}h^2w^2 - 84e^{2hw}h^2w^2 - 8e^{4hw}h^2w^2) \\
\beta_{44}^1 &= \frac{1}{6\xi_4}(18 - 24e^{hw} + 6e^{4hw} - h^2w^2 - 18e^{hw}h^2w^2 - 9e^{2hw}h^2w^2 - 8e^{3hw}h^2w^2)
\end{aligned}$$

$$B_{12} = B_{22} = 0.$$

$$C = (-1/h, \eta_1, -1, \eta_2, -2, \eta_3, -3, \eta_4, -1/h, \eta_5, -1/h, \eta_6, -1/h, \eta_7, -1/h, \eta_8, 0, 0, \dots, 0)$$

$$\begin{aligned} \beta_{11}^2 &= \frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{hw}hw - 323h^2w^2 - 882e^{2hw}h^2w^2 + 656e^{3hw}h^2w^2 - 171e^{4hw}h^2w^2) \\ \beta_{12}^2 &= \frac{1}{360\xi_4 h} (2160 - 2160e^{hw} + 2160e^{hw}hw - 264h^2w^2 - 882e^{hw}h^2w^2 + 102e^{3hw}h^2w^2 - 36e^{4hw}h^2w^2) \\ \beta_{13}^2 &= \frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{hw}hw - 159h^2w^2 - 656e^{hw}h^2w^2 + 102e^{2hw}h^2w^2 - 7e^{4hw}h^2w^2) \\ \beta_{14}^2 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{hw}hw - 38h^2w^2 - 171e^{hw}h^2w^2 + 36e^{2hw}h^2w^2 - 7e^{3hw}h^2w^2) \\ \beta_{21}^2 &= \frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{2hw}hw - 458h^2w^2 - 2832e^{2hw}h^2w^2 + 1496e^{3hw}h^2w^2 - 366e^{4hw}h^2w^2) \\ \beta_{22}^2 &= \frac{1}{360\xi_4 h} (2160 - 2160e^{hw} + 2160e^{2hw}hw + 21h^2w^2 - 2832e^{hw}h^2w^2 - 588e^{3hw}h^2w^2 + 159e^{4hw}h^2w^2) \\ \beta_{23}^2 &= \frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{2hw}hw - 84h^2w^2 - 1496e^{hw}h^2w^2 - 588e^{2hw}h^2w^2 + 8e^{4hw}h^2w^2) \\ \beta_{24}^2 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{2hw}hw - 23h^2w^2 - 366e^{hw}h^2w^2 - 159e^{2hw}h^2w^2 + 8e^{3hw}h^2w^2) \\ \beta_{31}^2 &= \frac{1}{360\xi_4 h} (1440 + 1440e^{hw} - 1440e^{3hw}hw + 443h^2w^2 + 3522e^{2hw}h^2w^2 - 656e^{3hw}h^2w^2 + 291e^{4hw}h^2w^2) \\ \beta_{32}^2 &= \frac{1}{360\xi_4 h} (2160 - 2160e^{hw} + 2160e^{3hw}hw + 216h^2w^2 - 3522e^{hw}h^2w^2 - 2538e^{3hw}h^2w^2 + 444e^{4hw}h^2w^2) \\ \beta_{33}^2 &= \frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{3hw}hw - 279h^2w^2 - 656e^{hw}h^2w^2 - 2538e^{2hw}h^2w^2 + 127e^{4hw}h^2w^2) \\ \beta_{34}^2 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{3hw}hw - 38h^2w^2 - 291e^{hw}h^2w^2 - 444e^{2hw}h^2w^2 - 127e^{3hw}h^2w^2) \\ \beta_{41}^2 &= -\frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{4hw}hw - 458h^2w^2 - 3312e^{2hw}h^2w^2 - 424e^{3hw}h^2w^2 - 846e^{4hw}h^2w^2) \\ \beta_{42}^2 &= -\frac{1}{360\xi_4 h} (2160 - 2160e^{hw} + 2160e^{4hw}hw + 141h^2w^2 - 3312e^{hw}h^2w^2 - 3948e^{3hw}h^2w^2 - 441e^{4hw}h^2w^2) \\ \beta_{43}^2 &= -\frac{1}{360\xi_4 h} (1440 - 1440e^{hw} + 1440e^{4hw}hw - 564h^2w^2 + 424e^{hw}h^2w^2 - 3948e^{2hw}h^2w^2 - 952e^{4hw}h^2w^2) \\ \beta_{44}^2 &= -\frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{4hw}hw + 97h^2w^2 - 846e^{hw}h^2w^2 + 441e^{2hw}h^2w^2 - 952e^{3hw}h^2w^2) \end{aligned}$$

where

$$\begin{aligned}
\eta_1 &= -\frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360hw + 502e^{hw}h^2w^2 - 621e^{2hw}h^2w^2 + 396e^{3hw}h^2w^2 \\
&\quad - 97e^{4hw}h^2w^2) \\
\eta_2 &= \frac{1}{12\xi_4} (12 - 24e^{hw} + 12e^{2hw} - 14e^{hw}h^2w^2 + 5e^{2hw}h^2w^2 \\
&\quad - 4e^{3hw}h^2w^2 + e^{4hw}h^2w^2) \\
\eta_3 &= \frac{1}{12\xi_4} (24 - 36e^{hw} + 12e^{3hw} - 29e^{hw}h^2w^2 - 9e^{3hw}h^2w^2 \\
&\quad + 2e^{4hw}h^2w^2) \\
\eta_4 &= \frac{1}{6\xi_4} (18 - 24e^{hw} + 6e^{4hw} - 22e^{hw}h^2w^2 - 3e^{2hw}h^2w^2 - 12e^{3hw}h^2w^2 \\
&\quad + e^{4hw}h^2w^2) \\
\eta_5 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{hw}hw - 323e^{hw}h^2w^2 + 264e^{2hw}h^2w^2 - 159e^{3hw}h^2w^2 \\
&\quad + 38e^{4hw}h^2w^2) \\
\eta_6 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{2hw}hw - 458e^{hw}h^2w^2 - 21e^{2hw}h^2w^2 - 84e^{3hw}h^2w^2 \\
&\quad + 23e^{4hw}h^2w^2) \\
\eta_7 &= \frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{3hw}hw - 443e^{hw}h^2w^2 - 216e^{2hw}h^2w^2 - 279e^{3hw}h^2w^2 \\
&\quad + 38e^{4hw}h^2w^2) \\
\eta_8 &= -\frac{1}{360\xi_4 h} (360 - 360e^{hw} + 360e^{4hw}hw - 458e^{hw}h^2w^2 - 141e^{2hw}h^2w^2 - 564e^{3hw}h^2w^2 \\
&\quad - 97e^{4hw}h^2w^2)
\end{aligned} \tag{3.75}$$

$$\tau(h) = (\tau_1, \dots, \tau_N, 0, 0, \dots, 0, h\tau'_1, \dots, h\tau'_N, 0, 0, \dots, 0)^T$$

τ_n are all the entries of C_{p+2}^k in (3.67).

$$Y = (y(x_1), \dots, y(x_N), hy^0(x_1), \dots, hy^0(x_N), 0, 0, \dots, 0)^T$$

$$F = (f(x_1), \dots, f(x_N), hf^0(x_1), \dots, hf^0(x_N), 0, 0, \dots, 0)^T$$

—

Theorem 3.4.2. (Jator and Manathunga, 2018) Let Y be an approximation of the solution vector Y for the system obtained on a partition $\pi_N = \{a = x_0 < x_1 < x_2 < \dots <$

$x_{N-1} < x_N = b\}$ and $E = Y - Y$. Define $e_i = |y(x_i) - y_i|$, $he'_i = |hy'(x_i) - hy'_i|$ for $i = 1, \dots, N$, where the exact solution $y(x) \in C^2[a, b]$. Then the block method (3.85) is a $k + 2$ -order convergent method. That is $\|E\|_\infty = O(h^{k+2})$.

Proof. The exact form of any of the systems (3.36), (3.41), (3.45)-(3.48) or (3.52)-(3.52) is defined as in (3.85), while the approximate form is defined by

$$AY - h^2BF(Y) + C = 0 \quad (3.76)$$

subtracting (3.85) from (3.76), one obtains

$$A(Y - Y) - h^2B(F(Y) - F(Y)) = \tau(h) \quad (3.77)$$

using the fact that $E = Y - Y$, we can write (3.77) as

$$AE - h^2B(F(Y) - F(Y)) = \tau(h) \quad (3.78)$$

$$F(Y) = F(Y) + J_F(Y)(Y - Y) + O\|Y - Y\| \quad (3.79)$$

where J_F is a Jacobian matrix whose approximate value is $\frac{F(\bar{Y}) - F(Y)}{\bar{Y} - Y} = J_F(Y)$. Without loss of generality, the first derivative term is used so that

$$F(Y) - \bar{F}(Y) = J_F(Y)(Y - \bar{Y}) = J_F(Y)E = J_F E \quad (3.80)$$

where

$$J_F = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as $J_{11} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \dots & \frac{\partial f_N}{\partial y_N} \end{pmatrix};$

$$J_{12} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1''} & \dots & \frac{\partial f_1}{\partial y_N''} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1''} & \dots & \frac{\partial f_N}{\partial y_N''} \end{pmatrix}$$

$$J_{21} = h^2 \begin{pmatrix} \frac{\partial f_1''}{\partial y_1'} & \dots & \frac{\partial f_1''}{\partial y_N'} \\ \vdots & & \vdots \\ \frac{\partial f_N''}{\partial y_1'} & \dots & \frac{\partial f_N''}{\partial y_N'} \end{pmatrix}$$

$$J_{22} = h^i \begin{pmatrix} \frac{\partial f_1''}{\partial y_1''} & \dots & \frac{\partial f_1''}{\partial y_N''} \\ \vdots & & \vdots \\ \frac{\partial f_N''}{\partial y_1''} & \dots & \frac{\partial f_N''}{\partial y_N''} \end{pmatrix}_N$$

Substituting (3.80) in (3.78) to obtain

$$AE - h^2 B J_F E = \tau(h) \quad (3.81)$$

$$(A - h^2 B(J_F))E = \tau(h) \quad (3.82)$$

Let $\Gamma = -h^2 B(J_F)$ be a matrix of dimension $2N$ so that (3.82) becomes

$$(A + \Gamma)E = \tau(h) \quad (3.83)$$

and for sufficiently small h , $A + \Gamma$ is a monotone matrix and thus nonsingular (Jain and Aziz, 1983). Hence

$$\begin{aligned} (A + \Gamma)^{-1} &= D = (d_{ij}) \geq 0 \\ \sum_{j=1}^{2\Gamma} d_{ij} &= O(h^{-2}), \\ E &= DL(h), \\ ||E|| &= ||DL(h)|| = O(h^{-2})O(h^{k+2}) \\ &= O(h^k) \end{aligned} \quad (3.84)$$

which shows that the method is k^{th} order convergent, that is, with a global error of order

$$O(h^k) \quad \square$$

Convergence of the HENIS

In this section, the convergence of the HENIS is discussed. The 2-step block methods (3.52)-(3.55) can be written in the form

$$AY - h^2 BF(Y) + C + \tau(h) = 0 \quad (3.85)$$

where A is a $2N \times 2N$ matrix defined by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Where the entries of A are $N \times N$ matrices given as

For the hybrid method where $k=2$

$$\begin{array}{c}
 \begin{array}{cc}
 \boxed{?} & \boxed{?} \\
 0 & \\
 \boxed{?} & \\
 \boxed{?} & \\
 & 0
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \boxed{?} \beta_{14ij} \\
 \boxed{?} \boxed{?} \\
 \boxed{?} \beta_{24ij}
 \end{array}$$

for i

$$= j = 1$$

$$\begin{aligned}\beta_{11}^{ij} &= \frac{(360-360e^{hw}+360hw+89h^2w^2+186e^{\frac{hw}{2}}h^2w^2-128e^{\frac{3hw}{2}}h^2w^2+33e^{2hw}h^2w^2)}{90(-1+e^{\frac{hw}{2}})^4hw^2} \\ \beta_{12}^{ij} &= \frac{(180-180e^{hw}+180hw+29h^2w^2+62e^{\frac{hw}{2}}h^2w^2-2e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2)}{30(-1+e^{\frac{hw}{2}})^4hw^2} \\ \beta_{13}^{ij} &= -\frac{(360-360e^{hw}+360hw+57h^2w^2+128e^{\frac{hw}{2}}h^2w^2-6e^{hw}h^2w^2+e^{2hw}h^2w^2)}{90(-1+e^{\frac{hw}{2}})^4hw^2} \\ \beta_{14}^{ij} &= \frac{(90-90e^{hw}+90hw+14h^2w^2+33e^{\frac{hw}{2}}h^2w^2-3e^{hw}h^2w^2+e^{\frac{3hw}{2}}h^2w^2)}{90(-1+e^{\frac{hw}{2}})^4hw^2} \\ \beta_{21}^{ij} &= \frac{(-12+24e^{hw}-12e^{2hw}+h^2w^2+10e^{hw}h^2w^2+e^{2hw}h^2w^2)}{3(-1+e^{\frac{hw}{2}})^4w^2} \\ \beta_{22}^{ij} &= \frac{(18-36e^{hw}+18e^{2hw}+h^2w^2-10e^{\frac{hw}{2}}h^2w^2-10e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2)}{3(-1+e^{\frac{hw}{2}})^4w^2} \\ \beta_{23}^{ij} &= -\frac{(-12+24e^{hw}-12e^{2hw}+h^2w^2+10e^{hw}h^2w^2+e^{2hw}h^2w^2)}{3(-1+e^{\frac{hw}{2}})^4w^2} \\ \beta_{24}^{ij} &= \frac{(3-6e^{hw}+3e^{2hw}-e^{\frac{hw}{2}}h^2w^2-e^{hw}h^2w^2-e^{\frac{3hw}{2}}h^2w^2)}{3(-1+e^{\frac{hw}{2}})^4w^2}\end{aligned}$$

for i

$$= 2, j = 1$$

$$\begin{aligned}\beta_{11}^{ij} &= \frac{\left(-360+360e^{hw}-360e^{hw}hw+31h^2w^2+234e^{\frac{hw}{2}}h^2w^2-112e^{\frac{3hw}{2}}h^2w^2+27e^{2hw}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{12}^{ij} &= \frac{\left(90-90e^{hw}+90e^{hw}hw+2h^2w^2-39e^{\frac{hw}{2}}h^2w^2-11e^{\frac{3hw}{2}}h^2w^2+3e^{2hw}h^2w^2\right)}{15\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{13}^{ij} &= -\frac{\left(360-360e^{hw}+360e^{hw}hw-3h^2w^2-112e^{\frac{hw}{2}}h^2w^2-66e^{hw}h^2w^2+e^{2hw}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{14}^{ij} &= \frac{\left(90-90e^{hw}+90e^{hw}hw-h^2w^2-27e^{\frac{hw}{2}}h^2w^2-18e^{hw}h^2w^2+e^{\frac{3hw}{2}}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{21}^{ij} &= \frac{\left(-360+360e^{hw}-360e^{2hw}hw+31h^2w^2+294e^{hw}h^2w^2+128e^{\frac{3hw}{2}}h^2w^2+87e^{2hw}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{22}^{ij} &= -\frac{\left(180+180e^{hw}-180e^{2hw}hw-9h^2w^2+98e^{\frac{hw}{2}}h^2w^2+162e^{\frac{3hw}{2}}h^2w^2+19e^{2hw}h^2w^2\right)}{30\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{23}^{ij} &= \frac{\left(360+360e^{hw}-360e^{2hw}hw+63h^2w^2-128e^{\frac{hw}{2}}h^2w^2+486e^{hw}h^2w^2+119e^{2hw}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{24}^{ij} &= \frac{\left(90-90e^{hw}+90e^{2hw}hw+14h^2w^2-87e^{\frac{hw}{2}}h^2w^2+57e^{hw}h^2w^2-119e^{\frac{3hw}{2}}h^2w^2\right)}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2}\end{aligned}$$

$$=, j = 2$$

for i

$$\begin{aligned}
\beta_{11}^{ij} &= -\frac{\left(-96+192e^{\frac{hw}{2}}-96e^{hw}+7h^2w^2+32e^{hw}h^2w^2-20e^{\frac{3hw}{2}}h^2w^2+5e^{2hw}h^2w^2\right)}{48\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{12}^{ij} &= -\frac{\left(288-576e^{\frac{hw}{2}}+288e^{hw}-5h^2w^2-64e^{\frac{hw}{2}}h^2w^2-4e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2\right)}{96\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{13}^{ij} &= -\frac{\left(48+96e^{\frac{hw}{2}}-48e^{hw}+h^2w^2+10e^{\frac{hw}{2}}h^2w^2+e^{hw}h^2w^2\right)}{24\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{14}^{ij} &= -\frac{\left(48+96e^{\frac{hw}{2}}-48e^{hw}+h^2w^2+10e^{\frac{hw}{2}}h^2w^2+e^{hw}h^2w^2\right)}{96\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{21}^{ij} &= -\frac{\left(-48+144e^{hw}-96e^{\frac{3hw}{2}}+4h^2w^2+39e^{hw}h^2w^2-10e^{\frac{3hw}{2}}h^2w^2+3e^{2hw}h^2w^2\right)}{24\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{22}^{ij} &= \frac{\left(96-288e^{hw}+192e^{\frac{3hw}{2}}+5h^2w^2-52e^{\frac{hw}{2}}h^2w^2-32e^{\frac{3hw}{2}}h^2w^2+7e^{2hw}h^2w^2\right)}{32\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{23}^{ij} &= \frac{\left(-96+288e^{hw}-192e^{\frac{3hw}{2}}+3h^2w^2+20e^{\frac{hw}{2}}h^2w^2+48e^{hw}h^2w^2+e^{2hw}h^2w^2\right)}{48\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\beta_{24}^{ij} &= -\frac{\left(1+2e^{\frac{hw}{2}}\right)\left(-48+96e^{\frac{hw}{2}}-48e^{hw}+h^2w^2+10e^{\frac{hw}{2}}h^2w^2+e^{hw}h^2w^2\right)}{96\left(-1+e^{\frac{hw}{2}}\right)^4w^2}
\end{aligned}$$

for i

$$= j = 2$$

$$\begin{aligned}\beta_{11}^{ij} &= \frac{\left(-2880+2880e^{hw}-2880e^{\frac{hw}{2}}hw+113h^2w^2-78e^{hw}h^2w^2-56e^{\frac{3hw}{2}}h^2w^2+21e^{2hw}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{12}^{ij} &= -\frac{\left(-1440+1440e^{hw}-1440e^{\frac{hw}{2}}hw+63h^2w^2-26e^{\frac{hw}{2}}h^2w^2-54e^{\frac{3hw}{2}}h^2w^2+17e^{2hw}h^2w^2\right)}{240\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{13}^{ij} &= -\frac{\left(2880+2880e^{hw}-2880e^{\frac{hw}{2}}hw+99h^2w^2+56e^{\frac{hw}{2}}h^2w^2-162e^{hw}h^2w^2+7e^{2hw}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{14}^{ij} &= -\frac{\left(-720+720e^{hw}-720e^{\frac{hw}{2}}hw+23h^2w^2+21e^{\frac{hw}{2}}h^2w^2-51e^{hw}h^2w^2+7e^{\frac{3hw}{2}}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{21}^{ij} &= \frac{\left(-2880+2880e^{hw}-2880e^{\frac{3hw}{2}}hw+233h^2w^2+2562e^{hw}h^2w^2-56e^{\frac{3hw}{2}}h^2w^2+141e^{2hw}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{22}^{ij} &= \frac{\left(1440-1440e^{hw}+1440e^{\frac{3hw}{2}}hw+97h^2w^2-854e^{\frac{hw}{2}}h^2w^2-826e^{\frac{3hw}{2}}h^2w^2+143e^{2hw}h^2w^2\right)}{240\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{23}^{ij} &= \frac{\left(-2880+2880e^{hw}-2880e^{\frac{3hw}{2}}hw+219h^2w^2+56e^{\frac{hw}{2}}h^2w^2+2478e^{hw}h^2w^2+127e^{2hw}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\ \beta_{24}^{ij} &= -\frac{\left(720+720e^{hw}-720e^{\frac{3hw}{2}}hw+23h^2w^2+141e^{\frac{hw}{2}}h^2w^2+429e^{hw}h^2w^2+127e^{\frac{3hw}{2}}h^2w^2\right)}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2}\end{aligned}$$

for i

$$C = (1/2, \eta_1, -1/2, \eta_2, -1/h, \eta_3, -1, \eta_4, -1/h, \eta_5, -1/h, \eta_6, -1/h, \eta_7, -1/h, \eta_8, 0, 0, \dots, 0)^T$$

where

$$\begin{aligned}
\eta_1 &= -\frac{\left(-90+90e^{hw}-90hw-89e^{\frac{hw}{2}}h^2w^2+87e^{hw}h^2w^2-57e^{\frac{3hw}{2}}h^2w^2+14e^{2hw}h^2w^2\right)f_n}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\
\eta_2 &= -\frac{\left(48-96e^{\frac{hw}{2}}+48e^{hw}-14e^{\frac{hw}{2}}h^2w^2+5e^{hw}h^2w^2-4e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2\right)f_n}{96\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\eta_3 &= \frac{\left(48-144e^{hw}+96e^{\frac{3hw}{2}}-16e^{\frac{hw}{2}}h^2w^2-15e^{hw}h^2w^2-6e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2\right)f_n}{96\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\eta_4 &= \frac{\left(3-6e^{hw}+3e^{2hw}-e^{\frac{hw}{2}}h^2w^2-e^{hw}h^2w^2-e^{\frac{3hw}{2}}h^2w^2\right)f_n}{3\left(-1+e^{\frac{hw}{2}}\right)^4w^2} \\
\eta_5 &= \frac{\left(720-720e^{hw}+720e^{\frac{hw}{2}}hw-113e^{\frac{hw}{2}}h^2w^2+189e^{hw}h^2w^2-99e^{\frac{3hw}{2}}h^2w^2+23e^{2hw}h^2w^2\right)f_n}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\
\eta_6 &= \frac{\left(90-90e^{hw}+90e^{hw}hw-31e^{\frac{hw}{2}}h^2w^2-12e^{hw}h^2w^2-3e^{\frac{3hw}{2}}h^2w^2+e^{2hw}h^2w^2\right)f_n}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\
\eta_7 &= \frac{\left(720-720e^{hw}+720e^{\frac{3hw}{2}}hw-233e^{\frac{hw}{2}}h^2w^2-291e^{hw}h^2w^2-219e^{\frac{3hw}{2}}h^2w^2+23e^{2hw}h^2w^2\right)f_n}{720\left(-1+e^{\frac{hw}{2}}\right)^4hw^2} \\
\eta_8 &= -\frac{\left(-90+90e^{hw}-90e^{2hw}hw+31e^{\frac{hw}{2}}h^2w^2+27e^{hw}h^2w^2+63e^{\frac{3hw}{2}}h^2w^2+14e^{2hw}h^2w^2\right)f_n}{90\left(-1+e^{\frac{hw}{2}}\right)^4hw^2}
\end{aligned} \tag{3.86}$$

$$\tau(h) = (\tau_1, \dots, \tau_N, 0, 0, \dots, 0, h\tau'_1, \dots, h\tau'_N, 0, 0, \dots, 0)^T$$

τ_n are all the entries of C_{p+2} in (3.71).

$$Y = (y(x_1), \dots, y(x_N), hy^0(x_1), \dots, hy^0(x_N), 0, 0, \dots, 0)^T$$

$$F = (f(x_1), \dots, f(x_N), hf^0(x_1), \dots, hf^0(x_N), 0, 0, \dots, 0)^T$$

Thus, the above Theorem 3.4.2 for the convergence is of the HENIS is proved as

follows.

Proof. Also following the same proof as above, the exact form of any of the systems (3.52)-(3.55) is defined as in (3.85), while the approximate form is defined by

$$AY - h^2 BF(Y) + C = 0 \quad (3.87)$$

subtracting (3.85) from (3.87), one obtains

$$A(Y - \bar{Y}) - h^2 B(F(Y) - F(\bar{Y})) = \tau(h) \quad (3.88)$$

using the fact that $E = Y - \bar{Y}$, we can write (3.88) as

$$AE - h^2 B(F(Y) - F(\bar{Y})) = \tau(h) \quad (3.89)$$

$$F(Y) = F(\bar{Y}) + J_F(\bar{Y})(Y - \bar{Y}) + O(\|Y - \bar{Y}\|) \quad (3.90)$$

where J_F is a Jacobian matrix whose approximate value is $\frac{F(\bar{Y}) - F(Y)}{\bar{Y} - Y} = J_F(Y)$. Without loss of generality, the first derivative term is used so that

$$F(Y) - F(\bar{Y}) = J_F(Y)(Y - \bar{Y}) = J_F(Y)E = J_F E \quad (3.91)$$

where

$$J_F = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as

$$J_{11} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{pmatrix}; \quad J_{12} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1''} & \cdots & \frac{\partial f_1}{\partial y_N''} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1''} & \cdots & \frac{\partial f_N}{\partial y_N''} \end{pmatrix}; \quad J_{21} = h^2 \begin{pmatrix} \frac{\partial f_1''}{\partial y_1'} & \cdots & \frac{\partial f_1''}{\partial y_N'} \\ \vdots & & \vdots \\ \frac{\partial f_N''}{\partial y_1'} & \cdots & \frac{\partial f_N''}{\partial y_N'} \end{pmatrix};$$

$$J_{22} = h^2 \begin{pmatrix} \frac{\partial f_1''}{\partial y_1''} & \cdots & \frac{\partial f_1''}{\partial y_N''} \\ \vdots & & \vdots \\ \frac{\partial f_N''}{\partial y_1''} & \cdots & \frac{\partial f_N''}{\partial y_N''} \end{pmatrix}$$

Substituting (3.91) in (3.89) to obtain

$$AE - h^2 B J_F E = \tau(h) \quad (3.92)$$

$$(A - h^2 B(J_F))E = \tau(h) \quad (3.93)$$

Let $\Gamma = -h^2 B(J_F)$ be a matrix of dimension $2N$ so that (3.93) becomes

$$(A + \Gamma)E = \tau(h) \quad (3.94)$$

and for sufficiently small h , $A + \Gamma$ is a monotone matrix and thus nonsingular (Jain and

Aziz, 1983). Hence

$$\begin{aligned} (A + \Gamma)^{-1} &= D = (d_{ij}) \geq 0 \\ \sum_{j=1}^{2\Gamma} d_{ij} &= O(h^{-2}) \\ E &= DL(h) \\ ||E|| &= ||DL(h)|| = O(h^{-2})O(h^7) \\ &= O(h^5) \end{aligned} \quad (3.95)$$

which shows that the method is 5th order convergent, that is, with a global error of order

$O(h^k)$ $\square$

3.4.7 Implementation of Method

The methods are implemented follows: Using method (3.59), $n = 0$, solve for the values of y_1 with the aid of Newton's Method on the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP (3.1). Next, for $n = 1$, the values of y_2 is obtained over the sub-interval $[x_1, x_2]$, as y_1 is known from the previous sub-interval. The process is continued for $n = 2, \dots, N - 1$ to obtain the numerical solution of (3.1) on the sub-intervals $[x_2, x_3]$, $[x_3, x_4], \dots, [x_{N-1}, x_N]$. The details of the implementation is given in Algorithm below.

In what follows in the next section, we shall discuss the MOL for the purpose of extending the result of this study to IBVPs in PDEs.

Algorithm 1 Block Algorithm for ENIS

- 1 begin procedure ENTER Partitions ($a, b, N, h, \text{variables}$)
 - 2 For $x_n = x_{n-1} + h, n = 1, \dots, N, h = \frac{b-a}{N}$
 - 3 Generate block system
 - 4 Solve [Sysytem, variables]
 - 5 Obtain y_n
 - 6 end procedure
-

3.5 Extension of Methods to IBVPs

3.5.1 Method of Lines (MOLs)

Using the Method of Lines (MOLs) approach in which for real numbers a, b, c, d , we seek a solution in the strip rectangle $[a, b] \times [c, d]$ by first fixing the grid in the spatial variable x ; then, approximating this spatial derivative using central difference methods, and finally solving the resulting system of second order time independent ODEs in the spatial variable t . We consider the two-dimensional PDE of the form

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} + p(t, x) \frac{\partial y}{\partial x} + q(t, x) \frac{\partial y}{\partial t} + r(t, x)y = g(t, x) \quad (3.96)$$

subject to Dirichlet or Neumann boundary conditions, where $y(t, x)$ is the dependent variable, x and t are the independent or spatial variables and $g(t, x)$ is a forcing term. When $r(t, x) = 0$, (3.96) becomes the two dimensional convection diffusion equation. For $p(t, x) = q(t, x) = r(t, x) = g(t, x) = 0$, (3.96) becomes the Laplace equation, for $p(t, x) = q(t, x) = r(t, x) = 0$, it is the Poisson equation and finally for $p(t, x) = q(t, x) = g(t, x) = 0$ it is the Helmholtz equation.

Specifically, we discretize the x variable such that with mesh spacings $\delta t = (c - d)/M$, $t_m = m\delta t$, $m = 0, 1, \dots, M$. We define $y_m(t) \approx u(t_m, x)$, $y = [y_1(x), \dots, y_m(x)]^T$, $g_m(t) \approx g(t_m, x)$, $g = [g_1(x), \dots, g_m(x)]^T$, and replace the partial derivatives $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ by their corresponding central difference approximations

$$\frac{\partial y(t_m, x)}{\partial t} = \frac{y(t_{m+1}, x) - y(t_{m-1}, x)}{2\delta x} \quad (3.97)$$

and

$$\frac{\partial^2 y(t_m, x)}{\partial t^2} = \frac{y(t_{m+1}, x) - 2y(t_m, x) + y(t_{m-1}, x))}{(\delta x)^2} \quad (3.98)$$

$m = 1, \dots, M-1$. The problem (3.96) then leads to the resulting semi-discrete problem

$$\frac{d^2 y_m}{dx^2} = -\frac{y_{m+1} - 2y_m + y_{m-1}}{(\delta t)^2} - \frac{q(t_m, x)y_{m+1} - y_{m-1}}{2\delta t} - p(t_m, x)\frac{dy_m}{dx} - r(x_m, x)y_m + g_m \quad (3.99)$$

which can be written in the form

$$y^{00} = f(x, y, y^0) \quad (3.100)$$

subject to the boundary conditions $y(c) = y_0$, $y(d) = y_M$, or $y'(c) = y'_0$, $y'(d) = y'_M$,

where $f(x, y, y^0) = Ay + g$, g is a vector of constants, and A is an $(M - 1) \times (M - 1)$ matrix arising from the semi-discretized system (3.99) which is expressed in the form

(3.100).

Chapter 4

NUMERICAL RESULTS AND DISCUSSION

In this section, we give numerical examples to illustrate the accuracy of the method. Let $y(x_n)$ be the exact solution and y_n the approximate solution on the partition π_N . We find the absolute error of the approximate solution as $|y(x_n) - y_n|$

4.1 Numerical Examples for IVPs

Example 1. We consider the singular IVP (Jator and Coleman, 2017).

$$\begin{cases} y' - y^2 = 1, & 0 \leq x \leq 1 \\ y(0) = 1 \end{cases} \quad (4.1)$$

The analytic solution is given by $y(x) = \tan\left(x + \frac{\pi}{4}\right)$. With $N = 20$.

Example 2. We consider the linear IVP, (Ayinde & Ibijola, 2015).

$$\begin{cases} y' - y = x^2, & 0 \leq x \leq 1 \\ y(0) = 1 \end{cases} \quad (4.2)$$

The analytic solution is given by $y(x) = -2 - 2x - x^2 + 3e^x$. With $N = 10$.

Example 3. We consider a linear system of IVP (Yakubu *et al.*, 2018).

$$\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} -2y_1(x) + y_2(x) \\ 998y_1(x) - 999y_2(x) \end{pmatrix} + \begin{pmatrix} 2\sin(x) \\ 999(\cos(x) - \sin(x)) \end{pmatrix}, \quad \begin{matrix} y_1(0) = 2 \\ y_2(0) = 1 \end{matrix} \quad (4.3)$$

The exact solution is

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 2e^{-x} + \sin(x) \\ 2e^{-x} - \cos(x) \end{pmatrix} \quad (4.4)$$

Example 4. We consider the following system of first order IVP (Ramos and Popescu, 2018)

$$\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2, y_1(0) = 1 \\ y_2' &= y_1 - y_2(1 + y_2), y_2(0) = 1 \end{aligned} \quad (4.5)$$

The exact solution are; $y_1(x) = e^{-2x}$, $y_2(x) = e^{-x}$.

Example 5. We consider a first order nonlinear singular IVP (Method in Kamrul *et al.*, 2014) ;

$$y' = -\frac{y^2}{\sqrt{x}}, \quad y(0) = 1, \quad 0 < x \leq 1 \quad (4.6)$$

The exact solution is $y = \frac{1}{1+2\sqrt{x}}$.

Solving the initial value problem, we adopt the same value for the step-size $h = 1/12000000$

Example 6. We consider a second order nonlinear singular IVP (Method in Kamrul *et al.*, 2014) ;

$$y'' + \frac{2}{x}y' + 4(2e^x + e^{\frac{y}{2}}) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad 0 < x \leq 1 \quad (4.7)$$

Decompose (4.7) into two first order initial value problems we have;

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 0, \\ y_2' &= \frac{2}{x}y_2 - 4(2e^x + e^{\frac{y_1}{2}}) \quad y_2(0) = 0, \quad 0 < x \leq 1 \end{aligned} \quad (4.8)$$

where the exact solutions are $y_1 = -2\ln(1 + x^2)$ and $y_2 = -\frac{4x}{(1+x^2)}$.

Example 7. We consider a system of first order linear singular IVP (Komashynska *et al.*, 2016) ;

$$\begin{aligned} y_1' + \left(\sin(x) - \frac{1}{x \sinh(x)} \right) y_1 + \left(\frac{2}{x \sinh(x)} - 3 \cos(x) \right) y_2 &= f_1(x), \quad y_1(0) = 0 \\ y_2' - \ln(x^2 + 1) y_1 + 4 \sin(e^x) y_2 &= f_2(x), \quad y_2(0) = 0 \end{aligned} \quad (4.9)$$

where $f_1 = ((1 + \sin(x))x^2 + 2x)e^x + \frac{2xe^x + \sin(x)}{\sinh(x)} - 3x \cos(x) \sin(x)$, $f_2 = (1 + 4x \sin(e^x)) \sin(x) + x \cos(x) - x^2 e^x \ln(x^2 + 1)$, $x \in [0, 2]$ The exact solution are; $y_1(x) = x^2 e^x$ and $y_2(x) = x \sin(x)$.

4.2 Numerical Examples for BVPs

Example 8. Consider a linear BVP with Neumann boundary conditions (Siraj-ullislam *et al.*, 2010).

$$\begin{cases} y'' + (2 - 4x^2)y = 0, & 0 \leq x \leq 1 \\ y(0) = 1, \quad y'(1) = \frac{-2}{e} \end{cases} \quad (4.10)$$

The analytic solution is given by $y(x) = e^{-x^2}$; With $N = 10$.

Example 9. Consider a nonlinear BVP with Neumann boundary conditions (Siraj-ullslam *et al.*, 2010).

$$\begin{cases} y'' - 2y^3 = 0, & 0 \leq x \leq 1 \\ y(0) = 1, & y'(1) = -\frac{1}{4} \end{cases} \quad (4.11)$$

The analytic solution is given by $y(x) = \frac{1}{1+x}$. With $N = 10$.

Example 10. Consider the following differential equation for the temperature distribution of the radiation fin of trapezoidal profile (Siraj-ul-Islam *et al.*, 2010).

$$\begin{cases} y'' - y - \sin(2\pi x)(-1 - 4\pi^2) \left(x^3 - \frac{4}{3}x^2 + \frac{x}{3} \right) \\ - 6x - \frac{8}{3} \sin(2\pi x) - 4\pi \cos(2\pi x) \left(3x^2 - \frac{8}{3}x + \frac{1}{3} \right) = 0, & 0 \leq x \leq 1 \\ y(0) = y(1), & y'(0) = y'(1) \end{cases} \quad (4.12)$$

The analytic solution is given by $y(x) = \left(x^3 - \frac{4}{3}x^2 + \frac{x}{3} \right) \sin(2\pi x)$.

Example 11. We consider the BVP with Neumann boundary conditions (Islam and Shirin, 2011).

$$\begin{cases} y'' + y = -1, & 0 \leq x \leq 1 \\ y'(0) = \frac{1-\cos(1)}{\sin(1)}, & y'(1) = (1 - \cos(1)) \cot(1) - \sin(1) \end{cases} \quad (4.13)$$

The analytic solution is given by $y(x) = \cos(x) + \frac{1-\cos(1)}{\sin(1)} \sin(x) - 1$. With $N = 10$.

Example 12. Consider the linear second order boundary value problem (Adeyeye and Omar, 2016).

$$\begin{cases} y'' - y = \cos(x), & 0 \leq x \leq 1 \\ y(0) = 0, & y(1) = 1 \end{cases} \quad (4.14)$$

The analytic solution is given by $y(x) = \cos(x) - \cot(1)\sin(x) + 2\csc(1)\sin(x) - 1$. With $N = 10$.

Example 13. Consider the nonlinear second order boundary value problem (Islam and Shirin, 2011).

$$\begin{cases} y'' + \frac{1}{8}yy' = 4 + \frac{1}{4}x^3, & 1 \leq x \leq 3 \\ y(1) = 17, \quad y(3) = \frac{43}{3} \end{cases} \quad (4.15)$$

The analytic solution is given by $y(x) = x^2 + \frac{16}{x}$. With $N = 30$.

4.3 Numerical Examples for IBVPs

Example 14. Consider the stiff parabolic equation (Ngwane and Jator, 2014).

$$\begin{aligned} \kappa \frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial t} &= 0 \\ y(0, t) = y(1, t) &= 0, \quad y(x, 0) = \sin \pi x + \sin \omega \pi x, \quad \omega > 1 \end{aligned} \quad (4.16)$$

The exact solution is $y(x, t) = e^{-\pi^2 \kappa t} \sin \pi x + e^{-\omega^2 \pi^2 \kappa t} \sin \omega \pi x$

Example 15. We consider the Telegraph equation (Jator and Oladejo, 2017):

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + 2\pi \frac{\partial y}{\partial t} + \pi^2 y &= \pi^2 \sin \pi x (\sin \pi t + 2 \cos \pi t); \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1 \\ y(0, t) = 0, \quad 0 \leq x \leq 1 \quad y_t(x, 0) &= \pi \sin \pi x, \quad 0 \leq x \leq 1 \quad y(0, t) = 0, \quad 0 \leq t \leq 1 \end{aligned} \quad (4.17)$$

The exact solution is $y(x, t) = \sin \pi x (\sin \omega \pi x)$.

Example 16. We consider the PDE (Biala, 2016):

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} &= -32\pi^2 \sin(4\pi x), \quad x \in [0, 1] \\ y(\pm 1, t) = y(x, \pm 1) &= 0, \quad t > 0 \end{aligned} \quad (4.18)$$

The exact solution is $y(x, t) = \sin 4\pi x \sin 4\pi t$.

Example 17. We consider the singular hyperbolic differential equation (Jator, 2016)

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial^2 y}{\partial x^2} - \sin(y), \quad x \in [-3, 3] \\
y(x, 0) &= 4 \arctan(e^{\frac{x}{\sqrt{1-c^2}}}), \\
y_t(x, 0) &= -\frac{4ce^{\frac{x}{\sqrt{1-c^2}}}}{\sqrt{1-c^2}(1+e^{\frac{x}{\sqrt{1-c^2}}})}, \quad 0 < t < 1
\end{aligned} \tag{4.19}$$

The exact solution is $y(x, t) = 4\arctan(\text{sech}(x)t)$, c is the velocity of the solitary wave.

The problem is solved for $c = 0.5$, $\Delta t = 0.125$

4.4 Table of Results

Table 4.1: Numerical Results for Example 1.

x	EXACT	ENIS1	ENIS2	ENIS3	ENIS4	ENIS5
0.	0.	0	0	0	0	0
0.1	0.10033467208	0.1002555774	0.1003341351	0.1003346248	0.1003346714	0.1003346903
0.2	0.20271003550	0.2025492223	0.2027096847	0.2027099349	0.2027100276	0.2027100262
0.3	0.30933624960	0.3090801434	0.3093369904	0.3093360764	0.3093362343	0.3093362333
0.4	0.42279321873	0.4224131751	0.4227964179	0.4227929321	0.4227931978	0.4227931925
0.5	0.54630248984	0.5457460467	0.5463104790	0.5463020081	0.5463024633	0.5463023153
0.6	0.68413680834	0.6833100183	0.6841539028	0.6841359652	0.6841367806	0.6841366258
0.7	0.84228838046	0.8410192278	0.8423231240	0.8422868186	0.8422883648	0.8422879875
0.8	1.02963855705	1.0275967012	1.0297091017	1.0296354487	1.0296385875	1.0296380068
0.9	1.26015821755	1.2566639787	1.2603065483	1.2601514458	1.2601584042	1.2601553992
1.	1.55740772465	1.5509260215	1.5577422636	1.5573911225	1.5574084704	1.5574039899

Table 4.2: Error of Methods for Example 1.

x	ENISK1	ENISK2	ENISK3	ENISK4	ENISK5	PCM	SSM	RKM
0.0	0	0	0	0	0	0.00	0.00	0.00
0.1	7.91E-5	1.65E-6	4.72E-8	6.71E-10	1.83E-8	1.06E-6	1.03E-6	2.15E-8
0.2	1.61E-4	2.44E-6	1.01E-7	7.88E-9	9.23E-9	2.71E-6	2.45E-6	2.80E-8
0.3	2.56E-4	2.63E-6	1.73E-7	1.52E-8	1.63E-8	5.37E-6	4.68E-6	5.21E-7

0.4	3.80E-4	1.54E-5	2.87E-7	2.09E-8	2.62E-8	1.03E-5	8.71E-6	3.63E-6
0.5	5.56E-4	5.21E-5	4.82E-7	2.65E-8	1.74E-7	2.13E-5	1.75E-5	2.60E-5
0.6	8.27E-4	5.90E-5	8.43E-7	2.77E-8	1.82E-7	5.54E-5	4.41E-5	2.91E-4
0.7	1.30E-3	1.47E-4	1.56E-6	1.56E-8	3.93E-7	2.80E-4	2.14E-4	1.34E-2
0.8	2.04E-3	4.15E-4	3.12E-6	3.05E-8	5.50E-7	1.02E-2	7.37E-3	1.39E-3
0.9	3.49E-3	5.21E-4	6.77E-6	1.87E-7	2.82E-6	1.58E-4	1.18E-4	-
1.0	6.48E-3	1.49E-3	1.66E-5	7.46E-7	3.73E-6	4.43E-5	3.22E-5	-

Table 4.3: Error of Methods for Example 2.

xy	app	y_{ex}	ENIS1ENIS2ENIS3ENIS4ENIS5Ayinde&Ibijola
0.11.1055127541.1055127548.93E-41.77E-154.66E-136.83E-122.04E-102.45E-5			
0.21.2242082741.2242082748.12E-41.73E-141.26E-136.77E-121.12E-102.71E-5			
0.31.3595764221.3595764221.79E-37.11E-159.83E-139.26E-123.95E-102.99E-5			
0.41.5154740921.5154740921.81E-32.66E-142.301-141.20E-123.05E-103.31E-5			
0.51.6961638121.6961638122.88E-34.79E-141.012-123.69E-123.93E-103.65E-5			
0.61.9063564011.9063564013.02E-31.51E-141.48E-122.14E-126.24E-104.04E-5			
0.72.1512581222.1512581224.22E-37.11E-153.90E-131.48E-112.53E-94.46E-5			
0.82.4366227852.4366227854.49E-33.91E-141.10E-121.16E-111.60E-94.93E-5			
0.92.7688093332.7688093335.86E-36.75E-142.07E-121.58E-112.31E-95.45E-5			
1.03.1548454853.1548454856.31E-38.17E-142.52E-123.94E-111.84E-96.03E-5			

Table 4.3 shows the approximate exact solutions for Example 2. It also shows the errors for different methods and

that of Ayinde & Ibijola (2015). The proposed method is superior over the method Ayinde & Ibijola (2015), for $k = 2, 3, 4, 5$.

Table 4.4: ENISK1 absolute errors for Example 3.

x	y_1 Approx	y_1 Exact	y_2 Approx	y_2 Exact	Error 1	Error 2
5.0	-0.9457141250	-0.9454482458	0.2968713029	0.2971382487	2.65E-3	2.67E-3
50.0	-0.2625209222	-0.2623756565	0.9648192868	0.9649664445	1.46E-3	1.47E-3

Table 4.5: ENISK2 absolute errors for Example 3.

x	y_1 Approx	y_1 Exact	y_2 Approx	y_2 Exact	Error 1	Error 2
5.0	-0.9454508041	-0.9454485485	0.2971356637	0.2971385548	2.42E-6	2.42E-6
50.0	-0.2623986957	-0.2623755455	0.9649422793	0.9649664545	2.38E-5	2.37E-5

Table 4.6: ENISK3 absolute errors for Example 3.

x	y_1 Approx	y_1 Exact	y_2 Approx	y_2 Exact	Error 1	Error 2
5.0	-0.9454481741	-0.9454485454	0.2971382870	0.2971384454	2.07E-7	2.08E-7
50.0	-0.2623739956	-0.2623755454	0.9649668935	0.9649665456	8.58E-7	8.65E-7

Table 4.7: ENISK4 absolute errors for Example 3.

x	y_1 Approx	y_1 Exact	y_2 Approx	y_2 Exact	Error 1	Error 2
5.0	-0.9454483785	-0.9454486554	0.2971380815	0.2971385445	2.11E-9	2.09E-9
50.0	-0.26237481171	-0.2623755454	0.9649660702	0.9649664515	4.19E-8	4.17E-8

Table 4.8: ENISK5 absolute errors for Example 3.

x	y_1 Approx	y_1 Exact	y_2 Approx	y_2 Exact	Error 1	Error 2
5.0	-0.9454483808	-0.9454483808	0.2971380792	0.2971380792	2.15E-10	2.17E-10
50.0	-0.26237485419	-0.2623755419	0.9649660279	0.9649660279	4.92E-10	4.97E-10

Table 4.9: Error of Results for Example 3.

x	Error 1 of Method 1	Error 2 of Method 1	Error 1 of Method 2	Error 2 of Method 2
5.0	1.27E-3	1.35E-6	1.19E-3	1.26E-6
50.0	3.70E-5	1.10E-7	3.27E-5	1.02E-7

Table 4.10: Numerical and Analytic solution for Example 4.

x	y_1 approx	y_1 exact	y_2 approx	y_2 exact
0.1	0.81873	0.818731	0.904837	0.904837
0.2	0.67031	0.670319	0.818730	0.818730
0.3	0.54881	0.548811	0.740818	0.740818
0.4	0.44932	0.449329	0.670320	0.670320
0.5	0.36787	0.367879	0.606530	0.606531

Table 4.11: ErrorsofMethodsforExample4.

ENISK1			ENISK2		ENISK3		ENISK4		ENISK5	
xy	y_1 error	y_2 error	y_1 error	y_2 error	y_1 error	y_2 error	y_1 error	y_2 error	y_1 error	y_2 error
0.12.68E-11	112.86E-11		6.40E-132.98E-14		8.67E-154.06E-16		4.75E-181.94E-18		1.21E-205.59E-20	
0.22.51E-11	113.89E-11		5.70E-135.20E-14		6.49E-156.91E-16		9.16E-183.45E-18		9.21E-209.52E-21	
0.32.25E-11	115.41E-11		4.97E-136.75E-14		4.92E-158.91E-16		3.60E-184.42E-19		7.33E-201.22E-21	
0.41.33E-11	115.78E-11		4.29E-137.78E-14		3.74E-161.02E-16		1.20E-185.12E-19		6.39E-201.41E-20	
0.51.76E-11	116.49E-11		3.68E-138.41E-14		2.84E-151.10E-16		2.34E-184.62E-19		6.43E-201.52E-20	

Table 4.12: Maximum error obtained for Example 4.

Methods	$Max\{y_1, y_2\}$
---------	-------------------

ENISK1	6.49E-11
ENISK2	6.40E-13
ENISK3	8.67E-15
ENISK4	9.16E-18
ENISK5	9.21E-20
BLOCK10SIMP	1.00E-14
BLOCK10	1.00E-15
BLOCK2SIMP	1.00E-6
BLOCK2	1.00E-6

Table 4.12 shows the maximum error of y_1 and y_2 , that is, $Max\{y_1, y_2\}$. Here, in Ramos and Popescu (2018), BLOCK2 is the 2-step block method, BLOCK2SIMP is the simplest 2-step block method, BLOCK10 is the 10-step block method, BLOCK10SIMP is the simplest 10-step block method,

Table 4.13: Exact solution, approximate solution and error for Example 5.

x	y_{ex}	y_{approx}	Err_{K3}	Err_{K4}	Err_{K5}	Method in Kamrul <i>et al.</i>
0.00000025	0.99900100	0.99900100	3.4E-4	5.5E-5	3.2E-7	2.2E-4
0.00000075	0.99827094	0.99866081	1.3E-4	4.5E-5	1.7E-7	3.8E-4
0.00000175	0.99736123	0.99764797	1.4E-4	2.5E-5	7.7E-7	2.9E-4
0.00000375	0.99614196	0.99658331	3.3E-4	4.2E-5	3.1E-7	4.4E-4
0.00000775	0.99446306	0.99480680	1.1E-4	1.3E-5	3.4E-7	3.4E-4
0.00001575	0.99212525	0.99263906	1.0E-4	2.9E-5	1.7E-7	5.1E-4

We remark here that our method became superior from $k=3$ compared to the method in Kamrul *et al.* (2014).

Table 4.14: Exact and approximate solution for example 6.

x	$y_1 ex$	$y_1 app$	$y_2 ex$	$y_2 app$
0.01638375	-0.00053678	-0.00053678	-0.06551741	- 0.06551741

0.03276775	-0.00214630	-0.00214626	-0.13093042	-	0.13093043
0.06553575	-0.00857148	-0.00857084	-0.26102193	-	0.26102237
0.13107175	-0.03406780	-0.03405850	-0.51543198	-	0.51544538
0.26214375	-0.13292212	-0.13281912	-0.98115095	-	0.98150148
0.52428775	-0.48570042	-0.48573794	-1.64498217	-	1.65021644

Table 4.14 shows the exact and approximate solutions for y_1 and y_2 .

Table 4.15: Error for Example 6.

x	y_{1erK4}	y_{2erK4}	y_{1erK5}	y_{2erK5}	y_{1Kam}	y_{2Kam}
0.01638375	2.34E-6	1.11E-6	2.34E-7	4.32E-7	0.0	0.0
0.03276775	4.43E-6	3.42E-6	1.13E-7	1.11E-7	1.86E-3	7.64E-6
0.06553575	4.61E-6	6.54E-6	4.55E-7	4.31E-7	7.47E-3	1.69E-4
0.13107175	2.11E-6	2.28E-6	3.33E-7	4.54E-7	2.73E-2	2.60E-3
0.26214375	8.76E-6	9.88E-6	3.23E-7	3.14E-7	7.75E-2	3.57E-2
0.52428775	1.21E-6	5.67E-6	4.52E-7	5.43E-7	7.72E-3	3.18E-1

y_{1erK4} and y_{2erK4} are the errors obtained with our methods for $k=4$, while y_{1erK5} and y_{2erK5} are the errors obtained with our methods for $k=5$. y_{1Kam} and y_{2Kam} are the errors obtained in Kamrul *et al.*, (2014) for the solutions y_1 and y_2 .

Table 4.16: Error in y_1 for Example 7.

x	app_{10th}	app_{15th}	app_{20th}	Err_{K4}	Err_{K5}
0.2	5.76 E-14	0	0	0	0
0.6	1.06 E-8	3.33E-15	0	0	0
1.0	3.06 E-6	1.23E-11	4.44E-16	0	0
1.4	1.29 E-4	2.75E-9	1.06E-14	2.22E-17	4.33E-19

1.8	2.15 E-3	1.58E-7	2.06E-12	1.01E-16	3.21E-17
2.0	7.01 E-3	8.66E-7	1.91E-11	0	0

Example
Table 4.17: Error in y_2 for 7.

x	app_{10th}	app_{15th}	app_{20th}	Err_{K4}	Err_{K5}
0.2	1.04 E-16	0	0	0	0
0.6	5.44 E-13	2.22E-16	0	0	0
1.0	2.49 E-8	7.62E-13	0	0	0
1.4	1.40 E-6	1.65E-9	4.44E-16	0	1.45E-19
1.8	2.83 E-5	9.17E-9	7.99E-15	5.61E-17	1.81E-18
2.0	1.00 E-4	4.94E-8	8.15E-14	5.59E-17	4.67E-17

Table Numerical Results for Example

4.18: 8.

x	Exact	ENISK2	ENISK3	ENISK4
0.0	1.	0.9999957355634368	1.0000004790817458	0.999999987761327
0.1	0.9900498337491681	0.9900502643308879	0.9900502993342012	0.9900498156681743
0.2	0.9607894391523232	0.9607940478900681	0.9607898667562876	0.9607894160282915
0.3	0.9139311852712282	0.9139421186157146	0.9139315567243309	0.9139311544522981
0.4	0.8521437889662113	0.8521606179371466	0.8521440941660271	0.8521437510491455
0.5	0.7788007830714049	0.7788224045290519	0.7788010203888239	0.7788007404088794
0.6	0.697676326071031	0.6977026244957398	0.6976765014398049	0.6976762787362764
0.7	0.6126263941844161	0.6126540151529665	0.6126265191690115	0.6126263475128291
0.8	0.5272924240430485	0.5273216960312714	0.5272925133569318	0.5272923773596764
0.9	0.4448580662229411	0.4448855177781156	0.4448581352258902	0.4448580242025574
1.0	0.36787944117144233	0.36790577970620997	0.36787950381074214	0.36787940246618767

ENISK2 the derived method for $k = 2$, ENISK3 the derived method for $k = 3$ and ENISK4 the derivedmethod for $k = 4$.

Table 4.19: Maximum Errors for Example 8.

N	ENISK2	ENISK3	ENISK4	Siraj-ul-Islam <i>et al.</i> , 2010
16	4.43096E-6	1.98605E-5 (N=15)	1.14593E-7	2.9051E-4
32	2.76803E-7	1.21303E-6 (N=30)	1.37876E-8	7.4812E-5
64	1.72171E-8	1.11162E-7 (N=63)	3.63628E-9	1.8956E-5
128	2.30748E-8	4.08719E-7 (N=126)	5.97405E-10	4.7694E-6

Table Numerical Results for Example

In the ENIS, the number of subinterval N must be a multiple of the value of the step number (k). Since for ENISk3 N is not divisible by k , the following values (indicated in parenthesis in the table) $N = 15,30,63,126$ are used for the subinterval of integrations. From the above table, our method become superior over the method in Siraj-ul-Islam *et al.* (2010), for $k = 2,3,4$.

4.20: 9.

x	Exact	ENISk2	ENISk3	ENISk4
0.0	1	0.9999402011691245	0.9999991208411336	0.9999998678883298
0.1	0.9090909090909091	0.9090496649624987	0.9090900540030349	0.9090908184382793
0.2	0.8333333333333334	0.8333100676099695	0.8333325308215846	0.8333332812407208
0.3	0.7692307692307693	0.7692143335232808	0.7692300245383745	0.7692307294157241
0.4	0.7142857142857143	0.7142759744800682	0.7142850231684167	0.7142856852211318
0.5	0.6666666666666666	0.6666598221954336	0.6666660212562535	0.6666666419174813
0.6	0.6250000000000000	0.6249960303619259	0.6249993913355142	0.6249999789412057
0.7	0.5882352941176471	0.5882327044828676	0.5882347132176327	0.5882352747808625
0.8	0.5555555555555556	0.5555543581513043	0.5555549938367671	0.5555555377157348
0.9	0.5263157894736842	0.5263153333741466	0.5263152388759219	0.5263157725162025
1.	0.5000000000000000	0.5000003071493399	0.4999994529819388	0.4999999837357609

ENISk2 the derived method for $k = 2$, ENISk3 the derived method for $k = 3$ and ENISk4 the derived

method for $k = 4$.

Table 4.21: Maximum Errors for Example 9.

N	ENISk2	ENISk3	ENISk4	Siraj-ul-Islam <i>et al.</i> , 2010
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Table Numerical Results for Example

16	9.56116E-6	1.35429E-5 (N=15)	4.72952E-7	5.9492E-4
32	6.11904E-7	8.79159E-7 (N=30)	2.86368E-8	1.5545E-4
64	3.85091E-8	1.49798E-7 (N=60)	2.53342E-7	3.9757E-5
128	7.93903E-9	1.88489E-6 (N=126)	3.89912E-8	1.0055E-5

In the ENIS, the number of subinterval N must be a multiple of the value of the step number (k). Since for ENISK3 N is not divisible by k , the following values (indicated in parenthesis in the table) $N = 15, 30, 63, 126$ are used for the subinterval of integrations. From the above table, our method become superior over the method in Siraj-ul-Islam *et al.*, (2010), for $k = 2, 3, 4$.

4.22: 10.

x	Exact	ENISK2	ENISK3	ENISK4
0.	0.	0.00001472873464832	0.00001481445564429	4.296418700233004E-6
0.1	0.01234349029814193	0.01235634918728121	0.01235822196278201	0.012347875473090
0.2	0.02028920568096327	0.02030157420680915	0.02030354139055667	0.020293704092649
0.3	0.00665739561406607	0.00667049650328752	0.00667108422270768	0.006662308926448
0.4	-0.0094045640366795	-0.0093905677313092	-0.0093914156916137	-0.00939895610169
0.5	0.00000000000000000	0.00001408777339147	0.00001306668469782	5.813275864601304E-6
0.6	0.03761825614671828	0.03763157239416460	0.03763176814287214	0.037624148269740
0.7	0.07323135175472682	0.07324374197808987	0.07324556117711475	0.073237515963318
0.8	0.07101221988337147	0.07102461780516127	0.07102696229037027	0.071018912458609
0.9	0.02997704786691613	0.02999108006590673	0.02999193011581659	0.029984066623221
1.	0.00000000000000000	0.000016686061540131	0.00001481665939768	7.557379210815681E-6

ENISK2 the derived method for $k = 2$, ENISK3 the derived method for $k = 3$ and ENISK4 the derived

method for $k = 4$.

Table Numerical Results for Example

Table 4.23: Maximum Errors for Example 10.

N	ENISk2	ENISk3	ENISk4	Siraj-ul-Islam <i>et al.</i> , 2010
16	1.64071E-3	4.16905E-3 (N=15)	2.18317E-4	-
32	9.99569E-5	2.42177E-4 (N=30)	3.11129E-6	5.1818E-3
64	6.2073E-6	1.48822E-5 (N=60)	1.27741E-6	1.3008E-3
128	3.86763E-7	1.13562E-6 (N=126)	1.07049E-6	3.2608E-4

As stated in the previous two examples, the number of subinterval N must be a multiple of the value of the step number (k) in the ENIS. Since for ENISk3 N is not divisible by k , the following values (indicated in parenthesis in the table) $N = 15, 30, 63, 126$ are used for the subinterval of integrations.

From the above table, our method become superior over the method in Siraj-ul-Islam *et al.* (2010), for $k = 2, 3, 4$.

for Example
Table 4.24: Numerical Solution 11.

x	Exact	ENISK2	ENISK3	ENISK4
0.0	0.0000000000000000	2.342968382000441 E-6	1.633472747794000E-8	9.220233027749154E-10
0.1	0.049543409361800	0.049541205071352	0.049543392935187	0.049543410441439
0.2	0.088600127910183	0.088598075590590	0.088600111248742	0.088600129458214
0.3	0.116779913823847	0.116778082056380	0.116779896855429	0.116779915211972
0.4	0.133801203996942	0.133799601793022	0.133801186667072	0.133801205686380
0.5	0.139493927324549	0.139492610277504	0.139493909614661	0.139493929459764
0.6	0.133801203996942	0.133800175786305	0.133801185965773	0.133801206678424
0.7	0.116779913823847	0.116779214239142	0.116779895517970	0.116779916700439
0.8	0.088600127910183	0.088599754650427	0.088600109339576	0.088600131179990
0.9	0.049543409361800	0.049543384427101	0.049543390611437	0.049543413010573
1.0	0.0000000000000000	3.149078593803995E-7	1.882405014677961E-8	3.694897066707002E-9

ENISK2 the derived method for $k = 2$, ENISK3 the derived method for $k = 3$ and ENISK4 the derived

method for $k = 4$.

x	Bernoullipolynomial=5Bernoullipolynomial=7Bernoullipolynomial=10ENISK2ENISK3ENISK4
0.01.153937E-100.0000000000.0000000000.0000000000.0000000000.0000000000	
0.12.654280E-78.177960E-101.932482E-142.2568759521E-141.6549415461E-154.9652365263E-16	
0.29.870426E-78.626597E-112.672862E-145.2415641545E-145.4454564151E-158.4716341654E-16	
0.36.883117E-71.222063E-91.992850E-141.4846545654E-148.7569459466E-156.5962369456E-16	
0.45.349698E-79.031828E-111.013079E-143.6546546546E-146.6432165495E-155.0421548956E-16	
0.51.173569E-61.280755E-92.942091E-144.5465465465E-134.4946525134E-151.5214584154E-16	
0.65.349698E-79.031831E-111.010303E-148.5454546546E-145.1248964165E-151.0054545415E-16	
0.76.883117E-71.222063E-91.981748E-147.7489789859E-141.4978846452E-155.4646546513E-16	
0.89.870426E-78.626594E-112.670086E-148.6313597426E-147.3598921323E-152.2323787893E-16	
0.92.654280E-78.177957E-101.942890E-141.4685979663E-136.7652631645E-158.5464049746E-16	
1.00.0000000000.0000000000.0000000000.0000000000.8216892315E-148.6526541634E-151.1111655968E-16	

Islam and Shirin (2011) used the Bernoulli polynomials = 5, 7 and 10.

From the above table, ENISK2 ($k = 2$) is very competitive with methods in Islam and Shirin (2011). Our method become superior over the method theirs for $k = 3$ and 4.

Table 4.26: Numerical Solutions for Example 12.

x	Exact	ENISK2	ENISK3	ENISK4
0	0	0	0	0
0.1	0.059343034025940	0.059343034022515	0.059343087672044	0.059343033921229
0.2	0.110134207176555	0.110134207170058	0.110134310516337	0.110134206998409
0.3	0.151024408862577	0.151024408853471	0.151024555908033	0.151024408745217
0.4	0.180475345562389	0.180475345551271	0.180475527849810	0.180475345616859
0.5	0.196734670143683	0.196734670131303	0.196734876217296	0.196734670289925
0.6	0.197807972378616	0.197807972365905	0.197808187186196	0.197807973070783
0.7	0.181427245522797	0.181427245510896	0.181427449709712	0.181427246011405
0.8	0.145015397537614	0.145015397527905	0.145015566623854	0.145015399115091
0.9	0.085646323767636	0.085646323761786	0.085646427190123	0.085646324668269
1.0	0	0	0	0

ENISK2 the derived method for $k = 2$, ENISK3 the derived method for $k = 3$ and ENISK4 the derived method for $k = 4$.

Table 4.29: Error of Methods for Example 13.

xErrorinRefA:MethodIRefA:MethodIIBernoullipolynomial=15ErrorinENISK2ErrorinENISK3ErrorinENISK4	
1.00.0000000000.00000000000.0000000.0000000.0000000.0000000	
1.28.415722E-84.008605E-85.461631E-55.465412E-95.451316E-10	
1.41.848369E-62.496207E-85.612321E-53.131366E-96.523132E-10	
1.61.086740E-54.420697E-81.321384E-51.189795E-82.354898E-10	
1.81.444889E-53.310400E-82.115485E-55.126787E-89.963523E-10	
2.04.773188E-62.605950E-88.764113E-56.541121E-99.532316E-10	
2.28.874283E-61.346861E-88.541214E-58.794123E-99.654656E-10	
2.41.280107E-59.009668E-92.221221E-53.654645E-92.125645E-10	
2.67.607374E-62.007236E-85.122185E-53.239464E-91.235445E-10	
2.84.452108E-65.415501E-85.212158E-55.157974E-94.546569E-10	
3.00.000000.000000.000000.000000.000000.000000	
RefA:MethodIIBernoullipolynomial=10;MethodIIBernoullipolynomial=15inIslamandShirin(2011).	

Table 4.30: Approximate and Numerical solution for Example 14 $\omega=10$, $t=0.2$, $\kappa=1$.

x	Approx	Exact	Error
0.1	0.04292590084900628	0.0429259008574795	8.4732E-12
0.2	0.08164991544057418	0.0816499154566912	1.6117E-11
0.3	0.11238146742040642	0.1123814674425896	2.2183E-11
0.4	0.13211233836140385	0.1321123383874817	2.6077E-11
0.5	0.13891113314280026	0.1389111331702202	2.7419E-11
0.6	0.13211233836140385	0.1321123383874817	2.6077E-11
0.7	0.11238146742040642	0.1123814674425896	2.2183E-11
0.8	0.08164991544057419	0.0816499154566912	1.6117E-11
0.9	0.04292590084900629	0.0429259008574795	8.4732E-12

Table 4.31: Approximate and Numerical solution for Example 14 $\omega=10$, $t=0.4$, $\kappa=1$.

x	Approx	Exact	Error
0.1	0.00596288552811095	0.00596289	2.3540E-12
0.2	0.01134208227486398	0.0113420822793416	4.4776E-12
0.3	0.01561103698361934	0.0156110369897823	6.1629E-12
0.4	0.01835187462392765	0.0183518746311726	7.2450E-12
0.5	0.01929630291101678	0.0192963029186346	7.6178E-12
0.6	0.01835187462392765	0.0183518746311726	7.2450E-12
0.7	0.01561103698361934	0.0156110369897823	6.1629E-12
0.8	0.01134208227486398	0.0113420822793416	4.4776E-12
0.9	0.00596288552811095	0.0059628855304650	2.3540E-12

Table 4.32: Approximate and Numerical solution for Example 14 $\omega=10$, $t=0.6$, $\kappa=1$.

x	Approx	Exact	Error
0.1	0.0008283111855106	0.00082831118600120	4.9050E-13
0.2	0.0015755415010002	0.00157554150193322	9.3299E-13
0.3	0.0021685468369287	0.00216854683821288	1.2841E-12
0.4	0.0025492796993043	0.00254927970081401	1.5096E-12
0.5	0.0026804713028360	0.00268047130442336	1.5873E-12
0.6	0.0025492796993043	0.00254927970081401	1.5096E-12
0.7	0.0021685468369287	0.00216854683821288	1.2841E-12
0.8	0.0015755415010002	0.00157554150193322	9.3299E-13

0.9 0.0008283111855106 0.00082831118600120 4.9050E-13

Table 4.33: Approximate and Numerical solution for Example 14 $\omega=10$, $t=0.8$, $\kappa=1$.

x	Approx	Exact	Error
0.1	0.0001150616453741	0.00011506164546499	9.0848E-14
0.2	0.0002188602552174	0.00021886025539025	1.7280E-13
0.3	0.0003012352983910	0.00030123529862885	2.3784E-13
0.4	0.0003541233317283	0.00035412333200791	2.7960E-13
0.5	0.0003723473060337	0.00037234730632770	2.9399E-13
0.6	0.0003541233317283	0.00035412333200791	2.7960E-13
0.7	0.0003012352983910	0.00030123529862885	2.3784E-13
0.8	0.0002188602552174	0.00021886025539025	1.7280E-13
0.9	0.0001150616453741	0.00011506164546499	9.0848E-14

Table 4.34: Approximate and Numerical solution for Example 14 $\omega=10$, $t=1$, $\kappa=1$.

x	Approx	Exact	Error
0.1	0.00003196668708039	0.00003196668714980	6.9409E-14
0.2	0.00006080425210435	0.00006080425223638	1.3202E-13
0.3	0.00008368987328420	0.00008368987346592	1.8171E-13
0.4	0.00009838334656536	0.00009838334677898	2.1362E-13
0.5	0.00010344637240762	0.00010344637263223	2.2461E-13
0.6	0.00009838334656536	0.00009838334677898	2.1362E-13
0.7	0.00008368987328420	0.00008368987346592	1.8171E-13
0.8	0.00006080425210435	0.00006080425223638	1.3202E-13
0.9	0.00003196668708039	0.00003196668714980	6.9409E-14

Table 4.35: Comparison of maximum errors obtained in different methods for Example

14 at $t = 1$.

ω	HENIS	BHSDA	Method in Cash, (1981)
1	1.01E-13	2.64E-6	3.7E-5
2	1.02E-13	1.32E-6	1.8E-5
3	1.01E-13	1.32E-6	1.9E-5
5	1.04E-13	1.32E-6	1.8E-5
10	1.30E-13	1.32E-6	1.8E-5

Table 4.36: Approximate and Numerical solution for Example 15, $t=0.2$.

x	Approx	Exact	Error
0.1	0.1816356320013402	0.1816356320010075	3.3265E-13
0.2	0.3454915028125263	0.3454915028119288	5.9746E-13
0.3	0.4755282581475768	0.4755282581468399	7.3685E-13
0.4	0.5590169943749475	0.5590169943742358	7.1165E-13
0.5	0.5877852522924731	0.5877852522919649	5.0826E-13
0.6	0.5590169943749475	0.5590169943748063	1.4110E-13
0.7	0.4755282581475768	0.4755282581479252	3.4844E-13
0.8	0.3454915028125264	0.3454915028134226	8.9628E-13
0.9	0.1816356320013402	0.1816356320027636	1.4233E-12

Table 4.37: Approximate and Numerical solution for Example 15, $t=0.4$.

x	Approx	Exact	Error
0.1	0.293892626146236	0.2938926261458324	4.0412E-13
0.2	0.559016994374947	0.5590169943742358	7.1165E-13
0.3	0.769420884293813	0.7694208842929724	8.4099E-13
0.4	0.904508497187473	0.9045084971867351	7.3852E-13
0.5	0.951056516295153	0.9510565162947653	3.8824E-13
0.6	0.904508497187473	0.9045084971876584	1.8474E-13
0.7	0.769420884293813	0.7694208842947284	9.1504E-13
0.8	0.559016994374947	0.5590169943766530	1.7054E-12
0.9	0.293892626146236	0.2938926261486738	2.4371E-12

Table 4.38: Approximate and Numerical solution for Example 15, $t=0.6$.

x	Approx	Exact	Error
0.1	0.2938926261462362	0.2938926261461324	1.0419E-13
0.2	0.5590169943749475	0.5590169943748065	1.4099E-13
0.3	0.7694208842938134	0.7694208842937578	5.5622E-14
0.4	0.9045084971874737	0.9045084971876585	1.8474E-13
0.5	0.9510565162951536	0.9510565162957361	5.8242E-13
0.6	0.9045084971874737	0.9045084971885816	1.1078E-12
0.7	0.7694208842938134	0.7694208842955138	1.7004E-12
0.8	0.5590169943749476	0.5590169943772235	2.2759E-12

0.9 0.2938926261462367 0.2938926261489737 2.7370E-12

Table 4.39: Approximate and Numerical solution for Example 15, $t=0.8$.

x	Approx	Exact	Error
0.1	0.181635632001340	0.1816356320017929	4.5266E-13
0.2	0.345491502812526	0.3454915028134226	8.9628E-13
0.3	0.475528258147576	0.4755282581488961	1.3192E-12
0.4	0.559016994374947	0.5590169943766530	1.7054E-12
0.5	0.587785252292473	0.5877852522945064	2.0331E-12
0.6	0.559016994374947	0.5590169943772235	2.2759E-12
0.7	0.475528258147576	0.4755282581499814	2.4045E-12
0.8	0.345491502812526	0.3454915028149165	2.3900E-12
0.9	0.181635632001340	0.1816356320035489	2.2086E-12

Table 4.40: Approximate and Numerical solution for Example ², $t=1$.

x	Approx	Exact	Error
0.1	0.09549150281252629	0.09549150281223548	2.90809 E - 13
0.2	0.18163563200134025	0.18163563200115658	1.83686 E - 13
0.3	0.25000000000000006	0.25000000000011511	1.15074 E - 13
0.4	0.29389262614623657	0.29389262614678545	5.48894 E - 13
0.5	0.30901699437494745	0.30901699437478521	1.62259 E - 13
0.6	0.29389262614623657	0.29389262614632658	9.00390 E - 14
0.7	0.25000000000000006	0.25000000000011111	1.11077 E - 13

² at $t = 1$ and $x \in [0,100]$

BHM is Block Hybrid Method of Order 11 in Jator and Oladejo 2017 for $N = 646$.

0.8	0.18163563200134028	0.18163563200178985	4.49557 E - 13
0.9	0.09549150281252632	0.09549150281212878	3.97529 E - 13

Table 4.41: Comparison of maximum errors obtained in different methods for Example

x	HENIS	BHM
10.0	3.5×10^{-14}	1.4×10^{-13}
20.0	3.2×10^{-13}	1.0×10^{-12}
30.0	4.8×10^{-13}	2.1×10^{-12}
40.0	7.5×10^{-13}	2.5×10^{-12}
50.0	1.2×10^{-13}	4.1×10^{-14}
60.0	1.5×10^{-14}	2.8×10^{-12}
70.0	2.4×10^{-14}	1.8×10^{-12}
80.0	3.8×10^{-15}	7.9×10^{-14}
90.0	1.4×10^{-14}	1.4×10^{-12}
100.0	1.9×10^{-14}	1.7×10^{-12}

Table 4.42: Approximate and Numerical solution for Example 16, $t=0.2$.

x	Approx	Exact	Error
-1.0	2.879317311223987E-16	7.386367012937644E-12	7.3860E-12
-0.8	0.345491502812526	0.345491502808941	3.5851E-12
-0.6	-0.559016994374940	-0.5590169943755117	5.6410E-13
-0.4	0.559016994374947	0.5590169943777943	2.8467E-12
-0.2	-0.345491502812526	-0.34549150281491675	2.3901E-12
0	4.100235330129947E-16	4.1002353301400294E-16	1.0082E-27
0.2	0.345491502812526	0.3454915028149161	2.3900E-12
0.4	-0.559016994374948	-0.559016994377794	2.8466E-12
0.6	0.559016994374947	0.5590169943755118	5.6410E-13
0.8	-0.345491502812526	-0.345491502808941	3.5851E-12
1.0	-2.879317311223987E-16	-7.386367012937644E-12	7.3860E-12

for $N = 10$, $x \in [-1,1]$

Table 4.43: Approximate and Numerical solution for Example 16, $t=0.4$.

x	Approx	Exact	Error
-1.0	-4.65883327395637E-16	-1.19513928802923E-11	1.1950E-11
-0.8	-0.5590169943749471	-0.5590169943681256	6.8215E-12
-0.6	0.9045084971874737	0.9045084971867351	7.3863E-13
-0.4	-0.9045084971874738	-0.9045084971904285	2.9546E-12
-0.2	0.5590169943749479	0.5590169943777945	2.8466E-12
0	-6.63432012602339E-16	-6.63432012602759E-16	4.2006E-28
0.2	-0.5590169943749468	-0.5590169943777934	2.8466E-12
0.4	0.9045084971874735	0.9045084971904279	2.9544E-12
0.6	-0.904508497187474	-0.9045084971867352	7.3874E-13
0.8	0.5590169943749471	0.5590169943681256	6.8215E-12
1.0	4.65883327395637E-16	1.19513928802923E-11	1.1950E-11

for $N = 10$, $x \in [-1, 1]$

Table 4.44: Approximate and Numerical solution for Example 16, $t=0.6$.

x	Approx	Exact	Error
-1.0	4.658833273956369E-16	1.1951392880243522E-11	1.1950E-11
-0.8	0.5590169943749471	0.5590169943658431	9.1040E-12
-0.6	-0.9045084971874736	-0.9045084971830418	4.4317E-12
-0.4	0.9045084971874737	0.9045084971867352	7.3852E-13
-0.2	-0.5590169943749478	-0.559016994375512	5.6421E-13
0	6.634320126023398E-16	6.634320126000511E-16	2.2886E-27
0.2	0.5590169943749468	0.5590169943755109	5.6410E-13
0.4	-0.9045084971874734	-0.9045084971867348	7.3863E-13
0.6	0.9045084971874738	0.904508497183042	4.4317E-12
0.8	-0.5590169943749471	-0.5590169943658431	9.1040E-12
1.0	-4.65883327395636E-16	-1.195139288024352E-11	1.1950E-11

for $N = 10$, $x \in [-1, 1]$

Table 4.45: Approximate and Numerical solution for Example 16, $t=0.8$.

x	Approx	Exact	Error
-1.0	-2.879317311223985E-16	-7.38636701280989E-12	7.3860E-12
-0.8	-0.3454915028125259	-0.3454915028029655	9.5604E-12

-0.6	0.5590169943749471	0.5590169943658431	9.1040E-12
-0.4	-0.5590169943749471	-0.5590169943681257	6.8214E-12
-0.2	0.3454915028125264	0.3454915028089412	3.5851E-12
0	-4.10023533012994E-16	-4.10023533006911E-16	6.0831E-27
0.2	-0.3454915028125257	-0.345492	3.5851E-12
0.4	0.559016994374947	0.5590169943681254	6.8216E-12
0.6	-0.5590169943749472	-0.5590169943658432	9.1040E-12
0.8	0.3454915028125259	0.3454915028029655	9.5604E-12
1.0	2.879317311223985E-16	7.386367012809891E-12	7.3860E-12

for $N = 10$, $x \in [-1, 1]$

Table 4.46: Approximate and Numerical solution for Example 16, $t=1$

x	Approx	Exact	Error
-1.0	4.6588332785E-16	4.658833278E-16	0.
-0.8	0.559016125	0.559016994	8.68E-7
-0.6	-0.904508478	-0.904508497	1.86E-8
-0.4	0.904508497	0.904508497	3.54E-10
-0.2	-0.559016954	-0.559016994	4.02E-8
0	6.634320126E-16	6.634320126E-16	0.
0.2	0.559016912	0.559016994	8.17E-8
0.4	-0.904508437	-0.904508497	5.93E-8
0.6	0.904508435	0.904508497	6.14E-8
0.8	-0.559016931	-0.559016994	6.25E-8
1.0	-4.658833273E-16	-4.658833273E-16	0.

for $N = 10$, $x \in [-1, 1]$

Table 4.47: Comparison of maximum errors obtained in different methods for Example

16 at $t = 1$

N	HENIS L_∞ error	CPU Time	BVM L_∞ error	CPU Time	BUM L_∞ error	CPU Time
16	8.852E-8	0.124	9.662E-0	0.483	1.251E-1	0.531
32	5.455E-8	0.865	2.582E-2	1.235	2.578E-2	1.031
64	1.154E-8	2.458	6.433E-3	5.358	6.459E-3	5.516
128	4.128E-8	12.788	1.607E-3	43.641	1.607E-3	46.923

Table 4.48: Approximate and Numerical solution for Example 17, $t=0.2$

x	Approx	Exact	Error
0.	0.789582239399523	0.789582239399523	0.
0.125	0.783609723493636	0.783609723612349	1.1871E-10
0.25	0.766127054955597	0.766127055413306	4.5770E-10
0.375	0.738365336623264	0.738365337593123	9.6985E-10
0.5	0.702153117867834	0.702153119457096	1.5892E-9
0.625	0.659648113504815	0.659648115750088	2.2452E-9
0.75	0.613065771983594	0.613065774858484	2.8748E-9
0.875	0.564457997772449	0.564458001203073	3.4306E-9
1.0	0.515569228077945	0.515569231961146	3.8832E-9

Table 4.49: Approximate and Numerical solution for Example 17, $t=0.4$

x	Approx	Exact	Error
0.	1.522025508449459	1.522025508449459	0.
0.125	1.511307909959082	1.511307910172274	2.1319E-10
0.25	1.479841066755971	1.479841067581628	8.2565E-10
0.375	1.429587394767977	1.429587396529648	1.7616E-9
0.5	1.363520898111824	1.363520901023905	2.9120E-9
0.625	1.285250828272383	1.285250832427022	4.1546E-9
0.75	1.198611208348374	1.198611213722394	5.3740E-9
0.875	1.107290730563408	1.107290737040291	6.4768E-9
1.0	1.014555643941174	1.014555651340818	7.3996E-9

Table 4.50: Approximate and Numerical solution for Example 17, $t=0.6$

x	Approx	Exact	Error
0.	2.161678001082336	2.161678001082336	0.
0.125	2.147952279614072	2.147952279887370	2.7329E-10
0.25	2.107497952728263	2.107497953792780	1.0645E-9
0.375	2.042412107130567	2.042412109422456	2.2918E-9
0.5	1.955959481252558	1.955959485085221	3.8326E-9
0.625	1.852254909499083	1.852254915040219	5.5411E-9
0.75	1.735876601604514	1.735876608873773	7.2692E-9

0.875	1.611465635076303	1.611465643962000	8.8856E-9
1.0	1.483369449653304	1.483369459942914	1.0289E-8

Table 4.51: Approximate and Numerical solution for Example 17, $t=0.8$

x	Approx	Exact	Error
0.	2.698963768894211	2.69896	0.
0.125	2.683772541689999	2.68377	3.0277E-10
0.25	2.638826424147494	2.63883	1.1861E-9
0.375	2.565973378950783	2.56597	2.5771E-9
0.5	2.468174107556518	2.46817	4.3616E-9
0.625	2.349315152325619	2.34932	6.3956E-9
0.75	2.213943573491044	2.21394	8.5206E-9
0.875	2.066940834773645	2.06694	1.0582E-8
1.0	1.913172971874078	1.91317	1.2447E-8

Table 4.52: Comparison of maximum errors obtained in different methods for Example 17.

x	HENIS	SBVM	Exact	SBVM Err	HENIS Err
0.125	0.121956	0.121956	0.121956	1.30×10^{-7}	5.15×10^{-9}
0.25	0.113469	0.113469	0.113469	2.94×10^{-7}	2.65×10^{-9}
0.375	0.105572	0.105573	0.105572	4.51×10^{-7}	8.56×10^{-9}
0.5	0.0982249	0.0982256	0.0982249	6.49×10^{-7}	3.25×10^{-9}
0.625	0.0913884	0.0913892	0.0913884	8.41×10^{-7}	1.38×10^{-9}
0.75	0.0850273	0.0850284	0.0850273	1.03×10^{-6}	1.54×10^{-9}
0.875	0.0791088	0.0791101	0.0791088	1.25×10^{-6}	1.11×10^{-9}
1.00	0.0736021	0.0736035	0.0736021	1.44×10^{-6}	2.29×10^{-9}

4.5 Results and Discussion

Table 4.1 shows the exact and approximate solutions for different methods for Example 1. In Table 4.2 NSDM is nonlinear one-step second derivative method with a variable step-size implementation based on continued fractions for the numerical solution of singular initial value problems (IVPs) in Jator and Colman 2017. The method

was implemented in Predictor-Corrector Mode (PCM) and Self-Starting Mode (SSM). Where RKM is the Runge-Kutta Method for the same problem. It can be clearly seen that our method compares favourably with $k = 2$. This shows the superiority of our method for $k = 3$ to 5.

Table 4.3 shows the approximate and exact solutions for example 2. It also shows the errors for different methods and that of Ayinde & Ibijola (2015). The proposed method is superior over the method of Ayinde & Ibijola (2015), for $k = 2, 3, 4, 5$.

Table 4.4-4.8 show the the approximate, exact solutions and errors for the methods ENISK1 to ENISK5. Table 4.9 show the errors in the method of Yakubu *et al.*. It is remarked that the proposed method with $k = 2$ is very competitive with both methods of Yakubu *et al.*. The proposed method becomes superior for $k = 3$ to $k = 5$. Hence, this shows the superiority of our methods in terms of the errors obtained.

Table 4.10 shows the exact and approximate solution of Example 4. Table 4.11 shows the error obtained for different methods. Table 4.12 shows the maximum error of y_1 and y_2 , that is, $Max\{y_1, y_2\}$.

Here, in Ramos and Popescu (2018), BLOCK2 is the 2-step block method, BLOCK2SIMP is the simplest 2-step block method, BLOCK10 is the 10-step block method, BLOCK10SIMP is the simplest 10-step block method. The proposed method performed better than the methods in Ramos and Popescu (2018).

Table 4.15 shows the exact and approximate solutions for y_1 and y_2 .

For Table 4.14, it can be seen that our method become superior for $k=4$ and 5 when compared to the method in Kamrul *et al.*, (2014).

Table 4.16 shows the comparison of errors obtained in our method for $k = 4, 5$ respectively with the error for 10^{th} , 15^{th} , and 20^{th} -order approximations of y_1 as illustrated in Komashynska *et al.* (2016). Similarly, Table 4.17 shows the comparison of

errors obtained in our method for $k = 4, 5$ respectively with the error for 10th, 15th, and 20th-order approximations of y_2 as illustrated in Komashynska *et al.*, (2016).

The proposed method for $k=4$ and 5 is superior to the methods of Komashynska *et al.*, (2016) when the errors obtained are compared in Tables 4.16 and 4.17.

Table 4.18 shows the exact and numerical solutions for Example 8. In Table 4.19 for the ENIS, the number of subinterval N must be a multiple of the value of the step number (k). Since for ENISk3 N is not divisible by k , the following values (indicated in parenthesis in the table) $N = 15, 30, 63, 126$ are used for the subinterval of integrations. The proposed method is superior over the method in Siraj-ul-Islam *et al.* (2010), for $k = 2, 3, 4$.

Similarly, Tables 4.20 and 4.22 shows the exact and numerical solutions for Example 9 and Example 10 respectively. Tables 4.21 and 4.23 shows that the proposed method is superior over the method in Siraj-ul-Islam *et al.* (2010), for $k = 2, 3, 4$.

Table 4.24 shows the exact and numerical solutions for Example 9. Table 4.25 shows the comparison of the proposed method and the methods in Islam and Shirin (2011).

From Table 4.25, ENISk2 ($k = 2$) is very competitive with methods in Islam and Shirin (2011). The proposed method become superior over the methods in Islam and Shirin (2011) for $k = 3$ and 4.

Table 4.26 shows the exact and numerical solutions for Example 12. Table 4.27 shows the comparison of the proposed methods and those in Majid *et al.* (2013) and Adeyeye and Omar (2016). It clearly shows that the proposed methods performed better in terms of the errors obtained.

Table 4.28 shows the exact and numerical solutions for Example 13. Table 4.29 shows the comparison of the proposed method and the methods in Islam and Shirin

(2011). Method II in Islam and Shirin (2011) is highly competitive with the proposed methods for $k = 3$ and superior for $k = 4$.

Table 4.30 shows the exact and numerical solutions for Example 14. Table 4.31 shows the comparison of the proposed methods and the methods in Ngwane and Jator 2014. The HENIS perform excellently when applied to this problem. Tables 4.32-4.35 , displayed the results for $\kappa = 1$ values for ω and t .

Table 4.36 shows the exact and numerical solutions for Example 15. Table 4.37 shows the comparison of the proposed methods and Block Hybrid Method of Order 11 in Jator and King 2018. Tables 4.38-4.41, displayed the results for values of t . The HENIS perform excellently for this problem.

Table 4.42 shows the exact and numerical solutions for Example 16 for $N = 10$ (N is the number of subintervals). Table 4.43 shows the comparison of the proposed methods and the Method in Biala (2016) for $N = 16, 32, 64, 128, 256$. Tables 4.44-4.47, displayed the results for values of t . The CPU Time was also compared. The proposed methods performed excellently in terms of the l_∞ -norm and the computation time for this problem.

Table 4.48 shows the exact and numerical solutions for Example 17 at the grid $x = 1/8(1/8)1$. The table also shows the comparison between the proposed method and the method in Jator 2016. Tables 4.49-4.52, displayed the results for values of t .

Chapter 5

CONCLUDING REMARKS

5.1 Summary

In this thesis, new classes of Exponentially-fitted Numerical Integration Schemes (ENIS) for solving first order Initial value problems and Second order boundary value problems in ODEs were developed. Also a class of Hybrid Exponentially-fitted Numerical Integration Schemes (HENIS) for solving second order Initial-boundary value problems in PDEs were developed. Exponential term was fitted into a power series which serves as the basis function in the trial solutions. The technique of collocation was employed to derive the continuous schemes. These schemes were then assembled in block form and were applied simultaneously to solve some test problems. The block form of the schemes are self-starting, and as such avoid the need for predictors. The Method of Lines was applied to convert the second order PDEs into system of second order ODEs before applying the derived schemes to solve the problems numerically. The proposed schemes are robust as they have wide scope of application including nonlinear as well as stiff and oscillatory problems. The schemes were developed with the aid of Mathematica 8.0 package. The tables of results show that the developed schemes can be adopted for solving such problems.

5.2 Conclusion

The development of some numerical schemes of exponentially-fitted form has been proposed in this work. This was developed via the interpolation and collocation

techniques exponential function was incorporated into power series function which in whole constitutes the trial solution that was adopted in the development process. The methods were effectively illustrated with many problems characterized by certain peculiarities and the results obtained were accurate. The analysis of the new methods showed that they all satisfy the desirable properties of numerical methods for solution of ODEs, namely, consistency, zero-stability, continuity and convergence.

5.3 Contribution to Knowledge

The following contributions are made to the body of knowledge;

- (i) development of k-classes of continuous exponentially-fitted numerical integration schemes (ENIS) for direct solution of first order IVPs;
- (ii) development of k-classes of continuous exponentially-fitted numerical integration schemes (ENIS) for direct solution of second order BVPs;
- (iii) development of 2-step continuous hybrid exponentially-fitted numerical integration schemes (HENIS) for solution of second order IBVPs via the method of lines;
- (iv) the derived methods were self-starting, consistent, zero-stable and convergent;
- (v) the proposed scheme have a wide scope of application; and
- (vi) the schemes perform better, in terms of accuracy than existing methods in literature.

5.4 Recommendation

The method proposed in this thesis have been confirmed to perform better in terms of accuracy than existing methods. They are therefore recommended for solution of first order IVPs, second order BVPs and second order IBVPs.

5.5 Suggestion for Further Work

In the course of this work, only one exponential function was fitted into the power series which serves as the basis function for the trial solution. It is hereby suggested that more exponential terms can be considered for fitting. Further more, other form of basis functions other than power series where the exponential function is to be fitted may also be considered. This in a way may improve this work.

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Appendices

Appendix I Computer Program for ENIS for solving first and second order ODEs and PDE

Code for implementation of ENIS for First Order IVPs and Non-Hybrid ENIS for Second Order BVPs and Second Order IBVPs

```
a=ini;b=fin;m=numbersteps;
h=(b-a)/m *stepsize*; x0=inivalx;y0=inivaly; w=1 ;

SystemOfEqs=Mapping[x, Range[1,m]];

YVec :=Mapping[Y, Range[0,m]];

YpVec      =      :=      Mapping[Yp,      Range[1,m]];***for the case
of second order** ***the range is adjusted for bvp***

YpdeVec      =      :=      Mapping[Yp,      Range[1,m]];***for the case
of second order pde** ***the range is adjusted for ibvp***

*****

** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** fn

:=(xn,yn),***insert the problem to be solved***      fn      =

Mapping[f, Range[0,m]];

*****

*****
```

This the Derived Block Method

```
BlockMethod={};

Eqns=Table[BlockMethod,{i,0,m}]/Flatten;

Length[Eqns]
```

Variables

```
YInitVecSol=YVec;
```

128

```
Length[YInitVecSol]
```

```
MakeGuessPair[p_]:= {p,1}
```

```
InitialGuesses =Map[MakeGuessPair,YInitVecSol]; sol=FindRoot[Eqns,InitialGuesses];
```

Appendix I Computer Program for ENIS for solving first and second order ODEs and PDE

Output of solutions in the table form

```
q0=Table[yi ,{i,0,m}]/Flatten;
```

Exact Solution

```
YExact[x_] := Tan[x];
```

```
YPEExact[x_] := YExact'[x];
```

```
UVectors=Table[YExact[x],{i,0,m}]/Flatten;
```

```
Error=Abs[val];
```

```
Max[Error]
```

```
RESULT=Table[{i,1,m+1}];
```

```
Prepend[RESULT,{"x","yapprox","yExact","yError"}]/TableForm
```

