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Group I. Theoretical Methods of Modern Classical and Quantum Physics

The Effects of Linearly Varying Distributed Moving Loads on Beams with Winkler Foundations

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The dynamic behaviour of a Bernoulli–Euler beam on a Winkler foundation traversed by a linearly varying distributed moving load is investigated. Using a series solution for the dynamic deflection in terms of normal modes, the equation governing the model is reduced to a set of ordinary differential equations whose solution is obtained in form of a Duhamel integral. Several numerical results are presented to illustrate these effects.

1 Introduction

The study of the response of structures under moving loads is an interesting problem due to its practical importance in the areas of transport and design of machine parts. Some branches of transport have recorded development that features increase in speed and weight of vehicles as a result of which higher stresses more than ever before are developed. Moreover, a moving mass produces greater deflection and stress on the structure over which it moves than an equivalent static mass. Thus, the analyses of the effects of moving loads on beams have been attracting the attention of considerable numbers of researchers in applied mathematics, science and engineering who are interested in road and rail transports [1–6].

Many of the publications on the dynamic response of beams under the influence of moving loads have been centred on concentrated loads. Reviews of ealier research work on the subject were documented by Kolousek [1]. An extended review was reported by Fryba [7]. Some recent studies on the subject are by Esmailzadel and Ghorashi [4], Gbadeyan and Dada [2, 5, 6], Mahmoud and Abouzaid [3], and Michaltsos and Kounadis [8].

It is pertinent to state here that most of the publications on the dynamic response of structures to moving masses are centred on concentrated masses and a limited number are available for uniformly distributed moving masses. The analysis was extended to linearly distributed masses by Gbadeyan and Dada [6].

The motivation for studying this problem originates from the fact that the structures of roadway, runway concrete and reinforced concrete rest on various foundation models. Consequently, the present work examines the effects of linearly varying distributed moving mass on beams resting on continuous elastic foundation.

2 Analytical Formulation

The considered model is a finite elastic uniform thin beam of length L, mass per unit length m and flexural rigidity EI. The equation of motion describing the lateral vibration of the beam carrying the time varying force $f(x,t)$ is

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = f(x,t) - h(x,t), \tag{1}$$

where x is the length coordinate with the origin at left hand end of the beam, t is the time coordinate with the origin at the instant of the force arriving on the beam, the sub-grade reaction due to the Winkler foundation is expressed as $h(x,t) = ky$, k being the modulus of the sub-grade reaction and y the deflection of the beam measured downward from its equilibrium position when the beam is loaded with its own weight.

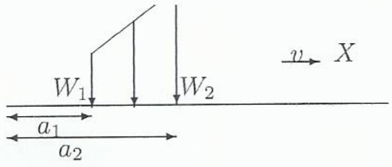

Figure 1: A beam of span L under a linearly distributed load.

The beam is under a force $f(x,t)$ with mass M_m that is linearly distributed partially on the beam as shown in Figure 1. The force [9] acting on the beam is,

$$f(x,t) = g\left\{M_1 < x - a_1 >^0 + \frac{M_2 - M_1}{a_2 - a_1} < x - a_1 >^1 - M_2 < x - a_2 >^0 - \frac{M_2 - M_1}{a_2 - a_1} < x - a_2 >^1\right\}, \quad (2)$$

where $W_2 = M_2 g$ and $W_1 = M_1 g$ are forces produced by masses M_2 and M_1 acting, respectively, at the right and left end points of the moving mass, $d = a_2 - a_1$ is the length of the distributed mass, $a_2 = vt + \frac{d}{2}$, $a_1 = vt - \frac{d}{2}$, v being the velocity of the moving mass, g is the acceleration due to gravity and the Macauly notation is defined as

$$< x - a >^n = \begin{cases} 0 & , \ x < a, \\ (x-a)^n & , \ x \geq a. \end{cases} \quad (3)$$

3 Method of Solution

Since the time and space functions may be separable for a modal motion, we seek for the overall response of the beam a series solution in terms of the normal modes in the form

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x) P_n(t), \quad (4)$$

where $X_n(x)$ is the modal shape eigen-funtion for the n-th mode of the freely vibrating beam with the corresponding generalised unknown function of time $P_n(t)$ that is to be calculated. Introducing (2) and (4) into (1), we have ($X_n^{(4)}(x)$ stands for the fourth order derivative of $X_n(x)$ with respect to x),

$$\begin{aligned} EI\sum_{n=1}^{\infty} X_n^{(4)}(x)P_n(t) + m\sum_{n=1}^{\infty} X_n(x)\ddot{P}_n(t) + k\sum_{n=1}^{\infty} X_n(x)P_n(t) = \\ = g\left\{M_1 < x - a_1 >^0 + \frac{M_2 - M_1}{a_2 - a_1} < x - a_1 >^1 - M_2 < x - a_2 >^0 - \frac{M_2 - M_1}{a_2 - a_1} < x - a_2 >^1\right\}. \quad (5) \end{aligned}$$

The n-th normal mode of vibration of a uniform beam satisfies

$$X_n(x) = A1_n \sin \Phi_n x + A2_n \cos \Phi_n x + A3_n \sinh \Phi_n x + A4_n \cosh \Phi_n x, \quad (6)$$

where the unkown constants $A1_n$, $A2_n$, $A3_n$, $A4_n$ and Φ_n are determined by applying the boundary conditions of the beam. For free vibration of the beam, we have

$$EIX_n^{(4)}(x) + kX_n(x) = m\omega_n^2 X_n(x), \quad (7)$$

with the natural frequencies $\omega_n^2 = (\Phi_n^4 EI + k)/m$ $(n = 1, 2, 3, \ldots)$. Substituting (7) into (5), multiplying the resultant equation by $X_r(x)$ and integrating both sides with respect to x from 0 to L, one obtains,

$$\begin{aligned} \ddot{P}_n(t) + \omega_n^2 P_n(t) &= \frac{1}{\kappa m}\int_{a_1}^{L}\left\{gX_n(x)\left(M_1 + \frac{M_2 - M_1}{a_2 - a_1}(x - a_1)\right)\right\}dx \\ &\quad - \frac{1}{\kappa m}\int_{a_2}^{L}\left\{gX_n(x)\left(M_2 + \frac{M_2 - M_1}{a_2 - a_1}(x - a_2)\right)\right\}dx, \quad (8) \end{aligned}$$

where

$$\int_0^L X_n(x)X_r(x)dx = \begin{cases} 0 & , \ n \neq s, \\ \kappa & , \ n = s. \end{cases}$$

and κ is a constant.

Equation (8) is a set of generalised ordinary differential equations that is solved subject to the boundary conditions of the beam. Many highways and railway bridges consist of simply supported girders [10]. Therefore, the dynamic response of a simply supported beam under moving mass is considered. The simply supported boundary conditions of a beam may be written in the form,

$$y(x,t)|_{x=0} = EI\frac{\partial^2 y(x,t)}{\partial x^2}|_{x=0} = 0, \tag{9a}$$

$$y(x,t)|_{x=L} = EI\frac{\partial^2 y(x,t)}{\partial x^2}|_{x=L} = 0, \tag{9b}$$

with initial conditions

$$y(x,t)|_{t=0} = EI\frac{\partial y(x,t)}{\partial t}|_{t=0} = 0.$$

Applying these end conditions, we have

$$X_n(x) = \sin\frac{n\pi x}{L}, \qquad \kappa = \frac{L}{2}, \qquad \omega_n^2 = \frac{(n\pi)^4 EI}{L^4 m} + \frac{k}{m}. \tag{10}$$

Substituting the set of equations (10) into (8), one obtains an equation in a generalised funtion of time for simply supported conditions. Thus

$$\ddot{P}_n(t) + \omega_n^2 P_n(t) = c_3\, c_1 \left\{ \left(M_1 \cos\frac{n\pi a_1}{L} - M_2 \cos\frac{n\pi a_2}{L} \right) + c_1\, Gr \left(\sin\frac{n\pi a_2}{L} - \sin\frac{n\pi a_1}{L} \right) \right\}, \tag{11}$$

where

$$Gr = \frac{M_2 - M_1}{d}, \qquad c_1 = \frac{L}{\pi n}, \qquad c_3 = \frac{2g}{mL}.$$

Using Duhamel's integral, the solution of equation (11) is expressed as,

$$P_n(t) = \frac{1}{\omega_n}\int_0^t q_n(\tau)\sin(\omega_n(t-\tau))d\tau, \tag{12}$$

where,

$$q_n(\tau) = c_3 c_1 \left[\left(M_1 \cos\frac{n\pi}{L}(v\tau - \frac{d}{2}) - M_2 \cos\frac{n\pi}{L}(v\tau + \frac{d}{2}) \right) + Gr c_1 \left(\sin\frac{n\pi}{L}(v\tau + \frac{d}{2}) - \sin\frac{n\pi}{L}(v\tau - \frac{d}{2}) \right) \right].$$

Using the non-dimensional quantities

$$\bar{x} = x/L, \qquad \bar{v} = vt_0/L, \qquad \bar{g} = gt_0^2/L, \qquad \bar{\omega_n} = \omega_n t_0,$$
$$\bar{M} = M_m/mL, \qquad \bar{M_1} = M_1/m, \qquad \bar{M_2} = M_2/m, \qquad \bar{Gr} = LGr/m,$$

the non-dimensional deflection is

$$\begin{aligned}\frac{X_n(x)P_n(t)}{L} = e_1 \sin n\pi\bar{x} \Big[\; & e_2\bar{M}_1(\cos n\pi\bar{a}_2 - \cos b_1) + e_3\bar{M}_1(\cos b_2 - \cos n\pi\bar{a}_2) \\ & - e_2\bar{M}_2(\cos n\pi\bar{a}_1 - \cos b_2) - e_3\bar{M}_2(\cos b_1 - \cos n\pi\bar{a}_1) \\ & + \frac{\bar{Gr}}{n\pi}\Big(e_2(\sin n\pi\bar{a}_2 - \sin n\pi\bar{a}_1 + \sin b_1 - \sin b_2) \\ & \quad + e_3(\sin b_2 - \sin b_1 + \sin n\pi\bar{a}_1 - \sin n\pi\bar{a}_2)\Big)\Big],\end{aligned} \tag{13}$$

where

$$e_1 = \bar{g}/(\bar{\omega}_n n\pi), \qquad e_2 = 1/(\bar{\omega}_n + n\pi\bar{v}), \qquad e_3 = 1/(-\bar{\omega}_n + n\pi\bar{v}),$$
$$b_1 = \bar{\omega}_n\bar{t} - n\pi\bar{d}/2, \qquad b_2 = \bar{\omega}_n\bar{t} + n\pi\bar{d}/2, \qquad t_0 = \pi\sqrt{mL^4/EI}.$$

4 Results and Discussion

In this Section, numerical results are presented in tabular and graphical forms. The illustrative example was computed for a simply supported beam of length $L = 50$ m, flexural rigidity $EI = 2.5 \times 10^5$ square metres, and mass per unit length $m = 4.5$ kg/metres.

Tables 1 and 2 show the effects of variation of dimensionless mass distribution gradient $\bar{G}$ and the ratio $\bar{M}$ on the maximum dimensionless deflection amplitude with modulus $k = 2$, and $db = 0.001$ which is defined as the ratio of the length of the moving load to that of the beam, $db = d/L$. For various values of $\bar{G} = 0, 100, 200, 300, 400$, the amplitude of deflection increases with an increase in the mass ratio $\bar{M}$. This fact is evidenced in the percentage comparison $P = (y_u - y_m)100/y_u$ where y_u and y_m are dimensionless deflections for uniformly and non-uniformly distributed loads, respectively. In the case of negative gradients $\bar{G}$, it shows that as the absolute value of $\bar{G}$ increases, the maximum amplitude of the deflection increases. Also, Table 1 as well as Table 2 show that mass ratio increase causes an increase in the maximum deflection amplitude. It is interesting to note that the uniformly distributed moving mass where $\bar{G} = 0$ produces the lowest maximum deflection amplitude as evidenced in all the Tables.

LGr/m		$M_m/mL = 0.4$	0.6	0.8	1.0
0	y_u	-0.5108	-0.7662	-1.0216	-1.2770
100	y_m	-0.51101	-0.76618	-1.0216	-1.277
	P	0.042884	1.92e-006	1.44e-006	1.15e-006
200	y_m	-0.51224	-0.76713	-1.022	-1.277
	P	0.28482	0.12353	0.043	2.31e-006
300	y_m	-0.51412	-0.76836	-1.0232	-1.2781
	P	0.65315	0.28482	0.16385	0.091272
400	y_m	-0.51659	-0.76995	-1.0245	-1.2794
	P	1.136	0.4922	0.28482	0.18805

Table 1: Deflection and its percentage comparisons (P) for varying positive gradient and mass ratio.

LGr/m		$M_m/mL = 0.4$	0.6	0.8	1.0
0	y_u	-0.51079	-0.76618	-1.0216	-1.277
-100	y_m	-0.51101	-0.76618	-1.0216	-1.277
	P	0.042878	-1.92e006	-1.4e-006	-1.15e-006
-200	y_m	-0.51224	-0.76713	-1.022	-1.277
	P	0.28481	0.12352	0.042878	-2.31e-006
-300	y_m	-0.51412	-0.76836	-1.0232	-1.2781
	P	0.65313	0.28481	0.16385	0.091265
-400	y_m	-0.51659	-0.76995	-1.0245	-1.2794
	P	1.136	0.49219	0.28481	0.18804

Table 2: Deflection and its percentage comparisons (P) for varying negative gradient and mass ratio.

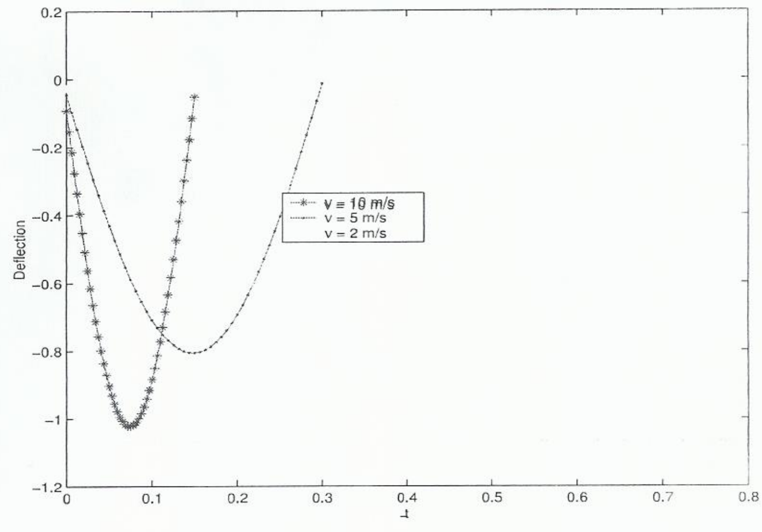

Figure 2: Mid-span dimensionless deflection as a function of dimensionless time.

As shown in Table 3 the effects of foundation were investigated for fixed values of $\bar{v} = 6.6643$ ($v = 10$ m/s) and $db = 0.001$. It can be seen that increase in the value of foundation parameter k decreases the maximum deflection amplitude. Figure 2 shows the variation of velocity $\bar{v}$ or v on the mid-span deflection at various times $\bar{t}$ with fixed modulus $k = 2$, $\bar{G} = 100$, $\bar{M} = 0.8$ and $db = 0.001$.

LGr/m		$k=0$	$k=1$	$k=2$	$k=4$
0	y_u	-2.1193	-1.0234	-0.51079	-0.19458
100	y_m	-2.1201	-1.0238	-0.51101	-0.19468
	P	0.038496	0.035857	0.042884	0.055442
200	y_m	-2.1247	-1.0259	-0.51224	-0.19529
	P	0.2557	0.23816	0.28482	0.36825
300	y_m	-2.1317	-1.029	-0.51412	-0.19622
	P	0.58638	0.54614	0.65315	0.84447
400	y_m	-2.1409	-1.0331	-0.51659	-0.19743
	P	1.0199	0.94988	1.136	1.4687

Table 3: Deflection and its percentage comparisons (P) for varying foundation parameter k and gradient.

5 Conclusion

The problem of assessing the dynamic response of a beam resting on a Winkler foundation to a linearly distributed moving load has been studied. The mathematical model was solved analytically using separation of variables coupled with Duhamel's integral techniques. On analysing the solution, one can draw the following conclusions.

(i) The presence of foundation has significant effects on beam vibration.

(ii) The modulus of mass distribution gradient of the moving load significantly affects the behaviour of the beam. As the absolute value of the mass distribution gradient $\bar{G}$ increases, the maximum amplitude of the deflection increases.

(iii) Uniformly distributed moving masses produce the lowest maximum deflection amplitude.

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