

**NUMERICAL METHODS FOR SOLVING
SINGULAR AND MULTI-ORDER FRACTIONAL
INTEGRO-DIFFERENTIAL EQUATIONS**

**OWOLANKE, Ayodele Olakiitan
(14/68DR002)**

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A THESIS SUBMITTED TO THE DEPARTMENT OF
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CERTIFICATION

This is to certify that this thesis has been read and approved as meeting the requirements of the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the Award of Doctor of Philosophy (Ph.D.) degree in Mathematics.

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DEDICATION

This research work is dedicated to Almighty God, the giver of divine wisdom, the beginning and the finisher of all things.

DECLARATION

I, Ayodele Olakiitan OWOLANKE, a Ph.D. student in the Department of Mathematics, University of Ilorin, Ilorin, hereby declare that this thesis entitled Numerical Methods for Solving Singular and Multi-Order Fractional Integro-Differential Equations , submitted by me is based on my actual and original work. Any materials obtained from other sources or work done by any other persons or institutions have been duly acknowledged. In addition, the research has been approved by the University of Ilorin Ethical Review Committee.

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Date

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ABSTRACT

This thesis considers the construction of canonical polynomials with the aid of Lanczos and Ortiz method. The research work is focussed on the approximate solution of a special class of Fractional Integro Differential Equations (FIDEs) of the singular and multi-order Volterra types. Fractional calculus as a branch of Mathematics deals with the integrals and derivatives of arbitrary orders, appearing in the fields of Engineering and Sciences. Many researchers that have worked in this area remarked that rigorous works were involved in obtaining the approximate solutions to the general class of problem considered (see (1) and (2)) and yet the results obtained were less accurate. Thus, this study was geared towards overcoming the drawbacks of the existing methods. The aim of this study, therefore was to propose three numerical methods and to construct canonical polynomials that provide less rigorous works in terms of computational costs with improved accuracy. The objectives of the study were to:

- (i) construct canonical polynomials to be used as basis functions for the general class of singular and multi-order fractional integro differential equations;
- (ii) demonstrate the formulation of the proposed methods on the general class of (1) and (2) using the constructed canonical polynomials as basis functions;
- (iii) carry out the formu-

lated three proposed methods on some numerical examples; and (iv) compare the results obtained with the available exact solutions in the literature.

The general form of the class of problems considered in the study is given as:

$$D^a y(x) + \sum_{i=0}^n p_i y^i(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x) \quad (1)$$

subject to the conditions

$$y^k(0) = a_k, k = 0, 1, 2, \dots, n-1, 0 \leq x, t \leq 1, \quad (2)$$

where $D^a y(x)$ is the a^{th} Caputo derivative of $y(x)$; $f(x)$ is a given smooth function; p_i s are constants; x and t are given real variables in the interval $[0, 1]$; $y(x)$ is the unknown function to be determined and (1) together with conditions (2) is referred to as singular and multi-order fractional integro differential equations. The methods considered for solving the problems were the Standard Collocation Method (SCM), Perturbation Collocation Method (PCM) and Exponentially Fitted Collocation Method (EFCM). In each case, the proposed methods (SCM, PCM, EFCM) assumed approximate solution using the constructed canonical polynomials as basis functions which is then substituted into the problems considered. For the case of the PCM, the right hand side of the problem considered is per-

turbed by adding a perturbation term denoted by $H_N(x)$, where N is the degree of the approximant used. Similarly, for the case of the EFCM, the problem considered is slightly perturbed by adding

$$\sum_{i=1}^n \tau_i T_{N-i+1}(x)$$

where n is the order of the problem considered, $\tau_i (i \geq 0)$ are the free tau parameters to be determined and the conditions are fitted with exponential of the form $\tau_n e^a$ which consists of one tau-parameter to be determined along with the unknown constants. Hence, all the three proposed methods resulted into algebraic linear system of equations after collocating at some equally spaced interior points by which the solutions of the unknown constants were obtained using the Gaussian elimination method or Maple 18 software.

The findings of the study were that:

- (i) canonical polynomials used as basis functions for the general class of singular and multi-order fractional integro differential equations were successfully constructed;
- (ii) three proposed numerical methods demonstrated on the general class of singular and multi-order fractional integro differential equations were successfully carried out;
- (iii) numerical examples were successfully solved by the three pro-

posed numerical methods; and

- (iv) results obtained using the three proposed numerical methods were comparatively better than those available in the literature in terms of accuracy.

The study concluded that SCM, PCM and EFCM are alternative methods of finding numerical solutions to singular and multi-order fractional integro differential equations. It is therefore recommended that the methods be adopted in solving singular and multi-order fractional integro differential equations for better accuracy.

Chapter 1

GENERAL INTRODUCTION

1.1 Background to the Study

Many physical laws and relations appear mathematically in the form of fractional calculus as some physical problems modelled in engineering, astronomy, sciences and social sciences usually give rise to differential equations. Therefore, a huge interest has risen towards fractional calculus due to its applicability in the fields of science and engineering, in which the transformation of physical problems are inevitable. These transformations are usually expressed either as fractional differential equations (ordinary and partial differential equations) or fractional integro differential equations. Hence, the need to provide good solutions to the differential equations are required, however not all could be solved analytically, and in that wise the importance of numerical methods come into play as an al-

ternative method of solution. The idea is that numerical schemes are constructed and their effectiveness in terms of accuracy and efficiency are verified by comparing the solutions of the methods with that of the existing methods before they are affirmed as good methods. Derivatives of non-integer order can be defined in different ways, e.g. Riemann Liouville, Grunwald Letnikov, Caputo and generalized functions approach (Podlubny, 1999). For the differential equations which involve Caputo differential operators of fractional order $q > 0$, which take the form

$$D^q y(x) = \frac{1}{\Gamma(m-q)} \int_0^x (x-\tau)^{m-q-1} [D^m f(\tau)] d\tau; \quad m-1 < q < m \quad (1.1)$$

$$\frac{d^m f(x)}{dx^m}; \quad q = m, m \in \mathbb{N}$$

where m is the integer defined by $m-1 < q \leq m$; D^q has m -dimensional kernel, and therefore there is need to specify m initial conditions in order to obtain a unique solution. Equations of the form 1 and 2 have recently proved to be valuable tools in the modelling of many physical phenomena, which includes Viscoelasticity, electromagnetic waves, chaotic system, optimization and heat transfer modelling. For instance, fractional derivatives without singular kernels have been proposed as mathematical tools to describe the mathematical models in viscoelasticity.

Non-integer derivatives can be traced back to the era of Leibnitz and L'Hopital, where in the question posed by L'Hopital to Leibnitz on the meaning of non-integer derivatives especially for the case $n = \frac{1}{2}$. The question was acknowledged by Leibnitz and considered $\frac{d^1}{dx^2}x$ to be equal to $x^2 \sqrt{\frac{d}{dx}} : x$. Afterwards, Lagrange came up with an idea which was well acknowledged in the field of fractional calculus in 1972; where the law of exponents for differential operators of integer order is developed as:

$$\frac{d^m}{dx^m} \frac{d^n}{dx^n} = \frac{d^{m+n}}{dx^{m+n}}, \quad m, n \in \mathbb{N}$$

These results are transferred to arbitrary choices of $n, m \in \mathbb{C}$ under certain conditions. The benefits of fractional calculus is made explicit by discussing some compulsory but relatively simple mathematical definitions that are appeared in the study of these concepts. The general forms of the class of problems considered in this study are given as:

$$D^a y(x) + \sum_{i=0}^n p_i y^i(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x) \quad (1.2)$$

subject to the conditions

$$y^k(0) = a_k, \quad k = 0, 1, 2, \dots, n-1, \quad 0 \leq x, t \leq 1, \quad (1.3)$$

where $D^a y(x)$ is the a^{th} Caputo derivative of $y(x)$; $f(x)$ is a given smooth function; p_i 's are constants; x and t are given real variables

in the interval $[0, 1]$; $y(x)$ is the unknown function to be determined and (1.2) together with conditions (1.3) is referred to as singular and multi-order fractional integro differential equation.

1.2 Definition of Relevant Terms

1.2.1 Differential Equations:

Differential equation is an equation in which the unknown function containing one or more variables relating the function values itself and its derivatives.

e.g

$$\frac{dy}{dx} = 6x \quad (1.4)$$

A differential equation is said to be linear if the dependent variables and all its derivative(s) involved occurs to the first degree only and no products of the dependent variable(s) and its derivatives occurs, otherwise it is said to be non-linear.

e.g

$$\frac{dy}{dx} - 2y = \ln(x) \quad (1.5)$$

$$\frac{1}{x} \frac{d^2y}{dx^2} - y^3 = 3x \quad (1.6)$$

Thus, equation (1.5) is linear and (1.6) is non-linear

1.2.2 Analytic Function:

A function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and every point in some neighbourhoods of z_0 . Analytic functions can be represented as a power series and it is infinitely differentiable.

1.2.3 Closed Form Solution:

This is a finite representation of a solution.

The following are few examples:

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 \quad (1.7)$$

$$y = \frac{2 + x^2}{1 - x} e^x \quad (1.8)$$

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 + \dots \quad (1.9)$$

Equations (1.7) and (1.8) are closed form solutions, while (1.9) is not a closed form.

1.2.4 Taylor Series:

Let a function $f(x)$ with its derivatives of all orders be defined on an interval $[a, b]$ containing an interior point x_0 . Then, Taylor series expansion of the function at point $x = x_0$ is written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} (x - x_0)^n$$

. This is equivalently written as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

If $x_0 = 0$, the series is known as Maclaurin's series.

1.2.5 Local Error:

This is the difference between the exact solution and the approximate solution in one step of the method

1.2.6 Global Error:

This is the magnitude of the difference between the exact solution say u^n and the approximate solution say V^n at the time step n of the method. It is known as the total or accumulated error of a method e.g.

$$\begin{aligned} u^n - V^n &= \sum_{n=1}^N (e^n - e^{n-1}) \\ &= (e^1 - e^0) + (e^2 - e^1) + (e^3 - e^2) + \dots + (e^N - e^{N-1}) \end{aligned}$$

1.2.7 Absolute Error:

It is defined as the absolute difference between the exact and the approximate solutions. It is mathematically expressed as

Absolute error = $|y(x) - y_N(x)|$, where $y(x)$ is the exact solution and $y_N(x)$ is the approximate solution of various values of N .

Chebyshev Polynomials:

The Chebyshev polynomials belong to a family of orthogonal poly-

nomials in the interval $[-1, 1]$. They are widely utilized for their good properties in approximating of functions. Chebyshev polynomials are of four kinds. The Chebyshev polynomial of the first kind of degree n denoted by $T_n(x)$ and valid in $[-1, 1]$ is defined as

$$T_n(x) = \cos n\theta, \quad (1.10)$$

$$\theta = \cos^{-1}x,$$

where $x = \cos\theta$, and the recurrence relation is given as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1 \quad (1.11)$$

The following are the few terms of the Chebyshev polynomials of the first kind

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$

$$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$$

The Chebyshev polynomials of the second kind denoted by $U_n(x)$ and valid in $[-1, 1]$ is defined as

$$U_n = \frac{\sin(n+1)\theta}{\sin\theta}; \quad n \geq 0,$$

and the recurrence relation is given as

$$U_n = 2\theta U_{n-1} - U_{n-2}; \quad n \geq 2,$$

where, $\theta = \cos^{-1}(x)$

The Chebyshev polynomials of the third kind denoted by $V_n(x)$ of degree n in x valid in $[-1, 1]$ is given as

$$V_n(\theta) = \frac{\cos(n + \frac{1}{2})\theta}{\cos(\frac{\theta}{2})},$$

where $x = \cos\theta$ and $\theta \in [0, \pi]$.

These polynomials $V_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the inner product:

$$\langle V_n(x), V_m(x) \rangle = \int_{-1}^1 \frac{1+x}{1-x} V_n(x) V_m(x) dx = \begin{cases} \pi; & \text{for } n = m \\ 0; & \text{for } n \neq m, \end{cases}$$

where $\frac{1+x}{1-x}$ is a weight function corresponding to $V_n(x)$. The

polynomials $V_n(x)$ may be generated using the recurrence relations

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad V_0(x) = 1, \quad V_1(x) = x, \quad n = 1, 2, \dots$$

Similarly, The Chebyshev polynomials of the fourth kind denoted by $W_n(x)$ and valid in $[-1, 1]$ is defined as

$$W_n(x) = \frac{\sin(n + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})}.$$

But for the purpose of this work, Chebyshev polynomials valid in interval $[a, b]$, is defined as

$$T_n(x) = \cos\{n \cos^{-1}[\frac{2x - a + b}{b - a}]\}, n \geq 0$$

and the recurrence relation is given as

$$T_{n+1}(x) = 2\{\frac{2x - a + b}{b - a}\}T_n(x) - T_{n-1}(x) \quad (1.12)$$

in the interval $[0, 1]$, the following terms are obtained from equation (1.12).

$$T_1(x) = 2x - 1$$

$$T_2(x) = 8x^2 - 8x + 1$$

$$T_3(x) = 32x^3 - 48x^2 + 18x - 1$$

$$T_4(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

$$T_5(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

$$T_6(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$$

$$T_7(x) = 8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1$$

$$T_8(x) = 1 - 128x + 2688x^2 - 21504x^3 + 84480x^4 - 180224x^5 + 212992x^6 - 131072x^7$$

$$+32768 x^8$$

1.2.9 Approximate Solution of Differential Equation:

This is the analytic expressions or numerical values obtained using numerical methods e.g. cubic spline method, finite difference method, e.t.c. when applied to solve problem such as fractional integro-differential equations.

1.2.10 Linearity and Homogeneity Concept of Integro Differential Equation

Integro differential equation is linear if the derivative of the unknown function is of degree one and the exponent of the unknown function inside the integral sign is one. Otherwise, it is nonlinear if the equation contains nonlinear functions of $u(x)$ such as e^u , $\sinh(u)$, $\ln(1 + u)$.

Examples

$$u' + u^2 - \int_0^x xtu(t)dt = f(x), \quad (1.13)$$

$$u' + u + \int_0^x xtu(t)dt = f(x), \quad (1.14)$$

$$u' + u^2 - \int_0^1 xtu(t)dt = f(x), \quad (1.15)$$

$$u' + u + \int_0^1 xtu(t)dt = f(x), \quad (1.16)$$

Equations (1.13) and (1.15) are nonlinear Volterra and Fredholm integro differential equations, whereas equations (1.13) and (1.15) are linear Volterra and Fredholm integro differential equations.

The concept of Homogeneity is usually associated with the second kind integro differential equations. If $f(x) = 0$ in equations (1.14)-(1.16), the equation is homogenous. Otherwise, it is nonhomogenous

1.3 Some Basic Integral Transforms

Another frequently used tool in the classical (integer) calculus are integral transforms. The most important integral transforms are the Laplace and Fourier transforms.

Let $f(x)$ be a function in a certain function space; the classical integral transform (Fourier transform) is given by

$$(kF(x)) = \int_{-\infty}^{\infty} k(x, t)f(t)dt + g(t),$$

where $k(x, t)$ is a given function called the kernel of the transform, and g is the transform of the function f . And the Laplace transform for the function $f(x)$, $0 < x < \infty$ is given as

$$k(x, t) = \begin{cases} e^{-xt}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

and is also written as

$$L(f(p)) = \int_0^{\infty} e^{-pt} f(t) dt$$

The advantage of integral transform is that it makes mathematical problems to be solved easily after first being transformed by the integral transform. However, the result is often needed in the original domain so that the existence of an inverse transform is essential.

The inverse Fourier transform of a given function $f(x)$ in a certain function space is defined as

$$F^{-1}(f(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{px} f(p) dp.$$

Also, the inverse of the Laplace transform is defined as

$$L^{-1}(f(x)) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{ixt} f(t) dt.$$

In association with the stated integral transform, the idea and properties of convolution of two functions are quite useful

1.3.1 Convolution of Fourier and Laplace Transform:

The Fourier convolution of f and g denoted by $f * g$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt, \quad x \in \mathbb{R} \quad (1.17)$$

The Laplace convolution of f and g denoted by $f * g$ is defined as

$$(f * g)(x) = \int_0^{\infty} f(x-t)g(t) dt, \quad x > 0 \quad (1.18)$$

The convolutions in equations (1.16) and (1.17) satisfy the following properties :

1. Commutativity, $f * g = g * f$
2. Associativity, $f * (g * h) = (f * g) * h$
3. Distributivity, $f * (g + h) = f * g + f * h$
4. Associativity with scalar multiplication, $a(f * g) = (af) * g = f * (a * g)$

1.4 Some Basic Special Functions of Fractional Calculus

Fractional calculus is associated with the generalization of special functions which are frequently used in many results of classical calculus. Such functions include Mittag-Leffler function, Beta function, Euler's Gamma function known as the generalization of factorials.

1.4.1 Gamma Function:

Gamma function denoted by $\Gamma(z)$ for $\mathbb{Z}^+ \setminus \{0, -1, -2, -3, \dots\}$

is defined as

$$\Gamma(z) = \begin{cases} \int_0^{\infty} t^{z-1} e^{-t} dt, & \text{if } \operatorname{Re}(z) > 0 \\ \frac{\Gamma(z+1)}{z}, & \text{if } \operatorname{Re}(z) \leq 0, \quad z \neq 0, -1, -2, \dots \end{cases}$$

This is highly defined in the whole complex plane except for zero and negative integers.

1.4.2 Properties of Gamma Function:

The properties of Gamma function are:

1. $\Gamma(1 + z) = z\Gamma(z)$ for $z \in \mathbb{C}$ except $\{0, -1, -2, \dots\}$;
2. $\Gamma(n) = (n - 1)!$ for $n \in \mathbb{N}$;
3. $\Gamma(1 - z) = -z\Gamma(-z)$ for $z \in \mathbb{C}$ except $\{0, -1, -2, \dots\}$;
4. for $\operatorname{Re}(z) > 0$, then the following holds

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1)(z+2) \cdots (z+n)}$$

5. Gamma function is never zero

The generalized binomial coefficients

$$\binom{a}{k} = \frac{(-1)^{k-1} \Gamma(k-a)}{\Gamma(1-a) \Gamma(k+1)} = \frac{a(a-1)(a-2) \cdots (a-k+1)}{k!}$$

1.4.3 Beta Function:

Beta function denoted by $B(m, n)$ is defined as

$$B(m, n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt, \quad \operatorname{Re}(m) > 0, \operatorname{Re}(n) > 0.$$

Relationship between Gamma and Beta functions is given as

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad \operatorname{Re}(m) > 0, \operatorname{Re}(n) > 0.$$

1.4.4 Mittag-Leffler Function:

The Mittag-Leffler function plays an essential role in fractional calculus and Mittag-Leffler function of one parameter a denoted by $E_a(x)$ is defined as

$$E_a(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(aj+1)}, \quad \operatorname{Re}(a) > 0$$

such that when $a = 1$ and $a = 2$, $E_1(x) = e^x$ and $E_2(x) = \cosh(\sqrt{x})$ are obtained. And the Mittag-Leffler function of two parameters a and β denoted by $E_{a,\beta}(x)$ is defined as

$$E_{a,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(aj+\beta)}, \quad (a > 0, \beta > 0)$$

1.5 Fractional Derivatives and Integrals

Here, Riemann-Liouville integral, Riemann-Liouville fractional derivatives and Caputo fractional derivative, which are fundamental to the theory of fractional derivatives and integrals are introduced in this section.

1.5.1 Riemann-Liouville Fractional Integral:

The Riemann-Liouville integral operator of order m denoted by J_a^m is defined by taking $m \in \mathbb{R}^+$, the operator J_a^m on $L^1(a, b)$ by (Diethelm) is represented as

$$J_a^m f(x) = \frac{1}{\Gamma(m)} \int_a^x (x - \tau)^{m-1} f(\tau) d\tau, \quad a \leq x \leq b$$

. Assumption: for $m = 0$, the identity operator is set as $J_a^0 f(x) = f(x)$.
for $m = 0$, the identity operator is set as $J_a^0 f(x) = 1$.

1.5.2 Riemann-Liouville Fractional derivative:

Riemann Liouville fractional derivative denoted by $D_a^q(f(t))$ is defined as follows

$$\begin{aligned} D_a^q f(t) &= D_a^m D_a^{q-m} = D_a^m J_a^{m-q}; & q > 0 \\ &= D_a^m \frac{1}{\Gamma(m-q)} \int_a^t (t - \tau)^{m-q-1} f(\tau) d\tau, & (1.19) \\ & & q > 0, m - 1 < q < m \end{aligned}$$

where, $D_a^m = \frac{d^m}{dt^m}$ is the derivative part and $D_a^{p-m} f(x) = J_a^{m-q}$ is the integral part.

1.5.3 Caputo Fractional Derivative:

Let $m-1 < q < m$, and $q > 0$, Caputo fractional derivative denoted by $D_x^q f(x)$ is defined as follows

$$D_x^q f(x) = D_x^{q-m} [D^m f(x)] = J^{m-q} [D^m f(x)]; \quad m-1 < q < m$$

$$= \frac{1}{\Gamma(m-q)} \int_0^x (x-\tau)^{m-q-1} [D^m f(\tau)] d\tau;$$

$$m-1 < a < m \quad (1.20)$$

$$= D^m f(x); \quad a = m; m \in \mathbb{N}$$

$$1. J^a J^v f(x) = J^{a+v} f(x), \quad a, v > 0$$

Hence, we have the following properties

2. $J^a x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+a+1)} x^{\beta+a}$
3. $D^a x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-a+1)} x^{\beta-a}$
4. $J^a D^a f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, m-1 < a \leq m$

1.5.4 Prove of the Linearity of the Method

Here, a Caputo fractional derivative operator D^a is shown as a linear operator such that

$$D^a(c_1 f_1 + c_2 f_2) = c_1 D^a f_1 + c_2 D^a f_2.$$

Proof: By definition if

$$D_t^a f(t) = \frac{1}{\Gamma(m-a)} \int_0^t \frac{f^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau, \quad m-1 < a < m$$

then,

$$\begin{aligned}
D^a(c_1 f_1 + c_2 f_2) &= \frac{1}{\Gamma(m-a)} \int_0^t \frac{(c_1 f_1 + c_2 f_2)^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau \\
D^a(c_1 f_1 + c_2 f_2) &= \frac{1}{\Gamma(m-a)} \int_0^t \frac{(c_1 f_1^{(m)} + c_2 f_2^{(m)}) (\tau)}{(x-\tau)^{1+a-m}} d\tau \\
&= \frac{1}{\Gamma(m-a)} \int_0^t \frac{c_1 f_1^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau + \frac{1}{\Gamma(m-a)} \int_0^t \frac{c_2 f_2^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau \\
&= \frac{c_1}{\Gamma(m-a)} \int_0^t \frac{f_1^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau + \frac{c_2}{\Gamma(m-a)} \int_0^t \frac{f_2^{(m)}(\tau)}{(x-\tau)^{1+a-m}} d\tau \\
&= c_1 D^a f_1 + c_2 D^a f_2
\end{aligned}$$

Prove

1.6 Applications of Fractional Differential Equations

Fractional diffusion equation, fractional viscoelastic differential equation and fractional advection-dispersion equation, which are fractional differential equations, are widely used as modelling tools in the field of science and engineering, for instance;

1. in the modelling of electrical circuits, it has been discovered that electrical circuits have substance representing an element with fractional order (Debnath, 2003);
2. in the viscoelasticity model of blood flow, it has been suggested that fractional order stress-strain relation seems to be more adequate in providing a qualitative description of heterogeneous

tissue with bio-mechanical properties in blood flow than the integer-order stress-strain model (Craiem and Magin, 2010). The Hooke's law for elastic solids and Newton's law for viscous liquids are integer-order stress-strain model;

3. the fractional diffusion model of cardiac electrical propagation has described the electrical excitation in cable equations (Bueno et al, 2014);
4. moreso, (Debnath, 2003) revealed that the fractional advection-dispersion equation is used to model the transport of passive tracers moved by fluid flow in porous medium

1.7 Aim and Objectives of the Study

The aim of this research is to apply the three proposed numerical method to solve singular and multi-order fractional integro-differential equations of the Volterra types.

The main objectives of the work are to :

1. construct canonical polynomials to be used as basis functions for the general class of singular and multi-order fractional integro-differential equations;
2. demonstrate the formulation of the proposed methods on the

general class of equations (1) and (2) using the constructed canonical polynomials as basis functions;

3. carry out the formulated three proposed methods on some numerical examples;
4. determine the absolute errors of the approximation obtained from the three proposed methods when applied to some numerical examples; and
5. compare the results obtained with the available results of methods in the literature.

1.8 Significance of the Study

Concisely, the three proposed numerical methods are highly significant as stated below:

1. solutions of the problems that cannot be solved analytically in the interval of consideration are easily solved;
2. the three proposed methods produce solutions to a high level of accuracy, even at lower degree of approximant;
3. the three proposed numerical methods are efficient in solving singular and multi-order fractional integro-differential equations; and

4. the proposed methods are efficient in solving varieties of singular and multi-order fractional integro-differential equations of the Volterra types.

Chapter 2

LITERATURE REVIEW

2.1 Discussion of Existing Methods

A few methods of solutions proposed by some authors as revealed in the literature are discussed. Each of these methods was independently used based on their unique properties for the solutions of both linear and nonlinear fractional order integro-differential equations. Several problems have been conveniently handled using these methods and the results of their solutions were found to be highly accurate and efficient.

2.1.1 Lanczos Tau Method

Truncating the series solution of a differential equation is sometimes similar to introducing a perturbation term to the equation (Lanczos, 1938). The truncated series can however be gotten by introducing a polynomial perturbation term. Tau method as a tool in the discussion

of differential equations in mathematical analysis which is inevitable was first introduced by Lanczos (1938), while its convergence analysis and error bounds were discussed in Lanczos (1956, 1973). The

recursive form of the method was formalized in Ortiz (1969) and extended to the case of system of ordinary differential equations in Freilich and Ortiz (1982). The Philosophy of the method was extended to the numerical solutions of linear and nonlinear initial value problems, boundary value problems and mixed problems as discussed in Onumanyi and Ortiz (1984); to stiff problems in Onumanyi and Ortiz (1984); to solve eigenvalue problems in Liu and Ortiz (1987); and to partial differential equations in Namasivayam and Ortiz (1985); Ortiz and Pham (1984, 1985); and Luke (1969)

where tau method was implemented to obtain polynomial approximations for hypergeometric functions. (Ortiz, 1969) confirmed it by suggesting a procedure for generating canonical polynomials, in which it was revealed that by means of recursive relation, elements of canonical polynomials are generated so as to formulate schemes for the solution of differential equations. Moreso, As discussed in (Issa et al, 2017), accuracy has been beautifully affirmed with the use Chebyshev polynomials as basis function, adopting the Tau method.

Furthermore, in Lanczos (1956), the concept of Canonical polynomials was proposed so as to express Tau method approximations

as weighted arithmetic mean of successive partial sums of the power series solutions and obtaining approximations high precision. Canonical polynomials have been very effective in the recursive implementation of Tau methods. To be precise, in Alexandros (2018), a complete basis of Canonical polynomials generated through Oritz's recurrence formula was applied to various operator equations associated with infinite matrices in the lower row echelon form. Furthermore, a generalized approximation technique for the recursive formulation of Tau method using Canonical polynomials was reported in Issa and Adeniyi (2015) to solve nonlinear ordinary differential equations. The work is an extension of the generalized recursive Tau method of Issa and Adeniyi (2013).

2.1.2 Collocation Method

Collocation method is highly essential in the accuracy, efficiency and convergence of any method of solution of both linear and nonlinear differential equations. The performance of the technique has been investigated on several classes of problems. The concept of collocation approximation was first carried out by a Russian Mathematician (Pafnuity Lyvoicle Chebyshev, 1821-1894) and was eventually efficient in solving classes of problems governed by differential equation in the field of Science, Engineering and Social Science (Economics). Line collocation method, spectra collocation method

and integral collocation method have similar concepts with the collocation method, which have made them to be closely related. For instance, the method of line collocation was introduced by Kantorovich (1934) to solve the boundary value problems of partial differential equations; (Abubakar and Taiwo, 2014) used standard integral collocation approximation method to obtain numerical solutions of specific higher order linear integro-differential equations. Moreover, the findings of [Brunner (2006), Mason and Handscomb (2003)] for the approximation of unknown functions were based on orthogonal functions, where the unknown coefficients are obtained within the interval of consideration using a collocation method, known as pseudo-spectra collocation method. Perturbed Collocation method is another technique for differential equation of any kind. It is suitable for solving both linear and nonlinear problems, and the idea is based on the addition of small parameters, though it has a few limitations, which include the expression of the approximate solution in terms of parameters which are valid for small values of the parameters.

Moreover, Chebyshev collocation method is also a technique which could be used for providing approximate solution to integro-differential equations. For instance, in Cerdik and Akyuz (2006), the method was considered to solve Fredholm-Volterra integro differen-

tial equation. The integro-differential equation was first transformed into a system of algebraic equations with unknown constants using the collocation points. Also, in Ezzati and Najafalizadeh (2011), a Fredholm-Volterra integral equation was solved using the collocation method, where Chebyshev polynomial was chosen as the basis function. **2.1.3 Least Squares Approximation Method with Orthogonal Polynomials**

The representation of functions in terms of series expansions using orthogonal polynomials is a fundamental concept in approximation theory which forms the basis of many methods of solutions of differential equations. Basically, the idea is to assume that the unknown function $y(x)$ say approximated by a linear combination of some basis functions called trial solutions, of which orthogonal polynomials are examples. These polynomials are chosen according to the property that make them suitable for a particular problem under consideration. These authors worked only on fractional order integro differential equations, involving no integer orders. Least Squares Method has been studied in such a way that it can be implemented as a method of solution when it is applied on basis functions of which orthogonal polynomials are examples. For instance, the works of Mohammed (2014), Mahdy and Mohammed (2016) were based on shifted Chebyshev polynomials with Least Squares Method. Similarly, Mahdy and Shywayye (2016) adopted

the Least Squares method, where shifted Laguerre polynomials were used as a basis function.

2.1.4 Laplace Transform Method

Laplace transform method is a method for providing numerical solution to fractional order integro-differential equation. It was implemented in Habibollah (2011), Aghili and Zeinali (2013), Abdul-Majid and Jafari (2011), and Atanackovic et.al (2014). In Habibollah (2011), the Haar wavelet approximating method is used to reduce the fractional Volterra and Abel integral equations, whereby singular solutions are considered to demonstrate the validity and applicability of the developed method. In Aghili and Zeinali (2013), the author implemented Laplace transform method to solve Volterra singular integral equations and partial differential equations which are non-homogenous time fractional heat equation.

2.2 Singular and Multi-Order Fractional Integro-Differential Equations

The behaviour of some physical systems in Science and Engineering are governed by singular and multi-order fractional integro-differential equations. The equations usually occur as mathematical models, but are later translated into functional equations, which can be of the

form :

$$D^a u(t) + u^n(t) + \lambda \sum_{i=0}^{n-1} \alpha_i(t) u^i(t) + \int_0^t u^i(s) k_i(t, s) ds = f(t)$$

where $k_i(t, s) = \frac{1}{(t-s)^\rho}$ is the kernel, u is the function to be determined, $n \in \mathbb{N}$ and $n - 1 \leq a \leq n$. The main difficulty is that the singularity behaviour occurs at $t = s$.

Singular and multi-order fractional integro-differential equation can be used to describe elasticity mechanism, heat conduction, Dirichlet problems e.t.c. where the need for numerical methods is required in cases the analytical solutions are not known. Hence, the attentions of many researchers have been drawn towards developing appropriate methods that provide the approximate solutions. For instance, Modified Adomian Decomposition Method was developed by Kumar and Singh (2010), Cubic B-Spline Collocation Method by Cheney and Kincaid (2008), Modified Adomian Decomposition Method by Yahaya and Liu (2008), and the Differential Transform Method by Yasir et al (2012). The authors reported that due to the singularity behaviour of the problem under consideration, the construction of Adomian and Cubic Spline polynomials were difficult to obtain in the Modified Adomian Decomposition and Cubic Spline Methods respectively. In view of this, Canonical polynomials with the aid of Lanczos and Ortiz method are constructed to overcome the

drawbacks.

2.3 The Existence and Uniqueness Solution of Fractional Differential Equation

The existence and uniqueness of solutions are very important for any differential equation. Thus, the existence and uniqueness of solutions of fractional differential equations are verified here.

Considering the initial value problem with $m - 1 < a < m, m \geq 1$

$$D^a u(x) = f(x, y(x)) \quad (2.1)$$

subject to

$$u^k(0) = u_0^k, k = 0, 1, 2, \dots, m - 1 \quad (2.2)$$

where, $D^a u(x)$ denotes the Caputo fractional derivative of order $a > 0$ with $m - 1 < a < m$ then

$$D^a u(x) = \frac{1}{\Gamma(m - a)} \int_0^x (x - s)^{m-a-1} \frac{ds}{du}$$

Here follows the theorem of Diethelm and Ford (2002) describing the uniqueness and existence of the solution.

Theorem 1.8.1 (Existence) : Assuming $K = [0, S^*] \times [u_0 - a, u_0 + a]$ with some $S^* > 0$ and $a > 0$ such that the function $f : K \rightarrow \mathbb{R}$ is continuous. Furthermore, defining $S = \min(S^*, (\frac{a\Gamma(q+1)}{\|f\|_\infty})^{\frac{1}{q}})$.

Then there exists a function $u : [0, S] \rightarrow \mathbb{R}$ solving the initial value

problem.

Diethelm and Ford (2002)

Theorem 1.8.2 (Uniqueness): Assuming $K = [0, S^*] \times [u_0 - a, u_0 + a]$ with some $S^* > 0$ and $a > 0$, and the function $f : K \rightarrow \mathbb{R}$ be bounded and fulfil the Lipschitz condition such that $|f(x, u) - f(x, u_1)| \leq L|u - u_1|, L > 0$. Then there exists at most one solution $u : [0, S] \rightarrow \mathbb{R}$

Diethelm and Ford (2002)

Lemma 1.8.3: If the function f is continuous, then the initial value problem under consideration is equivalent to the Volterra integral equation. In other words, every solution of the Volterra integral equation is the solution of the initial valueproblem.

$$u(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} u^{(k)}(0) + \frac{1}{\Gamma(a)} \int_0^x (x-z)^{a-1} f(z, u(z)) dz \quad (2.3)$$

Diethelm and Ford (2002)

Theorem 1.8.4: Let U be a nonempty subset of a Banach space E , and $a_n \geq 0$ for every n and such that $\sum_{n=0}^{\infty} a_n$ converges. Furthermore, let the mapping $A : U \rightarrow U$ satisfy the the following inequality

$$||H_n u_1 - H_n u_2|| \leq a_n ||u_1 - u_2||, n \in \mathbb{N}, u_1, u_2 \in U$$

which implies that H has a unique point u^* such that for any $u_0 \in U$,

the sequence $(H_n)_{n=1}^{\infty}$ converges to the unique point u^*

2.4 Error Analysis of Fractional Differential Equation

Considering an equation with $0 < \beta < 1$ of the form

$$D^\beta y(t) = By(t) + \int_0^t \frac{y(t)}{\sqrt{x-t}} dt + f(t), \quad y(0) = y_0 \quad (2.4)$$

Taking I^a to both sides of equation (2.4), that is

$$I^a D^\beta y(t) = I^a By(t) + I^a \int_0^t \frac{y(t)}{\sqrt{x-t}} dt + I^a f(t), \quad 0 \leq t \leq 1,$$

to give

$$y(t) = y_0 + I^a By(t) + I^a \int_0^t \frac{y(t)}{\sqrt{x-t}} dt + I^a f(t), \quad (2.5)$$

$$0 \leq t \leq 1,$$

where the order of the derivative is β ; the given function on the interval $[0, 1]$ is f , $\beta \leq 0$ and the unknown function is y .

The solution of the equation (1.24) is taken to be of the form

$$y(t)_i = \frac{1}{(a_{0i}) - t_i \Gamma(-a)\beta} [t_i \Gamma(-a) f(t)_i - \sum_{k=1}^i a_{ki} y(t - k)_i + y_0 \sum_{k=0}^i a_{ik} - R_i], \quad (2.6)$$

where,

$$R_j \leq C j^{a-2} t_j^2 \|y^\sharp\|_\infty$$

And,

$$y_i = \frac{1}{(a_{0i}) - t_i \Gamma(-a) \beta} [t_i \Gamma(-a) f(t_i) - \sum_{k=1}^i a_{ki} y_{i-k} + y_0 \sum_{k=0}^{i-1} a_{ik}], \quad (2.7)$$

The terms $y(t_i)$ and y_i are assumed to be taken as the exact and approximate solutions respectively. If the function is a smooth function, then there exists a constant C such that

$$|y(t_i) - y_i| \leq C h^{2-a} \|y^\sharp\|_\infty, \quad j = 1, 2, \dots, n \quad (2.8)$$

Let,

$$e_j = y(t_i) - y_i$$

Then the error equation is obtained by subtracting equation (2.6) from (2.7) to give

$$e_j = \frac{1}{(a_{0i}) - t_i \Gamma(-a) \beta} \left[\sum_{k=1}^i a_{ij} e_{i-j} - R_i \right], \quad (2.9)$$

where,

$$a_{0i} = \frac{1}{-a(1-a)j^{-a}} < 0, \quad \Gamma(-a) < 0, \quad \beta < 0, \quad a_{ki} < 0$$

and,

$$\begin{aligned}
|e_j| &\leq \sum_{k=1}^i a_{kj} e_{j-k} + R_i \\
&\leq a(1-a)j^{-a} \sum_{k=1}^i a_{kj} e_{j-k} + j^{a-2} t_i^2 \|y^\sharp\|_\infty
\end{aligned} \tag{2.10}$$

Expanding equation (2.10)

$$\begin{aligned}
&\leq a(1-a)j^{-a} \sum_{k=1}^i a_{kj} e_{j-k} + R_i \\
&\leq a(1-a)j^{-a} \sum_{k=1}^i a_{kj} e_{j-k} + a(1-a)j^{-a} j^{-2} t_i^2 \|y^\sharp\|_\infty \\
&\leq a(1-a)j^{-a} \sum_{k=1}^i a_{kj} e_{j-k} + a(1-a)j^{-a} h^2 \|y^\sharp\|_\infty
\end{aligned} \tag{2.11}$$

Let $b = a(1-a)j^{-a} h^2 \|y^\sharp\|_\infty$

Thus,

$$|e_j| \leq b + a(1-a)j^{-a} \sum_{k=1}^i a_{kj} |e_{j-k}| \tag{2.12}$$

2.5 Taylor's Series Method

Considering the one dimensional initial value problem

$$y'(x) = f(x, y) \quad y(x_0) = y_0$$

where f , is a function of two variables x and y and (x_0, y_0) is a known point on the solution curve, otherwise known as initial point. If the existence of all higher order partial derivatives is assumed for y at $x = x_0$, then by Taylor's series the value of y at any neighbouring point $x + h$ is written as

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \dots \quad (2.13)$$

where $y'(x)$ represents the derivative of y with respect to x since at x_0, y_0 is known, and $y'(x_0)$ is obtained by computing $f(x_0, y_0)$.

Similarly higher derivatives of y at x_0 is also computed by using the relation

$$y' = f(x, y) \quad (2.14)$$

$$y'' = f_x + f_y y' \text{ or } f'(x_i, y_i) \quad (2.15)$$

$$y''' = f_{xx} + 2f_{xy}y' + f_{yy}y'^2 + f_y y'' \quad (2.16)$$

then,

$$y(x_0 + h) = y(x_0) + hf + \frac{h^2}{2!}(f_x + f_y y') + \frac{h^3}{3!}(f_{xx} + 2f_{xy}y' + f_{yy}y'^2 + f_y y'') + 0h^4 \quad (2.17)$$

Hence the value of y at any neighbouring point $x_0 + h$ is obtained by assuming the above infinite series, the summation has to be terminated after the p^{th} derivative term. Thus, the approximated formula is called the Taylor's series approximation to y of order p and error is of order $p + 1$. The same process can be repeated to obtain y at other points of x in the interval $[x_0, x_n]$.

The error in the approximation is determined in such a way that the Taylor's series method of order p having the property that the final global error is of order $O(h^{p+1})$, hence p is chosen as large as possible to make the error as small as desired.

2.6 General Literature Review

Based on the significant roles of fractional order differential and integro differential equations which govern the behaviour of many physical systems in science and engineering (Bagley and Tarvik, 1984), considerable attentions were drawn towards the numerical solutions of differential equations which have no analytical solutions, especially the nonlinear fractional differential equations, which had led many researchers to propose several approximation methods of solutions of ordinary differential equations, partial differential equations, integro-differential equations, fractional order differential equa-

tions and fractional integro differential equations e.t.c., basically arising from the fields of science, engineering and social sciences. The differential equations mentioned above, mostly occur in form of mathematical models, which are later translated into functional equations, otherwise known as differential equations.

The interest of the researchers is to obtain efficient and accurate approximate solutions to functional equations having no analytical solutions. Explicitly, in Applied Mathematics such as gas dynamics, mechanics, fluid mechanics, reaction diffusion process, where some problems usually occur in form of singular differential equation as discussed in Chawla and Katti (1982). Similarly, Adomian Decomposition method was applied independently for the numerical solution of singular differential equation in Fillipova and Zdenk (2010), Kumar and Singh (2010), and Odetunde (2013). Moreso, numerical solution of fourth order fractional integro differential equations was investigated in Da Xu et al (2019), by constructing a compact finite difference. These numerical methods stated are highly advantageous but they require much computational works. Similarly, as regards singular integro differential equations, several researches have been carried out in this area due to its wide area of application, specifically in heat and mass diffusion processes, dispersive waves and population models e.t.c. And because most of the

equations of this form have no analytical solution, so the need for efficient methods to provide the approximate solution is necessitated. Consequently, Galerkin method by Mohammed (2014), Collocation method by Brunner (2007), Wavelet method by Bhattacharya and Mandal (2007), and Haar wavelet method by Ghanbari (2010). These methods were applied to find solutions to problems arising as singular integro differential equations, but were tedious, cumbersome and rigorous in terms of computational cost. Therefore these newly proposed methods had been developed to overcome the drawbacks.

Chapter 3

METHODOLOGY

3.1 Introduction

Canonical polynomials was introduced by Lanczos along with the Tau method and later used by Ortiz (1969) for the recursive formulation of the method. The extension of Ortiz recursive formulation of the Tau method to linear system of ordinary differential equations was carried out by Crisci and Russo (1983) in which a differential problem of the form

$$\begin{aligned} A(x)y'(x) + B(x)y(x) + F(x) &= 0, x \in [x_0, x_n], \\ y(x_0) &= y_0 \end{aligned} \tag{3.1}$$

was written as

$$Dy(x) + F(x) = 0, \quad x \in [x_0, x_n], y(x_0) = y_0, \tag{3.2}$$

where $D = A(x)\frac{d}{dx} + B(x)$

The idea of Lanczos (1956, 1973) was adopted in such a way that the solution to equation (3.1) was approximated by a polynomial of certain degree. Similarly, in Ortiz (1975), a step by step approach to Tau method was developed, and the concept was based on the division of the interval of integration of the range $[x_0, x_n]$ into the subintervals $[x_m, x_{m+1}]$, $m = 0, 1, \dots, n - 1$. Denoting with y_m the approximate value of $y(x)$ in x_m , the differential equation

$$Dy(x) + F(x) = 0, \quad x \in [x_n, x_{n+1}], \quad y(x_n) = y_n \quad (3.3)$$

has to be solved for each interval $[x_n, x_{n+1}]$ so as to give $y_{n+1} = y(x_{n+1})$. Hence, in order to facilitate the construction of the solution, canonical polynomials was introduced.

Ortiz (1969) pointed out that the advantage of introducing canonical polynomials was manifold in the sense that the solution $y(x)$ is easily expressed as a linear combination of polynomials which are independent of the integration range and initial conditions.

3.1.1 Description of the Problem Considered

In this section, a general singular and multi-order fractional integro-differential equation of the form:

$$D^a y(x) + \sum_{i=0}^{\infty} p_i y^i(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x); \quad (3.4)$$

$$a \leq x \leq b$$

is considered, subject to the conditions

$$y^k(0) = a_k, k = 0, 1, 2, \dots, n-1 \quad (3.5)$$

Equation (3.4) is equivalently written as

$$D^a y(x) + \sum_{i=0}^n p_i y^i(x) = f(x) - \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \quad (3.6)$$

Equation (3.6) is re-written as

$$D^a y(x) + \sum_{i=0}^n p_i y^i(x) = g(x), \quad (3.7)$$

where,

$$g(x) = f(x) - \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt$$

λ and $p_i (i \geq 0)$ are constant parameters. Also, $y(x)$ and $g(x)$ are functions of x , where x and t are chosen independent variables.

Expanding equation (3.7), we get

$$\begin{aligned} D^a y(x) + p_0 y(x) + p_1 y'(x) + p_2 y''(x) + \dots + \\ p_n y^n(x) = g(x), \end{aligned} \quad (3.8)$$

or,

$$\begin{aligned} D^a y(x) + p_0 y(x) + p_1 \frac{d}{dx} y(x) + p_2 \frac{d^2}{dx^2} y(x) + \dots + \\ p_n \frac{d^n}{dx^n} y(x) = g(x), \end{aligned} \quad (3.9)$$

or,

$$(D^a + p_0 + p_1 \frac{d}{dx} + p_2 \frac{d^2}{dx^2} + \dots + p_n \frac{d^n}{dx^n}) y(x) = g(x), \quad (3.10)$$

such that

$$L = (D^a + p_0 + p_1 \frac{d}{dx} + p_2 \frac{d^2}{dx^2} + \dots + p_n \frac{d^n}{dx^n}). \quad (3.11)$$

3.1.2 Construction of Canonical Polynomials

From the general singular and multi-order fractional order integro-differential equations stated in equations (3.4) and (3.5), we defined an operator denoted by L as follows:

$$L \equiv (D^a + p_0 + p_1 \frac{d}{dx} + p_2 \frac{d^2}{dx^2} + \dots + p_n \frac{d^n}{dx^n}), \quad (3.12)$$

or simply as:

$$L \equiv D^a + \sum_{i=0}^n p_i \frac{d^i}{dx^i}. \quad (3.13)$$

Following Lanczos (1956), we defined canonical polynomials $Q_j(x), j \leq 0$, which are uniquely associated with the operator L by

$$LQ_j(x) = x^j, \quad j = 0, 1, 2, 3, \dots \quad (3.14)$$

From equation (3.12) using equation (3.13), it follows that:

$$\begin{aligned} Lx^j &= \frac{\Gamma(j+1)}{\Gamma(j+1-a)} x^{j-a} + p_0 x^j + p_1 j x^{j-1} + \\ & p_2 j(j-1) x^{j-2} + \dots + p_n j(j-1)(j-2)\dots \\ & (j-n+1) x^{j-n}. \end{aligned} \quad (3.15)$$

This implies,

$$\begin{aligned} L[LQ_j(x)] &= \frac{\Gamma(j+1)}{\Gamma(j+1-a)} LQ_{j-a} + p_0 LQ_j(x) + \\ & p_1 j LQ_{j-1}(x) + p_2 j(j-1) LQ_{j-2}(x) + \dots + \\ & p_n j(j-1)(j-2)\dots(j-n+1) LQ_{j-n}(x) \end{aligned} \quad (3.16)$$

Hence,

$$\begin{aligned}
L^{-1}L[LQ_j(x)] &= \frac{\Gamma(j+1)}{\Gamma(j+1-a)}L^{-1}LQ_{j-a}+ \\
&p_0L^{-1}LQ_j(x) + p_1jL^{-1}LQ_{j-1}(x)+ \\
&p_2j(j-1)L^{-1}LQ_{j-2}(x) + \dots + \\
&p_nj(j-1)(j-2)\dots(j-n+1)L^{-1}LQ_{j-n}(x)
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
x^j &= \frac{\Gamma(j+1)}{\Gamma(j+1-a)}Q_{j-a} + p_0Q_j(x) + p_1jQ_{j-1}(x)+ \\
&p_2j(j-1)Q_{j-2}(x) + \dots + p_nj(j-1)(j-2)\dots \\
&(j-n+1)Q_{j-n}(x)
\end{aligned} \tag{3.18}$$

Which implies

$$\begin{aligned}
Q_j(x) &= \frac{1}{p_0} \left\{ x^j - \frac{\Gamma(j+1)}{\Gamma(j+1-a)}Q_{j-a} - p_1jQ_{j-1}(x) - \right. \\
&p_2j(j-1)Q_{j-2}(x) - \dots - p_nj(j-1)(j-2)\dots \\
&\left. (j-n+1)Q_{j-n}(x) \right\}; \quad j \geq 0.
\end{aligned} \tag{3.19}$$

Note here that a is a fraction, hence the term Q_{j-a} is further simplified using the definition

$$LQ_j(x) = x^j; j = 0, 1, 2, \dots$$

Therefore, by replacing $j = j - a$ in $(LQ_j(x) = x^j)$ above, and multiplying by L^{-1} , it leads to

$$\begin{aligned}
Q_{j-a}(x) &= L^{-1}x^{j-a} \\
LQ_{j-a}(x) &= LL^{-1}x^{j-a} \\
LQ_{j-a}(x) &= x^{j-a} \\
LQ_{j-a}(x) &= x^{j-1}.x^{1-a} \\
LQ_{j-a}(x) &= LQ_{j-1}(x)LQ_{1-a}(x)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{LQ_{j-a}(x)}{LQ_{j-1}(x)} &= LQ_{1-a}(x) \\
\Rightarrow L^{-1}\left[\frac{LQ_{j-a}(x)}{LQ_{j-1}(x)}\right] &= L^{-1}LQ_{1-a}(x)
\end{aligned}$$

such that

$$\begin{aligned}
\frac{L^{-1}LQ_{j-a}(x)}{L^{-1}LQ_{j-1}(x)} &= L^{-1}LQ_{1-a}(x) \\
\frac{Q_{j-a}(x)}{Q_{j-1}(x)} &= Q_{1-a}(x) \\
\Rightarrow Q_{j-a}(x) &= Q_{j-1}.x^{1-a}
\end{aligned}$$

From equation (3.14); that is,

$$LQ_j(x) = x^j$$

Implies,

$$LQ_{1-a}(x) = x^{1-a}.$$

Therefore,

$$Q_{j-a}(x) = Q_{j-1}(x).x^{1-a}.$$

Hence, on substitution into equation (3.19), it gives

$$Q_j(x) = \frac{1}{p_0} \left\{ x^j - \frac{\Gamma(j+1)}{\Gamma(j+1-a)} Q_{j-1}(x) \cdot x^{1-a} - p_1 j Q_{j-1}(x) - \right. \\ \left. p_2 j(j-1) Q_{j-2}(x) - \cdots - p_n j(j-1)(j-2) \cdots \right. \\ \left. (j-n+1) Q_{j-n}(x) \right\}; \quad j \geq 0.$$

Thus,

$$Q_j(x) = \frac{1}{p_0} \left\{ x^j - (x^{1-a} \frac{\Gamma(j+1)}{\Gamma(j+1-a)} + p_1 j) Q_{j-1}(x) - p_2 j(j-1) Q_{j-2}(x) - \right. \\ \left. \cdots - p_n j(j-1)(j-2) \cdots (j-n+1) Q_{j-n}(x) \right\}; \quad j \geq 0.$$

Hence, the following set of canonical polynomials are obtained:

$$\begin{aligned}
Q_0(x) &= \frac{1}{P_0} \\
Q_1(x) &= \frac{1}{P_0^2} \left(x p_0 - p_1 - \frac{x^{1-a}}{\Gamma(2-a)} \right) \\
Q_2(x) &= \frac{1}{P_0^3} \left(x^2 p_0^2 - 2 x p_0 p_1 - 2 p_0 p_2 + 2 p_1^2 - \frac{(p_1 a + x p_0 - 3 p_1) 2 x^{1-a}}{\Gamma(3-a)} \right) \\
Q_3(x) &= \frac{1}{P_0^4} \left(x^3 p_0^3 - 3 x^2 p_0^2 p_1 - 6 x p_0^2 p_2 + 6 x p_0 p_1^2 - 6 p_0^2 p_3 + 12 p_0 p_1 p_2 - 6 p_1^3 + \right. \\
&\quad \left(a^2 p_0 p_2 - a^2 p_1^2 - a x p_0 p_2 - a^2 p_0^2 - 5 a p_0 p_2 + 6 a p_1^2 + 5 x p_0 p_2 + 8 p_0 p_2 - \right. \\
&\quad \left. 11 p_1 \right) \frac{(6 x^{1-a})}{\Gamma(4-a)} + (x^{2-2a} a p_1 - 7 x^{2-2a} a p_1 - 2 x^{3-2a} p_0 + 12 p_1 x^{2-2a}) 6 \\
&\quad \left. (a-3) \Gamma(4-a)^2 + (a-2)(a-3)^2 \frac{12 x^{3-3a}}{\Gamma(4-a)^3} \right) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}$$

3.2 Discussion of Methods

3.2.1 Standard Collocation Method

Here, a single polynomial approximation $y_N(x)$ of the unique solution $y(x)$ of equations (3.4) and (3.5) in the form

$$y(x) \approx y_N(x) = \sum_{j=0}^N a_j Q_j(x), \quad a \leq x \leq b, \quad (3.20)$$

is assumed. The unknown constants a_j 's are to be determined and Q_j 's are the canonical polynomials constructed in subsection 3.1.2. On substituting (3.20) into the general singular and multi-order fractional order integro differential equation (3.4), we obtain

$$\begin{aligned} & D^a \left(\sum_{j=0}^N a_j Q_j(x) \right) + p_0 \sum_{j=0}^N a_j Q_j(x) + p_1 \sum_{j=0}^N a_j Q_j(x) + \\ & p_2 \sum_{j=0}^N a_j Q_j^1(x) + \dots + p_n \sum_{j=0}^N a_j Q_j^n(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \\ & = g(x) \end{aligned} \quad (3.21)$$

Collecting the like terms in equation (3.21), we obtain

$$\begin{aligned} & \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0(x) + \dots + p_n Q_0^n(x)\} a_0 + \\ & \{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1^1(x) + \dots + p_n Q_1^n(x)\} a_1 + \\ & \dots + \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N^1(x) + \dots + p_n Q_N^n(x)\} a_N \\ & + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = g(x) \end{aligned} \quad (3.22)$$

Thus, the integral in equation (3.22) was evaluated by adopting a Taylor series expansion of appropriate order as shown below,
Taking $t = x + h$, we obtain

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \dots \quad (3.23)$$

Hence,

$$y(t) = y(x) + (t - x)y'(x) + \frac{(t - x)^2}{2!}y''(x) + \dots \quad (3.24)$$

Substituting equation (3.24) into equation (3.22), we obtain

$$\begin{aligned} & [D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0'(x) + \dots + p_n Q_0^{(n)}(x)]a_0 + \\ & [D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1'(x) + \dots + p_n Q_1^{(n)}(x)]a_1 \\ & + \dots + [D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N'(x) + \dots + p_n Q_N^{(n)}(x)]a_N + \lambda \int_0^x \frac{1}{\sqrt{x-t}} \{ y_N(x) + (t-x)y_N'(x) + \\ & \frac{(t-x)^2}{2!}y_N''(x) + \dots + \frac{(t-x)^n}{n!}y_N^{(n)}(x) \} dt = g(x) \end{aligned} \quad (3.25)$$

Equation (3.25) is re-written as

$$\begin{aligned} & [D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0'(x) + \dots + p_n Q_0^{(n)}(x)]a_0 + \\ & [D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1'(x) + \dots + p_n Q_1^{(n)}(x)]a_1 + \\ & \dots + [D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N'(x) + \dots + p_n Q_N^{(n)}(x)]a_N + y_N(x) \int_0^x \frac{1}{\sqrt{x-t}} dt + y_N'(x) \lambda \int_0^x \frac{(t-x)}{\sqrt{x-t}} dt + \frac{y_N''(x)}{2!} \lambda \int_0^x \frac{(t-x)^2}{\sqrt{x-t}} dt + \dots + \frac{y_N^{(n)}(x)}{n!} \lambda \int_0^x \frac{(t-x)^n}{\sqrt{x-t}} dt = g(x) \end{aligned} \quad (3.26)$$

Hence, evaluating the integrals in equation (3.26), we get

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0'(x) + \dots + p_n Q_0^{(n)}(x)\} a_0 + \\
& \{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1'(x) + \dots + p_n Q_1^{(n)}(x)\} a_1 + \\
& \dots + \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N'(x) + \dots + p_n Q_N^{(n)}(x)\} a_N + \lambda \left\{ 2 \frac{x^1}{x^2} y_N'(x) - \frac{2}{3} \frac{x^3}{x^2} y_N''(x) + \frac{2}{5} \frac{x^5}{x^2} \frac{y_N'''}{2!} - \right. \\
& \left. \frac{2}{7} \frac{x^7}{x^2} \frac{y_N^{(4)}}{4!} + \dots \right\} = g(x)
\end{aligned} \tag{3.27}$$

where,

$$\begin{aligned}
y_N(x) &= \sum_{i=0}^N Q_i(x) = a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) + \\
&\quad \dots + a_n Q_N(x) \\
y_N'(x) &= \sum_{i=0}^N Q_i'(x) = a_0 Q_0'(x) + a_1 Q_1'(x) + a_2 Q_2'(x) + \\
&\quad \dots + a_n Q_N'(x) \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
y_N^n(x) &= \sum_{i=0}^N Q_i^{(n)}(x) = a_0 Q_0^{(n)}(x) + a_1 Q_1^{(n)}(x) + a_2 Q_2^{(n)}(x) + \\
&\quad \dots + a_n Q_N^{(n)}(x)
\end{aligned} \tag{3.28}$$

Hence, equation (3.27) becomes

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0(x) + \cdots + \\
& p_n Q_0^n(x)\} a_0 + \{D^a Q_1(x) + p_0 Q_1(x) + \\
& p_1 Q_1(x) + \cdots + p_n Q_1^n(x)\} a_1 + \cdots + \\
& D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N(x) + \cdots + \\
& p_n Q_N^n(x) a_N + 2\lambda x^{\frac{1}{2}} \{a_0 Q_0(x) + a_1 Q_1(x) + \\
& a_2 Q_2(x) + \cdots + a_n Q_N(x)\} - \frac{2}{3} \lambda x^{\frac{3}{2}} \{a_0 Q_0(x) + \\
& a_1 Q_1(x) + a_2 Q_2(x) + \cdots + a_n Q_N(x)\} + \frac{2}{10} \lambda x^{\frac{5}{2}} \{ \\
& a_0 Q_0^{\text{II}}(x) + a_1 Q_1^{\text{II}}(x) + a_2 Q_2^{\text{II}}(x) + \cdots + a_n Q_N^{\text{II}}(x)\} - \\
& \frac{2}{42} \lambda x^{\frac{7}{2}} \{a_0 Q_0^{\text{III}}(x) + a_1 Q_1^{\text{III}}(x) + a_2 Q_2^{\text{III}}(x) + \cdots + a_n Q_N^{\text{III}}(x) \\
& \} + \cdots = g(x)
\end{aligned} \tag{3.29}$$

Collecting like terms in equation (3.29), we obtain

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0(x) + \dots + \\
& p Q_0^n(x) + 2 \frac{1}{\lambda x^2} Q_0(x) - \frac{2}{3} \frac{1}{\lambda x^2} Q_0(x) + \\
& \frac{1}{2} \frac{1}{\lambda x^2} Q_0(x) - \frac{1}{10} \lambda x^2 Q_0(x) - \frac{1}{42} \lambda x^2 Q_0(x)\} a_0 + \{D Q_1(x) + \\
& p_0 Q_1(x) + p_1 Q_1(x) + \dots + p_n Q_1(x) + 2 \lambda x^2 Q_1(x) - \\
& \frac{2}{3} \lambda x^2 Q_1(x) + \frac{2}{10} \lambda x^2 Q_1(x) - \frac{2}{42} \lambda x^2 Q_1(x)\} a_1 + \\
& \dots + \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N(x) + \dots + \\
& p Q_N^n(x) + 2 \frac{1}{\lambda x^2} Q_N(x) - \frac{2}{3} \frac{1}{\lambda x^2} Q_N(x) + \\
& \frac{1}{2} \frac{1}{\lambda x^2} Q_N(x) - \frac{1}{10} \lambda x^2 Q_N(x) - \frac{1}{42} \lambda x^2 Q_N(x)\} a_N = g(x)
\end{aligned} \tag{3.30}$$

Thus, collocating equation (3.30) at point $x = x_i$, we obtain,

$$\begin{aligned}
& \{D^a Q_0(x_i) + p_0 Q_0(x_i) + p_1 Q_0(x_i) + \dots + \\
& p Q_0^n(x) + 2 \frac{1}{\lambda x_i^2} Q_0(x) - \frac{2}{3} \frac{1}{\lambda x_i^2} Q_0(x) + \\
& \frac{1}{2} \frac{1}{\lambda x_i^2} Q_0(x) - \frac{1}{10} \lambda x_i^2 Q_0(x) - \frac{1}{42} \lambda x_i^2 Q_0(x)\} a_0 + \{ \\
& D^a Q_1(x_i) + p_0 Q_1(x_i) + p_1 Q_1(x_i) + \dots + \\
& p Q_1^n(x) + 2 \frac{1}{\lambda x_i^2} Q_1(x) - \frac{2}{3} \frac{1}{\lambda x_i^2} Q_1(x) + \\
& \frac{1}{2} \frac{1}{\lambda x_i^2} Q_1(x) - \frac{1}{10} \lambda x_i^2 Q_1(x) - \frac{1}{42} \lambda x_i^2 Q_1(x)\} a_1 + \dots + \\
& \{D^a Q_N(x_i) + p_0 Q_N(x_i) + p_1 Q_N(x_i) + \dots + \\
& p Q_N^n(x) + 2 \frac{1}{\lambda x_i^2} Q_N(x) - \frac{2}{3} \frac{1}{\lambda x_i^2} Q_N(x) + \\
& \frac{1}{2} \frac{1}{\lambda x_i^2} Q_N(x) - \frac{1}{10} \lambda x_i^2 Q_N(x) - \frac{1}{42} \lambda x_i^2 Q_N(x)\} a_N = g(x_i)
\end{aligned} \tag{3.31}$$

where,

$$x_i = a + \frac{(b-a)i}{N-n+2}, i = 1, 2, \dots, N-n+1 \quad (3.32)$$

Thus, equation (3.31) gives $(N-n+1)$ algebraic linear system of equations in $N+1$ unknown constants. Additional n equations are obtained from equation (3.5). Altogether, $N+1$ algebraic linear system of equations in $N+1$ unknown constants is obtained.

3.2.2 Demonstration of Standard Collocation Method on Some Problems

Problem 3.2.2.1: Consider the problem

$$\begin{aligned} \frac{d^{0.5}y}{dx^{0.5}} + \left(\frac{-32}{35}x^{0.5}\right) \cdot \frac{dy}{dx} + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = \\ 1.504507437x^{1.5} + 1.805406667x^{2.5} - \frac{32}{15}x^{5/2} - \frac{16}{7}x^{3/2} + \frac{32}{35}x^{7/2}, \quad 0 \leq x \leq 1 \end{aligned} \quad (3.33)$$

subject to the condition

$$y(0) = 0 \quad (3.34)$$

The exact solution is given as :

$$y(x) = x^3 + x^2$$

The recursive canonical polynomials constructed for equation (3.33) is

$$Q_j = x^j - \frac{x^{1-a}\Gamma(j+1)}{\Gamma(j-a+1)} \frac{Q_{j-1}}{j-1} \quad j \geq 0 \quad (3.35)$$

$$Q_{-1} = 0 \text{ when } j = 0$$

Here, equations (3.33) – (3.34) are solved for case $N = 3$. Hence, equation (3.20) becomes

$$y_3(x) = \sum_{j=0}^3 a_j Q_j(x) \quad (3.36)$$

From equation (3.35), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 3$) are obtained

$$Q_0(x) = 1$$

$$Q_1(x) = x - 1.128379167 x^{0.5}$$

$$Q_2(x) = x^2 - 1.50450555600 x^{3/2} + 1.697652726 x \quad (3.37)$$

$$Q_3(x) = x^3 - 1.805406668 x^{5/2} + 2.716244363 x^2 - 3.064953551 x^{3/2}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 3$) in equation (3.37), gives

$$\begin{aligned} y_3(x) = & a_0 + (x - 1.1283791670000 x^{0.5})a_1 + (x^2 - \\ & 1.504505556000 x^{3/2} + 1.697652726 x)a_2 + \\ & (x^3 - 1.805406668 x^{5/2} + 2.716244363 x^2 - \\ & 3.064953551 x^{3/2})a_3 \end{aligned} \quad (3.38)$$

Thus, equation (3.38) is substituted into equation (3.33) to get

$$\begin{aligned} \frac{d^{0.5}y_3}{dx^{0.5}} + \left(\frac{-32}{35}x^{0.5}\right) \cdot \frac{dy_3}{dx} + \int_0^x y_3(t) \sqrt{x-t} dt = \\ 1.504507437000000 x^{1.5} + \frac{32}{35} x^{7/2} - \frac{32}{15} x^{5/2} - \\ \frac{16}{7} x^{3/2} + 1.805406667 x^{2.5} \end{aligned} \quad (3.39)$$

and $y_3(0) = 0$

Hence, evaluating the terms $\frac{d^{0.5}y_3}{dx^{0.5}}$, $\left(\frac{-32}{35}x^{0.5}\right) \cdot \frac{dy_3}{dx}$, and $\int_0^x y_3(t) \sqrt{x-t} dt$ in equation (3.39) gives

$$\begin{aligned} D^{0.5}y_3 = & \int_0^x (x-u)^{-0.5} (a_0 + (u - 1.1280000 u^{0.5})a_1 + \\ & (u^2 - 1.504505556 u^{3/2} + 1.697652726000 u)a_2 + (u^3 - \\ & 1.805406668 u^{5/2} + 2.716244363 u^2 - 3.064953551 u^{3/2} \\ &)a_3) du \times \frac{1}{\Gamma(0.5)} \end{aligned} \quad (3.40)$$

which implies

$$\begin{aligned} D^{0.5}y_3 = & (1.12837 x^{0.5} - 0.999997)a_1 + (0.655678 x^{1.5} + \\ & 1.915598535 x^{0.5})a_2 + (1.805406667 x^{2.5} + 2.35738 x^{1.5} - \\ & 2.999999124 x^{2.0})a_3, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \left(\frac{-32}{35}x^{0.5}\right) \frac{dy_3}{dx} = & -0.457142 a_0 x^{-0.5} - 1.37142 a_1 x^{0.5} - \\ & 2.2857142 x^{1.5} a_2 + 2.75109 x^{1.0} a_2 - 2.3282090 x^{0.5} a_2 - \\ & 3.20000 x^{2.5} a_3 + 4.9519725 x^{2.0} a_3 - 6.2085580 x^{1.5} a_3 + \\ & 5.604486494 x^{1.0} a_3 + 1.03166 a_1 \end{aligned} \quad (3.42)$$

and,

$$\begin{aligned} \int_0^x \frac{y_3(t)}{\sqrt{x-t}} dt = & 2.0 \sqrt{-x} a_0 - 1.77245000 x^{1.0} a_1 + \\ & 2.26354 x^{3/2} a_2 - 1.77245 x^2 a_2 + 1.06667 x^{5/2} a_2 - \\ & 3.61081 x^2 a_3 + 2.89733 x^{5/2} a_3 - 1.772450 x^3 a_3 + \\ & 0.914286 x^{7/2} a_3 \end{aligned} \quad (3.43)$$

Substituting equations (3.41 – 3.43) into equation (3.39), gives

$$\begin{aligned} & a_0 + 1.12837 x^{0.5} a_1 - 0.999 a_1 + 0.655678 x^{1.5} a_2 + \\ & 1.9155 x^{0.5} a_2 + 1.80540 x^{2.5} a_3 + 2.357380 x^{1.5} a_3 - \\ & 2.99999 x^{2.0} a_3 + 1.06660 x^{5/2} a_2 - 3.610810 x^2 a_3 + \\ & 2.897330 x^{5/2} a_3 - 1.77245 x^3 a_3 + 0.91428 x^{7/2} a_3 + \\ & 2.0 \sqrt{-x} a_0 - 1.77245 x^{1.0} a_1 + 1.3333 x^{3/2} a_1 - x a_1 - \\ & 0.4571 a_0 x^{-0.5} - 1.37142 a_1 x^{0.5} - 2.28571 x^{1.5} a_2 + \\ & 2.751095 x^{1.0} a_2 - 2.32820 x^{0.5} a_2 - 3.2000 x^{2.5} a_3 + \\ & 4.9519 x^{2.0} a_3 - 6.20855 x^{1.5} a_3 + 5.604481 x^{1.0} a_3 + \\ & x^2 a_2 - 1.50450 x^{3/2} a_2 + 1.697 x a_2 + 1.128 x^{0.5} a_1 + \\ & (x^3 - 1.80540 x^{5/2} + 2.7162 x^2 - 3.06495 x^{3/2}) a_3 + \\ & 1.03166 a_1 = 1.504507 x^{1.5} + 1.805 x^{2.5} - \frac{32}{15} x^{5/2} - \\ & \frac{16}{7} x^{3/2} + \frac{32}{35} x^{7/2} + x^3 + x^2 + 1.7724500000 x^2 a - \\ & 2.26354 x^{3/2} a_2 \end{aligned} \quad (3.44)$$

Collocating equation (3.44) at point $x = x_i$, gives

$$\begin{aligned}
& a_0 + 1.128371 x_i^{0.5} a_1 - 0.999997 a_1 + 0.655 x_i^{1.5} a_2 + \\
& 1.915500 x_i^{0.5} a_2 + 1.80540 x_i^{2.5} a_3 + 2.35738 x_i^{1.5} a_3 - \\
& 2.99999 x_i^{2.0} a_3 + 1.066600 x_i^{5/2} a_2 - 3.61081 x_i^2 a_3 + \\
& 2.89733 x_i^{5/2} a_3 - 1.77245 x_i^3 a_3 + 0.914286 x_i^{7/2} a_3 + \\
& 2 \sqrt{x_i} a_0 - 1.77245 x_i^{1.0} a_1 + 1.33333 x_i^{3/2} a_1 - x_i a_1 - \\
& 0.45714 a_0 x_i^{-0.5} - 1.37142 a_1 x_i^{0.5} - 2.28571 x_i^{1.5} a_2 + \\
& 2.7510950 x_i^{1.0} a_2 - 2.32820 x_i^{0.5} a_2 - 3.2000 x_i^{2.5} a_3 + \quad (3.45) \\
& 4.9519 x_i^{2.0} a_3 - 6.20855 x_i^{1.5} a_3 + 5.6044831 x_i^{1.0} a_3 + \\
& x_i^2 a_2 - 1.50450 x_i^{3/2} a_2 + 1.697 x_i a_2 + 1.128 x_i^{0.5} a_1 + \\
& (x_i^3 - 1.8054 x_i^{5/2} + 2.71624 x_i^2 - 3.06495 x_i^{3/2}) a_3 + \\
& 1.03166 a_1 = 1.504507 x_i^{1.5} + 1.8054 x_i^{2.5} - \frac{32}{15} x_i^{5/2} - \\
& \frac{16}{7} x_i^{3/2} + \frac{32}{35} x_i^{7/2} + x_i^3 + x_i^2 + 1.7724500040 x_i^2 a_1 - \\
& 2.26354 x_i^{3/2} a_2
\end{aligned}$$

where,

$$x_i = \frac{i}{4}; i = 1, 2, 3. \text{ Hence, for } i = 1, x_1 = \frac{1}{4}, \text{ thus equation (3.45)}$$

after simplification gives

$$\begin{aligned}
& 0.78206 a_2 + 0.58830 a_3 + 1.08571 a_0 - \\
& 0.6804900000000000 a_1 = -0.022630707 \quad (3.46)
\end{aligned}$$

For $i = 2$, $x_2 = \frac{2}{4}$, thus equation (3.45) after simplification gives

$$\begin{aligned} &1.620118 a_2 + 0.55212 a_3 + 1.76771 a_0 - \\ &0.85290000000000000000 a_1 = 0.12164408 \end{aligned} \quad (3.47)$$

For $i = 3$, $x_3 = \frac{3}{4}$, thus equation (3.45) after simplification gives

$$\begin{aligned} &2.49861 a_2 + 0.16672 a_3 + 2.20418 a_0 - \\ &0.86934000000000000000 a_1 = 0.65125867 \end{aligned} \quad (3.48)$$

Thus, from the given condition, $y_3(0) = 0$ implies

$$a_0 = 0 \quad (3.49)$$

Hence, solving equations (3.46) – (3.49) to obtain

$$\begin{aligned} &\square a_0 = 0 \\ &\square a_1 = -2.306627108, \\ &a_2 = -0.3964697218, \\ &\square a_3 = -2.1795024030 \end{aligned} \quad (3.50)$$

Substituting the values of a_0 , a_1 , a_2 , a_3 above into equation (3.38), and after simplification gives

$$\begin{aligned} y_3(x) = &2.602749975 x^{0.5} - 2.9796950120 x - \\ &6.316530838 x^2 + 7.276564528 x^{3/2} - \\ &2.1795024030 x^3 + 3.934888171 x^{5/2} \end{aligned} \quad (3.51)$$

Thus, equation (3.51) is the approximate solution of equation (3.33) together with the condition given in equation (3.34) for case $N = 3$

Problem 3.2.2.2: Considering a second order singular fractional integro-differential equation of the form

$$y'' + \frac{d^{1.5}y(x)}{dx^{1.5}} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 2.256758334 x^{0.5} + 7.2216266690 x^{2.5} - \frac{16}{15} x^{5/2} - \frac{256}{315} x^{9/2} + 2 + 12x^2 \quad (3.52)$$

$$0 \leq x \leq 1$$

subject to the conditions

$$y(0) = 0, y'(0) = 0 \quad (3.53)$$

Exact solution is given by $y(x) = x^4 + x^2$

The recursive canonical polynomials constructed for equation (3.52) is

$$Q_j(x) = \frac{x^{j+1} (j+1)(j) Q_{j-1} - \Gamma(j-a+2)}{x^{1-a} \Gamma(j+2)} \quad (3.54)$$

Here, equations (3.52) – (3.53) are solved for case $N = 7$. Hence, equation (3.20) becomes

$$y_7(x) = \sum_{j=0}^7 a_j Q_j(x) \quad (3.55)$$

From equation (3.54), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 7$)

are obtained

$$\begin{aligned}
& \square Q_0(x) = 1.772453851 x^{1.5} \\
& \square Q_1(x) = 0.4431134628 x^{2.5} - 1.57079632700 x^{2.0} \\
& \square Q_2(x) = 0.2215567313 x^{3.5} - 0.2945243112 x^{3.0} + \\
& \square \quad 1.04406149 x^{2.5} \\
& \square Q_3(x) = 0.1384729571 x^{4.5} - 0.1227184630 x^{4.0} + \\
& \square \quad 0.16313461 x^{3.5} - 0.5782971326000 x^{3.0} \\
& \square Q_4(x) = 0.09693107000 x^{5.5} - 0.0671116595 x^{5.0} + \\
& \square \quad 0.05947616 x^{4.5} - 0.0790640611700 x^{4.0} + \\
& \square \quad 0.2802747992 x^{3.5} \\
& \square Q_5(x) = 0.0726983025 x^{6.5} - 0.04228034547 x^{6.0} + \\
& \square \quad 0.0292734 x^{5.5} - 0.02594289506000 x^{5.0} + \\
& \square \quad 0.03448693820000 x^{4.5} - 0.12225301 x^{4.0} \\
& \square Q_6(x) = 0.05712009480 x^{7.5} - 0.02906773750 x^{7.0} + \\
& \square \quad 0.01690540 x^{6.5} - 0.01170470460000 x^{6.0} + \\
& \square \quad 0.01037302438 x^{5.5} - 0.01378928000 x^{5.0} + \\
& \square \quad 0.04888172573000 x^{4.5} \\
& \square Q_7(x) = 0.046410077030000 x^{8.5} - 0.02120758 x^{8.0} + \\
& \square \quad 0.01079228749 x^{7.5} - 0.006276647845 x^{7.0} + \\
& \square \quad 0.00434572900 x^{6.5} - 0.003851302884 x^{6.0} + \\
& \square \quad 0.0051196924730000 x^{5.5} - 0.01814884 x^{5.0}
\end{aligned} \tag{3.56}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 7$) in equation (3.56) gives

$$\begin{aligned}
 y_7(x) = & 1.772453851 x^{1.5} a_0 + (0.4431134600 x^{2.5} - \\
 & 1.57079633 x^{2.0}) a_1 + (0.2215567313 x^{3.5} - \\
 & 0.2945243 x^{3.0} + 1.044061499000 x^{2.5}) a_2 + \\
 & (0.138472957 x^{4.5} - 0.122718460000 x^{4.0} + \\
 & 0.1631346093 x^{3.5} - 0.578297132 x^{3.0}) a_3 + \\
 & (0.09693107 x^{5.5} - 0.0671116594900 x^{5.0} + \\
 & 0.05947615964 x^{4.5} - 0.07906406000 x^{4.0} + \\
 & 0.2802747992 x^{3.5}) a_4 + (0.072698300 x^{6.5} - \\
 & 0.04228034547 x^{6.0} + 0.029273420000 x^{5.5} - \\
 & 0.02594289506 x^{5.0} + 0.034486938000 x^{4.5} - \\
 & 0.122253013 x^{4.0}) a_5 + (0.05712009480 x^{7.5} - \\
 & 0.02906773750 x^{7.0} + 0.0169054000000 x^{6.5} - \\
 & 0.0117047046000 x^{6.0} + 0.01037302438 x^{5.5} - \\
 & 0.0137892800 x^{5.0} + 0.04888172573 x^{4.5}) a_6 + \\
 & (0.04641007703 x^{8.5} - 0.021207580000 x^{8.0} + \\
 & 0.010792287490 x^{7.5} - 0.006276647845 x^{7.0} + \\
 & 0.004345729000 x^{6.5} - 0.003851302884 x^{6.0} + \\
 & 0.005119692473 x^{5.5} - 0.01814884000 x^{5.0}) a_7
 \end{aligned} \tag{3.57}$$

Thus, equation (3.57) is substituted into equation (3.52) to get

$$y_7'' + \frac{d^{1.5}y_7(x)}{dx^{1.5}} - \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2 + 12x^2 - \frac{16}{15}x^{5/2} + 2.256758334x^{0.5} + 7.221626669000000x^{2.5} - \frac{256}{315}x^{9/2} \quad (3.58)$$

together with equation (3.53) to get

$$y_7(0) = 0, \quad y_7'(0) = 0$$

Hence, evaluating the terms $\frac{d^2y_7}{dx^2}$, $\frac{d^{1.5}y_7}{dx^{1.5}}$, and $\int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt$ in equation (3.58) gives

$$\begin{aligned} \frac{d^{1.5}y_7}{dx^{1.5}} = & 2.356196748002400 a_0 + (-3.5449103501322000000 x^{0.5} + \\ & 1.472619 x^{1.0}) a_1 + (1.288540011 x^{2.0} + 3.4697828640 x^{1.0} - \\ & 1.329337855 x^{1.5}) a_2 + (0.948769413 x^{2.0} - 0.8862289 x^{2.5} + \\ & 1.208008885 x^{3.0} - 2.610155405 x^{1.5}) a_3 + (-0.692362 x^{3.5} - \\ & 0.570971 x^{2.5} + 0.5188580000 x^{3.0} + 1.1627101030000 x^{4.0} + \\ & 1.630039619 x^{2.0}) a_4 + (0.3511408772 x^{4.0} - 0.2676430 x^{3.5} - \\ & 0.8828664278 x^{2.5} + 0.3008569 x^{3.0} - 0.581589100000 x^{4.5} + \\ & 1.133643056 x^{5.0}) a_5 + (0.4264342 x^{3.0} - 0.1422587100 x^{3.5} + \\ & 0.1244269000 x^{4.0} - 0.1610039098 x^{4.5} + 0.2636192755 x^{5.0} - \\ & 0.508888000 x^{5.5} + 1.113399934 x^{6.0}) a_6 + (-0.1872342 x^{3.5} + \end{aligned}$$

$$\begin{aligned}
& 0.061412 x^{4.0} - 0.05297649919 x^{4.5} + 0.067766500 x^{5.0} - \\
& 0.1098849484 x^{5.5} + 0.2103659849 x^{6.0} - 0.456960 x^{6.5} + \quad (3.59) \\
& 1.098488 x^{7.0}) a_7
\end{aligned}$$

$$\begin{aligned}
& \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2.08812 x^{2.0} a_0 + (0.43502006 x^{3.0} - \\
& 1.67552 x^{2.5}) a_1 + (0.190324 x^{4.0} + 1.02523000 x^{3.0} - \\
& 0.269279 x^{3.5}) a_2 + (0.140137 x^{4.0} - 0.0997331 x^{4.5} - \\
& 0.528729 x^{3.5} + 0.107057 x^{5.0}) a_3 + (0.2407640 x^{4.0} - \\
& 0.0642552 x^{4.5} + 0.0459826 x^{5.0} - 0.04958320 x^{5.5} + \\
& 0.068695 x^{6.0}) a_4 + (-0.099354 x^{4.5} + 0.02666 x^{5.0} - \\
& 0.019167 x^{5.5} + 0.020746 x^{6.0} - 0.0288345000 x^{6.5} + \quad (3.60) \\
& 0.047841 x^{7.0}) a_5 + (0.0377917 x^{5.0} - 0.010187 x^{5.5} + \\
& 0.00735135 x^{6.0} - 0.00798243 x^{6.5} + 0.0111250 x^{7.0} - \\
& 0.018502 x^{7.5} + 0.035240 x^{8.0}) a_6 + (-0.013409 x^{5.5} + \\
& 0.003628 x^{6.0} - 0.00262653 x^{6.5} + 0.002859830 x^{7.0} - \\
& 0.00399521 x^{7.5} + 0.00665828 x^{8.0} - 0.0127050 x^{8.5} + \\
& 0.0270419 x^{9.0}) a_7
\end{aligned}$$

and,

$$\begin{aligned}
\frac{d^2 y_7}{dx^2} = & 1.32934 a_0 x^{-0.5} + (1.661675 x^{0.5} - 3.141) a_1 + \\
& (1.938621399 x^{1.5} - 1.76710 x^{1.0} + 3.915231 x^{0.5}) a_2 + \\
& (2.1809490700 x^{2.5} - 1.472621 x^{2.0} + 1.4274278 x^{1.5} - \\
& 3.46978 x^{1.0}) a_3 + (2.399043 x^{3.5} - 1.342233190 x^{3.0} + \\
& 0.9367495144 x^{2.5} - 0.9487687 x^{2.0} + 2.4524 x^{1.5}) a_4 + \\
& (2.598964 x^{4.5} - 1.2684103641 x^{4.0} + 0.7245172 x^{3.5} - \\
& 0.518857 x^{3.0} + 0.5431693 x^{2.5} - 1.46703615 x^{2.0}) a_5 + \\
& (2.7846046 x^{5.5} - 1.220844975 x^{5.0} + 0.6043681 x^{4.5} - \\
& 0.351141138 x^{4.0} + 0.25673235 x^{3.5} - 0.2757850 x^{3.0} + \\
& 0.76988718 x^{2.5}) a_6 + (2.958642 x^{6.5} - 1.187624 x^{6.0} + \\
& 0.52612401 x^{5.5} - 0.263619209 x^{5.0} + 0.1553598 x^{4.5} - \\
& 0.1155391 x^{4.0} + 0.1267123887 x^{3.5} - 0.362976 x^{3.0}) a_7
\end{aligned} \tag{3.61}$$

Hence, substituting equations (3.59) – (3.61) into equation (3.58)

gives

$$\begin{aligned}
& 3584.0000 t_2 x^3 - 0.9487687341 a_4 x^{2.0} - 840 t_2 x^2 + 72 t_2 x - 1.883234863 a_1 x^{0.5} + \\
& 0.609283544 a_2 x^{1.5} - 1.767145 a_2 x^{1.0} + 3.915230622 a_2 x^{0.5} + 1.294720076 a_3 x^{2.5} - \\
& 1.472621556 a_3 x^{2.0} - 1.182727 a_3 x^{1.5} - 3.469782796 a_3 x^{1.0} + 1.706681809 a_4 x^{3.5} +
\end{aligned}$$

$$\begin{aligned}
& 0.3657783721 a_4 x^{2.5} - 1.34223 a_4 x^{3.0} - 0.3511411380 a_6 x^{4.0} - 0.33969700 a_5 x^{2.5} - \\
& 1.467036152 a_5 x^{2.0} + 2.285904137 a_6 x^{5.5} - 1.220844975 a_6 x^{5.0} + 0.443364 a_6 x^{4.5} + \\
& 2.452404493 a_4 x^{1.5} + 2.116729924 a_5 x^{4.5} - 1.268410364 a_5 x^{4.0} + 0.456874 a_5 x^{3.5} - \\
& 0.518857 a_5 x^{3.0} + 0.1144736425 a_6 x^{3.5} - 0.2757856052 a_6 x^{3.0} + 0.7698870 a_6 x^{2.5} + \\
& 2.504308101 a_7 x^{6.5} - 1.187624704 a_7 x^{6.0} + 0.4296477668 a_7 x^{5.5} - 0.26361 a_7 x^{5.0} + \\
& 0.102383346000 a_7 x^{4.5} - 0.11553000 a_7 x^{4.0} - 0.435026 a_1 x^{3.0} - 1.0250000 a_2 x^{3.0} + \\
& 0.269279 a_2 x^{3.5} - 0.0605218232 a_7 x^{3.5} - 0.3629767450 a_7 x^{3.0} - 0.1401370 a_3 x^{4.0} + \\
& 0.09973310 a_3 x^{4.5} + 0.528729 a_3 x^{3.5} - 0.107057 a_3 x^{5.0} + 0.921 a_4 x^{4.0} - 8192 t_1 x^7 + \\
& 0.06425520 a_4 x^{4.5} + 1.1069 a_5 x^{5.0} + 0.019167 a_5 x^{5.5} - 0.020746 a_5 x^{6.0} - \frac{256}{315} x^{9/2} - \\
& 0.04598 a_4 x^{5.0} + 0.04958 a_4 x^{5.5} - 0.068695 a_4 x^{6.0} - 0.0111251 a_6 x^{7.0} + 1568 t_1 x^2 + \\
& 1.106048584 a_6 x^{6.0} + 0.00798243 a_6 x^{6.5} + 0.0288345 a_5 x^{6.5} - 0.047841100 a_5 x^{7.0} + \\
& 0.22582757550 a_6 x^{5.0} + 0.003995210 a_7 x^{7.5} + 1.095628 a_7 x^{7.0} + 0.0185022 a_6 x^{7.5} - \\
& 0.0352401 a_6 x^{8.0} + 0.2067376649 a_7 x^{6.0} - 2.088120 x^{2.0} a_0 + 1.329340388 a_0 x^{-0.5} + \\
& 1.288541 a_2 x^{2.0} - 0.00665828 a_7 x^{8.0} + 0.012705000 a_7 x^{8.5} - 0.0270419000 a_7 x^{9.0} + \\
& 1.472619441 a_1 x^{1.0} + 1.63003 a_4 x^{2.0} + 3.469782864 a_2 x^{1.0} + 0.9487694131 a_3 x^{2.0} + \\
& 1.2080088 a_3 x^{3.0} + 0.51885808 a_4 x^{3.0} + 0.0614120 a_7 x^{4.0} + 0.35114080000 a_5 x^{4.0} + \\
& 0.30085691 a_5 x^{3.0} + 0.426434 a_6 x^{3.0} + 0.1244269349 a_6 x^{4.0} + 0.067766503 a_7 x^{5.0} + \\
& 2.3561960000 a_0 - 3.1415927000000000 a_1 + 6144 t_2 x^5 - 6912 t_2 x^4 + 28672 t_1 x^6 -
\end{aligned}$$

$$\begin{aligned}
& 39424 \, t_1 x^5 + 26880 \, t_1 x^4 - 9408 \, t_1 x^3 - 98 \, t_1 x + 12 \, x^2 - \\
& 2048 \, t_2 x^6 + t_1 - t_2 = 2 + 2.256758334 x^{0.5} - \frac{16}{15} x^{5/2} - \quad (3.62) \\
& 1.67552 \, a_1 x^{2.5} + 0.190324 \, a_2 x^{4.0}
\end{aligned}$$

Collocating equation (3.62) at the point $x = x_i$ to give

$$\begin{aligned}
& -0.9487687341 \, a_4 x_i^{2.0} + 3584 \, t_2 x_i^3 - 840 \, t_2 x_i^2 + 72 \, t_2 x_i - 1.883234863 \, a_1 x_i^{0.5} + \\
& 0.609283544 \, a_2 x_i^{1.5} - 1.767145 \, a_2 x_i^{1.0} + 3.915230622 \, a_2 x_i^{0.5} + 1.2947200 \, a_3 x_i^{2.5} - \\
& 1.472621556 \, a_3 x_i^{2.0} - 1.18272757300 \, a_3 x_i^{1.5} - 3.46978 \, a_3 x_i^{1.0} + 1.706681 \, a_4 x_i^{3.5} + \\
& 0.3657783721 \, a_4 x_i^{2.5} - 1.342230 \, a_4 x_i^{3.0} - 0.3511411380 \, a_6 x_i^{4.0} - 0.33969 \, a_5 x_i^{2.5} - \\
& 1.467036152 \, a_5 x_i^{2.0} + 2.2859041 \, a_6 x_i^{5.5} - 1.220844975 \, a_6 x_i^{5.0} + 0.443360 \, a_6 x_i^{4.5} + \\
& 2.45240440 \, a_4 x_i^{1.5} + 2.116729924 \, a_5 x_i^{4.5} - 1.268410364 \, a_5 x_i^{4.0} + 0.45687 \, a_5 x_i^{3.5} - \\
& 0.5188579012 \, a_5 x_i^{3.0} + 0.1144736425 \, a_6 x_i^{3.5} - 0.275780 \, a_6 x_i^{3.0} + 0.76988 \, a_6 x_i^{2.5} + \\
& 2.504308101 \, a_7 x_i^{6.5} - 1.187624704 \, a_7 x_i^{6.0} + 0.429647 \, a_7 x_i^{5.5} - 0.2636100 \, a_7 x_i^{5.0} + \\
& 0.102383346000 \, a_7 x_i^{4.5} - 0.1155390865 \, a_7 x_i^{4.0} - 0.435026 \, a_1 x_i^{3.0} - 1.025 \, a_2 x_i^{3.0} + \\
& 0.2692790 \, a_2 x_i^{3.5} - 0.0605218232 \, a_7 x_i^{3.5} - 0.36297674 \, a_7 x_i^{3.0} - 0.140137 \, a_3 x_i^{4.0} + \\
& 0.09970 \, a_3 x_i^{4.5} + 0.528 \, a_3 x_i^{3.5} - 0.10705 \, a_3 x_i^{5.0} + 0.9219461 \, a_4 x_i^{4.0} - 8192 \, t_1 x_i^7 + \\
& 0.0642500 \, a_4 x_i^{4.5} + 1.1069 \, a_5 x_i^{5.0} + 0.019167 \, a_5 x_i^{5.5} - 0.0207 \, a_5 x_i^{6.0} - \frac{256}{315} x_i^{9/2} - \\
& 0.04598 \, a_4 x_i^{5.0} + 0.0495832 \, a_4 x_i^{5.5} - 0.0686 \, a_4 x_i^{6.0} - 0.0111 \, a_6 x_i^{7.0} + 1568 \, t_1 x_i^2 + \\
& 1.1060485 \, a_6 x_i^{6.0} + 0.00798243 \, a_6 x_i^{6.5} + 0.0288345 \, a_5 x_i^{6.5} - 0.047841100 \, a_5 x_i^{7.0} + \\
& 0.2258275755 \, a_6 x_i^{5.0} + 0.003995210 \, a_7 x_i^{7.5} + 1.0956285 \, a_7 x_i^{7.0} + 0.01850 \, a_6 x_i^{7.5} -
\end{aligned}$$

$$\begin{aligned}
& 0.0352401 a_6 x_i^{8.0} + 0.2067376 a_7 x_i^{6.0} - 2.08812 x_i^{2.0} a_0 + \\
& 1.329340 a_0 x_i^{-0.5} + 1.2885 a_2 x_i^{2.0} - 0.00665828 a_7 x_i^{8.0} + \\
& 0.012705 a_7 x_i^{8.5} - 0.0270419 a_7 x_i^{9.0} + 1.472619 a_1 x_i^{1.0} + \\
& 1.630039 a_4 x_i^{2.0} + 3.469782 a_2 x_i^{1.0} + 0.9487694 a_3 x_i^{2.0} + \\
& 1.2080081 a_3 x_i^{3.0} + 0.5188 a_4 x_i^{3.0} + 0.06141203 a_7 x_i^{4.0} + \\
& 0.3511 a_5 x_i^{4.0} + 0.3008569 a_5 x_i^{3.0} + 0.42643424 a_6 x_i^{3.0} + \quad (3.63) \\
& 0.1244269349 a_6 x_i^{4.0} + 0.06776650 a_7 x_i^{5.0} + 6144 t_2 x_i^5 + \\
& 2.356196 a_0 - 3.1415927 a_1 - 6912 t_2 x_i^4 + 28672 t_1 x_i^6 - \\
& 39424 t_1 x_i^5 + 26880 t_1 x_i^4 - 9408 t_1 x_i^3 - 98 t_1 x_i + 1 t_1 - \\
& 2048 t_2 x_i^6 = 2 + 2.256758334 x_i^{0.5} - \frac{16}{15} x_i^{5/2} + 12 x_i^2 - \\
& 1.67552 a_1 x_i^{2.5} + 0.190324 a_2 x_i^{4.0} + t_2
\end{aligned}$$

where, $x_i = \frac{i}{n}$, $i = 1, 2, 3, \dots, 6$.

Hence, for $i = 1$, $x_1 = \frac{1}{n}$, thus equation (3.63) after simplification gives

$$\begin{aligned}
& 5.830686128 a_0 - 3.631358393 a_1 + 1.77947603700 a_2 - \\
& 0.55619301920 a_3 + 0.1490143774 a_4 - 0.03267458 a_5 + \quad (3.64) \\
& 0.006480912194 a_6 - 0.001132310517 a_7 = 3.089516701
\end{aligned}$$

For $i = 2$, $x_2 = \frac{2}{7}$, thus equation (3.63) after simplification gives

$$\begin{aligned} &4.672706114 a_0 - 3.664511540 a_1 + 2.75566586700 a_2 - \\ &1.124278144 a_3 + 0.4544787360 a_4 - 0.1304252483 a_5 + \quad (3.65) \\ &0.03963707026 a_6 - 0.008787220042 a_7 = 4.136441622 \end{aligned}$$

For $i = 3$, $x_3 = \frac{3}{7}$, thus equation (3.63) after simplification gives

$$\begin{aligned} &4.003265488 a_0 - 3.576111949 a_1 + 3.62720741100 a_2 - \\ &1.641155158 a_3 + 0.9122109167 a_4 - 0.2720666010 a_5 + \quad (3.66) \\ &0.126572985000 a_6 - 0.02303527619 a_7 = 5.535269346 \end{aligned}$$

For $i = 4$, $x_4 = \frac{4}{7}$, thus equation (3.63) after simplification gives

$$\begin{aligned} &3.432913684 a_0 - 3.391283402 a_1 + 4.4429403590 a_2 - \\ &2.058545002 a_3 + 1.559742062 a_4 - 0.3988991239 a_5 + \quad (3.67) \\ &0.32888300150 a_6 - 0.01245447334 a_7 = 7.295524209 \end{aligned}$$

For $i = 5$, $x_5 = \frac{5}{7}$, thus equation (3.63) after simplification gives

$$\begin{aligned} &2.8637261600 a_0 - 3.117396602 a_1 + 5.210228386 a_2 - \\ &2.332751497 a_3 + 2.455623149 a_4 - 0.4020046863 a_5 + \quad (3.68) \\ &0.7806974034 a_6 + 0.1426630646 a_7 = 9.39101586800 \end{aligned}$$

For $i = 6$, $x_6 = \frac{6}{7}$, thus equation (3.63) after simplification gives

$$\begin{aligned} & 2.257919495 a_0 - 2.757157089 a_1 + 5.9231717000 a_2 - \\ & 2.423218256 a_3 + 3.674457892 a_4 - 0.1106453576 a_5 + \\ & 1.736596254 a_6 + 0.750390673900 a_7 = 11.774004570 \end{aligned} \quad (3.69)$$

Thus, from the given conditions (3.53); that is $y_7(0) = 0$, equation (3.57) gives

$$\begin{aligned} & 0.00000005604991216 a_0 - 0.00000000015693950 a_1 + \\ & 3.29 \times 10^{-13} a_2 - 5.777 \times 10^{-16} a_3 + 8.85 \times 10^{-19} a_4 - \\ & 1.22 \times 10^{-21} a_5 + 1.544 \times 10^{-24} a_6 - 1.813 \times 10^{-27} a_7 \\ & = 0 \end{aligned} \quad (3.70)$$

and, $y_7^l(0) = 0$ equation (3.57) gives

$$\begin{aligned} & 0.008407486823122 a_0 - 0.00003138089534710 a_1 + \\ & 0.00000008245219679 a_2 - 0.000000000173309 a_3 + \\ & 3.098919445 \times 10^{-13} a_4 - 4.885225851 \times 10^{-16} a_5 + \\ & 6.94911484 \times 10^{-19} a_6 - 9.06553720 \times 10^{-22} a_7 = 0 \end{aligned} \quad (3.71)$$

Solving equations (3.64) – (3.71) to obtain

$$\begin{aligned}
 a_0 &= 0.9814154107e^{-5}, \\
 a_1 &= 0.6979476705e^{-2}, \\
 a_2 &= 1.639814103, \\
 a_3 &= -7.594666360, \\
 a_4 &= -40.86686269, \\
 a_5 &= -43.40797184, \\
 a_6 &= 92.16410776, \\
 a_7 &= -41.33097405
 \end{aligned} \tag{3.72}$$

Thus, substituting the values of $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ above into equation (3.57), and after simplification gives

$$\begin{aligned}
 y_7(x) &= -12.32959281 x^{3.5} + 0.0000173951352400 x^{1.5} - \\
 &0.474127362 x^{4.5} + 9.469861248 x^{4.0} + 0.91572848 x^{6.0} - \\
 &1.777227882 x^{6.5} - 4.487549960 x^{5.5} + 3.34800380 x^{5.0} + \\
 &4.81836680 x^{7.5} - 2.419582122 x^{7.0} - 1.918173689 x^{8.5} + \\
 &0.876530104 x^{8.0} + 3.90900866 x^{3.0} + 1.715159470 x^{2.5} - \\
 &0.01096333637 x^{2.0}
 \end{aligned} \tag{3.73}$$

Thus, equation (3.73) is the approximate solution of equation (3.52) together with the given conditions (3.53) for case $N = 7$

Problem 3.2.2.3: Consider a third order singular fractional integro-differential equation of the form

$$y'''(x) + \frac{d^{2.75}y}{dx^{2.75}} + xy(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 6 + 12x^2 + \frac{32}{35}x^{7/2} + 6.61957590800x^{0.25} \quad (3.74)$$

subject to the conditions

$$\begin{aligned} y(0) &= 0, & y'(0) &= 0, \\ y''(0) &= 0 \end{aligned} \quad (3.75)$$

The exact solution is $y(x) = x^3$

The recursive canonical polynomials constructed for equation (3.74) is

$$Q_j(x) = \frac{(x^{j+1} - (j+1)(j)(j-1)Q_{j-2})\Gamma(j-a+2)}{x^{1-a}\Gamma(j+2)} \quad (3.76)$$

$$j = 0, 1, \dots$$

Here, equations (3.74) – (3.75) are solved for case $N = 10$. Hence, equation (3.20) becomes

$$y_{10} = \sum_{j=0}^{10} a_j Q_j(x) \quad (3.77)$$

From equation (3.76), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 10$) are obtained

$$Q_0(x) = - 4.834146544 x^{11/4}$$

$$Q_1(x) = 1.812804954 x^{\frac{15}{4}}$$

$$Q_2(x) = 0.1510670796 x^{\frac{19}{4}} + 2.190841202000000000 x^{9/2}$$

$$Q_3(x) = 0.04720846234 x^{\frac{23}{4}} - 0.3423189376000000 x^{11/2}$$

$$Q_4(x) = 0.02124380806 x^{\frac{27}{4}} - 0.01604620022000 x^{13/2} - 0.2327090499 x^{\frac{25}{4}}$$

$$Q_5(x) = 0.011507062700 x^{\frac{31}{4}} - 0.0032593844160 x^{15/2} + 0.02363451287 x^{\frac{29}{4}}$$

$$Q_6(x) = 0.006986430925 x^{\frac{35}{4}} - 0.0010389287830 x^{17/2} + 0.00078473968580 x^{\frac{33}{4}} + 0.011380639920000 x^8$$

$$Q_7(x) = 0.004584845296 x^{\frac{39}{4}} - 0.0004220648180 x^{19/2} + 0.0001195501865 x^{\frac{37}{4}} - 0.0008668846814000 x^9$$

$$Q_8(x) = 0.003183920343 x^{\frac{43}{4}} - 0.0002001981550 x^{21/2} + 0.00002977079838 x^{\frac{41}{4}} - 0.0000224869378 x^{10} - 0.0003261154587 x^{\frac{39}{4}}$$

$$Q_9(x) = 0.002308342249 x^{\frac{47}{4}} - 0.00010583392100 x^{23/2} + 0.000009742700519 x^{\frac{45}{4}} - 0.000002759627 x^{11} + 0.00002001066535 x^{\frac{43}{4}}$$

$$\begin{aligned}
Q_{10}(x) = & 0.0017312566 x^{\frac{51}{4}} - 0.00006063401720 x^{\frac{25}{2}} + \\
& 0.00000381253 x^{\frac{49}{4}} - 0.0000005669498 x^{\frac{12}{2}} + \quad (3.78) \\
& 0.000000428237 x^{\frac{47}{4}} + 0.000006210485 x^{23/2}
\end{aligned}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 10$) in equation (3.78) gives

$$\begin{aligned}
y_{10}(x) = & -4.834 x^{11/4} a_0 + 1.81 x^{\frac{15}{4}} a_1 + (0.1510671 x^{\frac{19}{4}} + \\
& 2.19084 x^{9/2}) a_2 + (0.0472084 x^{\frac{23}{4}} - 0.3423189 x^{11/2}) a_3 + \\
& (0.02124381 x^{\frac{27}{4}} - 0.0160462 x^{13/2} - 0.2327090 x^{\frac{25}{4}}) a_4 + \\
& (0.011507 x^{\frac{31}{4}} - 0.00325938 x^{15/2} + 0.02363451 x^{\frac{29}{4}}) a_5 + \\
& (0.0069864 x^{\frac{35}{4}} - 0.001038928 x^{17/2} + 0.000784739 x^{\frac{33}{4}} + \\
& 0.0113806 x^8) a_6 + (0.0045848 x^{\frac{39}{4}} - 0.000422060 x^{19/2} + \\
& 0.00011955 x^{\frac{37}{4}} - 0.0008668 x^9) a_7 + (0.003183920 x^{\frac{43}{4}} - \\
& 0.00020019820 x^{21/2} + 0.0000297 x^{\frac{41}{4}} - 0.00002248 x^{10} - \\
& 0.0003261 x^{\frac{39}{4}}) a_8 + (0.002308 x^{\frac{47}{4}} - 0.000105830 x^{23/2} + \\
& 0.000009742 x^{\frac{45}{4}} - 0.000002759 x^{11} + 0.0000200 x^{\frac{43}{9}}) a_9 + \\
& (0.001731256 x^{\frac{51}{4}} - 0.000060634 x^{\frac{25}{2}} + 0.00000381 x^{\frac{49}{4}} - \\
& 0.00000056 x^{12} + 0.00000042 x^{\frac{47}{4}} + 0.0000062 x^{23/2}) a_{10}
\end{aligned} \quad (3.79)$$

Thus, equation (3.79) is substituted into equation (3.74) to get

$$\begin{aligned}
y_{10}'''(x) + \frac{d^{2.75} y_{10}(x)}{dx^{2.75}} + x y_{10}(x) + \int_0^x \frac{y_{10}(t)}{\sqrt{x-t}} dt = \quad (3.80) \\
6.00000000 + 12 x^2 + \frac{32}{35} x^{7/2} + 6.619575908 x^{0.25}
\end{aligned}$$

together with equation (3.75) to get

$$y_{10}(0) = 0, \quad y_{10}'(0) = 0, \quad y_{10}''(0) = 0$$

Hence, evaluating the terms $\frac{d^3 y_{10}}{dx^3}$, $\frac{d^{2.75} y_{10}}{dx^{2.75}}$, xy_{10} , and $\int_0^x \frac{y_{10}(t)}{\sqrt{x-t}} dt$ in equation (3.80) gives

$$\begin{aligned} \frac{d^{2.75} y_{10}}{dx^{2.75}} = & -21.38136809 a_0 + 30.06749285000 x a_1 + \\ & (5.95086083300 x^2 + 71.29917628 x^{7/4}) a_2 + \\ & (3.5643106470 x^3 - 22.28099604 x^{11/4}) a_3 + \\ & (2.706659086 x^4 - 23.115000000000 x^{7/2} - \\ & 1.8103298940 x^{15/4}) a_4 + (3.782287766 x^{9/2} - \\ & 0.58061679 x^{19/4} + 2.2724645530000 x^5) a_5 + \\ & (0.188374920000 x^{11/2} + 2.482230639 x^{21/4} + \\ & 2.012078018 x^6 - 0.273583762500 x^{23/4}) a_6 + \\ & (1.839166421 x^7 - 0.27226950080000 x^{25/4} + \\ & 0.04083919774 x^{13/2} - 0.15642416 x^{27/4}) a_7 + \\ & (-0.1308179 x^7 + 1.7162353800000000 x^8 - \\ & 0.00974158 x^{29/4} + 0.013898840000000 x^{15/2} - \end{aligned}$$

$$\begin{aligned}
& 0.1005246039 x^{\frac{31}{4}}) a_8 + (0.006020062 x^{17/2} - \\
& 0.06984369703 x^{\frac{35}{4}} + 1.6244632350000 x^9 + \\
& 0.01078637 x^8 - 0.0015940000000 x^{\frac{33}{4}}) a_9 + \\
& (0.003037723384 x^{19/2} - 0.051300610 x^{\frac{39}{4}} + \\
& 0.004098510 x^{\frac{35}{4}} - 0.00042483600000 x^{\frac{37}{4}} + \\
& 0.0003013672258 x^9 + 1.553393813 x^{10}) a_{10}
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
\frac{d^3 y_{10}}{dx^3} = & 32.71546441 x^{3/4} a_1 - 17.44824769 \frac{a_0}{x} + \\
& (7.399926 x^{7/4} + 86.264372300 x^{3/2}) a_2 + \\
& (4.835179 x^{11/4} - 29.65338000 x^{5/2}) a_3 + \\
& (3.9164950000 x^{15/4} - 2.581432460 x^{7/2} - \\
& 32.45200422 x^{13/4}) a_4 + (3.46128850 x^{19/4} - \\
& 0.873922 x^{9/2} + 5.62242903600 x^{17/4}) a_5 + \\
& (3.197929 x^{23/4} - 0.430506100000 x^{11/2} + \\
& 0.2933577 x^{21/4} + 3.82389500000 x^5) a_6 + \\
& (3.031370762000 x^{27/4} - 0.255613 x^{13/2} + \\
& 0.066143008 x^{25/4} - 0.436909879 x^6) a_7 + \\
& (2.920003196 x^{31/4} - 0.169743010 x^{15/2} + \\
& 0.023286800 x^{29/4} - 0.016190595250 x^7 - \\
& 0.21561800 x^{27/4}) a_8 + (2.842831684 x^{35/4} - \\
& 0.121404736 x^{17/2} + 0.0103919601 x^{33/4} - \\
& 0.00273203 x^8 + 0.0183519700 x^{31/4}) a_9 + \\
& (2.78816184300 x^{39/4} - 0.0915194 x^{19/2} + \\
& 0.0053855080 x^{37/4} - 0.00074737776 x^9 + \\
& 0.000527394 x^{35/4} + 0.00712420 x^{17/2}) a_{10}
\end{aligned} \tag{3.82}$$

and,

$$\begin{aligned}
\int_0^x \frac{y_{10}(t)}{(x-t)} dt = & 1.51352 x^{\frac{17}{4}} a_1 - 4.57418 x^{\frac{13}{4}} a_0 + \\
& (1.6938 x^5 + 0.1141140000 x^{\frac{21}{4}}) a_2 + \\
& (-0.242601 x^6 + 0.0328079 x^{\frac{25}{4}}) a_3 + \\
& (-0.155848 x^{\frac{27}{4}} - 0.0105596000 x^7 + \\
& 0.0137454 x^{\frac{29}{4}}) a_4 + (0.0148071 x^{\frac{31}{4}} - \\
& 0.00201087 x^8 + 0.00699417 x^{\frac{33}{4}}) a_5 + \\
& (0.00681788 x^{17/2} + 0.00046354 x^{\frac{35}{4}} - \\
& 0.00060535 x^9 + 0.00401692 x^{\frac{37}{4}}) a_6 + \\
& (-0.000492 x^{19/2} + 0.00006699 x^{\frac{39}{4}} - \\
& 0.000233 x^{10} + 0.002507500 x^{\frac{41}{4}}) a_7 + \\
& (-0.0001783 x^{\frac{41}{4}} - 0.0000121 x^{21/2} + \\
& 0.000015911 x^{\frac{43}{4}} - 0.000105780 x^{11} + \\
& 0.0016639 x^{\frac{45}{4}}) a_8 + (0.00001045 x^{\frac{45}{4}} - \\
& 0.000001426 x^{23/2} + 0.00000498 x^{\frac{47}{4}} - \\
& 0.00005359 x^{12} + 0.00115712 x^{\frac{49}{4}}) a_9 + \\
& (0.00000315 x^{12} + 0.0000002150 x^{\frac{49}{4}} - \\
& 0.00000028 x^{\frac{25}{2}} + 0.00000187400 x^{\frac{51}{4}} - \\
& 0.00002952 x^{13} + 0.00083508 x^{\frac{53}{4}}) a_{10}
\end{aligned} \tag{3.83}$$

Hence, substituting equations (3.81) – (3.83) into (3.80) gives

$$\begin{aligned}
& 18.69455109 \, x^{11/4} a_1 - 21.38136809 \, a_0 - 23.26433025 \, x^{7/4} a_0 - \\
& 17.44824769 \, \frac{a_0}{x} + 32.71546441 \, x^{3/4} a_1 - 4.574180000 \, x^{13/4} a_0 + \\
& 0.0000018740 \, a_{10} x^{51/4} - 0.00001215 \, a_8 x^{21/2} + 0.0000159 \, a_8 x^{43/4} - \\
& 0.00010578 \, a_8 x^{11} + 0.00166394 \, a_8 x^{45/4} + 0.00001045770 \, a_9 x^{45/4} - \\
& 0.000001426 \, a_9 x^{23/2} + 0.000004984 \, a_9 x^{47/4} - 0.0000535 \, a_9 x^{12} + \\
& 0.00115712 \, a_9 x^{49/4} + 0.00000314476 \, a_{10} x^{12} + 1.5135200 \, x^{17/4} a_1 + \\
& 30.06749285 \, x a_1 + 0.00046354900 \, a_6 x^{35/4} - 0.000605356 \, a_6 x^9 + \\
& 0.00401692 \, a_6 x^{37/4} - 0.00049199 \, a_7 x^{19/2} + 0.000066997 \, a_7 x^{39/4} - \\
& 0.000233629 \, a_7 x^{10} + 0.002507511 \, a_7 x^{41/4} - 0.000178357 \, a_8 x^{41/4} - \\
& 0.0000295219 \, a_{10} x^{13} + 0.000835088 \, a_{10} x^{53/4} + 5.9508608 \, a_2 x^2 + \\
& 3.56431064700 \, a_3 x^3 + 2.706659086 \, a_4 x^4 + 2.272464553 \, a_5 x^5 + \\
& 2.012078018112 \, a_6 x^6 + 1.83916612222 \, a_7 x^7 + 1.716235 \, a_8 x^8 - \\
& 0.002010871 \, a_5 x^8 + 1.6244632350 \, a_9 x^9 + 1.553393813 \, a_{10} x^{10} - \\
& 0.027821725 \, a_8 x^{35/4} + 0.0020973678 \, a_9 x^{39/4} + 0.000749 \, a_{10} x^{21/2} + \\
& 0.6373158355 \, a_6 x^7 - 0.06241569706 \, a_7 x^8 - 0.002023824 \, a_8 x^9 - \\
& 0.00030355 \, a_9 x^{10} - 0.000074837 \, a_{10} x^{11} + 34.50574893 \, a_2 x^{7/2} + \\
& 1.289381128 \, a_3 x^{19/4} + 0.000054091 \, a_{10} x^{43/4} + 2.690882355 \, a_2 x^{15/4} +
\end{aligned}$$

$$\begin{aligned}
& 0.4737673471 a_6 x^{\frac{31}{4}} - 8.472393 a_3 x^{9/2} + 0.82452 a_4 x^{\frac{23}{4}} - \\
& 0.57365165 a_4 x^{11/2} - 7.6357 a_4 x^{\frac{21}{4}} + 0.60196358 a_5 x^{\frac{27}{4}} - \\
& 0.15889499 a_5 x^{13/2} + 1.0709 a_5 x^{\frac{25}{4}} - 0.0199697 a_8 x^{19/2} + \\
& 0.0028226 a_8 x^{\frac{37}{4}} - 0.0662317 a_6 x^{15/2} + 0.046937 a_6 x^{\frac{29}{4}} + \\
& 0.3911446 a_7 x^{\frac{35}{4}} - 0.03408173 a_7 x^{17/2} + 0.00912 a_7 x^{\frac{33}{4}} + \\
& 0.33371465 a_8 x^{\frac{39}{4}} + 0.29157 a_9 x^{\frac{43}{4}} - 0.0127794 a_9 x^{21/2} + \\
& 0.00112 a_9 x^{\frac{41}{4}} + 0.259363 a_{10} x^{\frac{47}{4}} - 0.00871613 a_{10} x^{23/2} + \\
& 0.00052500 a_{10} x^{\frac{45}{4}} - 25.6964 a_4 x^{7/2} - 17.44581 a_3 x^{11/4} - \\
& 29.6533779711 a_3 x^{5/2} + 2.10616 a_4 x^{\frac{15}{4}} + 78.699 a_2 x^{7/4} + \\
& 86.26437232 a_2 x^{3/2} + 2.9243458 a_6 x^{\frac{23}{4}} - 0.242 a_6 x^{11/2} - \\
& 32.45200422 a_4 x^{\frac{13}{4}} + 2.88067 a_5 x^{\frac{19}{4}} + 2.908365 a_5 x^{9/2} + \\
& 5.622429036 a_5 x^{\frac{17}{4}} - 2.77298 a_9 x^{\frac{35}{4}} - 0.115384 a_9 x^{17/2} + \\
& 0.0080543 a_9 x^8 + 2.73686 a_{10} x^{\frac{39}{4}} - 0.0884817 a_{10} x^{19/2} + \\
& 0.00496 a_{10} x^{\frac{37}{4}} + 0.01835196879 a_9 x^{\frac{31}{4}} + 1.693811 a_2 x^5 + \\
& 0.114114 a_2 x^{\frac{21}{4}} - 0.0004470 a_{10} x^9 + 0.00462590 a_{10} x^{\frac{35}{4}} + \\
& 0.007124205 a_{10} x^{17/2} - 0.010559 a_4 x^7 - 0.242601 a_3 x^6 + \\
& 0.0328079 a_3 x^{\frac{25}{4}} - 0.155848 a_4 x^{\frac{27}{4}} + 6.619575911 x^{0.25} + \\
& 0.006994 a_5 x^{\frac{33}{4}} + 0.0137454 a_4 x^{\frac{29}{4}} + 0.01480717 a_5 x^{\frac{31}{4}} - \\
& 0.00000028 a_{10} x^{\frac{25}{2}} + 0.0000002 a_{10} x^{\frac{49}{4}} + 0.006 a_6 x^{17/2} = \\
& 6 + 12 x^2 + \frac{32}{35} x^{7/2}
\end{aligned} \tag{3.84}$$

Collocating equation (3.84) at the point $x = x_i$ gives

$$\begin{aligned}
& 0.007124202898 a_{10} x_i^{17/2} + 1.6938 a_2 x_i^5 - 0.0884817 a_{10} x_i^{19/2} + 2.880670 a_5 x_i^{19/4} + \\
& 1.07093 a_5 x_i^{25/4} + 2.908365 a_5 x_i^{9/2} + 0.00115712 a_9 x_i^{49} - 0.0000535904000 a_9 x_i^{12} + \\
& 0.0000049844 a_9 x_i^{47} - 0.2061264 a_7 x_i^{25} + 2.77558840 a_6 x_i^{21} + 0.01480710 a_5 x_i^{31} - \\
& 0.00201 a_5 x_i^8 + 2.8749460 a_7 x_i^{27} - 0.214773807 a_7 x_i^{13/2} + 0.47376734710 a_6 x_i^{31/4} + \\
& 2.772987 a_9 x_i^{35} + 0.2593638 a_{10} x_i^{47} + 2.73686123 a_{10} x_i^{39} - 0.0003035590 a_9 x_i^{10} - \\
& 0.4369098794 a_7 x_i^6 + 2.819478592 a_8 x_i^{31/4} - 0.242131 a_6 x_i^{11/2} - 8.472393 a_3 x_i^{9/2} + \\
& 2.690882355 a_2 x_i^{15/4} - 0.155848 a_4 x_i^{27} - 0.0105596 a_4 x_i^7 + 0.00083508800 a_0 x_i^{53} - \\
& 4.574180 x_i^{13/4} a_0 + 32.71546441 x_i^{3/4} a_1 + 0.002097367 a_9 x_i^{39} + 0.82452514 a_4 x_i^{23/4} - \\
& 0.02782170 a_8 x_i^{35} + 1.553393 a_{10} x_i^{10} - 0.1588949903 a_5 x_i^{13/2} - 0.0624156 a_7 x_i^8 - \\
& 0.0340817 a_7 x_i^{17/2} + 0.04693724 a_6 x_i^{29/4} + 0.00401692 a_6 x_i^{37} - 0.002023111 a_8 x_i^9 + \\
& 0.000525415 a_{10} x_i^{45} + 0.0000669 a_7 x_i^{39} - 0.000491997 a_7 x_i^{19/2} + 1.51352 x_i^{17/4} a_1 - \\
& 0.000605 a_6 x_i^9 + 0.00046133 a_6 x_i^{35/4} + 30.06749285 x_i a_1 + 0.013545222961 a_8 x_i^{29/4} - \\
& 32.45200422 a_4 x_i^{13/4} + 34.50574893 a_2 x_i^{7/2} - 21.38136809 a_0 - 0.0000748 a_{10} x_i^{11} + \\
& 2.92434 a_6 x_i^{23/4} + 0.114114 a_2 x_i^{21} - 0.0002336294 a_7 x_i^{10} - 17.445816818 a_3 x_i^{11/4} - \\
& 0.00871613 a_{10} x_i^{23/2} + 0.00112349 a_9 x_i^{41/4} - 0.00044 a_{10} x_i^9 + 3.8238950130 a_6 x_i^5 + \\
& 0.008790 a x_i^{33/9} - 17.44824769 a_0 - 23.26433 x_i^{7/4} a + 0.002822643821 a x_i^{37/8} + \\
& 0.33371465 a_8 x_i^{39/4} + 0.00699417 a_5 x_i^{33/4} - 0.0662317 a_6 x_i^{15/2} - 0.0127777 a_9 x_i^{21/2} - \\
& 7.63576570 a_4 x_i^{21/4} + 0.006817 a_6 x_i^{17/2} + 0.0328079 a_3 x_i^{25/4} + 18.69455160 x_i^{11/4} a_1 -
\end{aligned}$$

$$\begin{aligned}
& 0.0001 a_8 x_i^{11} - 0.0000002 a_{10} x_i^{\frac{25}{2}} + 0.0000002 a_{10} x_i^{\frac{49}{4}} - \\
& 0.57365 a_4 x_i^{11/2} + 0.0046259 a_{10} x_i^{\frac{35}{4}} - 0.21561 a_8 x_i^{\frac{27}{4}} - \\
& 0.0199 a_8 x_i^{19/2} + 0.0000549 a_{10} x_i^{\frac{43}{4}} + 78.6991 a_2 x_i^{7/4} - \\
& 0.000178 a_8 x_i^{\frac{41}{4}} + 0.002507 a_7 x_i^{\frac{41}{4}} + 0.0000159 a_8 x_i^{\frac{43}{4}} - \\
& 0.000012159 a_8 x_i^{21/2} - 0.1470 a_8 x_i^7 + 0.637315 a_6 x_i^7 + \\
& 0.000749 a_{10} x_i^{21/2} + 0.008054 a_9 x_i^8 + 0.018351 a_9 x_i^{\frac{31}{4}} - \\
& 0.1558441 a_8 x_i^{15/2} + 0.39114 a_7 x_i^{\frac{35}{4}} + 1.289381 a_3 x_i^{\frac{19}{4}} - \\
& 0.00008 a_{10} x_i^{13} + 0.000009 a_{10} x_i^{\frac{51}{4}} + 0.000003 a_{10} x_i^{12} - \\
& 0.115384673 a_9 x_i^{17/2} - 0.242 a_3 x_i^6 + 0.601964 a_5 x_i^{\frac{27}{4}} + \quad (3.85) \\
& 2.706659086 a_4 x_i^4 + 2.012078 a_6 x_i^6 + 2.272468 a_5 x_i^5 - \\
& 0.0000014 a_9 x_i^{23/2} + 0.000019 a_9 x_i^{\frac{45}{4}} + 0.0016 a_8 x_i^{\frac{45}{4}} + \\
& 1.624463235 a_9 x_i^9 + 1.71623 a_8 x_i^8 + 1.8391661 a_7 x_i^7 + \\
& 2.10616528 a_4 x_i^{\frac{15}{4}} - 29.653 a_3 x_i^{5/2} + 0.00496 a_{10} x_i^{\frac{37}{4}} + \\
& 0.0137454 a_4 x_i^{\frac{29}{4}} + 86.264372 a_2 x_i^{3/2} + 3.56431 a_3 x_i^3 + \\
& 5.950860833 a_2 x_i^2 + 0.2915724 a_9 x_i^{\frac{43}{4}} + 5.622 a_5 x_i^{\frac{17}{4}} + \\
& 0.0091231736 a_7 x_i^{\frac{33}{4}} - 25.6964 a_4 x_i^{7/2} = 6.619 x_i^{0.25} + \\
& 6 + 12 x_i^2 + \frac{32}{35} x_i^{7/2}
\end{aligned}$$

where, $x_i = \frac{-i}{16}$ $i = 1, 2, \dots, 8$.

Hence, for $i = 1$, $x_1 = \frac{1}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
& - 52.10371140 a_0 + 9.681476674 a_1 + 4.967811936 a_2 - \\
& 0.158984077 a_3 - 0.0365575460 a_4 + 0.0007667827 a_5 + \\
& 0.000103999820511 a_6 + 0.00000024501447841114 a_7 + \quad (3.86) \\
& 0.00000003529488847 a_8 + 0.000000016756349750 a_9 + \\
& 0.000000001812575078 a_{10} = 9.970380135
\end{aligned}$$

For $i = 2$, $x_2 = \frac{2}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
& - 48.50205482 a_0 + 17.5717251 a_1 + 15.180060312 a_2 - \\
& 0.9387832142 a_3 - 0.3662120404 a_4 + 0.0163657031 a_5 + \quad (3.87) \\
& 0.00381958259 a_6 + 0.0000797498 a_7 + 0.000020648 a_8 + \\
& 0.000007430633214 a_9 + 0.000001609 a_{10} = 11.1422506
\end{aligned}$$

For $i = 3$, $x_3 = \frac{3}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
& - 47.87523370 a_0 + 25.29992383 a_1 + 29.55989430 a_2 - \\
& 2.674328180 a_3 - 1.418950479 a_4 + 0.099781814140 a_5 + \quad (3.88) \\
& 0.03226532062 a_6 + 0.0016014589 a_7 + 0.000598736 a_8 + \\
& 0.00026581601 a_9 + 0.000086439019 a_{10} = 12.38267382
\end{aligned}$$

For $i = 4$, $x_4 = \frac{4}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
 & - 48.70712491 a_0 + 33.22967068 a_1 + 47.95368689 a_2 - \\
 & 5.66253238830 a_3 - 3.731518730 a_4 + 0.3638471323 a_5 + \\
 & 0.1489811826 a_6 + 0.01279953023 a_7 + 0.006235042 a_8 + \\
 & 0.00338366642 a_9 + 0.001467085552 a_{10} = 13.82874248
 \end{aligned} \tag{3.89}$$

For $i = 5$, $x_5 = \frac{5}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
 & - 50.58561764 a_0 + 41.59381692 a_1 + 70.49410377 a_2 - \\
 & 10.2042729910 a_3 - 7.944278264 a_4 + 1.001028887 a_5 + \\
 & 0.49368880 a_6 + 0.06321153155 a_7 + 0.0378708659 a_8 + \\
 & 0.02444479045 a_9 + 0.01324352868 a_{10} = 15.53549837
 \end{aligned} \tag{3.90}$$

For $i = 6$, $x_6 = \frac{6}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned}
 & 97.48270479 a_2 - 53.35845314 a_0 + 50.582268511 a_1 - \\
 & 16.62256999 a_3 - 14.81020583 a_4 + 2.30504967811 a_5 + \\
 & 1.325948508 a_6 + 0.2318393382 a_7 + 0.1646182293 a_8 + \\
 & 0.1234338108 a_9 + 0.08019838430 a_{10} = 17.535983870
 \end{aligned} \tag{3.91}$$

For $i = 7$, $x_7 = \frac{7}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned} & 60.36761396 a_1 - 56.96823221 a_0 + 129.3462172 a_2 - \\ & 25.27431453 a_3 - 25.20937801 a_4 + 4.695481417 a_5 + \\ & 3.080316282 a_6 + 0.6945587578 a_7 + 0.56959695 a_8 + \\ & 0.4868845667 a_9 + 0.3687728464 a_{10} = 19.85511385 \end{aligned} \quad (3.92)$$

For $i = 8$, $x_8 = \frac{8}{9}$, thus equation (3.85) after simplification gives

$$\begin{aligned} & 71.11571115 a_1 - 61.40141018 a_0 + 166.6162428 a_2 - \\ & 36.55926888 a_3 - 40.16687161 a_4 + 8.746109128 a_5 + \\ & 6.434989203 a_6 + 1.797206774 a_7 + 1.670123834 a_8 + \\ & 1.603253539 a_9 + 1.386501254 a_{10} = 22.51438868 \end{aligned} \quad (3.93)$$

Thus, from the given conditions (3.75); that is $y_{10}(0) = 0$, equation (3.79) gives

$$\begin{aligned} & 6.95 \times 10^{-23} a_2 - 8.59 \times 10^{-14} a_0 + 3.224 \times 10^{-19} a_1 - \\ & 1.07 \times 10^{-28} a_3 - 1.313 \times 10^{-32} a_4 + 1.32 \times 10^{-38} a_5 + \\ & 1.14 \times 10^{-42} a_6 - 8.607 \times 10^{-49} a_7 - 5.82 \times 10^{-53} a_8 + \\ & 3.537 \times 10^{-59} a_9 + 1.97 \times 10^{-63} a_{10} = 0 \end{aligned} \quad (3.94)$$

Thus, from the given conditions (3.75); that is $y_{10}^I(0) = 0$, equation (3.79) gives

$$\begin{aligned} & 1.0959 \times 10^{-11} a_2 - 0.0041370479 a_0 + 0.000000033 a_1 - \\ & 2.656 \times 10^{-17} a_3 - 4.31 \times 10^{-21} a_4 + 5.983 \times 10^{-27} a_5 + \\ & 6.39 \times 10^{-31} a_6 - 6.194 \times 10^{-37} a_7 - 4.966 \times 10^{-41} a_8 + \\ & 3.7068 \times 10^{-47} a_9 + 2.3809 \times 10^{-51} a_{10} = 0 \end{aligned} \quad (3.95)$$

Thus, from the given conditions (3.75); that is $y_{10}^{II}(0) = 0$, equation (3.79) gives

$$\begin{aligned} & 1.209 \times 10^{-13} a_1 - 0.0000000236 a_0 + 3.13 \times 10^{-17} a_2 - \\ & 5.94 \times 10^{-23} a_3 - 8.20 \times 10^{-27} a_4 + 9.574 \times 10^{-33} a_5 + \\ & 9.13 \times 10^{-37} a_6 - 7.79 \times 10^{-43} a_7 - 5.675 \times 10^{-47} a_8 + \\ & 3.8021 \times 10^{-53} a_9 + 2.2672 \times 10^{-57} a_{10} = 0 \end{aligned} \quad (3.96)$$

Solving equations (3.86) – (3.96) to obtain

$$\begin{aligned}
a_0 &= 4.914878900 \times 10^{-13}, \\
a_1 &= 4.806392439 \times 10^{-7}, \\
a_2 &= -0.2231742324 \times 10^{-2}, \\
a_3 &= -396.1775044, \\
a_4 &= 2287.338006, \\
a_5 &= 58534.77735, \\
a_6 &= -1.389945307 \times 10^5, \\
a_7 &= 8.993199179 \times 10^5, \\
a_8 &= -1.374708029 \times 10^6, \\
a_9 &= 9.307379679 \times 10^5, \\
a_{10} &= -2.543447053 \times 10^5
\end{aligned} \tag{3.97}$$

Thus, substituting the values of $a_i (i = 0, 1, 2, \dots, 10)$ above into equation (3.79), and after simplification gives

$$\begin{aligned}
y(x) = & 30.912974010000 x^{10} + 107.51386390000 x^{\frac{37}{4}} - \\
& 0.0003371427953000 x^{\frac{19}{4}} - 0.004889393036000 x^{9/2} + \\
& 135.61906 x^{11/2} + 48.59176957 x^{\frac{27}{4}} - 36.703083 x^{13/2} - \\
& 532.2842542 x^{\frac{25}{4}} + 1383.440949 x^{\frac{29}{4}} - 190.7873 x^{15/2} - \\
& 440.3359719 x^{\frac{51}{4}} - 2.568490058 x^{11} + 673.56335 x^{\frac{31}{4}} + \\
& 144.4054186 x^{17/2} - 971.0756877 x^{\frac{35}{4}} - 109.07452 x^{\frac{33}{4}} - \\
& 1581.846705 x^8 + 4571.55624 x^{\frac{39}{4}} - 379.5712977 x^{19/2} + \\
& 0.0000008713052024 x^{\frac{15}{4}} - 2.375924485 \times 10^{-12} x^{11/4} - \\
& 779.6066605 x^9 - 4358.336173 x^{\frac{43}{4}} + 275.214016 x^{21/2} - \\
& 40.92615556 x^{\frac{41}{4}} + 2148.352854 x^{\frac{47}{4}} - 100.08325 x^{23/2} + \\
& 9.067901283 x^{\frac{45}{4}} + 15.421941 x^{\frac{25}{2}} - 0.9696989451 x^{\frac{49}{4}} + \\
& 0.1442006878 x^{12} - 18.70293080 x^{\frac{23}{4}}
\end{aligned} \tag{3.98}$$

Thus, equation (3.98) is the approximate solution of equation (3.74) together with the given conditions (3.75) for case $N = 10$

3.3 A Perturbed Collocation Method

In this method, the derived canonical polynomials used in Standard Collocation Method as a new basis function for collocation approximations to the linear and nonlinear singular and multi-order fractional integro-differential equations (3.4) together with the given conditions in equation (3.5) are considered. The basic idea of the perturbed collocation method is the addition to equation (3.4) of a perturbation term $H_N(x)$ which causes the equation

$$D^a y_N(x) + \sum_{i=0}^n p_i y_N^i(x) + \lambda \int_0^x \frac{y(t)}{\sqrt[N]{x-t}} dt = f(x) + H_N(x) \quad a \leq x \leq b \quad (3.99)$$

to have an exact polynomial solution $y_N(x)$ which satisfies exactly the given conditions

$$y_N^i(0) = a_i, i = 0, 1, 2, \dots, n-1 \quad (3.100)$$

where,

$$H_N(x) = \sum_{i=1}^N \tau_i T_{N-k}^*(x); \quad x \in [a, b]$$

A useful choice of $T_{N-k}(x)$ is the set of shifted Chebyshev polynomials $\tau_i^*(x)$, defined for $x \in [a, b]$; ($i = 0, 1, 2, \dots$) which characterizes the Lanczos Tau method (Lanczos, (1956)). A form of collocation method used in Taiwo (1988) involving collocating (3.99)

at $(N + 1)$ equally spaced interior points on $[a, b]$ in addition to the $(n - 1)$ conditions in equation (3.100) to be satisfied by $y_N(x)$. These collocation equations lead to $(N + n + 1)$ algebraic equations for the unique determination of the $(N + n + 1)$ parameters

$$a_j; \tau_k; j = 0, 1, \dots, N; k = 1, \dots, n$$

3.4 Demonstration of Perturbed Collocation Method on Some Problems

Problem 3.4.1: Consider a second order singular fractional integro-differential equation of the form

$$y'' + \frac{d^{1.5}y(x)}{dx^{1.5}} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 2.256758334 x^{0.5} + 7.221626669 x^{2.5} - \frac{16}{15} x^{5/2} - \frac{256}{315} x^{9/2} + 2 + 12 x^2, \quad (3.101)$$

$$0 \leq x \leq 1$$

subject to the conditions

$$y(0) = 0, y'(0) = 0; \quad (3.102)$$

The exact solution is given by $y(x) = x^4 + x^2$

The recursive canonical polynomials obtained for equation (3.101)

is

$$Q_j(x) = \frac{x^{j+1} (j+1)(j) Q_{j-2} - \Gamma(j-a+2)}{x^{1-a} \Gamma(j+2)} \quad (3.103)$$

Here, equations (3.101)–(3.102) are solved for case $N = 7$. Hence, equation (3.20) becomes

$$y_7(x) = \sum_{j=0}^7 a_j Q_j(x) \quad (3.104)$$

From equation (3.103), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 7$) are obtained

$$\square Q_0(x) = 1.772453851 x^{1.5}$$

$$\square Q_1(x) = 0.4431134628 x^{2.5} - 1.57079632700 x^{2.0}$$

$$\square Q_2(x) = 0.2215567313 x^{3.5} - 0.2945243112 x^{3.0} + 1.04406149 x^{2.5}$$

$$\square Q_3(x) = 0.1384729571 x^{4.5} - 0.1227184630 x^{4.0} +$$

$$\square 0.16313461 x^{3.5} - 0.57829713260000 x^{3.0}$$

$$\square Q_4(x) = 0.09693107000 x^{5.5} - 0.0671116595 x^{5.0} +$$

$$\square 0.05947616 x^{4.5} - 0.0790640611700 x^{4.0} +$$

$$\square 0.2802747992 x^{3.5}$$

$$\square Q_5(x) = 0.0726983025 x^{6.5} - 0.04228034547 x^{6.0} +$$

$$\square 0.0292734 x^{5.5} - 0.02594289506000 x^{5.0} +$$

$$\square 0.0344869382 x^{4.5} - 0.1222530100000 x^{4.0}$$

$$\square Q_6(x) = 0.05712009480 x^{7.5} - 0.0290677375 x^{7.0} +$$

$$\begin{aligned}
& 0.01690540 x^{6.5} - 0.01170470460 x^{6.0} + \\
& 0.01037302438 x^{5.5} - 0.01378928 x^{5.0} + \\
& 0.04888172573 x^{4.5} \\
& Q_7(x) = 0.04641007703 x^{8.5} - 0.02120758 x^{8.0} + \quad (3.105) \\
& 0.010792287 x^{7.5} - 0.0062766478 x^{7.0} + \\
& 0.004345729 x^{6.5} - 0.0038513028 x^{6.0} + \\
& 0.005119692473 x^{5.5} - 0.01814884 x^{5.0}
\end{aligned}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 7$) in equation (3.105) gives

$$\begin{aligned}
y_7(x) = & 1.772453851 x^{1.5} a_0 + (0.4431134600 x^{2.5} - \\
& 1.57079633 x^{2.0}) a_1 + (0.2215567313 x^{3.5} - \\
& 0.2945243 x^{3.0} + 1.044061499000 x^{2.5}) a_2 + \\
& (0.138472957 x^{4.5} - 0.122718460000 x^{4.0} + \\
& 0.1631346093 x^{3.5} - 0.578297132 x^{3.0}) a_3 + \\
& (0.09693107 x^{5.5} - 0.0671116594900 x^{5.0} + \\
& 0.05947615964 x^{4.5} - 0.07906406000 x^{4.0} + \\
& 0.280274799 x^{3.5}) a_4 + (0.0726983024 x^{6.5} - \\
& 0.04228034547 x^{6.0} + 0.02927342000 x^{5.5} - \\
& 0.02594289506 x^{5.0} + 0.03448693800 x^{4.5} -
\end{aligned}$$

$$\begin{aligned}
& 0.122253013 x^{4.0})a_5 + (0.05712009480 x^{7.5} - \\
& 0.02906773750 x^{7.0} + 0.016905400000 x^{6.5} - \\
& 0.011704704600 x^{6.0} + 0.01037302438 x^{5.5} - \\
& 0.01378928 x^{5.0} + 0.048881725730 x^{4.5})a_6 + \\
& (0.0464100770300 x^{8.5} - 0.021207580 x^{8.0} + \\
& 0.01079228749 x^{7.5} - 0.006276647845 x^{7.0} + \\
& 0.004345729 x^{6.5} - 0.00385130288400 x^{6.0} + \\
& 0.005119692473 x^{5.5} - 0.0181488400 x^{5.0})a_7
\end{aligned} \tag{3.106}$$

Thus, equation (3.106) is substituted into a slightly perturbed equation (3.101), to give

$$y_7''(x) + \frac{d^{1.5}y_7(x)}{dx^{1.5}} - \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2 + 12x^2 - \frac{16}{15}x^{5/2} + 2.256758334x^{0.5} + 7.221626669x^{2.5} - \frac{256}{315}x^{9/2} + H_7(x) \quad (3.107)$$

together with equation (3.102) to get

$$y_7(0) = 0, \quad y_7'(0) = 0$$

where,

$$H_7(x) = \tau_1(8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1) + \tau_2(2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1) \quad (3.108)$$

Hence, evaluating the terms $\frac{d^2y_7(x)}{dx^2}$, $\frac{d^{1.5}y_7(x)}{dx^{1.5}}$, and $\int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt$ in equation (3.107) gives

$$\begin{aligned}
\frac{d^{1.5}y_7}{dx^{1.5}} = & 2.356196748 a_0 + (-3.54491035 x^{0.5} + \\
& 1.472619 x^{1.0}) a_1 + (1.28854 x^{2.0} + 3.46978 x^{1.0} - \\
& 1.32933 x^{1.5}) a_2 + (0.948769 x^{2.0} - 0.88622 x^{2.5} + \\
& 1.2080 x^{3.0} - 2.610155 x^{1.5}) a_3 + (-0.69236 x^{3.5} - \\
& 0.570971 x^{2.5} + 0.518858 x^{3.0} + 1.16271010 x^{4.0} + \\
& 1.630039 x^{2.0}) a_4 + (0.3511408 x^{4.0} - 0.2676 x^{3.5} - \\
& 0.8828664 x^{2.5} + 0.3008569 x^{3.0} - 0.581589 x^{4.5} + \quad (3.109) \\
& 1.133643 x^{5.0}) a_5 + (0.42643 x^{3.0} - 0.142258 x^{3.5} + \\
& 0.1244269 x^{4.0} - 0.1610039 x^{4.5} + 0.2636192 x^{5.0} - \\
& 0.508888 x^{5.5} + 1.1133 x^{6.0}) a_6 + (-0.187234 x^{3.5} + \\
& 0.061412 x^{4.0} - 0.0529764 x^{4.5} + 0.06776658 x^{5.0} - \\
& 0.1098849484 x^{5.5} + 0.2103 x^{6.0} - 0.4569608 x^{6.5} + \\
& 1.098488 x^{7.0}) a_7.
\end{aligned}$$

$$\begin{aligned}
& \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2.08812 x^{2.0} a_0 + (0.43502600 x^{3.0} - \\
& 1.67552 x^{2.5}) a_1 + (0.19032400 x^{4.0} + 1.025111 x^{3.0} - \\
& 0.269279 x^{3.5}) a_2 + (0.140137 x^{4.0} - 0.0997331 x^{4.5} - \\
& 0.528729 x^{3.5} + 0.107057 x^{5.0}) a_3 + (0.2407640 x^{4.0} - \\
& 0.06425520 x^{4.5} + 0.0459826 x^{5.0} - 0.0495832 x^{5.5} + \\
& 0.068695 x^{6.0}) a_4 + (-0.0993548 x^{4.5} + 0.02666 x^{5.0} - \\
& 0.019167 x^{5.5} + 0.020746444 x^{6.0} - 0.0288345 x^{6.5} + \\
& 0.047841 x^{7.0}) a_5 + (0.0377917 x^{5.0} - 0.010187 x^{5.5} + \\
& 0.00735135 x^{6.0} - 0.00798243 x^{6.5} + 0.0111251 x^{7.0} - \\
& 0.0185022 x^{7.5} + 0.03524 x^{8.0}) a_6 + (-0.013409 x^{5.5} + \\
& 0.003628 x^{6.0} - 0.00262653 x^{6.5} + 0.002859830 x^{7.0} - \\
& 0.00399521 x^{7.5} + 0.00665828 x^{8.0} - 0.0127050 x^{8.5} + \\
& 0.0270419 x^{9.0}) a_7
\end{aligned} \tag{3.110}$$

and,

$$\begin{aligned}
\frac{d^2 y_7}{dx^2} = & 1.32934 a_0 x^{-0.5} + (1.6616 x^{0.5} - 3.14159) a_1 + \\
& (1.938621 x^{1.5} - 1.767145867 x^{1.0} + 3.91523 x^{0.5}) a_2 + \\
& (2.180949 x^{2.5} - 1.472621556 x^{2.0} + 1.42742781 x^{1.5} - \\
& 3.469782 x^{1.0}) a_3 + (2.39904398 x^{3.5} - 1.342233 x^{3.0} + \\
& 0.9367495 x^{2.5} - 0.9487687 x^{2.0} + 2.4524044 x^{1.5}) a_4 + \\
& (2.598964314 x^{4.5} - 1.2684103 x^{4.0} + 0.7245172 x^{3.5} - \\
& 0.51885790 x^{3.0} + 0.5431693 x^{2.5} - 1.467036 x^{2.0}) a_5 + \\
& (2.784604622 x^{5.5} - 1.220844975 x^{5.0} + 0.60436 x^{4.5} - \\
& 0.35114113 x^{4.0} + 0.2567323 x^{3.5} - 0.275785680 x^{3.0} + \\
& 0.7698871803 x^{2.5}) a_6 + (2.958642 x^{6.5} - 1.18762 x^{6.0} + \\
& 0.52612401 x^{5.5} - 0.2636192095 x^{5.0} + 0.1553598 x^{4.5} - \\
& 0.1155391 x^{4.0} + 0.1267123 x^{3.5} - 0.362976745 x^{3.0}) a_7
\end{aligned} \tag{3.111}$$

Hence, substituting equations (3.108)–(3.111) into equation (3.107),
to give

$$\begin{aligned}
& 3584.0 t_2 x^3 - 0.9487687341000 a_4 x^{2.0} - 840 t_2 x^2 + 72 t_2 x - 1.883234863 a_1 x^{0.5} + \\
& 0.6092835 a_2 x^{1.5} - 1.7671458 a_2 x^{1.0} + 3.915230622 a_2 x^{0.5} + 1.294720076 a_3 x^{2.5} - \\
& 1.4726215 a_3 x^{2.0} - 1.1827275 a_3 x^{1.5} - 3.469782796 a_3 x^{1.0} + 1.706681809 a_4 x^{3.5} +
\end{aligned}$$

$$\begin{aligned}
& 0.3657783721 a_4 x^{2.5} - 1.342233 a_4 x^{3.0} - 0.3511411380 a_6 x^{4.0} - 0.3396970 a_5 x^{2.5} - \\
& 1.467036152 a_5 x^{2.0} + 2.285904137 a_6 x^{5.5} - 1.220844975 a_6 x^{5.0} + 0.443364 a_6 x^{4.5} + \\
& 2.452404493 a_4 x^{1.5} + 2.116729924 a_5 x^{4.5} - 1.268410364 a_5 x^{4.0} + 0.456874 a_5 x^{3.5} - \\
& 0.5188579012 a_5 x^{3.0} + 0.114473 a_6 x^{3.5} - 0.2757856052 a_6 x^{3.0} + 0.7698871 a_6 x^{2.5} + \\
& 2.504308101 a_7 x^{6.5} - 1.187624704 a_7 x^{6.0} + 0.42964776 a_7 x^{5.5} - 0.2636190 a_7 x^{5.0} + \\
& 0.10238334600 a_7 x^{4.5} - 0.1155390865 a_7 x^{4.0} - 0.435026 a_1 x^{3.0} - 1.025000 a_2 x^{3.0} + \\
& 0.2692790 a_2 x^{3.5} - 0.0605218232 a_7 x^{3.5} - 0.3629767450 a_7 x^{3.0} - 0.140137 a_3 x^{4.0} + \\
& 0.099733 a_3 x^{4.5} + 0.528729 a_3 x^{3.5} - 0.107057 a_3 x^{5.0} + 0.92194 a_4 x^{4.0} - 8192 t_1 x^7 + \\
& 0.0642552 a_4 x^{4.5} + 1.10698 a_5 x^{5.0} + 0.019167 a_5 x^{5.5} - 0.020746 a_5 x^{6.0} - \frac{256}{315} x^{9/2} - \\
& 0.04598 a_4 x^{5.0} + 0.04958 a_4 x^{5.5} - 0.068695 a_4 x^{6.0} - 0.0111251 a_6 x^{7.0} + 1568 t_1 x^2 + \\
& 1.106048584 a_6 x^{6.0} + 0.00798243 a_6 x^{6.5} + 0.0288345 a_5 x^{6.5} - 0.047841111 a_5 x^{7.0} + \\
& 0.2258275755 a_6 x^{5.0} + 0.003995210 a_7 x^{7.5} + 1.095628573 a_7 x^{7.0} + 0.01850 a_6 x^{7.5} - \\
& 0.0352401 a_6 x^{8.0} + 0.2067376649 a_7 x^{6.0} - 2.088120 x^{2.0} a_0 + 1.329340388 a_0 x^{-0.5} + \\
& 1.2885413060 a_2 x^{2.0} - 0.00665828 a_7 x^{8.0} + 0.012705 a_7 x^{8.5} - 0.027041900 a_7 x^{9.0} + \\
& 1.4726194 a_1 x^{1.0} + 1.6300396 a_4 x^{2.0} + 3.469782864 a_2 x^{1.0} + 0.9487694131 a_3 x^{2.0} + \\
& 1.208008885 a_3 x^{3.0} + 0.518858 a_4 x^{3.0} + 0.0614120 a_7 x^{4.0} + 0.35114087720 a_5 x^{4.0} + \\
& 0.30085691 a_5 x^{3.0} + 0.42643423 a_6 x^{3.0} + 0.1244269349 a_6 x^{4.0} + 0.0677665 a_7 x^{5.0} +
\end{aligned}$$

$$\begin{aligned}
& 2.3561967 a_0 - 3.141592 a_1 + 6144 t_2 x^5 - 6912 t_2 x^4 + \\
& 28672 t_1 x^6 - 39424 t_1 x^5 + 26880 t_1 x^4 - 9408.0 t_1 x^3 - \\
& 98 t_1 x - 2048 t_2 x^6 + t_1 - t_2 = 2 + 2.256758334 x^{0.5} - \\
& \frac{16}{15} x^{5/2} + 12 x^2 - 1.67552 a_1 x^{2.5} + 0.19032400 a_2 x^{4.0}
\end{aligned} \tag{3.112}$$

Collocating equation (3.112) at the point $x = x_i$ to give

$$\begin{aligned}
& 3584 t_2 x_i^3 - 0.94876873410 a_4 x_i^{2.0} - 840 t_2 x_i^2 + 72 t_2 x_i - 1.883234863 a_1 x_i^{0.5} + \\
& 0.609283544 a_2 x_i^{1.5} - 1.767145867 a_2 x_i^{1.0} + 3.915230 a_2 x_i^{0.5} + 1.294720 a_3 x_i^{2.5} - \\
& 1.472621556 a_3 x_i^{2.0} - 1.182727573 a_3 x_i^{1.5} - 3.469782 a_3 x_i^{1.0} + 1.706681 a_4 x_i^{3.5} + \\
& 0.3657783721 a_4 x_i^{2.5} - 1.342233 a_4 x_i^{3.0} - 0.3511411 a_6 x_i^{4.0} - 0.3396971 a_5 x_i^{2.5} - \\
& 1.467036152 a_5 x_i^{2.0} + 2.28590 a_6 x_i^{5.5} - 1.220844975 a_6 x_i^{5.0} + 0.4433641 a_6 x_i^{4.5} + \\
& 2.45240449 a_4 x_i^{1.5} + 2.116729924 a_5 x_i^{4.5} - 1.268410 a_5 x_i^{4.0} + 0.4568746 a_5 x_i^{3.5} - \\
& 0.5188579012 a_5 x_i^{3.0} + 0.1144736 a_6 x_i^{3.5} - 0.2757856 a_6 x_i^{3.0} + 0.769887 a_6 x_i^{2.5} + \\
& 2.504308101 a_7 x_i^{6.5} - 1.187624 a_7 x_i^{6.0} + 0.429647766 a_7 x_i^{5.5} - 0.263619 a_7 x_i^{5.0} + \\
& 0.10238334600 a_7 x_i^{4.5} - 0.1155390865 a_7 x_i^{4.0} - 0.435026 a_1 x_i^{3.0} - 1.025 a_2 x_i^{3.0} + \\
& 0.269279 a_2 x_i^{3.5} - 0.06052182 a_7 x_i^{3.5} - 0.3629767450 a_7 x_i^{3.0} - 0.140137 a_3 x_i^{4.0} + \\
& 0.09973 a_3 x_i^{4.5} + 0.528 a_3 x_i^{3.5} - 0.107057 a_3 x_i^{5.0} + 0.92194 a_4 x_i^{4.0} - 8192 t_1 x_i^7 + \\
& 0.06425 a_4 x_i^{4.5} + 1.106980 a_5 x_i^{5.0} + 0.01916 a_5 x_i^{5.5} - 0.0207 a_5 x_i^{6.0} - \frac{256}{315} x_i^{9/2} - \\
& 0.0459826 a_4 x_i^{5.0} + 0.04958 a_4 x_i^{5.5} - 0.068 a_4 x_i^{6.0} - 0.0111 a_6 x_i^{7.0} + 1568 t_1 x_i^2 + \\
& 1.106048584 a_6 x_i^{6.0} + 0.00798243 a_6 x_i^{6.5} + 0.028834 a_5 x_i^{6.5} - 0.0478411 a_5 x_i^{7.0} +
\end{aligned}$$

$$\begin{aligned}
& 0.22582 a_6 x_i^{5.0} + 0.003995 a_7 x_i^{7.5} + 1.09562 a_7 x_i^{7.0} + \\
& 0.01850 a_6 x_i^{7.5} - 0.03524 a_6 x_i^{8.0} + 0.206737 a_7 x_i^{6.0} - \\
& 2.088 x_i^{2.0} a_0 + 1.3293403 a_0 x_i^{-0.5} + 1.28854 a_2 x_i^{2.0} - \\
& 0.00665 a_7 x_i^{8.0} + 0.0127 a_7 x_i^{8.5} - 0.0270419 a_7 x_i^{9.0} + \\
& 1.47261 a_1 x_i^{1.0} + 1.630039 a_4 x_i^{2.0} + 3.46978 a_2 x_i^{1.0} + \\
& 0.94876941 a_3 x_i^{2.0} + 1.208 a_3 x_i^{3.0} + 0.51885 a_4 x_i^{3.0} + \\
& 0.061412 a_7 x_i^{4.0} + 0.351140 a_5 x_i^{4.0} + 0.3008 a_5 x_i^{3.0} + \\
& 0.42643 a_6 x_i^{3.0} + 0.12442 a_6 x_i^{4.0} + 0.067766 a_7 x_i^{5.0} + \\
& 2.3561967 a_0 - 3.14159 a_1 + 6144 t_2 x_i^5 - 6912 t_2 x_i^4 + \\
& 28672 t_1 x_i^6 - 39424 t_1 x_i^5 + 26880 t_1 x_i^4 - 9408 t_1 x_i^3 - \\
& 98 t_1 x_i - 2048 t_2 x_i^6 + t_1 - t_2 = 2 + 2.2567583 x_i^{0.5} - \\
& \frac{16}{15} x_i^{5/2} + 12 x_i^2 - 1.67552 a_1 x_i^{2.5} + 0.190324 a_2 x_i^{4.0}
\end{aligned} \tag{3.113}$$

where, $x_i = \frac{i}{9}$, $i = 1, 2, 3, \dots, 8$.

Hence, for $i = 1$, $x_1 = \frac{1}{9}$ thus equation (3.113) after simplification gives

$$\begin{aligned}
& 6.318438653 a_0 - 3.599414825 a_1 + 1.531420819 a_2 - \\
& 0.4285946593 a_3 + 0.1005398815 a_4 - 0.0196130 a_5 + \\
& 0.003413361915 a_6 - 0.0005295692 a_7 + 0.59265 t_2 + \\
& 0.0453122363 t_1 = 2.895970062
\end{aligned} \tag{3.114}$$

For $i = 2$, $x_2 = \frac{2}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned}
 & 5.073036520 \alpha_0 - 3.667878440 \alpha_1 + 2.341159914 \alpha_2 - \\
 & 0.8749837585 \alpha_3 + 0.3011584324 \alpha_4 - 0.0795838 \alpha_5 + \\
 & 0.02030271840 \alpha_6 - 0.004253672928 \alpha_7 - 0.9239 \tau_2 + \\
 & 0.8314008072 \tau_1 = 3.630673217
 \end{aligned} \tag{3.115}$$

For $i = 3$, $x_3 = \frac{3}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned}
 & 4.426668508 \alpha_0 - 3.646633078 \alpha_1 + 3.053878492 \alpha_2 - \\
 & 1.304772464 \alpha_3 + 0.5888005049 \alpha_4 - 0.17476341 \alpha_5 + \\
 & 0.06063133708 \alpha_6 - 0.01332279332 \alpha_7 - 0.451303 \tau_2 - \\
 & 0.690900759 \tau_1 = 4.562053930
 \end{aligned} \tag{3.116}$$

For $i = 4$, $x_4 = \frac{4}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned} & 3.937739182 \alpha_0 - 3.560132008 \alpha_1 + 3.720284279 \alpha_2 - \\ & 1.693274329 \alpha_3 + 0.9738722002 \alpha_4 - 0.28840424 \alpha_5 + \\ & 0.1416852158 \alpha_6 - 0.02423209514 \alpha_7 + 0.785033 \tau_2 - \\ & 0.70284348 \tau_1 = 5.713269383 \end{aligned} \quad (3.117)$$

For $i = 5$, $x_5 = \frac{5}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned} & 3.495212549 \alpha_0 - 3.416293799 \alpha_1 + 4.354674550 \alpha_2 - \\ & 2.018468906 \alpha_3 + 1.476752173 \alpha_4 - 0.388430619 \alpha_5 + \\ & 0.2973847455 \alpha_6 - 0.01740269513 \alpha_7 + 0.785033 \tau_2 + \\ & 0.7028427 \tau_1 = 7.082703733 \end{aligned} \quad (3.118)$$

For $i = 6$, $x_6 = \frac{6}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned} & 3.056246237 \alpha_0 - 3.218372735 \alpha_1 + 4.960047457 \alpha_2 - \\ & 2.259822357 \alpha_3 + 2.125090421 \alpha_4 - 0.423865683 \alpha_5 + \\ & 0.5897233605 \alpha_6 + 0.06119187569 \alpha_7 - 0.451302 \tau_2 + \\ & 0.6909017 \tau_1 = 8.657814276 \end{aligned} \quad (3.119)$$

For $i = 7$, $x_7 = \frac{7}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned} & 2.600343362 \alpha_0 - 2.967863047 \alpha_1 + 5.534416846 \alpha_2 - \\ & 2.397947136 \alpha_3 + 2.952518935 \alpha_4 - 0.321526041 \alpha_5 + \\ & 1.122426473 \alpha_6 + 0.3282756178 \alpha_7 - 0.92392010 \tau_2 - \\ & 0.831399 \tau_1 = 10.41817206 \end{aligned} \quad (3.120)$$

For $i = 8$, $x_8 = \frac{8}{9}$, thus equation (3.113) after simplification gives

$$\begin{aligned} & 2.116302560 \alpha_0 - 2.665506721 \alpha_1 + 6.073086748 \alpha_2 - \\ & 2.414528830 \alpha_3 + 3.997760971 \alpha_4 + 0.016989903 \alpha_5 + \\ & 2.056677689 \alpha_6 + 1.003127778 \alpha_7 + 0.592651300 \tau_2 - \\ & 0.045312 \tau_1 = 12.33622872 \end{aligned} \quad (3.121)$$

Thus, from the given conditions (3.102); that is $y_7(0) = 0$, equation (3.106) gives

$$\begin{aligned} & 5.604991216 \times 10^{-8} \alpha_0 - 1.56939507 \times 10^{-10} \alpha_1 + \\ & 3.298674117 \times 10^{-13} \alpha_2 - 5.777824785 \times 10^{-16} \alpha_3 + \\ & 8.855179697 \times 10^{-19} \alpha_4 - 1.221442139 \times 10^{-21} \alpha_5 + \\ & 1.544400234 \times 10^{-24} \alpha_6 - 1.8132685 \times 10^{-27} \alpha_7 = 0 \end{aligned} \quad (3.122)$$

and $y_7^1(0) = 0$, thus equation (3.106) gives

$$\begin{aligned} & 0.008407486823 \alpha_0 - 0.00003138089534000000 \alpha_1 + \\ & 8.245219679 \times 10^{-8} \alpha_2 - 1.7330907180 \times 10^{-10} \alpha_3 + \\ & 3.098919445 \times 10^{-13} \alpha_4 - 4.885225851 \times 10^{-16} \alpha_5 + \\ & 6.949114848 \times 10^{-19} \alpha_6 - 9.0655372 \times 10^{-22} \alpha_7 = 0 \end{aligned} \quad (3.123)$$

Solving equations (3.114) – (3.123) to obtain

$$\begin{aligned}a_0 &= 0.00006170000817, \\a_1 &= 0.04435015855, \\a_2 &= 10.75918619, \\a_3 &= 81.95384197, \\a_4 &= 292.4533372, \\a_5 &= 292.4371049, \\a_6 &= -633.0046401, \\a_7 &= 271.5674822, \\\tau_1 &= 0.00005327195165, \\\tau_2 &= 0.000005699649191\end{aligned}\tag{3.124}$$

Thus, substituting the values of $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ above into equation (3.106), and after simplification gives

$$\begin{aligned}
 y_7(x) = & 11.25290421 x^{2.5} - 0.06966506615 x^{2.0} - \\
 & 50.562513720 x^{3.0} + 97.720578470 x^{3.5} + \\
 & 7.88529337 x^{4.5} - 68.93111516000 x^{4.0} + \\
 & 31.732619200 x^{5.5} - 23.413649550 x^{5.0} + \\
 & 11.7386425300 x^{6.5} - 6.001098124 x^{6.0} - \\
 & 33.226450710 x^{7.5} + 16.695479260 x^{7.0} + \\
 & 12.60346777 x^{8.5} - 5.75929019000 x^{8.0} + \\
 & 0.000109360 x^{1.5}
 \end{aligned} \tag{3.125}$$

Thus, equation (3.125) is the approximate solution of equation (3.101) together with the given conditions (3.102) for case $N = 7$

Problem 3.4.2: Consider a third order singular fractional integro-differential equation of the form

$$y'''(x) + \frac{d^{2.75}y}{dx^{2.75}} + xy(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 6.00 + 12x^2 + \frac{32}{35}x^{7/2} + 6.619575908x^{0.25} \quad (3.126)$$

subject to the conditions

$$\begin{aligned} y(0) &= 0, & y'(0) &= 0, \\ y''(0) &= 0 \end{aligned} \quad (3.127)$$

The exact solution is $y(x) = x^3$

The recursive canonical polynomials obtained for equation (3.126) is

$$Q_j(x) = \frac{(x^{j+1} - (j+1)(j)(j-1)Q_{j-2})\Gamma(j-a+2)}{x^{1-a}\Gamma(j+2)} \quad (3.128)$$

$$j = 0, 1, \dots$$

Here, equations (3.126) – (3.127) are solved for case $N = 10$.

Hence, equation (3.20) becomes

$$y_{10}(x) = \sum_{j=0}^{10} a_j Q_j(x) \quad (3.129)$$

From equation (3.128), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 10$)

are obtained

$$Q_0(x) = - 4.834146544 x^{11/4}$$

$$Q_1(x) = 1.812804954 x^{\frac{15}{4}}$$

$$Q_2(x) = 0.1510670796 x^{\frac{19}{4}} + 2.19084120200000 x^{9/2}$$

$$Q_3(x) = 0.04720846234 x^{\frac{23}{4}} - 0.3423189376000 x^{11/2}$$

$$Q_4(x) = 0.02124380806 x^{\frac{27}{4}} - 0.01604620022 x^{13/2} - 0.2327090499 x^{\frac{25}{4}}$$

$$Q_5(x) = 0.01150706270 x^{\frac{31}{4}} - 0.00325938441 x^{15/2} + 0.02363451287 x^{\frac{29}{4}}$$

$$Q_6(x) = 0.00698643092 x^{\frac{35}{4}} - 0.001038928783 x^{17/2} + 0.0007847396858 x^{\frac{33}{4}} + 0.0113806399200 x^8$$

$$Q_7(x) = 0.0045848452 x^{\frac{39}{4}} - 0.0004220648183 x^{19/2} + 0.0001195501865 x^{\frac{37}{4}} - 0.0008668846814 x^9$$

$$Q_8(x) = 0.00318392034 x^{\frac{43}{4}} - 0.000200198155 x^{21/2} + 0.0000297707983 x^{\frac{41}{4}} - 0.00002248693 x^{10} - 0.0003261154587 x^{\frac{39}{4}}$$

$$\begin{aligned}
Q_9(x) = & 0.002308342249 x^{\frac{47}{4}} - 0.00010583392 x^{23/2} + \\
& 0.000009742700519 x^{\frac{45}{4}} - 0.0000027596 x^{11} + \\
& 0.000020010665 x^{\frac{43}{4}} \\
Q_{10}(x) = & 0.001731256687 x^{\frac{51}{4}} - 0.000060634017 x^{\frac{25}{2}} + \\
& 0.0000038125383 x^{\frac{49}{4}} - 0.00000056694 x^{12} + \\
& 0.0000004282372 x^{\frac{47}{4}} + 0.00000621048 x^{23/2}
\end{aligned} \tag{3.130}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 10$) in equation (3.130) gives

$$\begin{aligned}
y_{10}(x) = & -4.83414 x^{11/4} a_0 + 1.812804954 x^{\frac{15}{4}} a_1 + \\
& (0.1510670796 x^{\frac{19}{4}} + 2.19084120 x^{9/2}) a_2 + \\
& (0.0472084 x^{\frac{23}{4}} - 0.3423189376 x^{11/2}) a_3 + \\
& (0.02124380806 x^{\frac{27}{4}} - 0.016046200 x^{13/2} - \\
& 0.2327090499 x^{\frac{25}{4}}) a_4 + (0.011507062 x^{\frac{31}{4}} - \\
& 0.00325938 x^{15/2} + 0.0236345128 x^{\frac{29}{4}}) a_5 + \\
& (0.00698643092 x^{\frac{35}{4}} - 0.001038928 x^{17/2} + \\
& 0.000784739 x^{\frac{33}{4}} + 0.01138063992 x^8) a_6 + \\
& (0.004584840 x^{\frac{39}{4}} - 0.00042206482 x^{19/2} + \\
& 0.00011955018 x^{\frac{37}{4}} - 0.000866884 x^9) a_7 +
\end{aligned}$$

$$\begin{aligned}
& (0.00318392 x^{\frac{43}{4}} - 0.0002001981559 x^{21/2} + \\
& 0.000029770 x^{\frac{41}{4}} - 0.00002248000000 x^{10} - \\
& 0.00032611 x^{\frac{39}{4}}) a_8 + (0.00230834224 x^{474} - \\
& 0.0001058330 x^{23/2} + 0.000009742000 x^{\frac{45}{4}} - \\
& 0.0000027596 x^{11} + 0.0000200106 x^{\frac{43}{4}}) a_9 + \\
& (0.001731256687 x^{\frac{51}{4}} - 0.0000606340 x^{\frac{25}{2}} + \\
& 0.000003812538358 x^{\frac{49}{4}} - 0.00000056 x^{12} + \\
& 0.000000428 x^{\frac{47}{4}} + 0.00000621048 x^{23/2}) a_{10}
\end{aligned} \tag{3.131}$$

Thus, equation (3.131) is substituted into a slightly perturbed equation (3.126) to get

$$\begin{aligned}
& y_{10}'''(x) + \frac{d^{2.75} y_{14}}{dx^{2.75}} + x y_{14}(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = \\
& 6.000 + 12 x^2 + \frac{32}{35} x^{7/2} + 6.619575908 x^{0.25} + \\
& H_{10}(x)
\end{aligned} \tag{3.132}$$

together with the conditions (3.127) to get

$$y_{10}(0) = 0, \quad y_{10}'(0) = 0, \quad y_{10}''(0) = 0$$

where,

$$\begin{aligned}
 H_{10}(x) = & \tau_1(1 - 200x + 6600.00\tau_1x^2 - 84480x^3 + \\
 & 549120.0x^4 - 2050048x^5 + 4659200x^6 - \\
 & 6553600x^7 + 5570560x^8 - 2621440x^9 + \\
 & 524288x^{10}) + \tau_2(-1 + 162x - 4320x^2 + \\
 & 44352x^3 - 228096x^4 + 658944.0000x^5 - \quad (3.133) \\
 & 1118208x^6 + 1105920x^7 - 589824.0x^8 + \\
 & 131072x^9) + \tau_3(1 - 128x + 2688.00x^2 - \\
 & 21504x^3 + 84480x^4 - 180224.00000x^5 + \\
 & 212992.00000x^6 - 131072x^7 + 32768x^8)
 \end{aligned}$$

Hence, evaluating the terms $\frac{d^3y_{10}(x)}{dx^3}$, $\frac{d^{2.75}y_{10}(x)}{dx^{2.75}}$, $xy_{10}(x)$, and $\int_0^x \frac{y_{10}(t)}{\sqrt{x-t}} dt$ in equation (3.132) gives

$$\begin{aligned}
 \frac{d^{2.75}y_{10}}{dx^{2.75}} = & -21.38136809a_0 + 30.067492850xa_1 + \\
 & (5.950860833x^2 + 71.29917628x^{7/4})a_2 + \\
 & (3.564310647x^3 - 22.28099604x^{11/4})a_3 + \\
 & (2.706659086x^4 - 23.11500744000x^{7/2} - \\
 & 1.810329894x^{\frac{15}{4}})a_4 + (3.782287766x^{9/2} -
 \end{aligned}$$

$$\begin{aligned}
& 0.5806167936 x^{\frac{19}{4}} + 2.272464553 x^5) a_5 + \\
& (0.1883749265 x^{11/2} + 2.482230639 x^{\frac{21}{4}} + \\
& 2.012078018 x^6 - 0.2735837625 x^{\frac{23}{4}}) a_6 + \\
& (1.839166421 x^7 - 0.272269500800 x^{\frac{25}{4}} + \\
& 0.0408391 x^{13/2} - 0.1564241643 x^{\frac{27}{4}}) a_7 + \\
& (-0.13081798980 x^7 + 1.7162353800 x^8 - \\
& 0.00974158855 x^{\frac{29}{4}} + 0.013898847 x^{15/2} - \\
& 0.100524603 x^{\frac{31}{4}}) a_8 + (0.00602006 x^{17/2} - \\
& 0.06984369703 x^{\frac{35}{4}} + 1.62446323500 x^9 + \\
& 0.0107863782 x^8 - 0.0015940021 x^{\frac{33}{4}}) a_9 + \\
& (0.0030377233 x^{19/2} - 0.0513006100 x^{\frac{39}{4}} + \\
& 0.004098510 x^{\frac{35}{4}} - 0.0004248361073 x^{\frac{37}{4}} + \\
& 0.00030136722 x^9 + 1.5533938130 x^{10}) a_{10}.
\end{aligned} \tag{3.134}$$

$$\begin{aligned}
\frac{d^3 y_{10}}{dx^3} = & -17.44824769 \frac{a_0}{x^{3/4}} + 32.71546441 x^{3/4} a_1 + \\
& (7.399926476 x^{7/4} + 86.26437232 x^{3/2}) a_2 + \\
& (4.835179230 x^{11/4} - 29.65337797 x^{5/2}) a_3 + \\
& (3.916495176 x^{15/4} - 2.5814324600000 x^{7/2} - \\
& 32.452004220 x^{13/4}) a_4 + (3.4612885010 x^{19/4} - \\
& 0.8739224466 x^{9/2} + 5.6224290360 x^{17/4}) a_5 + \\
& (3.197929593 x^{23/4} - 0.430506114500 x^{11/2} + \\
& 0.2933577654 x^{21/4} + 3.82389501300 x^5) a_6 + \\
& (3.03137076200 x^{27/4} - 0.2556130056 x^{13/2} + \\
& 0.06614300864 x^{25/4} - 0.4369098794 x^6) a_7 + \\
& (2.920003196 x^{31/4} - 0.169743011400 x^{15/2} + \\
& 0.0232868115200 x^{29/4} - 0.01619059525 x^7 - \\
& 0.21561836930 x^{27/4}) a_8 + (2.842831684 x^{35/4} - \\
& 0.1214047367 x^{17/2} + 0.010391960170 x^{33/4} - \\
& 0.00273203118 x^8 + 0.0183519687 x^{31/4}) a_9 + \\
& (2.788161843 x^{39/4} - 0.091519469760 x^{19/2} + \\
& 0.005385508286 x^{37/4} - 0.000747377760 x^9 + \\
& 0.00052739428 x^{35/4} + 0.0071242098 x^{17/2}) a_{10}
\end{aligned} \tag{3.135}$$

and,

$$\begin{aligned}
\int_0^x \frac{y_{10}(t)}{(x-t)} dt = & -4.574 x^{\frac{13}{4}} a_0 + 1.513500 x^{\frac{17}{4}} a_1 + \\
& (1.6938 x^5 + 0.1141140 x^{\frac{21}{4}}) a_2 + (-0.24260120 x^6 + \\
& 0.03280790 x^{\frac{25}{4}}) a_3 + (-0.155848 x^{\frac{27}{4}} - 0.01055 x^7 + \\
& 0.01374540 x^{\frac{29}{4}}) a_4 + (0.0148071 x^{\frac{31}{4}} - 0.002010 x^8 + \\
& 0.0069949 x^{\frac{33}{4}}) a_5 + (0.0068178 x^{17/2} + 0.000464 x^{\frac{35}{4}} - \\
& 0.0006059 x^9 + 0.004014 x^{\frac{37}{4}}) a_6 + (-0.000499 x^{19/2} + \\
& 0.0000669973 x^{\frac{39}{4}} - 0.00023 x^{10} + 0.002599 x^{\frac{41}{4}}) a_7 + \quad (3.136) \\
& (-0.000179 x^{\frac{41}{4}} - 0.0000121 x^{21/2} + 0.0000159 x^{\frac{43}{4}} - \\
& 0.00010578 x^{11} + 0.001668 x^{\frac{45}{4}}) a_8 + (0.0000104 x^{\frac{45}{4}} - \\
& 0.00000142 x^{23/2} + 0.00000498 x^{\frac{47}{4}} - 0.0000536 x^{12} + \\
& 0.00115 x^{\frac{49}{4}}) a_9 + (0.0000031 x^{12} - 0.000000282 x^{\frac{25}{4}} + \\
& 0.000001874 x^{\frac{51}{4}} - 0.0000295 x^{13} + 0.00083508 x^{\frac{53}{4}} + \\
& 0.000000214665 x^{\frac{49}{4}}) a_{10}
\end{aligned}$$

Hence, substituting equations (3.133)–(3.136) into (3.132) gives

$$\begin{aligned}
& 0.0016639 a_8 x^{\frac{45}{4}} + 0.0000104577 a_9 x^{\frac{45}{4}} - 0.000012000 a_8 x^{21/2} + \\
& 0.000015907900 a_8 x^{\frac{43}{4}} - 0.00010 a_8 x^{11} - 0.000001426 a_9 x^{23/2} + \\
& 0.0000049840 a_9 x^{\frac{47}{4}} - 0.0000535904 a_9 x^{12} + 0.0011571 a_9 x^{\frac{49}{4}} + \\
& 0.0000018740 a_{10} x^{\frac{51}{4}} + 0.000003144760 a_{10} x^{12} + 200 t_1 x + t_2 - \\
& t_3 + 5.950860833 a_2 x^2 + 0.000835088 a_{10} x^{\frac{53}{4}} + 1.55339 a_{10} x^{10} - \\
& 0.00002 a_{10} x^{13} - 6600 t_1 x^2 + 84480 t_1 x^3 - 549120 t_1 x^4 + 0.0 + \\
& 2050048 t_1 x^5 - 4659200 t_1 x^6 + 6553600 t_1 x^7 - 5570560 t_1 x^8 + \\
& 2621440 t_1 x^9 - 524288 t_1 x^{10} + 4320 t_2 x^2 - 44352.0 t_2 x^3 - t_1 + \\
& 228096 t_2 x^4 - 658944 t_2 x^5 + 1118208.00 t_2 x^6 - 1105920 t_2 x^7 - \\
& 131072 t_2 x^9 + 128 t_3 x - 2688 t_3 x^2 + 21504 t_3 x^3 - 84480 t_3 x^4 - \\
& 212992 t_3 x^6 + 131072 t_3 x^7 - 32768 t_3 x^8 + 2.0120780184 a_6 x^6 + \\
& 3.5643106473 a_3 x^3 + 2.706659086 a_4 x^4 + 2.27246455311 a_5 x^5 - \\
& 21.3813680902 a_0 - 0.02782172507 a_8 x^{\frac{35}{4}} + 1.839166421 a_7 x^7 + \\
& 1.716235380 a_8 x^8 + 1.624463235 a_9 x^9 - 0.00007483737 a_{10} x^{11} + \\
& 0.002097367862 a_9 x^{\frac{39}{4}} + 0.00005409 a_{10} x^{\frac{43}{4}} + 0.000749 a_{10} x^{21/2} - \\
& 0.002023824406 a_8 x^9 + 180224 t_3 x^5 - 0.00030355902100 a_9 x^{10} + \\
& 0.6373158355 a_6 x^7 - 0.06241569706 a_7 x^8 + 1.289381128 a_3 x^{\frac{19}{4}} + \\
& 589824 t_2 x^8 - 162 t_2 x - 8.472393706 a_3 x^{9/2} + 2.6908823 a_2 x^{\frac{15}{4}} +
\end{aligned}$$

$$\begin{aligned}
& 34.50574893 a_2 x^{7/2} + 0.82452530031 a_4 x^4 - 0.5736516577 a_4 x^{11/2} - \\
& 7.635765700 a_4 x^4 + 0.6019632175 a_5 x^4 - 0.158894990340 a_5 x^{13/2} + \\
& 1.070938864 a_5 x^4 + 0.4737673471 a_6 x^4 - 0.066231709921 a_6 x^{15/2} + \\
& 0.04693724246 a_6 x^4 + 0.3911446144 a_7 x^4 + 0.333714651061 a_8 x^4 - \\
& 0.03408173408 a_7 x^{17/2} + 0.009123173606 a_7 x^4 + 0.291572480 a_9 x^4 - \\
& 0.0196976605 a_8 x^{19/2} + 0.00282264382 a_8 x^4 + 0.2593638924 a_{10} x^4 - \\
& 0.012779 a_9 x^{21/2} + 0.001123455153 a_9 x^4 - 0.008716139977 a_{10} x^{23/2} + \\
& 0.0005254154425 a_{10} x^4 + 0.00000021 a_{10} x^4 - 0.0000002813 a_{10} x^2 + \\
& 30.067492850 x a_1 - 23.26433025 x^{7/4} a_0 + 18.6945510900000 x^{11/4} a_1 - \\
& 17.44824769000 \sqrt[4]{\frac{a_0}{x}} + 32.715464410000 x^{3/4} a_1 - 4.574180000 x^4 a_0 + \\
& 1.513520000 x^4 a_1 + 5.62242903600 a_5 x^4 + 78.699102760000 a_2 x^{7/4} + \\
& 86.26437232 a_2 x^{3/2} - 17.445816810000 a_3 x^{11/4} - 29.65337797 a_3 x^{5/2} + \\
& 2.106165282 a_4 x^4 - 25.6964399000 a_4 x^{7/2} - 32.452004220000 a_4 x^4 + \\
& 2.8806717070000 a_5 x^4 + 2.90836531900 a_5 x^{9/2} - 0.4369098794 a_7 x^6 + \\
& 2.9243458300000 a_6 x^4 - 0.2421311880 a_6 x^{11/2} + 2.775588404 a_6 x^4 + \\
& 3.823895013 a_6 x^5 + 2.8749465986 a_7 x^4 - 0.214773807900000 a_7 x^{13/2} - \\
& 0.2061264922000 a_7 x^4 + 2.819478592 a_8 x^4 - 0.1558441637 a_8 x^{15/2} + \\
& 0.0135452229600 a_8 x^4 - 0.1470085850 a_8 x^7 - 0.215618369300 a_8 x^4 + \\
& 2.7729879870 a_9 x^4 - 0.1153846737 a_9 x^{17/2} + 0.004960672179 a_{10} x^4 + \\
& 2.736861233 a_{10} x^4 - 0.08848174638 a_{10} x^{19/2} + 0.008797958017 a_9 x^4 +
\end{aligned}$$

$$\begin{aligned}
& 0.00805434 a_9 x^8 + 0.01835 a_9 x^{\frac{31}{4}} - 0.000233 a_7 x^{10} - \\
& 0.000447 a_{10} x^9 + 0.0046 a_{10} x^{\frac{35}{4}} + 0.007124 a_{10} x^{17/2} - \\
& 0.000499 a_7 x^{19/2} + 0.0000669 a_7 x^{\frac{39}{4}} - 0.1558 a_4 x^{\frac{27}{4}} + \\
& 0.002599 a_7 x^{\frac{41}{4}} + 1.6938 a_2 x_5 + 0.114114000 a_2 x^{\frac{21}{4}} + \\
& 0.0328 a_3 x^{\frac{25}{4}} + 0.006999 a_5 x^{\frac{33}{4}} + 0.00681788 a_6 x^{17/2} - \quad (3.137) \\
& 0.0105596 a_4 x^7 + 0.0137454 a_4 x^{\frac{29}{4}} + 0.014807 a_5 x^{\frac{31}{4}} - \\
& 0.00201 a_5 x^8 - 0.0001789 a_8 x^{\frac{41}{4}} + 0.00401692 a_6 x^{\frac{37}{4}} - \\
& 0.000605356 a_6 x^9 + 0.000463549 a_6 x^{\frac{35}{4}} = 6 + 12 x^2 + \\
& 6.619575908 x^{0.25} + \frac{32}{35} x^{7/2} + 0.242601 a x^6
\end{aligned}$$

Collocating equation (3.137) at the point $x = x_i$ gives

$$\begin{aligned}
& 0.00166 a_8 x_i^{\frac{45}{4}} + 0.0000104577 a_9 x_i^{\frac{45}{4}} - 0.00001215460 a_8 x_i^{21/2} + \\
& 0.0000159 a_8 x_i^{\frac{43}{4}} - 0.00010578 a_8 x_i^{11} - 0.00000142678 a_9 x_i^{23/2} + \\
& 0.00000498 a_9 x_i^{\frac{47}{4}} - 0.0000535904 a_9 x_i^{12} + 0.001157127 a_9 x_i^{\frac{49}{4}} + \\
& 0.000001874030 a_{10} x_i^{\frac{51}{4}} + 0.000003144760000 a_{10} x_i^{12} - \tau_1 + \tau_2 - \\
& \tau_3 + 5.950860833 a_2 x_i^2 + 0.000835 a_{10} x_i^{\frac{53}{4}} + 1.55339381 a_{10} x_i^{10} - \\
& 0.0000295219 a_{10} x_i^{13} - 6600 \tau_1 x_i^2 + 84480 \tau_1 x_i^3 - 549120 \tau_1 x_i^4 + \\
& 2050048 \tau_1 x_i^5 - 4659200 \tau_1 x_i^6 + 6553600 \tau_1 x_i^7 - 5570560 \tau_1 x_i^8 + \\
& 2621440 \tau_1 x_i^9 - 524288 \tau_1 x_i^{10} - 162 \tau_2 x_i + 4320 \tau_2 x_i^2 + 0.0000 + \\
& 228096 \tau_2 x_i^4 - 658944 \tau_2 x_i^5 + 1118208 \tau_2 x_i^6 - 1105920.00 \tau_2 x_i^7 + \\
& 589824 \tau_2 x_i^8 - 131072 \tau_2 x_i^9 + 128 \tau_3 x_i - 2688 \tau_3 x_i^2 + 200.0 \tau_1 x_i + \\
& 21504 \tau_3 x_i^3 - 84480 \tau_3 x_i^4 + 180224 \tau_3 x_i^5 - 212992 \tau_3 x_i^6 + 0.000 + \\
& 131072 \tau_3 x_i^7 - 32768 \tau_3 x_i^8 + 2.012078018 a_6 x_i^6 + 3.564311 a_3 x_i^3 + \\
& 2.70665908600000 a_4 x_i^4 + 2.27246455300 a_5 x_i^5 - 21.38136809 a_0 - \\
& 0.02782172507 a_8 x_i^{\frac{35}{4}} - 44352 \tau_2 x_i^3 + 0.0007499160945 a_{10} x_i^{21/2} + \\
& 1.716235380 a_8 x_i^8 + 1.624463235 a_9 x_i^9 - 0.000074837377 a_{10} x_i^{11} + \\
& 0.002097367862 a_9 x_i^{\frac{39}{4}} + 0.000054091 a_{10} x_i^{\frac{43}{4}} + 1.839166421 a_7 x_i^7 -
\end{aligned}$$

$$\begin{aligned}
& 0.0003035590 a_9 x_i^{10} + 0.6373158355 a_6 x_i^7 - 0.0624156970600 a_7 x_i^8 - \\
& 0.0020238244060 a_8 x_i^9 + 1.289381128 a_3 x_i^{\frac{19}{4}} - 8.472393706 a_3 x_i^{9/2} + \\
& 2.6908823550 a_2 x_i^{\frac{15}{4}} + 34.50574893 a_2 x_i^{7/2} + 0.824525300383 a_4 x_i^{\frac{23}{4}} - \\
& 0.5736516577 a_4 x_i^{11/2} - 7.635765700 a_4 x_i^{\frac{21}{4}} + 0.60196321755 a_5 x_i^{\frac{27}{4}} - \\
& 0.1588949903 a_5 x_i^{13/2} + 1.070938864 a_5 x_i^{\frac{25}{4}} + 0.47376734718 a_6 x_i^{\frac{31}{4}} - \\
& 0.06623170992 a_6 x_i^{15/2} + 0.04693724246 a_6 x_i^{\frac{29}{4}} + 0.39114465 a_7 x_i^{\frac{35}{4}} + \\
& 0.3337146510 a_8 x_i^{\frac{39}{4}} - 0.03408173408 a_7 x_i^{17/2} + 0.009123173 a_7 x_i^{\frac{33}{4}} + \\
& 0.2915724804 a_9 x_i^{\frac{43}{4}} - 0.01996976605 a_8 x_i^{19/2} + 0.002822649 a_8 x_i^{\frac{37}{4}} + \\
& 0.25936399 a_{10} x_i^{\frac{47}{4}} - 0.0127794 a_9 x_i^{21/2} + 0.00112345515342 a_9 x_i^{\frac{41}{4}} - \\
& 0.00871613997 a_{10} x_i^{23/2} + 0.0005254 a_{10} x_i^{\frac{45}{4}} + 0.0000002146 a_{10} x_i^{\frac{49}{4}} - \\
& 0.0000002813990000 a_{10} x_i^{\frac{25}{2}} + 30.067492 x_i a_1 - 23.26433025 x_i^{7/4} a_0 + \\
& 18.69455109 x_i^{11/4} a_1 - 17.44824769 \sqrt[4]{\frac{a_0}{x_i}} + 32.715464410000 x_i^{3/4} a_1 - \\
& 4.57418 x_i^{\frac{13}{4}} a_0 + 1.5135 x_i^{\frac{17}{4}} a_1 + 5.622429 a_5 x_i^{\frac{17}{4}} + 78.699102 a_2 x_i^{7/4} + \\
& 86.26437232012 a_2 x_i^{3/2} - 17.44581681 a_3 x_i^{11/4} - 29.6533779 a_3 x_i^{5/2} + \\
& 2.106165282000 a_4 x_i^{\frac{15}{4}} - 25.69643990 a_4 x_i^{7/2} - 32.452004221 a_4 x_i^{\frac{13}{4}} + \\
& 2.880671707 a_5 x_i^{\frac{19}{4}} + 2.908365319000 a_5 x_i^{9/2} - 0.43690987945 a_7 x_i^6 + \\
& 2.924345830 a_6 x_i^{\frac{23}{4}} - 0.2421311880 a_6 x_i^{11/2} + 2.775588404780 a_6 x_i^{\frac{21}{4}} + \\
& 3.8238950130 a_6 x_i^5 + 2.874946598 a_7 x_i^{\frac{27}{4}} - 0.2147738079445 a_7 x_i^{13/2} - \\
& 0.20612649220 a_7 x_i^{\frac{25}{4}} + 2.819478592 a_8 x_i^{\frac{31}{4}} - 0.15584416376 a_8 x_i^{15/2} +
\end{aligned}$$

$$\begin{aligned}
& 0.01354522 a_8 x_i^{\frac{29}{4}} - 0.14706 a_8 x_i^7 - 0.2156181 a_8 x_i^{\frac{27}{4}} + \\
& 2.77298 a_9 x_i^{\frac{35}{4}} - 0.1153846 a_9 x_i^{17/2} + 0.00491 a_{10} x_i^{\frac{37}{4}} + \\
& 2.73686 a_{10} x_i^{\frac{39}{4}} - 0.0881 a_{10} x_i^{19/2} + 0.0087979 a_9 x_i^{\frac{33}{4}} + \\
& 0.0080541 a_9 x_i^8 + 0.01835 a_9 x_i^{\frac{31}{4}} - 0.00023361 a_7 x_i^{10} - \\
& 0.00044 a_{10} x_i^9 + 0.004629 a_{10} x_i^{\frac{35}{4}} + 0.00717 a_{10} x_i^{17/2} - \\
& 0.000499 a_7 x_i^{19/2} + 0.0000669 a_7 x_i^{\frac{39}{4}} - 0.15581 a_4 x_i^{\frac{27}{4}} + \\
& 0.002599 a_7 x_i^{\frac{41}{4}} + 1.6938 a_2 x_i^5 + 0.1141143161 a_2 x_i^{\frac{21}{4}} + \\
& 0.0328 a_3 x_i^{\frac{25}{4}} + 0.006999 a_5 x_i^{\frac{33}{4}} + 0.00689951 a_6 x_i^{17/2} - \\
& 0.01055 a_4 x_i^7 + 0.0137454 a_4 x_i^{\frac{29}{4}} + 0.01480711 a_5 x_i^{\frac{31}{4}} - \\
& 0.002010 a_5 x_i^8 - 0.000178 a_8 x_i^{\frac{41}{4}} + 0.00401681 a_6 x_i^{\frac{37}{4}} - \\
& 0.000605356 a_6 x_i^9 + 0.000463561 a_6 x_i^{\frac{35}{4}} = 6 + 12 x_i^2 + \\
& 6.619575908 x_i^{0.25} + \frac{32}{35} x_i^{7/2} + 0.2426011153311 a_3 x_i^6
\end{aligned} \tag{3.138}$$

where,

$$x_i = \frac{i}{12} \quad i = 1, 2, \dots, 11.$$

Hence, for $i = 1$, $x_1 = \frac{1}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 7.599994338 \alpha_1 - 54.15833153 \alpha_0 + 3.13972791 \alpha_2 - \\
& 0.07628217978 \alpha_3 - 0.01408011 \alpha_4 + 0.00021705 \alpha_5 + \\
& 0.0000235962455043 \alpha_6 - 0.00000000352796111 \alpha_7 - \\
& 0.0000000001604993710 \alpha_8 + 0.00000000134752 \alpha_9 + \\
& 0.0000000001093429324 \alpha_{10} - 0.9104895875111 \tau_1 + \\
& 0.5301529266 \tau_2 + 0.02690138334 \tau_3 = 9.640084687
\end{aligned} \tag{3.139}$$

For $i = 2$, $x_2 = \frac{2}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 13.68119608 \alpha_1 - 49.71428903 \alpha_0 + 9.5249525360 \alpha_2 - \\
& 0.4485970263 \alpha_3 - 0.1405613783 \alpha_4 + 0.00457677 \alpha_5 + \\
& 0.0008510642799125 \alpha_6 + 0.000008595510861222 \alpha_7 + \\
& 0.0000017255051996 \alpha_8 + 0.000000590274318211 \alpha_9 + \\
& 0.00000009584872551 \alpha_{10} + 0.52839172512133340 \tau_1 + \\
& 0.28054681 \tau_2 - 0.9024538966 \tau_3 = 10.564593750004
\end{aligned} \tag{3.140}$$

For $i = 3$, $x_3 = \frac{3}{12}$, thus equation (3.138) after simplification gives

$$\begin{aligned}
& 19.50081225 a_1 - 48.16375035 a_0 + 18.397233230 a_2 - \\
& 1.271297924 a_3 - 0.5423745805 a_4 + 0.027625728 a_5 + \\
& 0.007082782204 a_6 + 0.000192937 a_7 + 0.0000555 a_8 + \quad (3.141) \\
& 0.0000209856345744 a_9 + 0.00000511613239600 a_{10} + \\
& 0.500000000 t_1 - 1.0 t_2 + 0.500000 t_3 = 11.43788987
\end{aligned}$$

For $i = 4$, $x_4 = \frac{4}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 25.29992383 a_1 - 47.87523370 a_0 + 29.5598943010 a_2 - \\
& 2.674328180 a_3 - 1.418950479 a_4 + 0.09978181414 a_5 + \\
& 0.03226532062 a_6 + 0.001601458917 a_7 + 0.082967 t_2 + \quad (3.142) \\
& 0.0002658160176555 a_9 + 0.000086439015601440 a_{10} - \\
& 0.967217347 t_1 + 0.91190325 t_3 + 0.00059873623 a_8 = \\
& 12.38267382
\end{aligned}$$

For $i = 5$, $x_5 = \frac{5}{12}$, thus equation (3.138) after simplification gives

$$\begin{aligned}
& 31.21456085 a_1 - 48.39159073 a_0 + 42.9750589710 a_2 - \\
& 4.781861491 a_3 - 3.002078810 a_4 + 0.27192455230 a_5 + \\
& 0.1055605642 a_6 + 0.008048078351 a_7 + 0.0036936 a_8 + \quad (3.143) \\
& 0.00191128240 a_9 + 0.000776930 a_{10} - 0.10350606 \tau_1 + \\
& 0.997963000000 \tau_2 - 0.22915710000 \tau_3 = 13.44438036
\end{aligned}$$

For $i = 6$, $x_6 = \frac{6}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 37.34498419 a_1 - 49.52827661 a_0 + 58.68999686 a_2 - \\
& 7.719696621 a_3 - 5.556102862 a_4 + 0.6200855145 a_5 + \\
& 0.2800079550 a_6 + 0.02977249316 a_7 + 0.0161748 a_8 + \quad (3.144) \\
& 0.009604194184 a_9 + 0.00468400071 a_{10} + 1.0 \tau_1 - \tau_3 \\
& = 14.64718985
\end{aligned}$$

For $i = 7$, $x_7 = \frac{7}{12}$, thus equation (3.138) after simplification gives

$$\begin{aligned}
& 43.77547287 a_1 - 51.19827589 a_0 + 76.80950107 a_2 - \\
& 11.61963363 a_3 - 9.379947372 a_4 + 1.250482833 a_5 + \\
& 0.6425383401 a_6 + 0.089531291 a_7 + 0.05613069 a_8 + \quad (3.145) \\
& 0.03768992055 a_9 + 0.0214374796 a_{10} - 0.103505 t_1 - \\
& 0.99797000000 t_2 - 0.22915500000 t_3 = 16.00702205
\end{aligned}$$

For $i = 8$, $x_8 = \frac{8}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 50.58226851 a_1 - 53.35845314 a_0 + 97.48270478 a_2 - \\
& 16.62256999 a_3 - 14.81020583 a_4 + 2.305049678 a_5 + \\
& 1.325948508 a_6 + 0.2318393382 a_7 + 0.16461822 a_8 + \quad (3.146) \\
& 0.1234338108 a_9 + 0.08019838430 a_{10} - 0.967316 t_1 - \\
& 0.08290000000 t_2 + 0.91189900000 t_3 = 17.53598387
\end{aligned}$$

For $i = 9$, $x_9 = \frac{9}{12}$ thus equation (3.138) after simplification gives

$$\begin{aligned}
& 57.83744433 a_1 - 55.98857032 a_0 + 120.8959165 a_2 - \\
& 22.88093839 a_3 - 22.22494652 a_4 + 3.967570548 a_5 + \\
& 2.523084953 a_6 + 0.5361660400 a_7 + 0.42499566 a_8 + \quad (3.147) \\
& 0.3521150692 a_9 + 0.2572266102 a_{10} + 0.4999000 t_1 + \\
& 1.00000000 t_2 + 0.50001000000000 t_3 = 19.24424787
\end{aligned}$$

For $i = 10$, $x_{10} = \frac{10}{12}$, thus equation (3.138) after simplification gives

$$\begin{aligned}
& 65.61105013 a_1 - 59.08167717 a_0 + 147.2684188 a_2 - \\
& 30.56076605 a_3 - 32.04805994 a_4 + 6.470521464 a_5 + \\
& 4.503519987 a_6 + 1.134994585 a_7 + 0.992877798 a_8 + \quad (3.148) \\
& 0.9009035999 a_9 + 0.7307653923 a_{10} + 0.5284700 t_1 - \\
& 0.2806000000 t_2 - 0.9024500000 t_3 = 21.14095991
\end{aligned}$$

For $i = 11$, $x_{11} = \frac{11}{12}$, thus equation (3.138) after simplification gives

$$\begin{aligned}
& 73.97241974 a_1 - 62.63918643 a_0 + 176.8499027 a_2 - \\
& 39.84350124 a_3 - 44.75404549 a_4 + 10.10270038 a_5 + \\
& 7.633035199 a_6 + 2.237772885 a_7 + 2.140355364 a_8 + \quad (3.149) \\
& 2.110940145 a_9 + 1.882122965 a_{10} - 0.909600000 t_1 - \\
& 0.530200000 t_2 + 0.026940000000 t_3 = 23.23472107
\end{aligned}$$

Thus, from the given conditions (3.127); that is $y_{10}(0) = 0$, equation (3.131) gives

$$\begin{aligned}
& 3.22 \times 10^{-19} a_1 - 8.59 \times 10^{-14} a_0 + 6.95 \times 10^{-23} a_2 - \\
& 1.07 \times 10^{-28} a_3 - 1.31 \times 10^{-32} a_4 + 1.32 \times 10^{-38} a_5 + \\
& 1.14 \times 10^{-42} a_6 - 8.61 \times 10^{-49} a_7 - 5.82 \times 10^{-53} a_8 + \\
& 3.53709598 \times 10^{-59} a_9 + 1.97130669 \times 10^{-63} a_{10} = 0 \quad (3.150)
\end{aligned}$$

Thus, from the given conditions (3.127); that is $y_{10}^I(0) = 0$, equation (3.131) gives

$$\begin{aligned} & 0.0000000332 a_1 - 0.004137 a_0 + 1.09595 \times 10^{-11} a_2 - \\ & 2.656 \times 10^{-17} a_3 - 4.31 \times 10^{-21} a_4 + 5.98 \times 10^{-27} a_5 + \\ & 6.39 \times 10^{-31} a_6 - 6.19 \times 10^{-37} a_7 - 4.966 \times 10^{-41} a_8 + \\ & 3.706811592 \times 10^{-47} a_9 + 2.38079461 \times 10^{-51} a_{10} = 0 \end{aligned} \quad (3.151)$$

Thus, from the given conditions (3.127); that is $y_{10}^{II}(0) = 0$, equation (3.131) gives

$$\begin{aligned} & 1.28 \times 10^{-13} a_1 - 0.000000023 a_0 + 3.13 \times 10^{-17} a_2 - \\ & 5.90 \times 10^{-23} a_3 - 8.20 \times 10^{-27} a_4 + 9.57 \times 10^{-33} a_5 + \\ & 9.13 \times 10^{-37} a_6 - 7.74 \times 10^{-43} a_7 - 5.67 \times 10^{-47} a_8 + \\ & 3.8021 \times 10^{-53} a_9 + 2.267200000000 \times 10^{-57} a_{10} = 0 \end{aligned} \quad (3.152)$$

Solving equations (3.139) – (3.152) to obtain

$$\begin{aligned}
a_0 &= 8.569477874 \times 10^{-13}, \\
a_1 &= 8.380383887 \times 10^{-7}, \\
a_2 &= -0.3891288698e-2, \\
a_3 &= -690.8121622, \\
a_4 &= 4136.395497, \\
a_5 &= 1.049524505 \times 10^5, \\
a_6 &= -2.416598979 \times 10^5, \\
a_7 &= 1.375262998 \times 10^6, \\
a_8 &= -1.868362279 \times 10^6, \\
a_9 &= 1.123793646 \times 10^6, \\
a_{10} &= -2.703958248 \times 10^5, \\
\tau_1 &= -1.957885665, \\
\tau_2 &= -6.840884700, \\
\tau_3 &= -1.292842922
\end{aligned} \tag{3.153}$$

Thus, substituting the values of $\alpha_i (i = 0, 1, 2, \dots, 10)$ above into equation (3.131), and after simplification gives

$$\begin{aligned}
 y_{10}(x) = & 2480.500042 x^{\frac{29}{4}} - 342.0803816000 x^{15/2} + \\
 & 1207.69442800 x^{\frac{31}{4}} + 6914.669909100 x^{\frac{39}{4}} - \\
 & 580.450127400 x^{19/2} + 164.412947900 x^{\frac{37}{4}} - \\
 & 1192.194426 x^9 + 42.01374645110000 x^{10} - \\
 & 468.1245798 x^{\frac{51}{4}} + 16.3951851000000 x^{\frac{25}{2}} - \\
 & 962.5766661 x^{\frac{25}{4}} - 55.6226367110000 x^{\frac{41}{4}} - \\
 & 5926.2288090 x^{\frac{43}{4}} + 374.0426828000 x^{21/2} - \\
 & 0.0005878456111 x^{\frac{19}{4}} - 0.0085251950 x^{9/2} - \\
 & 1.030894454 x^{\frac{49}{4}} + 0.153300867300111 x^{12} - \\
 & 66.37343033 x^{13/2} + 87.872792000001 x^{\frac{27}{4}} + \\
 & 0.00000151920 x^{\frac{15}{4}} - 1688.3401840000 x^{\frac{35}{4}} + \\
 & 251.06742360 x^{17/2} - 189.64011230000 x^{\frac{33}{4}} - \\
 & 2750.244281 x^8 - 32.612179940000000 x^{\frac{23}{4}} + \\
 & 236.478085400 x^{11/2} - 120.614777311 x^{23/2} + \\
 & 10.9487849400000 x^{\frac{45}{4}} - 3.10125180900 x^{11} - \\
 & 4.142600 \times 10^{-12} x^{11/4} + 2593.984558000 x^{\frac{47}{4}}
 \end{aligned} \tag{3.154}$$

Thus, equation (3.154) is the approximate solution of equation (3.126) together with conditions (3.127) for case $N = 10$

Consider the problem 3.2.2.1 of section 3.2.2 which is of the form

$$\frac{d^{0.5}y(x)}{dx^{0.5}} + \left(\frac{-32}{35}x^{0.5}\right) \cdot \frac{dy(x)}{dx} + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 1.504507437x^{1.5} + 1.805406667x^{2.5} - \frac{32}{15}x^{5/2} - \frac{16}{7}x^{3/2} + \frac{32}{35}x^{7/2}, \quad 0 \leq x \leq 1 \quad (3.155)$$

subject to the condition

$$y(0) = 0 \quad (3.156)$$

The exact solution is given as :

$$y(x) = x^3 + x^2$$

The recursive canonical polynomials obtained for equation (3.155) is

$$Q_j(x) = x^j - \frac{x^{1-a}\Gamma(j+1)}{\Gamma(j-a+1)} \frac{Q_{j-1}}{j-1} \quad j \geq 0 \quad (3.157)$$

Here, equations (3.155)–(3.156) are solved for case $N = 8$. Hence, equation (3.20) becomes

$$y_8(x) = \sum_{j=0}^8 a_j Q_j(x) \quad (3.158)$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 8$) in equation (3.157) gives

$$\begin{aligned}
y_8(x) = & a_0 + (x - 1.12837 x^{0.5})a_1 + (x^2 - 1.504 x^{3/2} + \\
& 1.69765 x)a_2 + (x^3 - 1.805406668 x^{5/2} + 2.716240 x^2 - \\
& 3.06495 x^{3/2})a_3 + (x^4 - 2.0633219 x^{7/2} + 3.725135 x^3 - \\
& 5.60448 x^{5/2} + 6.32398 x^2)a_4 + (x^5 - 2.29257900 x^{9/2} + \\
& 4.7303303 x^4 - 8.540169000 x^{7/2} + 12.8487330600 x^3 - \\
& 14.498242 x^{5/2})a_5 + (x^6 - 2.500996 x^{11/2} + 5.7330 x^5 - \\
& 11.8305 x^{9/2} + 21.3589328812 x^4 - 32.134633190 x^{7/2} + \\
& 36.26005 x^3)a_6 + (x^7 - 2.693380 x^{13/2} + 6.736134 x^6 - \\
& 15.44312 x^{11/2} + 31.86414228 x^5 - 57.527734940 x^{9/2} + \\
& 86.55079685 x^4 - 97.662 x^{7/2})a_7 + (x^8 - 2.8720 x^{15/2} + \\
& 7.737918 x^7 - 19.35250 x^{13/2} + 44.3671662500001 x^6 - \\
& 91.54374604 x^{11/2} + 165.2736895 x^5 - 248.65511 x^{9/2} + \\
& 280.5773294 x^4)a_8
\end{aligned} \tag{3.159}$$

Thus, equation (3.159) is substituted into a slightly perturbed equation (3.155) to get

$$\begin{aligned}
& \frac{d^{0.5} y_8}{dx^{0.5}} + \left(\frac{-32}{35} x^{0.5} \right) \cdot \frac{dy_8}{dx} + \sum_0^x \frac{y_8(t)}{\sqrt{x-t}} dt = \\
& 1.504507437 x^{1.5} + 1.8054066670000000 x^{2.5} - \\
& \frac{32}{15} x^{5/2} - \frac{16}{7} x^{3/2} + \frac{32}{35} x^{7/2} + H_8(x)
\end{aligned} \tag{3.160}$$

together with the equation (3.156) to get

$$y_8(0) = 0$$

where,

$$\begin{aligned} H_8(x) = & \tau_1(1 - 128x + 2688x^2 - 21504x^3 + \\ & 84480x^4 - 180224x^5 + 212992x^6 - \\ & 131072x^7 + 32768x^8) \end{aligned} \quad (3.161)$$

Hence, evaluating the terms $\frac{d^{0.5}y_8}{dx^{0.5}}, (\frac{-32}{35}x^{0.5})(\frac{dy_8}{dx})$, and $\int_0^x \frac{y_8(t)}{\sqrt{x-t}} dt$ in equation (3.160) gives

$$\begin{aligned} \frac{d^{0.5}y_8}{dx^{0.5}} = & (1.1283791670000x^{0.5} - 0.99999782730000)a_1 + \\ & (0.6556785664x^{1.5} + 1.915598535x^{0.5})a_2 + (1.8054x^{2.5} + \\ & 2.357387188x^{1.5} - 2.99999x^{2.0})a_3 + (9.5144931360x^{1.5} + \\ & 2.063320293x^{3.5} - 9.312851779x^{2.00} + 6.725365511x^{2.5} - \\ & 4.0000025930x^{3.0})a_4 + (23.197162x^{2.5} - 16.5561430x^{3.0} - \\ & 5.000000420x^{4.0} + 2.292578731x^{4.5} + 9.7601977000x^{3.5} - \\ & 24.09134690x^{2.0})a_5 + (44.07037168x^{3.5} - 5.9999811x^{5.0} + \\ & 2.500996005x^{5.5} - 25.80180013x^{4.0} + 13.145053110x^{4.5} + \\ & 65.464049x^{2.5} - 62.29668543x^{3.00})a_6 + (-7.000013x^{6.0} + \\ & 2.693379011x^{6.5} + 73.0512672700x^{4.5} - 189.329612x^{3.0} + \end{aligned}$$

$$\begin{aligned}
& 178.58236 x^{3.5} - 125.465043 x^{4.0} - 37.04875022 x^{5.0} + \\
& 16.84703948 x^{5.5}) a_7 + (578.9205 x^{3.5} - 542.3041 x^{4.0} + \\
& 378.902954 x^{4.5} - 219.6175657 x^{5.0} + 110.961986 x^{5.5} - \quad (3.162) \\
& 50.296485 x^{6.0} + 20.8411632 x^{6.5} - 7.999982618 x^{7.0} + \\
& 2.872937988 x^{7.5}) a_8
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{-32}{35} x^{0.5} \right) \frac{dy_8}{dx} = 1.0316609530 a_1 - 19.461930440 x^{3.5} a_5 - \\
& 1.3714285710 a_1 x^{0.5} - 2.2857142 x^{1.5} a_2 + 10.4803 x^{4.0} a_5 - \\
& 3.2000 x^{2.5} a_3 + 4.951972575 x^{2.0} a_3 - 4.114285714 x^{3.5} a_4 + \\
& 7.5458 x^{3.0} a_4 - 11.92043240 x^{2.5} a_4 - 5.028571429 x^{4.5} a_5 + \\
& 54.08246115 x^{4.0} a_6 + 13.719 x^{5.0} a_6 - 28.83248955 x^{4.5} a_6 + \\
& 31.232621 x^{3.0} a_5 - 5.942857143 x^{5.5} a_6 - 160.23111 x^{4.5} a_7 - \\
& 40.03188632 x^{5.5} a_7 + 84.71658252 x^{5.0} a_7 - 87.8760 x^{3.5} a_6 - \\
& 6.857142857 x^{6.5} a_7 + 17.23763569 x^{6.0} a_7 + 262.983 x^{4.0} a_7 - \\
& 7.7714285 x^{7.5} a_8 + 21.0134987 x^{7.0} a_8 - 53.060015 x^{6.5} a_8 + \\
& 123.85603 x^{6.0} a_8 + 2.75109587 x^{1.0} a_2 - 263.66773 x^{5.5} a_8 + \\
& 502.1828354 x^{5.0} a_8 + 117.52094 x^{3.0} a_6 - 831.09055 x^{4.5} a_8 - \\
& 0.4571429 a_0 x^{-0.5} - 2.32820945 x^{0.5} a_2 - 6.208550 x^{1.5} a_3 + \\
& 5.60448649 x^{1.0} a_3 + 15.3723058 x^{2.0} a_4 - 116.03219 x^{2.5} a_6 -
\end{aligned}$$

$$\begin{aligned}
& 14.454824 x^{1.5} a_4 - 41.115945 x^{2.5} a_5 + 39.76660 x^{2.0} a_5 - \\
& 356.0947 x^{3.5} a_7 + 357.1643 x^{3.0} a_7 + 1136.70940 x^{4.0} a_8 - \quad (3.163) \\
& 1154.375298 x^{3.5} a_8
\end{aligned}$$

and,

$$\begin{aligned}
& \int_0^x \frac{y_8(t)}{\sqrt{x-t}} dt = 2 \sqrt{-x} a_0 + (-1.77 x^{1.0} + 1.33 x^{3/2}) a_1 + \\
& (2.263 x^{3/2} - 1.772 x^2 + 1.06667 x^{5/2}) a_2 + (-3.61 x^2 + \\
& 2.89733 x^{5/2} - 1.772 x^3 + 0.914 x^{7/2}) a_3 + (6.745 x^{5/2} - \\
& 5.5021 x^3 + 3.40584 x^{7/2} - 1.772 x^4 + 0.8120 x^{9/2}) a_4 + \\
& (-14.2336 x^3 + 11.7474 x^{7/2} - 7.3362 x^4 + 3.844 x^{9/2} - \\
& 1.77245 x^5 + 0.7388 x^{11/2}) a_5 + (33.152 x^{7/2} - 27.6 x^4 + \quad (3.164) \\
& 17.3584 x^{9/2} - 9.1465 x^5 + 4.23618 x^{11/2} - 1.77245 x^6 + \\
& 0.6819 x^{13/2}) a_6 + (-83.89 x^4 + 70.339 x^{9/2} - 44.47 x^5 + \\
& 23.5418 x^{11/2} - 10.94 x^6 + 4.59394 x^{13/2} - 1.77245 x^7 + \\
& 0.63 x^{15/2}) a_7 + (228.02 x^{9/2} - 192.24 x^5 + 122.1 x^{11/2} - \\
& 64.877 x^6 + 30.2577 x^{13/2} - 12.7355 x^7 + 4.9250 x^{15/2} - \\
& 71.77245 x^8 + 0.599077 x^{17/2}) a_8
\end{aligned}$$

Substituting equations (3.161 – 3.164) into equation (3.160), gives

$$\begin{aligned}
& 0.03166312571 a_1 + a_0 - \tau_1 - 16.5561433334 a_5 x^{3.0} - 5.00000042000000 a_5 x^{4.0} - \\
& 4.0000025 a_4 x^{3.0} - 9.3128517 a_4 x^{2.0} - 2.999999124 a_3 x^{2.0} - 24.09134657 a_5 x^{2.0} - \\
& 5.99998 a_6 x^{5.0} - 25.80180013 a_6 x^{4.0} - 62.29668543 a_6 x^{3.0} - 7.000013000 a_7 x^{6.0} - \\
& 189.329 a_7 x^{3.0} - 125.4650438 a_7 x^{4.0} - 37.04875022 a_7 x^{5.0} - 542.3041054 a_8 x^{4.0} - \\
& 219.617 a_8 x^{5.0} - 7.999982618 a_8 x^{7.0} + a_1 x - 1.37142857 a_1 x^{0.5} - 0.772451 a_2 x^2 - \\
& 50.296485830 a_8 x^{6.0} + 6.32398580 a_4 x^2 + 0.75903444 a_2 x^{3/2} + 1.697652726 a_2 x + \\
& 1.091923332 a_3 x^{5/2} - 0.894565637 a_3 x^2 - 3.064953551 a_3 x^{3/2} - 0.7724500 a_4 x^4 + \\
& 1.342518095 a_4 x^{7/2} - 1.777054874 a_4 x^3 + 1.141093506000 a_4 x^{5/2} - 844800 \tau_1 x^4 + \\
& 180224 \tau_1 x^5 - 212992 \tau_1 x^6 + 131072 \tau_1 x^7 - 32768 \tau_1 x^8 - 9.701732740000 x^{3.5} a_5 - \\
& 0.77245 a_5 x^5 + 128 \tau_1 x - 2688 \tau_1 x^2 + 21504 \tau_1 x^3 - 14.4982 a_5 x^{5/2} - 0.7724 a_3 x^3 - \\
& 0.77245000 a_6 x^6 - 0.77245 a_7 x^7 + 1.551750105 a_5 x^{9/2} - 2.60592968400000 a_5 x^4 + \\
& 3.207230104 a_5 x^{7/2} - 1.38486694 a_5 x^3 + 1.900559424 a_7 x^{13/2} + 1.735183 a_6 x^{11/2} - \\
& 3.412766282 a_6 x^5 + 5.52786162 a_6 x^{9/2} - 6.24566712 a_6 x^4 + 1.0173668100 a_6 x^{7/2} + \\
& 36.26005061 a_6 x^3 - 97.6621160 a_7 x^{7/2} - 4.208365282 a_7 x^6 + 8.09867298 a_7 x^{11/2} - \\
& 12.61205772 a_7 x^5 + 12.81196506 a_7 x^{9/2} + 2.65619685 a_7 x^4 - 0.7724500000 a_8 x^8 + \\
& 2.052390718 a_8 x^{15/2} - 1.63003572 x^{1.5} a_2 - 4.997581142 a_8 x^7 + 10.90519 a_8 x^{13/2} + \\
& 10.48036524 x^{4.0} a_5 - 20.509833 a_8 x^6 + 30.56325396 a_8 x^{11/2} - 20.6301842 a_8 x^{9/2} + \\
& 280.5773294 a_8 x^4 - 26.9683105 a_8 x^5 - 1.394593333 x^{2.5} a_3 + 4.951972575 x^{2.0} a_3 - \\
& 2.050965421 x^{3.5} a_4 + 7.545862 x^{3.0} a_4 - 15.68743644 x^{4.5} a_6 - 5.195066889 x^{2.5} a_4 -
\end{aligned}$$

$$\begin{aligned}
& 262.984 x^{4.0} a_7 - 4.898490 x^{7.5} a_8 - 152.7057446 x^{5.5} a_8 + \\
& 123.85 x^{6.0} a_8 + 2.751095874 x^{1.0} a_2 - 32.218851 x^{6.5} a_8 - \\
& 0.41261 x^{0.5} a_2 + 502.1828 x^{5.0} a_8 - 452.1875992 x^{4.5} a_8 - \\
& 0.4571428 a_0 x^{-0.5} - 3.851179 x^{1.5} a_3 + 5.60448 x^{1.0} a_3 + \\
& 15.37230581 x^{2.0} a_4 + 39.76660 x^{2.0} a_5 - 50.5682 x^{2.5} a_6 - \\
& 4.940331 x^{1.5} a_4 - 17.91878328 x^{2.5} a_5 + 357.164 x^{3.0} a_7 + \\
& 117.5209442 x^{3.0} a_6 - 177.5123 x^{3.5} a_7 + 1136.71 x^{4.0} a_8 - \\
& 575.4547 x^{3.5} a_8 + 0.914286 a_3 x^{7/2} + 0.599077 a_8 x^{17/2} - \\
& 1.7724 a_1 x^{1.0} + 1.33333000 a_1 x^{3/2} + 1.0666700 a_2 x^{5/2} + \\
& 0.681985 a_6 x^{13/2} + 0.81269 a_4 x^{9/2} + 0.738800 a_5 x^{11/2} + \\
& 2.735992 x^{4.5} a_5 + 54.08246115 x^{4.0} a_6 + 13.7197 x^{5.0} a_6 - \\
& 87.17984813 x^{4.5} a_7 + 31.2326 x^{3.0} a_5 - 3.441861 x^{5.5} a_6 - \\
& 23.184846 x^{5.5} a_7 + 84.716582 x^{5.0} a_7 - 43.80638 x^{3.5} a_6 - \\
& 4.16376384 x^{6.5} a_7 + 21.01349 x^{7.0} a_8 + 17.2376 x^{6.0} a_7 + \\
& 0.636519 a_7 x^{15/2} + 2.0 \sqrt{x} a_0 = 1.5045074370000 x^{1.5} + \\
& 1.8054060 x^{2.5} - \frac{32}{15} x^{5/2} - \frac{16}{7} x^{3/2} + \frac{32}{35} x^{7/2} + x^3 + x^2
\end{aligned} \tag{3.165}$$

Collocating equation (3.165) at the $x = x_i$ gives

$$\begin{aligned}
& 0.0316631257 a_1 - 16.5561433300 a_5 x_i^{3.0} - 5.000000420 a_5 x_i^{4.0} - \\
& 4.000002593 a_4 x_i^{3.0} - 9.312851779 a_4 x_i^{2.0} - 2.999999124 a_3 x_i^{2.0} - \\
& 24.09134657 a_5 x_i^{2.0} - 5.999986964 a_6 x_i^{5.0} - 25.80180013 a_6 x_i^{4.0} - \\
& 62.29668543 a_6 x_i^{3.0} - 7.000013000 a_7 x_i^{6.0} - 189.3296121 a_7 x_i^{3.0} - \\
& 125.4650438 a_7 x_i^{4.0} - 37.04875022 a_7 x_i^{5.0} - 542.3041054 a_8 x_i^{4.0} - \\
& 219.6175657 a_8 x_i^{5.0} - 7.999982618 a_8 x_i^{7.0} - 1.371428571 a_1 x_i^{0.5} - \\
& 0.77245 a_2 x_i^2 - 50.296485831 a_8 x_i^{6.0} + 6.3239858000 a_4 x_i^2 - \tau_1 + \\
& 0.7590344441 a_2 x_i^{3/2} + 1.697652726 a_2 x_i + 1.091923332 a_3 x_i^{5/2} - \\
& 0.89456563700 a_3 x_i^2 - 3.064953551 a_3 x_i^{3/2} - 0.772450000 a_4 x_i^4 + \\
& 1.3425180950 a_4 x_i^{7/2} - 1.777054874 a_4 x_i^3 + 1.141093506 a_4 x_i^{5/2} - \\
& 84480.0 \tau_1 x_i^4 + 180224 \tau_1 x_i^5 - 212992 \tau_1 x_i^6 + 131072 \tau_1 x_i^7 + a_0 - \\
& 32768 \tau_1 x_i^8 - 9.701732740000 x_i^{3.5} a_5 - 0.77245 a_5 x_i^5 + 128 \tau_1 x_i - \\
& 2688 \tau_1 x_i^2 + 21504 \tau_1 x_i^3 - 14.498242700 a_5 x_i^{5/2} - 0.77245 a_3 x_i^3 - \\
& 0.772450 a_6 x_i^6 - 0.77245 a_7 x_i^7 + 1.5517501050000 a_5 x_i^{9/2} + a_1 x_i - \\
& 2.605929684 a_5 x_i^4 + 3.20723010400 a_5 x_i^{7/2} - 1.3848669433 a_5 x_i^3 + \\
& 1.900559424 a_7 x_i^{13/2} + 1.735183751 a_6 x_i^{11/2} - 3.412766282 a_6 x_i^5 + \\
& 3.412766282 a_6 x_i^5 + 5.527861620000 a_6 x_i^{9/2} - 6.245667120 a_6 x_i^4 +
\end{aligned}$$

$$\begin{aligned}
& 1.01736681 a_6 x_i^{7/2} + 36.26005061 a_6 x_i^3 - 97.66211600 a_7 x_i^{7/2} - \\
& 4.208365282 a_7 x_i^6 + 8.09867298 a_7 x_i^{11/2} - 12.61205772 a_7 x_i^5 + \\
& 12.81196506 a_7 x_i^{9/2} + 2.65619685 a_7 x_i^4 - 0.7724500000 a_8 x_i^8 + \\
& 2.052390718 a_8 x_i^{15/2} - 1.63003572 x_i^{1.5} a_2 - 4.997581142 a_8 x_i^7 + \\
& 10.9051939 a_8 x_i^{13/2} + 10.48036524 x_i^{4.0} a_5 - 20.50983375 a_8 x_i^6 + \\
& 30.56325396 a_8 x_i^{11/2} - 20.6301842 a_8 x_i^{9/2} + 280.5773294 a_8 x_i^4 - \\
& 26.9683105 a_8 x_i^5 - 1.394593333 x_i^{2.5} a_3 + 4.95197257500 x_i^{2.0} a_3 - \\
& 2.050965421 x_i^{3.5} a_4 + 7.545862968 x_i^{3.0} a_4 - 15.68743644 x_i^{4.5} a_6 - \\
& 5.195066889 x_i^{2.5} a_4 - 2.735992698 x_i^{4.5} a_5 + 54.08246115 x_i^{4.0} a_6 + \\
& 13.71975085 x_i^{5.0} a_6 - 87.17984813 x_i^{4.5} a_7 + 31.23262134 x_i^{3.0} a_5 - \\
& 3.441861138 x_i^{5.5} a_6 - 23.18484684 x_i^{5.5} a_7 + 84.71658252 x_i^{5.0} a_7 - \\
& 43.80638072 x_i^{3.5} a_6 - 4.163763846 x_i^{6.5} a_7 + 21.01349875 x_i^{7.0} a_8 + \\
& 17.23763569 x_i^{6.0} a_7 + 262.9840 x_i^{4.0} a_7 - 4.898490583000 x_i^{7.5} a_8 - \\
& 152.7057446 x_i^{5.5} a_8 + 123.8560 x_i^{6.0} a_8 + 2.751095874000 x_i^{1.0} a_2 - \\
& 32.218851810 x_i^{6.5} a_8 - 0.41261 x_i^{0.5} a_2 + 502.1828354000 x_i^{5.0} a_8 - \\
& 452.1875992 x_i^{4.5} a_8 - 0.4571428571 a_0 x_i^{-0.5} - 3.8511790 x_i^{1.5} a_3 + \\
& 5.6044800 x_i^{1.0} a_3 + 15.37230581 x_i^{2.0} a_4 + 39.7666085400 x_i^{2.0} a_5 - \\
& 50.56820000 x_i^{2.5} a_6 - 4.940331544 x_i^{1.5} a_4 - 17.91878328 x_i^{2.5} a_5 +
\end{aligned}$$

$$\begin{aligned}
& 357.164 x_i^{3.0} a_7 + 117.5209 x_i^{3.0} a_6 - 177.512 x_i^{3.5} a_7 + \\
& 1136.71 x_i^{4.0} a_8 - 575.45472 x_i^{3.5} a_8 + 0.9142 a_3 x_i^{7/2} + \\
& 0.5990 a_8 x_i^{17/2} - 1.77245 a_1 x_i^{1.0} + 1.333330 a_1 x_i^{3/2} + \\
& 1.06667 a_2 x_i^{5/2} + 0.68198 a_6 x_i^{13/2} + 0.81269 a_4 x_i^{9/2} + \quad (3.166) \\
& 0.738817 a_5 x_i^{11/2} + 0.636519 a_7 x_i^{15/2} + 2 \sqrt{x_i a_0} = x_i^3 \\
& 1.504507 x_i^{1.5} + 1.805406 x_i^{2.5} - \frac{32}{15} x_i^{5/2} - \frac{16}{7} x_i^{3/2} + \\
& x_i^2 + \frac{32}{35} x_i^{7/2}
\end{aligned}$$

where, $x_i = \frac{i}{10}$, $i = 1, 2, \dots, 9$. Hence, for $i = 1$, $x_1 = \frac{1}{10}$, thus

equation (3.166) after simplification gives

$$\begin{aligned}
& 0.0927907907 a_7 - 0.1072115729 a_8 - 0.4371020 a_1 + \\
& 0.1868428880 a_0 + 0.2825009594 a_2 + 0.06572345 a_5 - \\
& 0.0437191624 a_4 + 0.3508752007 a_3 - 0.08003784 a_6 - \\
& 0.42197248 \tau_1 = -0.01445180230 \quad (3.167)
\end{aligned}$$

For $i = 2$, $x_2 = \frac{2}{10}$ thus equation (3.166) after simplification gives

$$\begin{aligned}
& 0.538624065 a_7 - 0.9299764696 a_8 - 0.6168917158 a_1 + \\
& 0.8722246870 a_0 + 0.6155029589 a_2 + 0.133824807 a_5 - \\
& 0.0080973487 a_4 + 0.5362737175 a_3 - 0.296711644 a_6 - \\
& 0.4219725 \tau_1 = -0.02446834719 \quad (3.168)
\end{aligned}$$

For $i = 3$, $x_3 = \frac{3}{10}$, thus equation (3.166) after simplification gives

$$\begin{aligned} & 1.336103342 a_7 - 3.030359997 a_8 - 0.732145763 a_1 + \\ & 1.260820266 a_0 + 0.9485691893 a_2 + 0.09274004 a_5 + \\ & 0.1375191372 a_4 + 0.6178220149 a_3 - 0.5141525 a_6 + \\ & 0.9886899 \tau_1 = -0.0140096161 \end{aligned} \quad (3.169)$$

For $i = 4$, $x_4 = \frac{4}{10}$ thus equation (3.166) after simplification gives

$$\begin{aligned} & 2.26703858 a_7 - 6.550261277 a_8 - 0.80737568630 a_1 + \\ & 1.542104742 a_0 + 1.282540968 a_2 - 0.1332597688 a_5 + \\ & 0.399212286 a_4 + 0.6182661095 a_3 - 0.562858325 a_6 + \\ & 0.040057 \tau_1 = 0.0301924010 \end{aligned} \quad (3.170)$$

For $i = 5$, $x_5 = \frac{5}{10}$ thus equation (3.166) after simplification gives

$$\begin{aligned} & 2.96025320 a_7 - 11.14550073 a_8 - 0.8529049743 a_1 + \\ & 1.767715934 a_0 + 1.620118766 a_2 - 0.604843994 a_5 + \\ & 0.778661563 a_4 + 0.552126766 a_3 - 0.265236443 a_6 - \\ & 1.0 \tau_1 = 0.1216440811 \end{aligned} \quad (3.171)$$

For $i = 6$, $x_6 = \frac{6}{10}$, thus equation (3.166) after simplification gives

$$\begin{aligned}
& 2.92207676 a_7 - 15.96776391 a_8 - 0.8744350913 a_1 + \\
& 1.959024447 a_0 + 1.964201356 a_2 - 1.374096427 a_5 + \\
& 1.276514795 a_4 + 0.430755960 a_3 + 0.556855950 a_6 + \\
& 0.04006 \tau_1 = 0.2744557176
\end{aligned} \tag{3.172}$$

For $i = 7$, $x_7 = \frac{7}{10}$ thus equation (3.166) after simplification gives

$$\begin{aligned}
& 1.55339424 a_7 - 19.66353420 a_8 - 0.8755906003 a_1 + \\
& 2.126929832 a_0 + 2.317591768 a_2 - 2.487325916 a_5 + \\
& 1.893725465 a_4 + 0.264101391 a_3 + 2.078554870 a_6 + \\
& 0.98869 \tau_1 = 0.5034154700
\end{aligned} \tag{3.173}$$

For $i = 8$, $x_8 = \frac{8}{10}$ thus equation (3.166) after simplification gives

$$\begin{aligned}
& 2.277753130 a_0 - 1.84056905 a_7 - 20.38244204 a_8 - \\
& 0.8588865933 a_1 + 2.682940214 a_2 - 3.9863190 a_5 + \\
& 2.632297897 a_4 + 0.061495516 a_3 + 4.47002221 a_6 - \\
& 0.42198 \tau_1 = 0.8239918516
\end{aligned} \tag{3.174}$$

For $i = 9$, $x_9 = \frac{9}{10}$, thus equation (3.166) after simplification gives

$$\begin{aligned}
 & 2.415495715 a_0 - 8.0449456 a_7 - 15.79269032 a_8 - \\
 & 0.8261761423 a_1 + 3.062743159 a_2 - 5.908755 a_5 + \\
 & 3.495796523 a_4 - 0.167939434 a_3 + 7.89539614 a_6 - \\
 & 0.42192 \tau_1 = 1.252315042
 \end{aligned} \tag{3.175}$$

Thus, from the given conditions (3.156); that is $y_8(0) = 0$, equation (3.159) gives

$$a_0 = 0 \tag{3.176}$$

Solving equations (3.167) – (3.176) to obtain

$$\begin{aligned}
 a_0 &= 0, \\
 a_1 &= -0.04156005027, \\
 a_2 &= 0.05768554845, \\
 a_3 &= -0.1095542543, \\
 a_4 &= -0.05944554009, \\
 a_5 &= -0.2394131870, \\
 a_6 &= -0.07254943678, \\
 a_7 &= -0.03924425607, \\
 a_8 &= -0.004633078085 \\
 \tau_1 &= -2.378478559 \times 10^{-7}
 \end{aligned} \tag{3.177}$$

Thus, substituting the values of $\alpha_i (i = 0, 1, 2, \dots, 8)$ above into equation (3.159), and after simplification gives

$$\begin{aligned}
 y_8(x) = & 0.046895494900 x^{0.5} + 0.01331055203 x^{15/2} - \\
 & 0.615823328700 x^2 + 0.2489904726000 x^{3/2} + \\
 & 0.05636997831000 x + 8.331311205000 x^{7/2} - \\
 & 6.037799303000 x^3 + 4.8168448010000 x^{9/2} - \\
 & 7.438085859000 x^4 + 1.2116292240000 x^{11/2} - \\
 & 2.671602807000 x^5 + 0.1953613886000 x^{13/2} - \\
 & 0.54246057820000 x^6 + 4.002022199000 x^{5/2} - \\
 & 0.075094638350 x^7 - 0.00463307808500000 x^8
 \end{aligned} \tag{3.178}$$

Thus, equation (3.178) is the approximate solution of equation (3.155) together with the condition (3.156) for case $N = 8$

3.5 Exponentially Fitted Collocation Method

3.5.1 Introduction

The attention of many Mathematicians, Physicists and Engineers have been drawn towards the study of singular initial and boundary value problems. Owing to its wide range of application in various fields, most especially in the heat and mass diffusion process, and population models, a lot of work has been done in this area of study. Among the authors who have worked in this area include Sahu and Saha (2016), Brunner (2006), Mehdi et al (2011), where each of them came up with numerical solutions to this class of problem, for instance Sahu and Saha (2016) developed a sinc-Galerkin method which was based on Sinc function to approximate the solution of fractional Volterra Fredholm integro-differential equations with singular kernels. The interest of most of these researchers is to obtain efficient and accurate approximate solutions to functional equations having no analytical solutions. Explicitly, in applied Mathematics such as gas dynamics, mechanics, fluid mechanics, reaction diffusion process, where some problems usually occur in form of singular differential equation as discussed in Chawla and Katti (1982). Similarly, Adomian Decomposition method was applied independently for the numerical solution of singular differ-

ential equation in Filippova and Zdenek (2010), Kumar and Singh (2010). These numerical methods stated, are highly advantageous but they require much computational works.

Several researches dealing with the solution of singular and multi-order fractional integro differential equations have been carried out due to its wide area of application, and because most of the equations that occur in this form are difficult to solve analytically, the need for efficient methods to provide the approximate solution is necessitated. Consequently, Variational Iterational method by Omar (2014), Adomian Decomposition method by Mittal and Nigam (2008), Homotopy Analysis method by Ibtisan (2011), and Haar wavelet method by Ghanbari (2010) among others, were developed and found to be very effective.

The concept of exponential fitting method has been investigated by many authors, and have individually come up with the idea describing it as a highly effecient method of solution. For instance, Taiwo (2000) investigated the numerical solution of two point boundary value problems by the application of the Cubic Spline Collocation Method using exponential fitting. Among other authors that have also adopted the technique as method of solution are Raptis (1982), Simos (1999, 2006), and Daele and Berghe (2007). Thus, in this work, the Exponential Fitting with collocation method is intro-

duced to solve singular and multi-order Volterra fractional integro-differential equations. So, it is our desire to implement the approach with the constructed canonical polynomials as basis function for the numerical solution of singular and multi-order fractional integro-differential equation of the form:

$$D^a y(x) + \sum_{i=0}^n p_i y^i(x) = f(x) - \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \quad (3.179)$$

subject to the conditions

$$y^{(k)}(a) = a_k; \quad k = 0, 1, \dots, n-1 \quad (3.180)$$

All the parameters involved have been defined in section 3.3

3.6 Perturbed Collocation Method with Exponential Fitting

The perturbed Collocation Method with exponential fitting is examined in this section. This method is an attempt to improve the accuracy and efficiency of the perturbed and standard collocation methods to handle singular and multi-order fractional integro-differential equations. Numerical results are given which show that there is increased accuracy in the results obtained without extra computational effort.

The assumed approximate solution of the form given in equation (3.20) is considered in this section. Hence, equation (3.20) is substituted into a slightly perturbed equation (3.179) to give

$$D^a y_N(x) + \sum_{i=0}^n p_i y_N^i(x) + \lambda \int_0^x \frac{y(t)}{\sqrt[N]{x-t}} dt = f(x) + H_N(x) \quad a \leq x \leq b \quad (3.181)$$

where $H_N(x)$ is an orthogonal polynomial (Shifted Chebyshev Polynomial) given as

$$H_N(x) = \sum_{i=1}^n \tau_i T_{N-i+1}(x)$$

and $T_N(x)$ is the Chebyshev polynomial defined in equation (1.10).

Thus, equation (3.181) is further simplified to obtain

$$\begin{aligned}
& D^a \left(\sum_{j=0}^N a_j Q_j(x) \right) + p_0 \sum_{j=0}^N a_j Q_j(x) + p_1 \\
& \sum_{j=0}^N a_j Q_j(x) + p_2 \sum_{j=0}^N a_j Q_j^1(x) + \dots + p_n \\
& \sum_{j=0}^N a_j Q_j^n(x) + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x) + \\
& \tau_1 T_N(x) + \tau_2 T_{N-1}(x) + \tau_3 T_{N-2}(x) + \dots \\
& + \tau_{n-1} T_{N-i+2}(x),
\end{aligned} \tag{3.182}$$

where, N is the degree of approximation and n is the highest order singular and multi-order fractional integro differential equation, $\tau_i, i = 1(1)n - 1$ are the free Tau parameters to be determined.

Hence, equation (3.182) is further simplified to give

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0^1(x) + \dots + p_n Q_0^n(x)\} a_0 \\
& + \{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1^1(x) + \dots + p_n Q_1^n(x) \\
& \} a_1 + \\
& \cdot \\
& \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N^1(x) + \dots + p_n Q_N^n(x) \\
& \} a_N + \lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x) + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) \\
& + \tau_3 T_{N-2}(x) + \dots + \tau_{n-1} T_{N-n+2}(x)
\end{aligned} \tag{3.183}$$

The integral in equation (3.183) is evaluated by adopting a Taylor series expansion of appropriate order as discussed for equations (3.23 – 3.24) to obtain

$$\begin{aligned}
& [D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0'(x) + \dots + p_n Q_0^{(n)}(x)] a_0 + \\
& [D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1'(x) + \dots + p_n Q_1^{(n)}(x)] a_1 + \\
& \dots \\
& [D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N'(x) + \dots + p_n Q_N^{(n)}(x)] \quad (3.184) \\
& a_N + \lambda \int_0^x \frac{1}{\sqrt{x-t}} \{ y_N(x) + (t-x) y_N'(x) + \frac{(t-x)^2}{2!} y_N''(x) + \dots + \frac{(t-x)^n}{n!} y_N^{(n)}(x) \} dt = f(x) + \tau_1 T_N(x) + \\
& \tau_2 T_{N-1}(x) + \tau_3 T_{N-2}(x) + \dots + \tau_{n-1} T_{N-n+2}(x)
\end{aligned}$$

Hence, equation (3.184) is re-written as

$$\begin{aligned}
& [D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0(x) + \dots + p_n Q^n(x)] \\
& a_0 + [D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1(x) + \dots + \\
& p_n Q_1^n(x)] a_1 + \\
& \cdot \\
& [D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N(x) + \dots + p_n Q_N^n(x)] \quad (3.185) \\
& \int_0^x \frac{1}{\sqrt{x-t}} dt + \frac{y_N^j(x) \lambda}{2!} \int_0^x \frac{(t-x)^2}{\sqrt{x-t}} dt + \frac{y_N^j(x) \lambda}{n!} \int_0^x \frac{(t-x)^n}{\sqrt{x-t}} dt + \dots + \\
& = f(x) + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) + \tau_3 T_{N-2}(x) + \dots + \\
& \tau_{n-1} T_{N-n+2}(x)
\end{aligned}$$

Equation (3.185) is further simplified to give

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0(x) + \dots + p_n Q^n(x)\} a_0 + \\
& \{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1(x) + \dots + p_n Q^n(x)\} a_1 + \\
& \cdot \\
& \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N(x) + \dots + p_n Q_N^n(x)\} \quad (3.186) \\
& a_N + \lambda \left\{ 2 x^{\frac{1}{2}} y_N(x) - \frac{2}{3} x^{\frac{3}{2}} y_N^j(x) + \frac{2}{5} x^{\frac{5}{2}} \frac{y_N^{jj}(x)}{2!} - \frac{2}{7} x^{\frac{7}{2}} \frac{y_N^{jjj}(x)}{4!} + \dots \right\} = f(x) + \tau_1(x) T_N + \tau_2 T_{N-1}(x) + \tau_3 \\
& T_{N-2}(x) + \dots + \tau_{n-1} T_{N-n+2}(x)
\end{aligned}$$

where,

$$\begin{aligned}
y_N(x) &= \sum_{i=0}^N Q_i(x) = a_0 Q_0(x) + a_1 Q_1(x) + \dots + a_n Q_N(x) \\
y_N^I(x) &= \sum_{i=0}^N Q_i^I(x) = a_0 Q_0^I(x) + a_1 Q_1^I(x) + \dots + a_n Q_N^I(x) \\
&\cdot \\
&\cdot \\
y_N^n(x) &= \sum_{i=0}^N Q_i^n(x) = a_0 Q_0^n(x) + a_1 Q_1^n(x) + \dots + a_n Q_N^n(x)
\end{aligned} \tag{3.187}$$

Hence, substituting equation (3.187) into equation (3.186) gives

$$\begin{aligned}
&\{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0^I(x) + \dots + p_n Q_0^n(x)\}a_0 + \\
&\{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1^I(x) + \dots + p_n Q_1^n(x)\}a_1 + \\
&\dots + \{D^a Q_N(x) + p_0 Q_N(x) + p_1 Q_N^I(x) + \dots + p_n Q_N^n(x) \\
&(x)\}a_N + 2\lambda x^2 \{a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) + \dots + \\
&a_n Q_N(x)\} - \frac{2}{3} \lambda x^2 \{a_0 Q_0(x) + a_1 Q_1(x) + a_2 Q_2(x) + \dots \\
&+ a_n Q_N(x)\} + \frac{2}{10} \lambda x^5 \{a_0 Q_0^II(x) + a_1 Q_1^II(x) + a_2 Q_2^II(x) + \\
&\dots + a_n Q_N^II(x)\} - \frac{7}{42} \lambda x^2 \{a_0 Q_0 + a_1 Q_1(x) + a_2 Q_2(x) + \\
&\dots + a_n Q_N^II(x)\} + \dots = f(x) + t_1 T_N(x) + t_2 T_{N-1}(x) + \\
&t_3 T_{N-2}(x) + \dots + t_{n-1} T_{N-n+2}(x)
\end{aligned} \tag{3.188}$$

Collecting like terms in equation (3.188), gives

$$\begin{aligned}
& \{D^a Q_0(x) + p_0 Q_0(x) + p_1 Q_0^I(x) + \dots + p_n Q_0^n(x) + \\
& 2\lambda x^{\frac{1}{2}} Q_0(x) - \frac{2}{3} \lambda x^{\frac{3}{2}} Q_0^I(x) + \frac{2}{10} \lambda x^{\frac{5}{2}} Q_0^{II}(x) - \frac{2}{42} \lambda x^{\frac{7}{2}} \\
& Q_0^{III}(x)\} a_0 + \{D^a Q_1(x) + p_0 Q_1(x) + p_1 Q_1^I(x) + \dots + \\
& p_n Q_1^n(x) + 2\lambda x^{\frac{1}{2}} Q_1(x) - \frac{2}{3} \lambda x^{\frac{3}{2}} Q_1^I(x) + \frac{2}{10} \lambda x^{\frac{5}{2}} Q_1^{II}(x) \\
& - \frac{2}{42} \lambda x^{\frac{7}{2}} Q_1^{III}(x)\} a_1 + \dots + \{D^a Q_N(x) + p_0 Q_N(x) + \\
& p_1 Q_N^I(x) + \dots + p_n Q_N^n(x) + 2\lambda x^{\frac{1}{2}} Q_N(x) - \frac{2}{3} \lambda x^{\frac{3}{2}} Q_N^I(x) \\
& + \frac{2}{10} \lambda x^{\frac{5}{2}} Q_N^{II}(x) - \frac{2}{42} \lambda x^{\frac{7}{2}} Q_N^{III}(x)\} a_N = f(x) + \tau_1 \\
& T_N(x) + \tau_2 T_{N-1}(x) + \tau_3 T_{N-2}(x) + \dots + \tau_{n-1} T_{N-n+2} \\
& (x)
\end{aligned} \tag{3.189}$$

Equation (3.189) is collocated at point $x = x_i$ to obtain,

$$\begin{aligned}
& \{D^a Q_0(x_i) + p_0 Q_0(x_i) + p_1 Q_0^I(x_i) + \dots + p_n Q_0^n(x_i) + \\
& 2\lambda x_i^{\frac{1}{2}} Q_0(x_i) - \frac{2}{3} \lambda x_i^{\frac{3}{2}} Q_0^I(x_i) + \frac{2}{10} \lambda x_i^{\frac{5}{2}} Q_0^{II}(x_i) - \frac{2}{42} \lambda \\
& x_i^{\frac{7}{2}} Q_0^{III}(x_i)\} a_0 + \{D^a Q_1(x_i) + p_0 Q_1(x_i) + p_1 Q_1^I(x_i) + \\
& \dots + p_n Q_1^n(x_i) + 2\lambda x_i^{\frac{1}{2}} Q_1(x_i) - \frac{2}{3} \lambda x_i^{\frac{3}{2}} Q_1^I(x_i) + \frac{2}{10} \lambda \\
& x_i^{\frac{5}{2}} Q_1^{II}(x_i) - \frac{2}{42} \lambda x_i^{\frac{7}{2}} Q_1^{III}(x_i)\} a_1 + \dots + \{D^a Q_N(x_i) +
\end{aligned}$$

$$\begin{aligned}
& p_0 Q_N(x_i) + p_1 Q_N(x_i) + \cdots + p_n Q_N(x_i) + \frac{2\lambda x_i^2}{3} Q_N(x_i) + \frac{2\lambda x_i^2}{10} Q_N(x_i) + \frac{2\lambda x_i^2}{42} Q_N(x_i) \\
& (x_i) \} a_N = f(x_i) + \tau_1 T_N(x_i) + \tau_2 T_{N-1}(x_i) + \tau_3 T_{N-2}(x_i) \\
& (x_i) + \cdots + \tau_{n-1} T_{N-n+2}(x_i)
\end{aligned} \quad (3.190)$$

where,

$$x_i = a + \frac{(b-a)i}{N+2}, \quad i = 1, 2, \dots, N+1$$

For the perturbed collocation method with exponential fitting, an approximation to the conditions is sought in the form

$$y_N^*(x) = \tau_n e^x$$

where τ_n is an additional free Tau parameter to be determined such that

$$y_N^{(k)}(a) + \tau_n e^{(a)} = a_k; \quad k = 0, \dots, n-1 \quad (3.191)$$

Thus, from equations (3.190) and (3.191), $(N+n+1)$ equations are obtained which give the unique values of the constants $a_0, a_1, a_2, \dots, a_N, \tau_1, \tau_2, \dots, \tau_n$ to obtain a single polynomial approximation

$$y(x) \approx y_N(x) + y_N^*(x); \quad a \leq x \leq b, \quad (3.192)$$

Hence, equation (3.192) is the required exponentially fitted perturbed collocation approximation.

3.7 Demonstration of Perturbed Collocation Method with Exponential Fitting

Problem 3.7.1: Consider a second order singular fractional integro-differential equation of the form

$$y'' + \frac{d^{1.5}y(x)}{dx^{1.5}} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 2.256758334 x^{0.5} + 7.221626669 x^{2.5} - \frac{16}{15} x^{5/2} - \frac{256}{315} x^{9/2} + 2 + 12 x^2 \quad (3.193)$$

subject to the conditions

$$y(0) = 0, \quad y'(0) = 0 \quad (3.194)$$

The exact solution is given as $y(x) = x^4 + x^2$

The recursive canonical polynomials obtained for equation (3.193) is given as

$$Q_j(x) = \frac{x^{j+1} (j+1)(j) Q_{j-1} \Gamma(j-a+2)}{x^{1-a} \Gamma(j+2)}; \quad (3.195)$$

$$Q_{-1} = 0$$

Here, equations (3.193)–(3.194) are solved for case $N = 7$. Hence, equation (3.20) becomes

$$y_7(x) = \sum_{j=0}^7 a_j Q_j(x) \quad (3.196)$$

From equation (3.195), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 7$) are obtained

$$Q_0(x) = 1.772453851 x^{1.5}$$

$$Q_1(x) = 0.4431134628 x^{2.5} - 1.570796327 x^{2.0}$$

$$Q_2(x) = 0.2215567313 x^{3.5} - 0.2945243112 x^{3.0} + 1.04406149900000 x^{2.5}$$

$$Q_3(x) = 0.1384729571 x^{4.5} - 0.1227184630 x^{4.0} + 0.1631346093000 x^{3.5} - 0.5782971326 x^{3.0}$$

$$Q_4(x) = 0.09693107000 x^{5.5} - 0.06711165949 x^{5.0} + 0.05947615964 x^{4.5} - 0.07906406117 x^{4.0} + 0.2802747992 x^{3.5}$$

$$Q_5(x) = 0.07269830247 x^{6.5} - 0.04228034547 x^{6.0} + 0.02927342232 x^{5.5} - 0.02594289506 x^{5.0} + 0.03448693820 x^{4.5} - 0.122253012700 x^{4.0}$$

$$Q_6(x) = 0.05712009480 x^{7.5} - 0.02906773750 x^{7.0} + 0.01690540139 x^{6.5} - 0.01170470460 x^{6.0} + 0.01037302438 x^{5.5} - 0.01378928026 x^{5.0} + 0.04888172573 x^{4.5}$$

$$Q_7(x) = 0.04641007703 x^{8.5} - 0.02120758400 x^{8.0} + 0.01079228749 x^{7.5} - 0.006276647845 x^{7.0} + 0.004345729937 x^{6.5} - 0.003851302 x^{6.0} + 0.005119692473 x^{5.5} - 0.01814883725 x^{5.0}$$

Making use of $Q_j(x)$; ($j = 0, 1, \dots, 7$) in equation (3.196) gives

$$\begin{aligned}
 y_7(x) = & 1.77245 x^{1.5} a_0 + (0.443 x^{2.5} - 1.5707 x^{2.0}) a_1 + \\
 & (0.221556 x^{3.5} - 0.2945243 x^{3.0} + 1.04406149 x^{2.5}) a_2 + \\
 & (0.13847295 x^{4.5} - 0.1227184 x^{4.0} + 0.163134600 x^{3.5} - \\
 & 0.5782971 x^{3.0}) a_3 + (0.09693107 x^{5.5} - 0.067111 x^{5.0} + \\
 & 0.05947615 x^{4.5} - 0.0790640 x^{4.0} + 0.2802747 x^{3.5}) a_4 + \\
 & (0.0726983024 x^{6.5} - 0.0422803 x^{6.0} + 0.029273 x^{5.5} - \\
 & 0.025942 x^{5.0} + 0.0344869 x^{4.5} - 0.12225300 x^{4.0}) a_5 + \\
 & (0.05712009 x^{7.5} - 0.029067737 x^{7.0} + 0.0169054 x^{6.5} - \\
 & 0.011704704 x^{6.0} + 0.01037302 x^{5.5} - 0.0137892 x^{5.0} + \\
 & 0.04888 x^{4.5}) a_6 + (0.04640 x^{8.5} - 0.02120750000 x^{8.0} + \\
 & 0.01079228 x^{7.5} - 0.006276647 x^{7.0} + 0.00434572 x^{6.5} - \\
 & 0.00385130 x^{6.0} + 0.00511969 x^{5.5} - 0.0181488 x^{5.0}) a_7
 \end{aligned} \tag{3.197}$$

Thus, equation (3.197) is substituted into a slightly perturbed equation (3.193) to get

$$y_7''(x) + \frac{d^{1.5}y_7(x)}{dx^{1.5}} - \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2 + 2.256758334x^{0.5} - \frac{4}{3}x^{3/2} - \frac{16}{15}x^{5/2} + H_7(x) \quad (3.198)$$

together with equation (3.194) to get

$$y_7(0) = 0, \quad y_7'(0) = 0$$

where,

$$H_7(x) = (8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1.00) \quad (3.199)$$

Hence, evaluating the terms $\frac{d^2y_7(x)}{dx^2}$, $\frac{d^{1.5}y_7(x)}{dx^{1.5}}$, and $\int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt$ in equation (3.198) gives

$$\begin{aligned}
\frac{d^2 y_7}{dx^2} = & 1.3293403880 a_0 x^{-0.5} + (1.661675486 x^{0.5} - \\
& 3.14159265) a_1 + (1.9386214 x^{1.5} - 1.7671458 x^{1.0} + \\
& 3.915230 x^{0.5}) a_2 + (2.180949 x^{2.5} - 1.4726215 x^{2.0} + \\
& 1.4274278 x^{1.5} - 3.4697827 x^{1.0}) a_3 + (2.39904 x^{3.5} - \\
& 1.3422332 x^{3.0} + 0.9367495 x^{2.5} - 0.94876873 x^{2.0} + \\
& 2.452404 x^{1.5}) a_4 + (2.598964 x^{4.5} - 1.2684109 x^{4.0} + \\
& 0.72451720 x^{3.5} - 0.5188579 x^{3.0} + 0.54316900 x^{2.5} - \\
& 1.4670361 x^{2.0}) a_5 + (2.7846046 x^{5.5} - 1.220844 x^{5.0} + \\
& 0.60436809 x^{4.5} - 0.3511411 x^{4.0} + 0.256732350 x^{3.5} - \\
& 0.27578560 x^{3.0} + 0.7698871 x^{2.5}) a_6 + (2.958642 x^{6.5} - \\
& 1.187624704 x^{6.0} + 0.52612401 x^{5.5} - 0.2636192 x^{5.0} + \\
& 0.1553598 x^{4.5} - 0.11553908 x^{4.0} + 0.126712380 x^{3.5} - \\
& 0.3629767450 x^{3.0}) a_7
\end{aligned} \tag{3.200}$$

$$\begin{aligned}
& \int_0^x \frac{y_7(t)}{\sqrt{x-t}} dt = 2.08812 x^{2.0} a_0 + (0.435026 x^{3.0} - \\
& 1.67552 x^{2.5}) a_1 + (0.190324 x^{4.0} + 1.025000 x^{3.0} - \\
& 0.269279 x^{3.5}) a_2 + (0.140137 x^{4.0} - 0.09973 x^{4.5} - \\
& 0.5287 x^{3.5} + 0.107 x^{5.0}) a_3 + (0.2407640000 x^{4.0} - \\
& 0.0642552 x^{4.5} + 0.0459826 x^{5.0} - 0.0495800 x^{5.5} + \\
& 0.068695 x^{6.0}) a_4 + (-0.09935 x^{4.5} + 0.02666 x^{5.0} - \\
& 0.019167 x^{5.5} + 0.020746000 x^{6.0} - 0.028830 x^{6.5} + \\
& 0.047840 x^{7.0}) a_5 + (0.037791 x^{5.0} - 0.010188 x^{5.5} + \\
& 0.0073514 x^{6.0} - 0.00798243 x^{6.5} + 0.011100 x^{7.0} - \\
& 0.018500 x^{7.5} + 0.0352401 x^{8.0}) a_6 - 0.012705 x^{8.5} + \\
& (-0.0134087 x^{5.5} + 0.003628 x^{6.0} - 0.002626 x^{6.5} + \\
& 0.00285983 x^{7.0} - 0.003995 x^{7.5} + 0.00665828 x^{8.0} + \\
& 0.0270419 x^{9.0}) a_7
\end{aligned} \tag{3.201}$$

and,

$$\begin{aligned}
\frac{d^{1.5}y_7}{dx^{1.5}} = & 2.356196748 a_0 + (-3.544910349 x^{0.5} + \\
& 1.472619441 x^{1.0}) a_1 + (1.2885413060 x^{2.0} + \\
& 3.469782864 x^{1.0} - 1.32933785000 x^{1.5}) a_2 + \\
& (0.9487694131 x^{2.0} - 0.886228900000 x^{2.5} + \\
& 1.208008885 x^{3.0} - 2.61015540500 x^{1.5}) a_3 + \\
& (-0.6923621731 x^{3.5} - 0.5709710000 x^{2.5} + \\
& 0.5188580788 x^{3.0} + 1.1627101000000 x^{4.0} + \\
& 1.6300396190 x^{2.0}) a_4 + (0.3511408770 x^{4.0} - \\
& 0.2676425114 x^{3.5} - 0.8828664000000 x^{2.5} + \\
& 0.3008569 x^{3.0} - 0.5815891903000000 x^{4.5} + \\
& 1.1336431 x^{5.0}) a_5 + (0.42643400000000 x^{3.0} - \\
& 0.14225870000 x^{3.5} + 0.124426934900 x^{4.0} - \\
& 0.1610039098 x^{4.5} + 0.263619270000000 x^{5.0} - \\
& 0.5088882847 x^{5.5} + 1.11339990000 x^{6.0}) a_6 + \\
& (-0.1872342 x^{3.5} + 0.06141203616000 x^{4.0} - \\
& 0.05297649919 x^{4.5} + 0.06776690000000 x^{5.0} - \\
& 0.10988495000000000 x^{5.5} + 0.21036590 x^{6.0} - \\
& 0.4569608400000 x^{6.5} + 1.0984884030 x^{7.0}) a_7
\end{aligned} \tag{3.202}$$

Hence, substituting equations (3.199)–(3.202) into equation (3.198) gives

$$\begin{aligned}
& 0.102383 a_7 x^{4.5} + 0.067766503 a_7 x^{5.0} + 0.4296477668 a_7 x^{5.5} - 0.0605218 a_7 x^{3.5} + \\
& 0.06141203616 a_7 x^{4.0} - 3.141592654 a_1 - 1.182727 a_3 x^{1.5} + 1.106048584 a_6 x^{6.0} + \\
& 1.706681809 a_4 x^{3.5} + 0.3657783721 a_4 x^{2.5} + 0.5188580 a_4 x^{3.0} + 0.921946 a_4 x^{4.0} + \\
& 1.63003 a_4 x^{2.0} - 1.883234863 a_1 x^{0.5} + 0.609283544 a_2 x^{1.5} - 1.767145867 a_2 x^{1.0} + \\
& 3.915230622 a_2 x^{0.5} + 1.294720076 a_3 x^{2.5} - 1.472621556 a_3 x^{2.0} + 0.20673 a_7 x^{6.0} + \\
& 2.50430 a_7 x^{6.5} + 1.095628573 a_7 x^{7.0} + 0.42643423 a_6 x^{3.0} + 0.1144736425 a_6 x^{3.5} + \\
& 0.124426 a_6 x^{4.0} + 0.4433641897 a_6 x^{4.5} + 0.2258275 a_6 x^{5.0} + 2.285904137 a_6 x^{5.5} - \\
& 0.351141 a_6 x^{4.0} - 3.469782796 a_3 x^{1.0} - 1.342233190 a_4 x^{3.0} - 0.94876870 a_4 x^{2.0} + \\
& 2.4524 a_4 x^{1.5} + 2.116729924 a_5 x^{4.5} - 1.268410364 a_5 x^{4.0} - 0.5188579012 a_5 x^{3.0} - \\
& 0.3396 a_5 x^{2.5} - 1.467036152 a_5 x^{2.0} - 1.220844975 a_6 x^{5.0} + 0.4568746912 a_5 x^{3.5} + \\
& 1.106980256 a_5 x^{5.0} - 0.2757856052 a_6 x^{3.0} + 0.7698871 a_6 x^{2.5} - 1.187624 a_7 x^{6.0} - \\
& 0.263619 a_7 x^{5.0} - 0.1155390865 a_7 x^{4.0} - 0.36297674 a_7 x^{3.0} + 0.35114087 a_5 x^{4.0} + \\
& 0.3008569163 a_5 x^{3.0} + 0.9487694131 a_3 x^{2.0} + 1.208008 a_3 x^{3.0} - 0.435026 a_1 x^{3.0} + \\
& 1.67552 a_1 x^{2.5} - 0.190324 a_2 x^{4.0} - 1.025 a_2 x^{3.0} + 0.26927 a_2 x^{3.5} + 0.0495 a_4 x^{5.5} - \\
& 0.14013 a_3 x^{4.0} + 0.09973 a_3 x^{4.5} + 0.5287 a_3 x^{3.5} - 0.10705 a_3 x^{5.0} + 0.0642 a_4 x^{4.5} - \\
& 0.0459826 a_4 x^{5.0} + 0.00798243 a_6 x^{6.5} - 0.068695 a_4 x^{6.0} a_5 x^{5.5} + 0.01916700000 -
\end{aligned}$$

$$\begin{aligned}
& 0.02074600 a_5 x^{6.0} + 0.0288345 a_5 x^{6.5} - 2 - 12 x^2 - \\
& 0.04784 a_5 x^{7.0} + 0.012705 a_7 x^{8.5} - 0.01112 a_6 x^{7.0} + \\
& 0.01850 a_6 x^{7.5} - 0.03524 a_6 x^{8.0} + 0.003995 a_7 x^{7.5} - \\
& 0.006658 a_7 x^{8.0} - 0.0270419 a_7 x^{9.0} + 1.472 a_1 x^{1.0} + \\
& 1.2885413 a_2 x^{2.0} + 3.46978 a_2 x^{1.0} + 1.3293 a_0 x^{-0.5} - \quad (3.203) \\
& 2.088 x^2 a_0 - 8192 t_1 x^7 + 28672 t_1 x^6 - 39424 t_1 x^5 + \\
& 26880 t_1 x^4 - 9408 t_1 x^3 + 1568 t_1 x^2 - 98 t_1 x + t_1 + \\
& 2.3561967480000 a_0 = 2.256758334 x^{0.5} - \frac{16}{15} x^{5/2} - \\
& \frac{256}{315} x^{9/2}
\end{aligned}$$

Collocating equation (3.203) at the point $x = x_i$ to give

$$\begin{aligned}
& 0.1023833460 a_7 x_i^{4.5} + 0.0677665034 a_7 x_i^{5.0} + 0.4296470 a_7 x_i^{5.5} - \\
& 0.0605218232 a_7 x_i^{3.5} + 0.06141203616 a_7 x_i^{4.0} - 3.141592654 a_1 - \\
& 1.182727573 a_3 x_i^{1.5} + 1.106048584 a_6 x_i^{6.0} + 1.706681809 a_4 x_i^{3.5} + \\
& 0.3657783721 a_4 x_i^{2.5} + 0.5188580788 a_4 x_i^{3.0} + 0.9219461 a_4 x_i^{4.0} + \\
& 1.630039619 a_4 x_i^{2.0} - 1.883234863 a_1 x_i^{0.5} + 0.609283544 a_2 x_i^{1.5} - \\
& 1.767145867 a_2 x_i^{1.0} + 3.915230622 a_2 x_i^{0.5} + 1.2947200760 a_3 x_i^{2.5} - \\
& 1.472621556 a_3 x_i^{2.0} + 0.2067376649 a_7 x_i^{6.0} + 2.504308101 a_7 x_i^{6.5} +
\end{aligned}$$

$$\begin{aligned}
& 1.095628573 a_7 x_i^{7.0} + 0.42643423380 a_6 x_i^{3.0} + 0.11447364 a_6 x_i^{3.5} + \\
& 0.12442693 a_6 x_i^{4.0} + 0.4433641897 a_6 x_i^{4.5} + 0.2258275755 a_6 x_i^{5.0} + \\
& 2.285904137 a_6 x_i^{5.5} - 0.3511411380 a_6 x_i^{4.0} - 3.469782796 a_3 x_i^{1.0} - \\
& 1.342233190 a_4 x_i^{3.0} - 0.9487687341 a_4 x_i^{2.0} + 2.452404493 a_4 x_i^{1.5} + \\
& 2.116729924 a_5 x_i^{4.5} - 1.268410364 a_5 x_i^{4.0} - 0.5188579012 a_5 x_i^{3.0} - \\
& 0.3396971512 a_5 x_i^{2.5} - 1.467036152 a_5 x_i^{2.0} - 1.220844975 a_6 x_i^{5.0} + \\
& 0.4568746912 a_5 x_i^{3.5} + 1.106980256 a_5 x_i^{5.0} - 0.275785605 a_6 x_i^{3.0} + \\
& 0.7698871803 a_6 x_i^{2.5} - 1.187624704 a_7 x_i^{6.0} - 0.263619209 a_7 x_i^{5.0} - \\
& 0.1155390865 a_7 x_i^{4.0} - 0.36297674 a_7 x_i^{3.0} + 0.3511408772 a_5 x_i^{4.0} + \\
& 0.30085 a_5 x_i^{3.0} + 0.94876 a_3 x_i^{2.0} + 0.02883 a_5 x_i^{6.5} - 0.435 a_1 x_i^{3.0} + \\
& 1.6755 a_1 x_i^{2.5} - 0.19032 a_2 x_i^{4.0} - 1.025 a_2 x_i^{3.0} + 0.269279 a_2 x_i^{3.5} + \\
& 0.0495 a_4 x_i^{5.5} - 0.14013 a_3 x_i^{4.0} + 0.09973 a_3 x_i^{4.5} + 0.5287 a_3 x_i^{3.5} - \\
& 0.107057 a_3 x_i^{5.0} + 0.064255 a_4 x_i^{4.5} - 0.0459826 a_4 x_i^{5.0} + 0.01916 + \\
& 0.0079824 a_6 x_i^{6.5} - 0.068695 a_4 x_i^{6.0} a_5 x_i^{5.5} + 1.208008885 a_3 x_i^{3.0} - \\
& 0.02074 a_5 x_i^{6.0} - 0.04784 a_5 x_i^{7.0} + 0.0127 a_7 x_i^{8.5} - 0.0111 a_6 x_i^{7.0} + \\
& 0.0185 a_6 x_i^{7.5} - 0.0352 a_6 x_i^{8.0} + 0.00399 a_7 x_i^{7.5} - 0.00665 a_7 x_i^{8.0} - \\
& 0.02704 a_7 x_i^{9.0} + 1.47261 a_1 x_i^{1.0} + 1.2885 a_2 x_i^{2.0} + 3.4697 a_2 x_i^{1.0} +
\end{aligned}$$

$$\begin{aligned}
& 1.3293000 a_0 x_i^{-0.5} - 2.088 x_i^{2.0} a_0 - 8192 t_1 x_i^7 + \\
& 28672.0 t_1 x_i^6 - 39424 t_1 x_i^5 + 26880 t_1 x_i^4 + t_1 - \\
& 9408 t_1 x_i^3 + 1568 t_1 x_i^2 - 98 t_1 x_i + 2.35619 a_0 = \\
& 2 + 12 x_i^2 + 2.256758 x_i^{0.5} - \frac{16}{15} x_i^{5/2} - \frac{256}{315} x_i^{9/2}
\end{aligned} \tag{3.204}$$

where, $x_i = \frac{i}{9}$, $i = 1, 2, \dots, 8$. Hence, for $i = 1$, $x_1 = \frac{1}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned}
& 0.04531223 t_1 + 6.318438653 a_0 - 3.599414825 a_1 + \\
& 1.531420820 a_2 - 0.4285946593 a_3 + 0.1005398 a_4 - \\
& 0.019613065 a_5 + 0.003413361 a_6 - 0.00052956 a_7 = \\
& 2.895970062
\end{aligned} \tag{3.205}$$

For $i = 2$, $x_2 = \frac{2}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned}
& 0.83140085 t_1 + 5.073036520 a_0 - 3.667878440 a_1 + \\
& 2.341159915 a_2 - 0.87498375 a_3 + 0.301158432 a_4 - \\
& 0.079583834 a_5 + 0.02030271 a_6 - 0.004253672 a_7 = \\
& 3.630673217
\end{aligned} \tag{3.206}$$

For $i = 3$, $x_3 = \frac{3}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned}
& 4.426668508 \, a_0 - 0.69090066 \, t_1 - 3.646633078 \, a_1 + \\
& 3.053878492 \, a_2 - 1.304772465 \, a_3 + 0.58880050 \, a_4 - \\
& 0.1747634121 \, a_5 + 0.060631337 \, a_6 - 0.0133227 \, a_7 = \\
& 4.562053930
\end{aligned} \tag{3.207}$$

For $i = 4$, $x_4 = \frac{4}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned}
& 3.937739182 \, a_0 - 0.70284355 \, t_1 - 3.560132008 \, a_1 + \\
& 3.720284279 \, a_2 - 1.693274330 \, a_3 + 0.97387220 \, a_4 - \\
& 0.2884042 \, a_5 + 0.1416852159 \, a_6 - 0.024232095 \, a_7 = \\
& 5.713269383
\end{aligned} \tag{3.208}$$

For $i = 5$, $x_5 = \frac{5}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned}
& 0.70284295 \, t_1 + 3.495212549 \, a_0 - 3.416293798 \, a_1 + \\
& 4.354674550 \, a_2 - 2.018468906 \, a_3 + 1.47675217 \, a_4 - \\
& 0.3884306198 \, a_5 + 0.2973847 \, a_6 - 0.01740269 \, a_7 = \\
& 7.082703733
\end{aligned} \tag{3.209}$$

For $i = 6$, $x_6 = \frac{6}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned} &0.69090166 \tau_1 + 3.056246237 a_0 - 3.218372735 a_1 + \\ &4.960047456 a_2 - 2.259822357 a_3 + 2.12509042 a_4 - \\ &0.423865683 a_5 + 0.58972336 a_6 + 0.06119187 a_7 = \\ &8.657814276 \end{aligned} \quad (3.210)$$

For $i = 7$, $x_7 = \frac{7}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned} &2.600343362 a_0 - 0.83139932 \tau_1 - 2.967863047 a_1 + \\ &5.534416846 a_2 - 2.39794713 a_3 + 2.952518935 a_4 - \\ &0.32152604 a_5 + 1.122426473 a_6 + 0.32827569 a_7 = \\ &10.41817206 \end{aligned} \quad (3.211)$$

For $i = 8$, $x_8 = \frac{8}{9}$, thus equation (3.204) after simplification gives

$$\begin{aligned} &2.116302560 a_0 - 0.04531611 \tau_1 - 2.665506721 a_1 + \\ &6.073086747 a_2 - 2.414528829 a_3 + 3.99776097 a_4 + \\ &0.016989904 a_5 + 2.056677689 a_6 + 1.00312777 a_7 = \\ &12.33622872 \end{aligned} \quad (3.212)$$

For the perturbed collocation method with exponential fitted, an approximation to the conditions are fitted with exponentials to give

$$y_7(0) + \tau_2 e^0 = 0 \quad (3.213)$$

and,

$$y_7^l(0) + \tau_2 e^0 = 0 \quad (3.214)$$

Therefore, equation (3.213) after simplification gives

$$\begin{aligned} & 5.61 \times 10^{-11} a_0 - 1.57 \times 10^{-14} a_1 + 3.301 \times 10^{-18} a_2 - \\ & 5.78 \times 10^{-22} a_3 + 8.86 \times 10^{-26} a_4 - 1.222 \times 10^{-29} a_5 + \\ & 1.5456000000 \times 10^{-33} a_6 - 1.8147 \times 10^{-37} a_7 + \tau_2 = 0 \end{aligned} \quad (3.215)$$

Also, equation (3.214) after simplification gives

$$\begin{aligned} & 0.00084074 a_0 - 0.0003313761 a_1 + 8.25 \times 10^{-11} a_2 - \\ & 1.73 \times 10^{-14} a_3 + 0.0000000014 a_4 - 4.9 \times 10^{-22} a_5 + \\ & 6.955300000 \times 10^{-26} a_6 - 9.0735 \times 10^{-30} a_7 + \tau_2 = 0 \end{aligned} \quad (3.216)$$

Hence, solving equations (3.205) – (2.216) to obtain

$$\begin{aligned}
a_0 &= -0.2415736320e^{-3}, \\
a_1 &= -0.6465085523, \\
a_2 &= 0.2266297359, \\
a_3 &= -0.08214116959e^{-1}, \\
a_4 &= 1.152359280, \\
a_5 &= -3.226612394, \\
a_6 &= 2.537745036, \\
a_7 &= -.7303013072, \\
t_1 &= -0.3887195604e^{-8}, \\
t_2 &= 3.385005618 \times 10^{-15}
\end{aligned} \tag{3.217}$$

Thus, substituting the values of $a_i (i = 0, 1, 2, \dots, 7)$ and t_2 into equation (3.192), that is

$$y_7(x) = \sum_{j=0}^7 a_j Q_j(x) + t_2 e^x$$

After simplification gives

$$\begin{aligned}
y_7(x) &= 0.03983060375000000 x^{5.5} - 0.194841337600 x^{6.5} - \\
&0.00042817810 x^{1.5} + 0.1370746153 x^{7.5} - 0.06918266 x^{7.0} - \\
&0.033893992 x^{8.5} + 0.015487926 x^{8.0} - 0.01536863483 x^{5.0} + \\
&0.069936948 x^{4.5} + 0.3134331195 x^{4.0} - 0.0498612615 x^{2.5} + \\
&1.015533259 x^{2.0} +
\end{aligned}$$

$$0.35978854x^{3.5} - 0.019245964020000x^{3.0} + 0.1095313422x^{6.0} + 3.385005618 \times 10^{-15}e^x \quad (3.218)$$

Thus, equation (3.218) is the approximate solution of equation (3.193) together with conditions (3.194) for case $N = 7$

Problem 3.7.2: Considering a third order singular fractional integro-differential equation of the form

$$y'''(x) + \frac{d^{2.75}y}{dx^{2.75}} + xy(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 6 + 12x^2 + \frac{32}{35}x^{7/2} + 6.619575908x^{0.25} \quad 0 \leq x \leq 1 \quad (3.219)$$

subject to the conditions

$$\begin{aligned} y(0) &= 0, & y'(0) &= 0, \\ y''(0) &= 0 \end{aligned} \quad (3.220)$$

The exact solution is $y(x) = x^3$

The recursive canonical polynomials obtained for equation (3.219) is given as

$$Q_j(x) = \frac{(x^{j+1} - (j+1)(j)(j-1)Q_{j-2})\Gamma(j-a+2)}{x^{1-a}\Gamma(j+2)} \quad j = 0, 1, \dots \quad (3.221)$$

Here, equations (3.219) – (3.220) are solved for case $N = 10$.

Hence, equation (3.20) becomes

$$y_{10}(x) = \sum_{j=0}^{10} a_j Q_j(x) \quad (3.222)$$

From equation (3.221), the following values of $Q_j(x)$; ($j = 0, 1, \dots$) are obtained

$$Q_0(x) = -4.834146544 x^{11/4}$$

$$Q_1(x) = 1.812804954 x^{\frac{15}{4}}$$

$$Q_2(x) = 0.1510670796 x^{\frac{19}{4}} + 2.1908412020000 x^{9/2}$$

$$Q_3(x) = 0.04720846234 x^{\frac{23}{4}} - 0.342318937600 x^{11/2}$$

$$Q_4(x) = 0.02124380806 x^{\frac{27}{4}} - 0.01604620022 x^{13/2} - 0.2327090499 x^{\frac{25}{4}}$$

$$Q_5(x) = 0.01150706270 x^{\frac{31}{4}} - 0.003259384416 x^{15/2} + 0.02363451287 x^{\frac{29}{4}}$$

$$Q_6(x) = 0.006986430925 x^{\frac{35}{4}} - 0.001038928783 x^{17/2} + 0.0007847396858 x^{\frac{33}{4}} + 0.01138063992000 x^8$$

$$Q_7(x) = 0.004584845296 x^{\frac{39}{4}} - 0.0004220648183 x^{19/2} + 0.0001195501865 x^{\frac{37}{4}} - 0.0008668846814000 x^9$$

$$Q_8(x) = 0.003183920343 x^{\frac{43}{4}} - 0.0002001981559 x^{21/2} + 0.00002977079838 x^{\frac{41}{4}} - 0.00002248693785 x^{10} - 0.0003261154587 x^{\frac{39}{4}}$$

$$Q_9(x) = 0.002308342249 x^{\frac{47}{4}} - 0.00010583392100 x^{23/2} + 0.000009742700519 x^{\frac{45}{4}} - 0.0000027596270 x^{11} + 0.00002001066535 x^{\frac{43}{4}}$$

$$\begin{aligned}
Q_{10}(x) = & 0.001731256687 x^{\frac{51}{4}} - 0.000060634010 x^{\frac{25}{2}} + \\
& 0.00000381253 x^{\frac{49}{4}} - 0.0000005669498 x^{\frac{12}{2}} + \quad (3.223) \\
& 0.00000042823727 x^{\frac{47}{4}} + 0.0000062104 x^{23/2}
\end{aligned}$$

Making use of the values of $Q_j(x)$; ($j = 0, 1, \dots, 10$) in equation (3.223), gives

$$\begin{aligned}
y_{10}(x) = & 1.8128 x^{\frac{15}{4}} a_1 - 4.834 x^{11/4} a_0 + (0.15106 x^{\frac{19}{4}} + \\
& 2.19084 x^{9/2}) a_2 + (0.047208 x^{\frac{23}{4}} - 0.3423180 x^{11/2}) a + \\
& (0.02124381 x^{\frac{27}{4}} - 0.0160462 x^{13/2} - 0.232709 x^{\frac{25}{4}}) a_4 + \\
& (0.011507 x^{\frac{31}{4}} - 0.00325938 x^{15/2} + 0.0236340 x^{\frac{29}{4}}) a_5 + \\
& (0.00698640 x^{\frac{35}{4}} - 0.00103892 x^{17/2} + 0.00078470 x^{\frac{33}{4}} + \\
& 0.011380 x^8) a_6 + (0.00458480 x^{\frac{39}{4}} - 0.00042206 x^{19/2} + \\
& 0.000119550 x^{\frac{37}{4}} - 0.0008668 x^9) a + (0.0031839 x^{\frac{43}{4}} - \\
& 0.0002001982 x^{21/2} + 0.00002977 x^{\frac{41}{4}} - 0.0000220 x^{10} - \\
& 0.0003261 x^{\frac{39}{4}}) a_8 + (0.0023080 x^{47/4} - 0.0001058 x^{23/2} + \\
& 0.000009742 x^{\frac{45}{4}} - 0.000002759 x^{11} + 0.000020 x^{\frac{43}{4}}) a + \\
& (0.001731256 x^{\frac{51}{4}} - 0.000060634 x^{\frac{25}{2}} + 0.0000038 x^{\frac{49}{4}} - \\
& 0.00000056 x^{12} + 0.00000042 x^{\frac{47}{4}} + 0.0000062 x^{23/2}) a_{10}
\end{aligned} \quad (3.224)$$

Thus, equation (3.224) is substituted into a slightly perturbed equation (3.219) to get

$$y_{10}^{(3)}(x) + \frac{d^{2.75}y_{10}}{dx^{2.75}} + xy_{10}(x) + \int_0^x \frac{y_{10}(t)}{\sqrt{x-t}} dt = 6.000 + 12x^2 + \frac{32}{35}x^{7/2} + 6.619575908x^{0.25} + \quad (3.225)$$

$$H_{10}(x)$$

together with the equation (3.220) to get

$$y_{10}(0) = 0, \quad y_{10}'(0) = 0, \quad y_{10}''(0) = 0$$

where,

$$H_{10}(x) = \tau_1(1 - 200x + 6600x^2 - 84480.00000x^3 + 549120x^4 - 2050048x^5 + 4659200x^6 - 6553600x^7 + 5570560x^8 - 2621440x^9 + 524288x^{10}) + \tau_2(-1.00 + 162x - 4320x^2 + 44352x^3 - 228096x^4 + 658944x^5 - 1118208x^6 + 1105920x^7 - 589824x^8 + 131072.0x^9) \quad (3.226)$$

Hence, evaluating the terms $\frac{d^3y_{10}(x)}{dx^3}$, $\frac{d^{2.75}y_{10}(x)}{dx^{2.75}}$, $xy_{10}(x)$, and $\int_0^x \frac{y_{10}(t)}{\sqrt{x-t}} dt$ in equation (3.225) gives

$$\begin{aligned}
\frac{d^{2.75}y_{10}}{dx^{2.75}} = & -21.381368090 a_0 + 30.0674 x a_1 + \\
& (5.950860833 x^2 + 71.299176280000 x^{7/4}) a_2 + \\
& (3.564310647 x^3 - 22.280996040000 x^{11/4}) a_3 + \\
& (2.706659086 x^4 - 23.115007440000000 x^{7/2} - \\
& 1.8103298940000 x^{\frac{15}{4}}) a_4 + (3.782287766 x^{9/2} - \\
& 0.5806167936 x^{\frac{19}{4}} + 2.27246455300000 x^5) a_5 + \\
& (0.1883749265 x^{11/2} + 2.48223063900000 x^{\frac{21}{4}} + \\
& 2.012078018 x^6 - 0.273583762500000 x^{\frac{23}{4}}) a_6 + \\
& (1.839166421 x^7 - 0.27226950080000000 x^{\frac{25}{4}} + \\
& 0.04083919774 x^{13/2} - 0.15642416430 x^{\frac{27}{4}}) a_7 + \\
& (-0.130817900000 x^7 + 1.7162353800000 x^8 - \\
& 0.0097415800000 x^{\frac{29}{4}} + 0.013898840000 x^{15/2} - \\
& 0.1005246039 x^{\frac{31}{4}}) a_8 + (0.006020062000 x^{17/2} - \\
& 0.06984369703 x^{\frac{35}{4}} + 1.6244632350000000 x^9 + \\
& 0.0107863700000 x^8 - 0.00159400000 x^{\frac{33}{4}}) a_9 + \\
& (0.003037723384 x^{19/2} - 0.051300610020 x^{\frac{39}{4}} + \\
& 0.004098510 x^{\frac{35}{4}} - 0.00042483600000000 x^{\frac{37}{4}} + \\
& 0.0003013672258000 x^9 + 1.553393813 x^{10}) a_{10}
\end{aligned} \tag{3.227}$$

$$\begin{aligned}
\frac{d^3 y_{10}}{dx^3} = & -17.44824769 \frac{a_0}{x^{3/4}} + 32.71546441 x^{3/4} a_1 + \\
& (7.399926 x^{7/4} + 86.26437 x^{3/2}) a_2 + (4.83517 x^{11/4} - \\
& 29.65338 x^{5/2}) a_3 + (3.916495 x^{15/4} - 2.5814324 x^{7/2} - \\
& 32.45200422 x^{13/4}) a_4 + (3.4612 x^{19/4} - 0.8739220 x^{9/2} + \\
& 5.622429036 x^{17/4}) a_5 + (3.197929 x^{23/4} - 0.4305 x^{11/2} + \\
& 0.2933577 x^{21/4} + 3.823895 x^5) a_6 + (3.03137000 x^{27/4} - \\
& 0.255613 x^{13/2} + 0.0661430 x^{25/4} - 0.43690987 x^6) a_7 + \quad (3.228) \\
& (2.9200030 x^{31/4} - 0.16974301 x^{15/2} + 0.02328680 x^{29/4} - \\
& 0.0161905 x^7 - 0.215618 x^{27/4}) a_8 + (2.842831684 x^{35/4} - \\
& 0.1214047367 x^{17/2} + 0.0103914 x^{33/4} - 0.00273203 x^8 + \\
& 0.01835197 x^{31/4}) a_9 + (2.78816 x^{39/4} - 0.0915194 x^{19/2} + \\
& 0.005385508 x^{37/4} - 0.00074737776 x^9 + 0.000527 x^{35/4} + \\
& 0.0071242098 x^{17/2}) a_{10}
\end{aligned}$$

and,

$$\begin{aligned}
& \int_0^x \frac{y_{10}(t)}{(x-t)} dt = -4.57418 x^{\frac{13}{4}} a_0 + 1.513520 x^{\frac{17}{4}} a_1 + \\
& (1.6938 x^5 + 0.114114 x^{\frac{21}{4}}) a_2 + (-0.24260100000 x^6 + \\
& 0.0328079 x^{\frac{25}{4}}) a_3 + (-0.1558480 x^{\frac{27}{4}} - 0.0105596 x^7 + \\
& 0.0137454 x^{\frac{29}{4}}) a_4 + (0.0148071 x^{\frac{31}{4}} - 0.002010870 x^8 + \\
& 0.00699417 x^{\frac{33}{4}}) a_5 + (0.006817 x^{17/2} + 0.0004635 x^{\frac{35}{4}} - \\
& 0.00060535 x^9 + 0.004016 x^{\frac{37}{4}}) a_6 + (-0.000492 x^{19/2} + \\
& 0.000066997 x^{\frac{39}{4}} - 0.0002337 x^{10} + 0.0025075 x^{\frac{41}{4}}) a_7 + (3.229) \\
& (-0.00017835 x^{\frac{41}{4}} - 0.0000121 x^{21/2} + 0.0000159 x^{\frac{43}{4}} - \\
& 0.00010578 x^{11} + 0.0016639 x^{\frac{45}{4}}) a_8 + (0.00001045 x^{\frac{45}{4}} - \\
& 0.00000142678 x^{23/2} + 0.00000498 x^{\frac{47}{4}} - 0.000053 x^{12} + \\
& 0.00115712 x^{\frac{49}{4}}) a_9 + (0.0000031 x^{12} + 0.00000021 x^{\frac{49}{4}} - \\
& 0.000000282 x^{\frac{25}{4}} + 0.000001874 x^{\frac{51}{4}} - 0.00002952 x^{13} + \\
& 0.000835088 x^{\frac{53}{4}}) a_{10}
\end{aligned}$$

Hence, substituting equations (3.226) – (3.229) into (3.225) gives

$$\begin{aligned}
& 78.69910276 a_2 x^{7/4} - 0.00010578 a_8 x^{11} + 86.264372 a_2 x^{3/2} + 1.6244632350 a_9 x^9 - \\
& 21.38136809 a_0 + 0.0000049844 a_9 x^{\frac{47}{4}} + 0.000463549 a_6 x^{\frac{35}{4}} - 0.000053590 a_9 x^{12} + \\
& 0.04693724 a_6 x^{\frac{29}{4}} - 0.06623170992 a_6 x^{15/2} - 0.242601 a_3 x^6 + 0.032807900 a_3 x^{\frac{25}{4}} -
\end{aligned}$$

$$\begin{aligned}
& 0.000178357 a_8 x^{\frac{41}{4}} + 2.908365319 a_5 x^{9/2} + 0.00250751 a_7 x^{\frac{41}{4}} - 0.115384 a_9 x^{17/2} - \\
& 0.0003035 a_9 x^{10} + 0.0011 a_9 x^{\frac{49}{4}} + 0.00000314476 a_{10} x^{12} - 0.00007483730 a_{10} x^{11} - \\
& 0.000491 a_7 x^{19/2} + 0.004016 a_6 x^{\frac{37}{4}} - 0.000000281 a_{10} x^{\frac{25}{4}} + 0.0000002146 a_{10} x^{\frac{49}{4}} - \\
& 0.06241569706 a_7 x^8 + 0.001123455153 a_9 x^{\frac{41}{4}} - 7.6357657 a_4 x^{\frac{21}{4}} - 32.4520 a_4 x^{\frac{13}{4}} + \\
& 0.00496067 a_{10} x^{\frac{37}{4}} + 5.950860830 a_2 x^2 - 0.0000121 a_8 x^{21/2} + 1.070938864 a_5 x^{\frac{25}{4}} + \\
& 0.4737673 a_6 x^{\frac{31}{4}} + 0.00879795 a_9 x^{\frac{33}{4}} + 0.2915724804 a_9 x^{\frac{43}{4}} - 0.206126492 a_7 x^{\frac{25}{4}} - \\
& 0.2147738 a_7 x^{13/2} + 3.823895013 a_6 x^5 + 2.874946598 a_7 x^{\frac{27}{4}} - 0.436909879 a_7 x^6 - \\
& 0.0000295 a_{10} x^{13} + 0.0000540 a_{10} x^{\frac{43}{4}} + 0.000001874 a_{10} x^{\frac{51}{4}} + 0.002822640 a_8 x^{\frac{37}{4}} - \\
& 0.1588949903 a_5 x^{13/2} + 0.6019632175 a_5 x^{\frac{27}{4}} - 0.000447 a_{10} x^9 + 0.004625 a_{10} x^{\frac{35}{4}} - \\
& 29.65337797 a_3 x^{5/2} + 2.106165282 a_4 x^{\frac{15}{4}} - 0.0278217 a_8 x^{\frac{35}{4}} + 0.00209736 a_9 x^{\frac{39}{4}} + \\
& 0.00912317 a_7 x^{\frac{33}{4}} - 0.03408173 a_7 x^{17/2} + 2.775588 a_6 x^{\frac{21}{4}} - 0.088481750 a_{10} x^{19/2} - \\
& 17.44824769000 \sqrt[4]{a_0} - 0.00201087 a_5 x^8 - 4.57418 x^{\frac{13}{4}} a_0 - 0.21561836930 a_8 x^{\frac{27}{4}} + \\
& 3.564310647 a_3 x^3 + 0.2593638924 a_{10} x^{\frac{47}{4}} - 8.47239370 a_3 x^{9/2} - 0.1470085 a_8 x^7 + \\
& 1.289381128 a_3 x^{\frac{19}{4}} - 0.019969766 a_8 x^{19/2} + 0.0148071 a_5 x^{\frac{31}{4}} + 0.00006699 a_7 x^{\frac{39}{4}} - \\
& t_1 + 6553600 t_1 x^7 - 5570560 t_1 x^8 + 2621440 t_1 x^9 - 549120 t_1 x^4 - 524288 t_1 x^{10} + \\
& 84480 t_1 x^3 - 6600 t_1 x^2 + 2050048 t_1 x^5 - 4659200 t_1 x^6 + 200 t_1 x - 162.000 t_2 x + \\
& 589824 t_2 x^8 - 131072 t_2 x^9 + 228096 t_2 x^4 - 44352 t_2 x^3 + 4320 t_2 x^2 + 30.06 x a_1 + \\
& 1118208 t_2 x^6 - 1105920 t_2 x^7 + t_2 + 1.5135 x^{\frac{17}{4}} a_1 - 0.1558 a_4 x^{\frac{27}{4}} - 658944 t_2 x^5 +
\end{aligned}$$

$$\begin{aligned}
& 0.000525415 a_{10} x^{\frac{45}{4}} + 0.007124 a_{10} x^{17/2} - 0.0020 a_8 x^9 - \\
& 0.000605356 a_6 x^9 + 2.27246455 a_5 x^5 + 0.008054 a_9 x^8 + \\
& 0.39114461 a_7 x^{\frac{35}{4}} + 0.00166394 a_8 x^{\frac{45}{4}} + 2.77298 a_9 x^{\frac{35}{4}} + \\
& 0.00681 a_6 x^{17/2} - 17.4458168 a_3 x^{11/4} + 0.82452 a_4 x^{\frac{23}{4}} + \\
& 0.000835 a_{10} x^{\frac{53}{4}} + 1.55339 a_{10} x^{10} + 0.000749 a_{10} x^{21/2} + \\
& 0.63731583 a_6 x^7 + 2.706659 a_4 x^4 + 2.012078018 a_6 x^6 + \\
& 0.00699417 a_5 x^{\frac{33}{4}} - 0.01277944 a_9 x^{21/2} + 1.6938 a_2 x^5 + \\
& 0.114114 a_2 x^{\frac{21}{4}} + 1.839166421 a_7 x^7 + 1.7162353 a_8 x^8 + \\
& 0.0135 a_8 x^{\frac{29}{4}} + 0.0000104 a_9 x^{\frac{45}{4}} - 0.00000142 a_9 x^{23/2} + \\
& 2.69 a_2 x^{\frac{15}{4}} + 34.50574893 a_2 x^{7/2} - 25.6964399 a_4 x^{7/2} + \\
& 0.0183519 a_9 x^{\frac{31}{4}} + 2.73686 a_{10} x^{\frac{39}{4}} = 6.61957590 x^{0.25} + \\
& 6 + 12 x^2 + \frac{32}{35} x^{7/2}
\end{aligned} \tag{3.230}$$

Collocating equation (3.230) at the point $x = x_i$ gives

$$\begin{aligned}
& 5.95086083 a_2 x_i^2 - 0.0000121546 a_8 x_i^{21/2} + 1.0709388 a_5 x_i^{\frac{25}{4}} + 2.819478 a_8 x_i^{\frac{31}{4}} - \\
& 0.155844 a_8 x_i^{15/2} + 2.924345830 a_6 x_i^{\frac{23}{4}} + 78.69910 a_2 x_i^{7/4} + 86.2643723 a_2 x_i^{3/2} + \\
& 1.62446 a_9 x_i^9 - 549120 t_1 x_i^4 - 524288 t_1 x_i^{10} + 84480 t_1 x_i^3 + 0.00250751 a_7 x_i^{\frac{41}{4}} - \\
& 0.115384 a_9 x_i^{17/2} - 0.000303559 a_9 x_i^{10} + 0.0011571 a_9 x_i^{\frac{49}{4}} + 0.00000314 a_{10} x_i^{12} - \\
& 0.00007483737776 a_{10} x_i^{11} - 6600.000000 t_1 x_i^2 + 2050048 t_1 x_i^5 - 4659200 t_1 x_i^6 + \\
& 2.874946 a_7 x_i^{\frac{27}{4}} - 0.436909879 a_7 x_i^6 - 0.0000295219 a_{10} x_i^{13} - 0.2156183 a_8 x_i^{\frac{27}{4}} +
\end{aligned}$$

$$\begin{aligned}
& 3.564310647 a_3 x_i^3 + 0.25936389 a_{10} x_i^{\frac{47}{4}} - 7.6357657 a_4 x_i^{\frac{21}{4}} - 32.4520040 a_4 x_i^{\frac{13}{4}} + \\
& 0.0049606721790 a_{10} x_i^{\frac{37}{4}} + 0.00005409172102 a_{10} x_i^{\frac{43}{4}} + 0.0000018740300 a_{10} x_i^{\frac{51}{4}} + \\
& 0.002822640 a_8 x_i^{\frac{37}{4}} - 0.01996976 a_8 x_i^{19/2} + 0.014807 a_5 x_i^{\frac{31}{4}} + 0.00006690 a_7 x_i^{\frac{39}{4}} + \\
& 0.03280790 a_3 x_i^{\frac{25}{4}} - 0.000178357 a_8 x_i^{\frac{41}{4}} + 2.9083653 a_5 x_i^{9/2} + 0.00000490 a_9 x_i^{\frac{47}{4}} + \\
& 0.0004635490 a_6 x_i^{\frac{35}{4}} - 0.00005359 a_9 x_i^{12} - 21.38136809 a_0 + 2.272464553 a_5 x_i^5 + \\
& 0.0080543471010 a_9 x_i^8 + 0.39114461 a_7 x_i^{\frac{35}{4}} + 0.00166 a_8 x_i^{\frac{45}{4}} + 2.7729870 a_9 x_i^{\frac{35}{4}} + \\
& 0.006817880 a_6 x_i^{17/2} - 17.44581681 a_3 x_i^{11/4} + 0.82452 a_4 x_i^{\frac{23}{4}} + 0.000835 a_{10} x_i^{\frac{53}{4}} + \\
& 1.553390 a_{10} x_i^{10} + 0.0007499 a_{10} x_i^{21/2} + 0.6373158355 a_6 x_i^7 + 2.70665908 a_4 x_i^4 + \\
& 2.012078 a_6 x_i^6 + 0.0069941 a_5 x_i^{\frac{33}{4}} - 0.01277944597 a_9 x_i^{21/2} + 1.693800001 a_2 x_i^5 + \\
& 0.114114 a_2 x_i^{\frac{21}{4}} + 1.839166421 a_7 x_i^7 + 1.716235380 a_8 x_i^8 + 0.0135452221 a_8 x_i^{\frac{29}{4}} + \\
& 0.0000104577 a_9 x_i^{\frac{45}{4}} - 0.00000142 a_9 x_i^{23/2} + 2.690882 a_2 x_i^{\frac{15}{4}} + 34.505742 a_2 x_i^{7/2} - \\
& 25.6964 a_4 x_i^{7/2} + 0.018351 a_9 x_i^{\frac{31}{4}} + 2.73686 a_{10} x_i^{\frac{39}{4}} - \tau_1 + \tau_2 - 8.472393 a_3 x_i^{9/2} - \\
& 0.147008585 a_8 x_i^7 + 1.2893811 a_3 x_i^{\frac{19}{4}} - 0.155848 a_4 x_i^{\frac{27}{4}} + 5.62242903600 a_5 x_i^{\frac{17}{4}} + \\
& 0.0000159 a_8 x_i^{\frac{43}{4}} + 0.00000021466 a_{10} x_i^{\frac{49}{4}} - 0.06241569 a_7 x_i^8 + 0.001123 a_9 x_i^{\frac{41}{4}} + \\
& 0.0071242 a_{10} x_i^{17/2} - 0.0020238 a_8 x_i^9 - 0.0006053 a_6 x_i^9 - 0.000491997 a_7 x_i^{19/2} + \\
& 0.004016 a_6 x_i^{\frac{37}{4}} - 0.00000028 a_{10} x_i^{\frac{25}{2}} - 0.1588949 a_5 x_i^{13/2} + 0.601963210 a_5 x_i^{\frac{27}{4}} - \\
& 0.000447 a_{10} x_i^9 + 2.8806717 a_5 x_i^{\frac{19}{4}} - 0.000233629 a_7 x_i^{10} - 23.264330251 x_i^{7/4} a_0 - \\
& 0.2061264922 a_7 x_i^{\frac{25}{4}} - 0.214779 a_7 x_i^{13/2} + 3.823895013000 a_6 x_i^5 - 131072 \tau_2 x_i^9 + \\
& 228096 \tau_2 x_i^4 - 44352 \tau_2 x_i^3 - 0.02782172507 a_8 x_i^{\frac{35}{4}} + 0.002097367862000 a_9 x_i^{\frac{39}{4}} +
\end{aligned}$$

$$\begin{aligned}
& 0.00912 a_7 x_i^{\frac{33}{4}} + 0.3337 a_8 x_i^{\frac{39}{4}} - 0.00870 a_{10} x_i^{23/2} + \\
& 0.0005254150 a_{10} x_i^{\frac{45}{4}} + 4320 t_2 x_i^2 - 658940 t_2 x_i^5 + \\
& 1118208 t_2 x_i^6 + 0.04693 a_6 x_i^{\frac{29}{4}} - 0.06623 a_6 x_i^{15/2} - \\
& 0.2426 a_3 x_i^6 - 0.0001057 a_8 x_i^{11} - 17.4482476 a_0 - \sqrt[4]{x_i^{\frac{35}{4}}} \\
& 0.002010 a_5 x_i^8 - 4.57418 x_i^{\frac{13}{4}} a_0 + 0.004625 a_{10} x_i^{\frac{35}{4}} - \\
& 29.6533 a_3 x_i^{5/2} + 2.106165 a_4 x_i^{\frac{15}{4}} + 0.47376 a_6 x_i^{\frac{31}{4}} + \\
& 0.00879795 a_9 x_i^{\frac{33}{4}} + 0.291572480 a_9 x_i^{\frac{43}{4}} - 162 t_2 x_i + \quad (3.231) \\
& 1.51352 x_i^{\frac{17}{4}} a_1 + 30.06749 x_i a_1 + 18.69455 x_i^{11/4} a_1 - \\
& 0.5736 a_4 x_i^{11/2} + 0.0137 a_4 x_i^{\frac{29}{4}} - 0.034081 a_7 x_i^{17/2} + \\
& 2.77558 a_6 x_i^{\frac{21}{4}} - 0.0884817463 a_{10} x_i^{19/2} + 200 t_1 x_i - \\
& 1105920 t_2 x_i^7 + 589824 t_2 x_i^8 - 0.24213118 a_6 x_i^{11/2} + \\
& 32.715464 x_i^{3/4} a_1 - 0.010559 a_4 x_i^7 + 6553600 t_1 x_i^7 - \\
& 5570560 t_1 x_i^8 + 2621440 t_1 x_i^9 = 6.6195759080 x_i^{0.25} \\
& + 6 + 12 x_i^2 + \frac{32}{35} x_i^{7/2}
\end{aligned}$$

where, $x_i = \frac{i}{12}$, $i = 1, 2, \dots, 11$. Hence, for $i = 1$, $x_1 = \frac{1}{12}$, thus equation (3.231) after simplification gives

$$\begin{aligned}
& 7.599994338 \alpha_1 - 54.15833153 \alpha_0 + 3.1397279320 \alpha_2 - \\
& 0.07628217977 \alpha_3 - 0.01408011607 \alpha_4 + 0.000217 \alpha_5 + \\
& 0.00002359624551 \alpha_6 - 0.00000000352796242100 \alpha_7 - \\
& 0.0000000001604993745 \alpha_8 + 0.000000001347526 \alpha_9 + \\
& 0.0000000001093429324 \alpha_{10} - 0.91048958300000 \tau_1 + \\
& 0.530152933 \tau_2 = 9.640084687
\end{aligned} \tag{3.232}$$

For $i = 2$, $x_2 = \frac{2}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
& 13.68119607 \alpha_1 - 49.71428903 \alpha_0 + 9.5249525000 \alpha_2 - \\
& 0.4485970263 \alpha_3 - 0.1405613784 \alpha_4 + 0.00457677 \alpha_5 + \\
& 0.0008510642799 \alpha_6 + 0.000008595510857000000 \alpha_7 + \\
& 0.000001725505199 \alpha_8 + 0.0000005902743181000 \alpha_9 + \\
& 0.00000009584872551 \alpha_{10} + 0.52839171000000000 \tau_1 + \\
& 0.28054676 \tau_2 = 10.56459375
\end{aligned} \tag{3.233}$$

For $i = 3$, $x_3 = \frac{3}{12}$, thus equation (3.231) after simplification gives

$$\begin{aligned}
& 19.50081225 a_1 - 48.16375035 a_0 + 18.397233000 a_2 - \\
& 1.271297924 a_3 - 0.5423745804 a_4 + 0.027625728 a_5 + \\
& 0.00708278220 a_6 + 0.000192937 a_7 + 0.00005553 a_8 + \quad (3.234) \\
& 0.00002098563457 a_9 + 0.0000051161323 a_{10} + 0.5 t_1 - \\
& 1.0 t_2 = 11.43788987
\end{aligned}$$

For $i = 4$, $x_4 = \frac{4}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
& - 47.87523370 a_0 + 25.29992383 a_1 + 29.5598943 a_2 - \\
& 2.674328179 a_3 - 1.418950479 a_4 + 0.0997818141 a_5 + \\
& 0.03226532063 a_6 + 0.00160145 a_7 + 0.000598736 a_8 + \quad (3.235) \\
& 0.0002658160178 a_9 + 0.000086439 a_{10} - 0.967216 t_1 + \\
& 0.082966 t_2 = 12.38267382
\end{aligned}$$

For $i = 5$, $x_5 = \frac{5}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
& 31.21456085 a_1 - 48.3915907 a_0 + 42.9750589700 a_2 - \\
& 4.781861491 a_3 - 3.002078810 a_4 + 0.2719245521 a_5 + \\
& 0.1055605641 a_6 + 0.00804807 a_7 + 0.0036936459 a_8 + \quad (3.236) \\
& 0.001911282400 a_9 + 0.00077693029 a_{10} - 0.10351 t_1 + \\
& 0.997968 t_2 = 13.44438036
\end{aligned}$$

For $i = 6$, $x_6 = \frac{6}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
 & 37.34498419 a_1 - 49.52827661 a_0 + 58.6899968 a_2 - \\
 & 7.71969662 a_3 - 5.5561028610 a_4 + 0.62008551 a_5 + \\
 & 0.2800079 a_6 + 0.02977249316 a_7 + 0.01617431 a_8 + \quad (3.237) \\
 & 0.009604194184 a_9 + 0.004684000774 a_{10} + 1.0 t_1 = \\
 & 14.64718985
 \end{aligned}$$

For $i = 7$, $x_7 = \frac{7}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
 & 43.77547287 a_1 - 51.19827589 a_0 + 76.80950107 a_2 - \\
 & 11.61963363 a_3 - 9.379947372 a_4 + 1.250482833 a_5 + \\
 & 0.6425383401 a_6 + 0.08953129174 a_7 + 0.056130 a_8 + \quad (3.238) \\
 & 0.03768992054 a_9 + 0.02143747962 a_{10} - 0.1035 t_1 - \\
 & 0.99798 t_2 = 16.00702205
 \end{aligned}$$

For $i = 8$, $x_8 = \frac{8}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
 & 50.58226851 a_1 - 53.35845314 a_0 + 97.48270479 a_2 - \\
 & 16.62256999 a_3 - 14.81020583 a_4 + 2.305049679 a_5 + \\
 & 1.325948508 a_6 + 0.2318393382 a_7 + 0.16461822 a_8 + \quad (3.239) \\
 & 0.1234338108 a_9 + 0.08019838432 a_{10} - 0.96730 t_1 - \\
 & 0.08295 t_2 = 17.53598387
 \end{aligned}$$

For $i = 9$, $x_9 = \frac{9}{12}$ thus equation (3.231) after simplification gives

$$\begin{aligned}
& 57.83744433 a_1 - 55.98857032 a_0 + 120.8959165 a_2 - \\
& 22.88093839 a_3 - 22.22494651 a_4 + 3.967570547 a_5 + \\
& 2.523084953 a_6 + 0.5361660402 a_7 + 0.42499566 a_8 + \quad (3.240) \\
& 0.3521150691 a_9 + 0.2572266103 a_{10} + 0.5003000 t_1 + \\
& t_2 = 19.24424787
\end{aligned}$$

For $i = 10$, $x_{10} = \frac{10}{12}$, thus equation (3.231) after simplification gives

$$\begin{aligned}
& 65.61105013 a_1 - 59.08167717 a_0 + 147.2684187 a_2 - \\
& 30.56076604 a_3 - 32.04805993 a_4 + 6.470521465 a_5 + \\
& 4.503519987 a_6 + 1.134994585 a_7 + 0.992877797 a_8 + \quad (3.241) \\
& 0.9009035999 a_9 + 0.7307653923 a_{10} + 0.5290000 t_1 - \\
& 0.2806 t_2 = 21.14095991
\end{aligned}$$

For $i = 11$, $x_{11} = \frac{11}{12}$, thus equation (3.231) after simplification gives

$$\begin{aligned}
& 73.97241974 a_1 - 62.63918643 a_0 + 176.8499026 a_2 - \\
& 39.84350124 a_3 - 44.75404549 a_4 + 10.10270038 a_5 + \\
& 7.633035199 a_6 + 2.237772885 a_7 + 2.140355365 a_8 + \quad (3.242) \\
& 2.110940145 a_9 + 1.882122964 a_{10} - 0.910000000 t_1 - \\
& 0.5302 t_2 = 23.23472107
\end{aligned}$$

For the perturbed collocation method with exponential fitted, an

approximation to the conditions are fitted with exponentials to give

$$y_{10}(0) + \tau_3 e^0 = 0 \quad (3.243)$$

$$y_{10}^j(0) + \tau_3 e^0 = 0 \quad (3.244)$$

and,

$$y_{10}^{jj}(0) + \tau_3 e^0 = 0 \quad (3.245)$$

Therefore, equation (3.243) after simplification gives

$$\begin{aligned} & 3.2 \times 10^{-19} a_1 - 8.5 \times 10^{-14} a_0 + 6.95 \times 10^{-23} a_2 - \\ & 0.14651929 a_3 - 1.31 \times 10^{-32} a_4 + 1.32 \times 10^{-38} a_5 + \\ & 1.14 \times 10^{-42} a_6 - 8.6 \times 10^{-49} a_7 - 5.82 \times 10^{-53} a_8 + \\ & 3.53 \times 10^{-59} a_9 + 1.97 \times 10^{-63} a_{10} + \tau_3 = 0 \end{aligned} \quad (3.246)$$

Also, equation (3.244) after simplification gives

$$\begin{aligned} & 1.21 \times 10^{-13} a_1 - 0.0000000236 a_0 + 3.13 \times 10^{-17} a_2 - \\ & 5.90 \times 10^{-23} a_3 - 8.20 \times 10^{-27} a_4 + 9.57 \times 10^{-33} a_5 + \\ & 9.13 \times 10^{-37} a_6 - 7.74 \times 10^{-43} a_7 - 5.67 \times 10^{-47} a_8 + \\ & 3.80 \times 10^{-53} a_9 + 2.26 \times 10^{-57} a_{10} + \tau_3 = 0 \end{aligned} \quad (3.247)$$

And, equation (3.246) after simplification gives

$$\begin{aligned}
& 0.00000003324413528 a_1 - 0.004137047947000 a_0 + \\
& 1.09 \times 10^{-11} a_2 - 2.65 \times 10^{-17} a_3 - 4.31 \times 10^{-21} a_4 + \\
& 5.98 \times 10^{-27} a_5 + 6.39 \times 10^{-31} a_6 - 6.19 \times 10^{-37} a_7 - \\
& 4.96 \times 10^{-41} a_8 + 3.70 \times 10^{-47} a_9 + 2.38 \times 10^{-51} a_{10} + \\
& \tau_3 = 0
\end{aligned} \tag{3.248}$$

Hence, solving equations (3.232) – (3.248) to obtain

$$\begin{aligned}
a_0 &= 1.08496571 \times 10^{-9}, \\
a_1 &= 4.951677395e^{-4}, \\
a_2 &= -1.094279913, \\
a_3 &= -750.7010332, \\
a_4 &= 4231.532015, \\
a_5 &= 1.028480062 \times 10^5, \\
a_6 &= -2.324134876 \times 10^5, \\
a_7 &= 1.28130362 \times 10^6, \\
a_8 &= -1.710670955 \times 10^6, \\
a_9 &= 1.01538687 \times 10^6, \\
a_{10} &= -2.415284433 \times 10^5, \\
\tau_1 &= -1.757257792 \times 10^{-5}, \\
\tau_2 &= -5.752181387 \times 10^{-8} \\
\tau_3 &= 9.668287931 \times 10^{-24}
\end{aligned} \tag{3.249}$$

Thus, substituting the values of $\alpha_i (i = 0, 1, 2, \dots, 10)$, and τ_3 into equation (3.192), that is

$$y_{10}(x) = \sum_{j=0}^{10} a_j Q_j(x) + \tau_3 e^x$$

and after simplification gives

$$\begin{aligned} y_{10}(x) = & 153.1800869000 x^{\frac{37}{4}} - 0.00000000524400000000 x^{11/4} - \\ & 108.9623831 x^{23/2} + 256.9791801 x^{11/2} + 6432.455122 x^{\frac{39}{4}} + \\ & 342.4731705 x^{21/2} + 9.66 \times 10^{-24} e^x - 1110.742481000 x^9 - \\ & 0.1653096707 x^{\frac{19}{4}} - 35.43944145 x^{\frac{23}{4}} + 2430.76252600 x^{\frac{29}{4}} - \\ & 418.14773260 x^{\frac{51}{4}} - 984.7157948 x^{\frac{25}{4}} - 2.3973935200 x^{9/2} + \\ & 241.4610618 x^{17/2} - 2.802089507 x^{11} - 67.90000995 x^{13/2} - \\ & 5426.321487 x^{\frac{43}{4}} + 38.46775145 x^{10} - 2645.0142150000 x^8 + \\ & 14.6448397900 x^{\frac{25}{2}} + 2343.756992 x^{\frac{47}{4}} - 540.7931800 x^{19/2} + \\ & 1183.478456 x^{\frac{31}{4}} + 9.89261023400 x^{\frac{45}{4}} + 0.1369345102 x^{12} + \\ & 0.0008976425312 x^{\frac{15}{4}} - 50.92804010 x^{\frac{41}{4}} - 335.22110 x^{15/2} - \\ & 182.3840872 x^{\frac{33}{4}} - 1623.74077 x^{\frac{35}{4}} + 89.8938539 x^{\frac{27}{4}} - \\ & 0.9208364546 x^{\frac{49}{4}} \end{aligned} \quad (3.250)$$

Thus, equation (3.250) is the approximate solution of equation (3.219) together with conditions (3.220) for case $N = 10$

3.8 Fourth Order Singular Fractional Integro-Differential Equation

Here, in order to test the efficiency of the method on higher order, we consider a fourth order singular fractional integro differential equation of the form

Problem 3.8.1 :

$$\frac{d^{3.5}y}{dx^{3.5}} + y + \frac{d^3y}{dx^3} + \left(\frac{1}{x^2}\right) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + \int_0^x \frac{y(t)}{\sqrt{t}} dt = 27.08110001 x^{0.5} + 26 + 24x - 5x - \frac{315}{315} x^{9/2} \quad 0 \leq x \leq 1 \quad (3.251)$$

subject to conditions

$$\begin{aligned} y(0) &= 0 & y'(0) &= 0 \\ y''(0) &= 0 & y'''(0) &= 0 \end{aligned} \quad (3.252)$$

The exact solution is given as $y(x) = x^4$

The recursive canonical polynomials obtained for equation (3.251)

is

$$Q_j(x) = \frac{x^{j+1} - (j+1)j(j-1)(j-2)Q_{j-3} - (j+1)j(j-1) \sum_{k=0}^{j-2} Q_{j-2-k} \Gamma(j-a+2)}{x^{1-a} \Gamma(j+2)} \quad (3.253)$$

Here, equations (3.251) – (3.252) are solved for case $N = 13$.

Hence, equation (3.20) becomes

$$y_{13}(x) = \sum_{j=0}^{13} \alpha_j Q_j(x) \quad (3.254)$$

From equation (3.253), the following values of $Q_j(x)$; ($j = 0, 1, \dots, 13$) are obtained

$$Q_0(x) = 2.363271801 x^{7/2}$$

$$Q_1(x) = -1.772453851 x^{9/2}$$

$$Q_2(x) = 0.2954089752 x^{11/2} - 4.188790206 x^6$$

$$Q_3(x) = 0.0369261218900 x^{13/2} - 2.094395101 x^6 + 1.570796327 x^7$$

$$Q_4(x) = 0.01107783657 x^{15/2} + 2.356194490 x^7 - 0.1963495409 x^8 + 2.784164001 x^{17/2}$$

$$Q_5(x) = 8.125094094 x^{17/2} - 0.4908738523 x^8 - 0.02045307717 x^9 - 0.8700512495 x^{19/2}$$

$$Q_6(x) = 4.060239162 x^{17/2} - 4.184813758 x^{19/2} - 0.0715857701 x^9 - 0.0053689327 x^{10} + 0.09516185544 x^{21/2} - 1.349359978 x^{11}$$

$$Q_7(x) = 0.6436407082 x^{21/2} - 0.0241601974 x^{10} - 5.13874019 x^{19/2} - 9.616203190 x^{11} + 0.008921423945 x^{23/2} + 0.3795074935 x^{12}$$

$$\begin{aligned}
Q_8(x) &= 0.0784840681 x^{23/2} - 21.11590681 x^{11} + 1.17762796 x^{21/2} + \\
&\quad 3.76054992 x^{12} + 0.00214671763 x^{\frac{25}{2}} - 0.03804957943 x^{13} + \\
&\quad 0.5395289892 x^{\frac{27}{2}} \\
Q_9(x) &= 0.0234395451 x^{\frac{25}{2}} + 12.78409624 x^{12} + 0.1860488619 x^{23/2} - \\
&\quad 10.55241669 x^{11} - 0.486293585 x^{13} + 7.077248277000 x^{\frac{27}{2}} - \\
&\quad 0.003312351780 x^{14} - 0.1409037760 x^{\frac{29}{2}} \\
Q_{10}(x) &= 34.12760689 x^{\frac{27}{2}} - 2.202188424 x^{13} + 0.067276597400 x^{\frac{25}{2}} + \\
&\quad 14.309359690 x^{12} - 0.05216102571 x^{14} - 2.365733908 x^{\frac{29}{2}} - \\
&\quad 0.0007472199823 x^{15} + 0.013244129 x^{\frac{31}{2}} - 0.187796864 x^{16} \\
Q_{11}(x) &= -15.32849226 x^{\frac{29}{2}} - 0.2933676318 x^{14} + 65.94339820 x^{\frac{27}{2}} - \\
&\quad 3.484180030 x^{13} - 0.01405684011 x^{15} + 0.27243830211 x^{\frac{31}{2}} - \\
&\quad 3.922834118 x^{16} + 0.00108889605 x^{\frac{33}{2}} + 0.046320431931 x^{17} \\
Q_{12}(x) &= 2.206628384 x^{\frac{31}{2}} - 0.094212525791 x^{15} - 44.393708110 x^{\frac{29}{2}} - \\
&\quad 0.58103272281 x^{14} + 32.95531797 x^{\frac{27}{2}} - 32.7604179300 x^{16} + \\
&\quad 0.02663446045 x^{\frac{33}{2}} + 1.1788655081 x^{17} + 0.0002333571 x^{\frac{35}{2}} - \\
&\quad 0.004136156736 x^{18} + 0.05864917554 x^{\frac{37}{2}}
\end{aligned}$$

$$\begin{aligned}
Q_{13}(x) = & 0.2586354937000 x^{\frac{33}{2}} - 131.5680506 x^{16} + \\
& 8.259986315 x^{\frac{31}{2}} - 0.2206108621000 x^{15} - \\
& 46.92270861 x^{\frac{29}{2}} + 12.327134690000 x^{17} + \\
& 0.006640678649 x^{\frac{35}{2}} - 0.1246450393 x^{18} + \\
& 1.785235306 x^{\frac{37}{2}} - 0.0003246060506 x^{19} - \\
& 0.01380838185 x^{\frac{39}{2}}
\end{aligned} \tag{3.255}$$

Making use of the values of $Q_j(x)$, ($j = 0, 1, \dots, 13$) in equation (3.254) gives

$$\begin{aligned}
y_{13}(x) = & 2.363271801 x^{7/2} a_0 - 1.772453851 x^{9/2} a_1 + (0.2954089752 x^{11/2} - \\
& 4.188790206 x^6) a_2 + (0.03692612189 x^{13/2} - 2.094395101000 x^6 + \\
& 1.570796327 x^7) a_3 + (0.01107783657000 x^{15/2} + 2.356194490 x^7 - \\
& 0.19634954090 x^8 + 2.784164001 x^{17/2}) a_4 + (8.1250940940 x^{17/2} - \\
& 0.4908738523 x^8 - 0.02045307717 x^9 - 0.870051249500 x^{19/2}) a_5 + \\
& (-4.184813758 x^{19/2} - 0.07158577010 x^9 + 4.06023916200 x^{17/2} - \\
& 0.005368932760 x^{10} + 0.09516185544 x^{21/2} - 1.349359978 x^{11}) a_6 + \\
& (0.6436407082 x^{21/2} - 0.02416019742 x^{10} - 5.13874019200 x^{19/2} - \\
& 9.6162031900 x^{11} + 0.008921423945 x^{23/2} + 0.3795074935 x^{12}) a_7 + \\
& (0.07848406816 x^{23/2} - 21.11590681 x^{11} + 1.1776279610000 x^{21/2} +
\end{aligned}$$

$$\begin{aligned}
& 3.760549923 x^{12} + 0.00214672 x^{\frac{25}{2}} - 0.03804950 x^{13} + \\
& 0.5395280 x^{\frac{27}{2}}) a_8 + (0.023439 x^{\frac{25}{2}} + 12.7840962 x^{12} + \\
& 0.1860488 x^{23/2} - 10.552416690 x^{11} - 0.486293 x^{13} + \\
& 7.07724820 x^{\frac{27}{2}} - 0.003312 x^{14} - 0.14090371 x^{\frac{29}{2}}) a_9 + \\
& (34.12760689 x^{\frac{27}{2}} - 2.202188 x^{13} + 0.067276500 x^{\frac{25}{2}} + \\
& 14.309359 x^{12} - 0.05216102000 x^{14} - 2.3657339 x^{\frac{29}{2}} - \\
& 0.00074721 x^{15} + 0.013244 x^{\frac{31}{2}} - 0.18779680 x^{16}) a_{10} + \\
& (15.32849226 x^{\frac{29}{2}} - 0.29336760 x^{14} + 65.94339 x^{\frac{27}{2}} - \\
& - \quad \quad \quad 2 \quad \quad \quad 2 \\
& 3.484180030 x^{13} - 0.014056 x^{15} + 0.27243830000 x^{\frac{31}{2}} - \\
& 3.922834 x^{16} + 0.001088 x^{\frac{33}{2}} + 0.0463204300 x^{17}) a_{11} + \\
& (2.206628384 x^{\frac{31}{2}} - 0.0942125200 x^{15} - 44.39370 x^{\frac{29}{2}} - \\
& 0.5810327228 x^{14} + 32.95531797 x^{\frac{27}{2}} - 32.760417 x^{16} + \\
& 0.02663446 x^{\frac{33}{2}} + 1.178865508 x^{17} + 0.000233357 x^{\frac{35}{2}} - \\
& 0.00413 x^{18} + 0.0586491 x^{\frac{37}{2}}) a_{12} + (0.2586354937 x^{\frac{33}{2}} - \\
& 131.5680506 x^{16} + 8.259986315 x^{\frac{31}{2}} - 0.22061086 x^{15} - \\
& 46.92270861 x^{\frac{29}{2}} + 12.32713469 x^{17} + 0.00664067 x^{\frac{35}{2}} - \\
& 0.1246450393 x^{18} + 1.785235306 x^{\frac{37}{2}} - 0.0003246 x^{19} - \\
& 0.01380838185 x^{\frac{39}{2}}) a_{13}
\end{aligned} \tag{3.256}$$

Thus, equation (3.256) is substituted into a slightly perturbed equation (3.251) to get

$$\frac{d^{3.5}y_{13}}{dx^{3.5}} + y_{13} + \frac{d^3y_{13}}{dx^3} + \left(\frac{1}{x^2}\right) \frac{d^2y_{13}}{dx^2} - x \frac{dy_{13}}{dx} + \int dx y_{13}(t) \sqrt{x} dt = 27.08110001 x^{0.5} + 26 + 24 x - \quad (3.257)$$

$$5 x^4 + \frac{256}{315} x^{9/2} + H_{13}(x)$$

together with equation (3.252) to get

$$y_{13}(0) = 0, \quad y'_{13}(0) = 0, \quad y''_{13}(0) = 0, \quad y'''_{13}(0) = 0$$

where,

$$H_{13}(x) = t_1(33554432 x^{13} - 218103808 x^{12} + 627048448 x^{11} - 1049624576 x^{10} + 1133117440 x^9 - 825556992 x^8 + 412778496 x^7 - 141213696 x^6 + 32361472.000 x^5 - 4759040 x^4 + 416416 x^3 - 18928.000000000000 x^2 + 338 x - 1) + t_2(8388608 x^{12} - 50331648.0000 x^{11} + 132120576 x^{10} - 199229440 x^9 + 190513152.00 x^8 - 120324096 x^7 + 50692096 x^6 - 14057472.00000 x^5 +$$

$$\begin{aligned}
& 2471040 x^4 - 256256 x^3 + 13728 x^2 - 288 x \\
& + 1) + {}_{\mathbb{B}}(2097152 x^{11} - 11534336.00 x^{10} + \\
& 27394048 x^9 - 36765696 x^8 + 30638080 x^7 - \\
& 16400384.0 x^6 + 5637632 x^5 - 1208064 x^4 + \\
& 151008 x^3 - 9680 x^2 + 242 x - 1.000000000)
\end{aligned} \tag{3.258}$$

Hence, evaluating the terms $\frac{d^{3.5}y(x)}{dx^{3.5}}$, y , $\frac{d^3y(x)}{dx^3}$, $\left(\frac{1}{x^2}\right)\frac{d^2y(x)}{dx^2}$, $x\frac{dy(x)}{dx}$, and $\int_0^x \frac{y_N(t)}{\sqrt{x-t}} dt$ in equation (3.257) gives

$$\begin{aligned}
\frac{d^{3.5}y_{13}}{dx^{3.5}} = & 27.48895266 a_0 - 92.77533511 x a_1 + (42.52195337.0000000 x - \\
& 907.4989451 x^{5/2}) a_2 + (-453.7483442 x^{5/2} + 11.5163506200 x^3 + \\
& 680.6213878 x^{7/2}) a_3 + (6.477968379 x^4 - 151.24963610000 x^{9/2} + \\
& 2767.750482 x^5 + 1020.934903 x^{7/2}) a_4 + (-25.7811507900 x^{11/2} - \\
& 1369.457376 x^6 + 8077.163753 x^5 - 378.1232440000000 x^{9/2}) a_5 + \\
& (4036.296909 x^{5.0} - 90.23422523 x^{5.5} + 224.6766536000000 x^{7.0} - \\
& 3837.865790 x^{7.5} - 6586.913387 x^{6.0} - 10.411610990000 x^{6.5}) a_6 + \\
& (1519.633359 x^{7.0} - 27350.50053 x^{7.5} - 8088.391126000000 x^{6.0} - \\
& 46.85227768 x^{6.5} + 30.27870089 x^{8.0} + 1523.8591390000 x^{8.5}) a_7 + \\
& (2780.377045 x^{7.0} - 60057.98116 x^{7.5} + 266.3696997000000 x^{8.0} + \\
& 15099.97001 x^{8.5} + 10.11919153 x^{9.0} - 209.07116980000000 x^{9.5} + \\
& 3433.358785 x^{10.0}) a_8 + (45036.94127 x^{10.0} - 24.2671480400 x^{10.5} -
\end{aligned}$$

$$\begin{aligned}
& 1181.960252 x^{11.0} - 2672.035 x^{9.5} - 30013.306 x^{7.5} + \\
& 631.4353 x^{8.0} + 51332.67642 x^{8.5} + 110.4891 x^{9.0}) a_9 + \\
& (-19844.74799 x^{11.0} - 7.1404397 x^{11.5} + 143.5 x^{12.0} - \\
& 2297.0696 x^{12.5} + 57457.06718 x^{8.5} + 317.12814 x^{9.0} - \\
& 12100.343 x^{9.5} + 217175.188 x^{10.0} - 382.14 x^{10.5}) a_{10} + \\
& (-19144.532 x^{9.5} + 419639.134 x^{10.0} - 2149.28 x^{10.5} - \\
& 128581.62 x^{11.0} - 134.32733 x^{11.5} + 2951.87900 x^{12.0} - \\
& 47982.80 x^{12.5} + 14.97466351 x^{13.0} + 713.46 x^{13.5}) a_{11} + \\
& (-79.086 x^{14.5} + 1243.4343 x^{15.0} + 209715.4743 x^{10.0} - \\
& 4256.7991 x^{10.5} - 372392.7704 x^{11.0} - 900.2942 x^{11.5} + \\
& 23908.887 x^{12.0} - 400714.5233 x^{12.5} + 366.2814 x^{13.0} + \\
& 18157.87 x^{13.5} + 4.0114 x^{14.0}) a_{12} + (89497.3930 x^{12.0} - \\
& 1609300.0 x^{12.5} + 3556.797 x^{13.0} + 189872.9266 x^{13.5} + \\
& 114.15 x^{14.0} - 2383.311699 x^{14.5} + 37849.22240 x^{15.0} - \\
& 7.6082 x^{15.5} - 356.7951852 x^{16.0} - 393606.8629 x^{11.0} - \\
& 2108.156 x^{11.5}) a_{13}
\end{aligned} \tag{3.259}$$

$$\begin{aligned}
\frac{d^4 y_{13}}{dx^4} = & 15.50897 \frac{a_0}{x} - 104.6855556 \sqrt{x} a_1 + (63.9745 x^{3/2} - 1507.96 x^2) a_2 + \\
& (20.79171451 x^{5/2} - 753.9822366 x^2 + 1319.468915 x^3) a_3 + (13.366102 x^{7/2} + \\
& 1979.203372 x^3 - 329.8672287 x^4 + 6345.283769 x^{9/2}) a_4 + (18517.5972 x^{9/2} - \\
& 824.66807 x^4 - 61.85010534 x^5 - 3425.0111 x^{11/2}) a_5 + (-16473.78091 x^{11/2} - \\
& 216.4753688 x^5 + 9253.538816 x^{9/2} - 27.05942111 x^6 + 605.140186400 x^{13/2} - \\
& 10686.93102 x^7) a_6 + (4092.951490 x^{13/2} - 121.7673950 x^6 - 20228.971 x^{11/2} - \\
& 76160.32927 x^7 + 86.98890182 x^{15/2} + 4508.549023 x^8) a_7 + (765.26381 x^{15/2} - \\
& 167237.98190 x^7 + 7488.609806 x^{13/2} + 44675.33309 x^8 + 30.781918380 x^{17/2} - \\
& 652.93078 x^9 + 10993.74617 x^{19/2}) a_8 + (336.1011031 x^{17/2} + 151875.0634 x^8 + \\
& 1814.081057 x^{15/2} - 83575.14016 x^7 - 8344.797919 x^9 + 144209.991900 x^{19/2} - \\
& 79.57593917 x^{10} - 3964.9001590000 x^{21/2}) a_9 + (695403.31470000000000 x^{19/2} - \\
& 37789.55335000 x^9 + 964.6833349 x^{17/2} + 169995.1931 x^8 - 1253.116481 x^{10} - \\
& 66569.53429 x^{21/2} - 24.47892661 x^{11} + 502.3042992 x^{23/2} - 8202.967 x^{12}) a_{10} + \\
& (-431329.4018 x^{21/2} - 7047.86398 x^{10} + 1343699.777 x^{19/2} - 59788.52932 x^9 - \\
& 460.5020819 x^{11} + 10332.6482900 x^{23/2} - 171349.3943 x^{12} + 54.51347104 x^{25} + \\
& 2645.823071 x^{13}) a_{11} +
\end{aligned}$$

$$\begin{aligned}
& (83689.82 x^{23/2} - 3086.4023 x^{11} - 1249197.327 x^{21/2} - \\
& 13958.730 x^{10} + 671516.096 x^{19/2} - 1430975.056 x^{12} + \\
& 1333.402 x^{\frac{25}{2}} + 67336.79782 x^{13} + 15.144109630 x^{\frac{27}{2}} - \\
& 303.7593 x^{14} + 4856.09674 x^{\frac{29}{2}}) a_2 + (12948.084 x^{\frac{25}{2}} - \\
& 5746892.451 x^{12} + 313272.8 x^{23/2} - 7227.211843 x^{11} - \\
& 1320361.030 x^{21/2} + 704125.9333 x^{13} + 430.9572 x^{\frac{27}{2}} - \\
& 9153.931 x^{14} + 147815.80970 x^{\frac{29}{2}} - 30.19615325 x^{15} - \\
& 1438.371671 x^{\frac{31}{2}}) a_{13}
\end{aligned} \tag{3.260}$$

$$\begin{aligned}
\frac{d^3 y_{13}}{dx^3} = & 31.01794 \sqrt{-x} a_0 - 69.79037 x^{3/2} a_1 + (25.58980 x^{5/2} - 502.65482 x^3) a_2 + \\
& (5.940489861 x^{7/2} - 251.3274122 x^3 + 329.8672287 x^4) a_3 + (2.970244930 x^{9/2} + \\
& 494.8008429 x^4 - 65.97344574 x^5 + 1153.687958 x^{11/2}) a_4 + (3366.835865 x^{11/2} - \\
& 164.9336144 x^5 - 10.30835089 x^6 - 526.9247880 x^{13/2}) a_5 + (-2534.4278 x^{13/2} - \\
& 36.0792281300 x^6 + 1682.461603 x^{11/2} - 3.865631587 x^7 + 80.685358180 x^{15/2} - \\
& 1335.866378 x^8) a_6 + (545.7268654 x^{15/2} - 17.39534214 x^7 - 3112.149529 x^{13/2} - \\
& 9520.041159 x^8 + 10.23398845 x^{17/2} + 500.9498914 x^9) a_7 + (90.03103668 x^{17/2} - \\
& 20904.74774 x^8 + 998.4813074 x^{15/2} + 4963.925899 x^9 + 3.240201935000 x^{19/2} - \\
& 65.29307830000 x^{10} + 1047.023445 x^{21/2}) a_8 + (35.379063 x^{19/2} + 16875.007 x^9 +
\end{aligned}$$

$$\begin{aligned}
& 213.4213 x^{17/2} - 10446.8925 x^8 - 834.4797919 x^{10} + \\
& 13734.284 x^{21/2} - 7.234176 x^{11} - 344.773 x^{23/2}) a_9 + \\
& (66228.88711 x^{21/2} - 3778.95 x^{10} + 101.5456 x^{19/2} + \\
& 18888.35479 x^9 - 113.919680 x^{11} - 5788.655 x^{23/2} - \\
& 2.0399 x^{12} + 40.18434394 x^{\frac{25}{2}} - 630.99746 x^{13}) a_{10} + \\
& (-37506.90 x^{23/2} - 640.714 x^{11} + 127971.407 x^{21/2} - \\
& 5978.85293 x^{10} - 38.37517349 x^{12} + 826.61186 x^{\frac{25}{2}} - \\
& 13180.72264 x^{13} + 4.038034 x^{\frac{27}{2}} + 188.9873 x^{14}) a_{11} + \quad (3.261) \\
& (6695.186346 x^{\frac{25}{2}} - 257.2001 x^{12} - 108625.85 x^{23/2} - \\
& 1268.9754 x^{11} + 63953.91 x^{21/2} - 110075.00430 x^{13} + \\
& 98.77056 x^{\frac{27}{2}} + 4809.7712730 x^{14} + 1.044421354 x^{\frac{29}{2}} - \\
& 20.25062338 x^{15} + 313.2965 x^{\frac{31}{2}}) a_{12} + (959.1173 x^{\frac{27}{2}} - \\
& 442068.6501 x^{13} + 25061.83099 x^{\frac{25}{2}} - 602.26765 x^{12} - \\
& 114814.0026 x^{23/2} + 50294.7095 x^{14} + 29.721187 x^{\frac{29}{2}} - \\
& 610.2621123 x^{15} + 9536.50385 x^{\frac{31}{2}} - 1.88725957 x^{16} - \\
& 87.17404068 x^{\frac{33}{2}}) a_{13}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{x^2}\right)\frac{d^2y_{13}}{dx^2} = & -0.437230435 a_9 x^{10} - 6.885255394 a_{10} x^{10} - 242.240721 a_{10} x^9 - \\
& 0.11656630 a_{10} x^{11} - 2.192867057 a_{11} x^{11} - 38.724527400 a_{11} x^{10} + \\
& 99.53386306 a_4 x^{9/2} - 23945.38520 a_{13} x^{12} - 29.91480944 a_{13} x^{14} - \\
& 34.17902935 a_{10} x^{12} + 9.727290706 a_{11} x^{13} - 0.992677616 a_{12} x^{14} - \\
& 0.08829284576 a_{13} x^{15} - 5962.396063 a_{12} x^{12} - 14.697154 a_{12} x^{11} - \\
& 97.15391840 a_6 x^7 + 34.15567442 a_7 x^8 + 47.123889800000 a_4 x^3 + \\
& 15.02344 a_9 x^{15/2} + 6.710840591 a_{10} x^{17/2} + 7962.66534 a_{11} x^{19/2} + \\
& 3979.354640 a_{12} x^{19/2} - 6745.1393 a_{13} x^{21/2} - 204.00967 a_6 x^{11/2} + \\
& 0.7204049836 a_7 x^{15/2} + 41.03209515 a_7 x^{13/2} - 53.4922943 a_9 x^9 + \\
& 338.4494931 a_8 x^8 - 4.1854537370 a_8 x^9 - 692.3666297000 a_7 x^7 - \\
& 34.41529448 a_{13} x^{11} + 2.584828532 a_2 x^{3/2} + 0.581586419 a_3 x^{5/2} + \\
& 0.274176450 a_4 x^{7/2} + 0.21413508 a_8 x^{17/2} + 2.23494680 a_{10} x^{23/2} + \\
& 45.97396348 a_{11} x^{23/2} - 2203.470762 a_{11} x^{21/2} - 5.89048622 a_4 x^4 - \\
& 0.85902924 a_5 x^5 - 0.3006602346 a_6 x^6 - 340.07424900 a_{10} x^{21/2} + \\
& 4120.908 a_{10} x^{19/2} - 42.4149 a_5 x^{11/2} + 6.0665600000000 a_6 x^{13/2} - \\
& 3.0066020000000 a_6 x^5 - 1.352971056 a_7 x^6 - 1520.345290 a_8 x^7 - \\
& 759.7740000000000 a_9 x^7 + 1287.842372 a_{10} x^8 - 383.25980 a_{11} x^9 -
\end{aligned}$$

$$\begin{aligned}
& 76.696 a_{12}x^{10} - 50.265400 a_2x^2 - 25.132741210 a_3x^2 - \\
& 14.726200 a_5x^4 + 290.4721 a_5x^{9/2} + 145.15351 a_6x^{9/2} - \\
& 250.513500 a_7x^{11/2} + 1393.8 a_{13} x^{23/2} + 0.0520 a_{12} x^{\frac{27}{2}} - \\
& 713.9550 a_{11}x^{12} + 372.368 a_{12}x^{23/2} + 6.33700 a_8x^{15/2} + \\
& 65.1481200 a_8x^{19/2} + 854.577 a_9x^{19/2} + 2.338 a_9x^{17/2} - \\
& 20.25491 a_9 x^{21/2} + 5.213000 a_{12} x^{\frac{25}{2}} + 247.5617 a_{12} x^{13} + (3.262) \\
& 2588.698 a_{13}x^{13} - 6381.5955 a_{12} x^{21/2} + 1.4924 a_{13} x^{\frac{27}{2}} + \\
& 1.772453 \sqrt[3]{\frac{a_0}{x}} - 6.646701942 \sqrt[3]{xa_1} + 50.62789 a_{13}x^{\frac{25}{2}} + \\
& 0.2131514 a_{11}x^{\frac{25}{2}} + 75.07378 a_8x^{13/2} + 1150.568 a_9 x^8 + \\
& 31.41592 a_3 x^3 + 456.57392 a_{13} x^{\frac{29}{2}} + 14.999520 a_{12} x^{\frac{29}{2}} - \\
& 3.987170260 a_{13}x^{\frac{31}{2}}
\end{aligned}$$

$$\begin{aligned}
x \frac{dy_{13}}{dx} = & 186.0216760 a_{10} x^{12} - 48.778520420000000 a_{11} x^{13} - \\
& 8.71549 a_{12}x^{14} - 3.529773 a_{13} x^{15} + 494.85 a_{10} x^{\frac{27}{2}} + \\
& 0.9082 a_{10}x^{\frac{25}{2}} - 36.66887 a_{10}x^{\frac{29}{2}} + 0.21852 a_{10}x^{\frac{31}{2}} - \\
& 237.5916 a_{11}x^{\frac{29}{2}} + 0.8337 a_{11} x^{17} - 1.50740 a_{12} x^{15} + \\
& 0.4661030 a_{12}x^{\frac{33}{2}} + 0.00431 a_{12}x^{\frac{35}{2}} + 1.143 a_{12}x^{\frac{37}{2}} + \\
& 4.526 a_{13}x^{\frac{33}{2}} + 0.02898068 a_8x^{\frac{25}{2}} + 7.823170 a_8x^{\frac{27}{2}} +
\end{aligned}$$

$$\begin{aligned}
& 13.54272 a_8 x^{21/2} + 77.1883 a_5 x^{17/2} - 9.13500 a_5 x^{19/2} - \\
& 43.9405 a_6 x^{19/2} + 477.852111 a_{12} x^{27} + 38.570 a_6 x^{17/2} + \\
& 1.094361 a_6 x^{21/2} + 7.401 a_7 x^{21/2} - 53.956711 a_7 x^{19/2} + \\
& 0.1115177 a_7 x^{23/2} + 0.9810 a_8 x^{23/2} - 30.83000 a_{10} x^{13} - \\
& 0.782415 a_{10} x^{14} - 0.0119555 a_{10} x^{15} - 3.192511 a_{10} x^{16} - \\
& 4.400514 a_{11} x^{14} - 0.22490 a_{11} x^{15} - 66.6881800 a_{11} x^{16} - \\
& 0.006492 a_{13} x^{19} + 221.8884 a_{13} x^{17} - 2.368250 a_{13} x^{18} + \\
& 0.1228525 a_{13} x^{35} + 34.81208 a_{13} x^{37} - 0.28300 a_{13} x^{39} + \quad (3.263) \\
& 0.3164338 a_9 x^{25} + 2.32561 a_9 x^{23/2} + 102.62010 a_9 x^{27} - \\
& 126.629 a_9 x^{11} - 6.8081101 a_9 x^{13} - 0.04968527 a_9 x^{14} - \\
& 556.92710 a_{12} x^{16} + 21.21957 a_{12} x^{17} - 0.07858 a_{12} x^{18} - \\
& 2236.656 a_{13} x^{16} + 956.179274 a_{11} x^{27} + 1.920 a_2 x^{11/2} + \\
& 0.2769 a_3 x^{13/2} + 0.09416161 a_4 x^{15/2} + 26.449 a_4 x^{17/2} + \\
& 4.4952319 a_{11} x^{31} + 0.019055 a_{11} x^{33} + 36.4093 a_{12} x^{31} - \\
& 727.3019 a_{13} x^{29} - 688.1024 a_{12} x^{29} + 136.289774 a_{13} x^{31}
\end{aligned}$$

and,

$$\begin{aligned}
\int_0^x \frac{y_{13}(t)}{\sqrt{x-t}} dt = & 2.03 x^4 a_0 - 1.3 x^5 a_1 + (0.21 x^6 - 2.850000000 x^{13/2}) a_2 + \\
& (-1.4283500000 x^{13/2} + 0.0243003 x^7 + 0.999842 x^{15/2}) a_3 + \\
& (0.00683 x^8 + 1.62226 x^9 - 0.117628 x^{17/2} + 1.499 x^{15/2}) a_4 +
\end{aligned}$$

$$\begin{aligned}
& (-0.294071 x^{17/2} + 4.73427 x^9 - 0.01160811 x^{19/2} - \\
& 0.4816 x^{10})a_5 + (2.36579 x^9 - 0.0406282000 x^{19/2} - \\
& 2.31646 x^{10} - 0.00290202 x^{21/2} + 0.050281500 x^{11} - \\
& 0.6976 x^{23/2})a_6 + (-2.84449 x^{10} - 0.013050 x^{21/2} + \\
& 0.3400861 x^{11} - 4.971 x^{23/2} + 0.004517480000 x^{12} + \\
& 0.18836400 x^{25})a_7 + (0.6222 x^{11} - 10.91731 x^{23/2} + \\
& 0.03974 x^{12} + 1.86651 x^{25} + 0.0010452100000 x^{13} - \\
& 0.01818 x^{27} + 0.253 x^{14})a_8 + (-5.455810000 x^{23/2} + \\
& 0.0942082 x^{12} + 6.34524 x^{25} + 0.011412400000 x^{13} - \\
& 0.232427 x^{27} + 3.322760000 x^{14} - 0.001528570 x^{29} - \\
& 0.063949 x^{15})a_9 + (7.10229 x^{25} + 0.0327561000 x^{13} - \\
& 1.05255000 x^{27} + 16.02291 x^{14} - 0.0240710000 x^{29} - \\
& 1.0736800 x^{15} - 0.0003337 x^{31} + 0.0058229900 x^{16} - \\
& 0.081 x^{33})a_{10} + (-1.66 x^{27} + 30.9603000000000 x^{14} - \\
& 0.1353820000000 x^{29} - 6.95681 x^{15} - 0.006277 x^{31} + \\
& 0.119782 x^{16} - 1.698800000000 x^{33} + 0.0004646 x^{17} + \\
& 0.0194862 x^{35})a_{11} + (15.4725 x^{14} - 0.26813200 x^{29} - \\
& 20.1480000 x^{15} - 0.0420742 x^{31} + 0.970179000 x^{16} - \\
& 14.1871000 x^{33} + 0.0113658 x^{17} + 0.495928000 x^{35} + \\
& 0.00009681 x^{18} - 0.0016 x^{37} + 0.023692100 x^{19})a_{12} + \\
& (-21.3 x^{15} -
\end{aligned}$$

$$\begin{aligned}
& 0.0985222 x^{\frac{31}{2}} + 3.63163 x^{16} - 56.97620000 x^{\frac{33}{2}} + \\
& 0.110369000 x^{17} + 5.18581 x^{\frac{35}{2}} + 0.00275509 x^{18} - \\
& 0.0510188 x^{\frac{37}{2}} + 0.721168 x^{19} - 0.000129459 x^{\frac{39}{2}} - \quad (3.264) \\
& 0.00543862 x^{20}) a_{13}
\end{aligned}$$

Hence, substituting equations (3.257) – (3.263) into (3.256) gives

$$\begin{aligned}
& 2.03012 x^4 a_0 - 1.37033 x^5 a_1 - 10.63472310 x^{7/2} a_0 + 9.748496180000000 x^{9/2} a_1 + \\
& 1.62226 a_4 x^9 - 0.4816080 a_5 x^{10} + 0.2533 a_8 x^{14} - 0.0639 a_9 x^{15} + 0.023691 a_{12} x^{19} - \\
& - 2.85669 a_2 x^{13/2} + 0.999842 a_3 x^{15/2} - 0.697645 a_6 x^{23/2} + 0.1883640000 a_7 x^{25/2} + \\
& 17.2814250500 \sqrt{\frac{a_0}{x}} - 111.33225750 \sqrt{x} a_1 - 1515.708 a_2 x^2 - 779.1149778 a_3 x^2 + \\
& 2026.327262 a_4 x^3 - 839.39428 a_5 x^4 - 219.4819711 a_6 x^5 - 123.1203600000 a_7 x^6 - \\
& 84334.914 a_9 x^7 + 1829.1045030 a_9 x^{15/2} - 60171.789 a_{11} x^9 - 14035.42645 a_{12} x^{10} - \\
& 7261.627137 a_{13} x^{11} + 66.559330 a_2 x^{3/2} - 432.3750433 a_3 x^{5/2} + 1034.575 a_4 x^{7/2} + \\
& 18429.9461 a_5 x^{9/2} + 9398.690 a_6 x^{9/2} - 20479.48 a_7 x^{11/2} + 7563.6830000 a_8 x^{13/2} - \\
& 6586.913387 a_6 x^{6.0} + 971.394 a_{10} x^{17/2} - 92.77533511 x a_1 - 90.23422523 a_6 x^{5.5} + \\
& 224.676 a_6 x^{7.0} - 3837.865790 a_6 x^{7.5} + 4036.296909 a_6 x^{5.0} - 10.41161099 a_6 x^{6.5} + \\
& 1519.633359 a_7 x^{7.0} - 27350.50053 a_7 x^{7.5} - 8088.391126 a_7 x^{6.0} - 46.8522 a_7 x^{6.5} + \\
& 30.27870089 a_7 x^{8.0} + 1523.859139 a_7 x^{8.5} + 2780.3770 a_8 x^{7.0} + 45036.941 a_9 x^{10.0} - \\
& 60057.981 a_8 x^{7.5} + 266.36969 a_8 x^{8.0} + 15099.97001 a_8 x^{8.5} + 10.119191531 a_8 x^{9.0} - \\
& 209.07116 a_8 x^{9.5} + 3433.3587 a_8 x^{10.0} - 1181.960252 a_9 x^{11.0} - 2672.03571 a_9 x^{9.5} -
\end{aligned}$$

$$\begin{aligned}
& 30013.30611 a_9 x^{7.5} + 51332.6764 a_9 x^{8.5} - 24.2671 a_9 x^{10.5} + 631.4353400 a_9 x^{8.0} + \\
& 57457.067 a_{10} x^{8.5} + 110.48 a_9 x^{9.0} - 19844.74799 a_{10} x^{11.0} - 7.140439788 a_{10} x^{11.5} + \\
& 143.50 a_{10} x^{12.0} - 2297.06968 a_{10} x^{12.5} + 317.12814 a_{10} x^{9.0} - 12100.34325 a_{10} x^{9.5} - \\
& 382.1453515 a_{10} x^{10.5} - 19144.532 a_{11} x^{9.5} + 2951.879 a_{11} x^{12.0} - 2149.285 a_{11} x^{10.5} - \\
& 134.3273 a_{11} x^{11.5} - 47982.8007 a_{11} x^{12.5} - 168758.327 a_8 x^7 + 171283.0355 a_{10} x^8 + \\
& 1351662.4 a_{11} x^{19/2} - 1327106.1 a_{13} x^{21/2} + 675495.4 a_{12} x^{19/2} + 217175.1 a_{10} x^{10.0} + \\
& 419639.1345 a_{11} x^{10.0} - 128581.627 a_{11} x^{11.0} - 76870.091 a_7 x^7 - 13065.441 a_{11} x^{10} - \\
& 1923.221077 a_6 x^{13/2} + 686.56 a_3 x^{7/2} - 126.37140 a_5 x^{11/2} + 10.23398845 a_7 x^{17/2} + \\
& 31.01794239 \sqrt{-x} a_0 + 121.027 a_8 x^{17/2} - 526.92478 a_5 x^{13/2} + 9749.129863 a_9 x^{21/2} + \\
& 1021.8340 a_7 x^{13/2} + 41110.86 a_{13} x^{14} - 5284.115910 a_{10} x^{23/2} + 4306.809662 a_8 x^9 + \\
& 264645.981 a_{13} x^{13} - 5771440.10 a_{13} x^{12} + 199852.7 a_{13} x^{23/2} - 1609300.0 a_{13} x^{12.5} - \\
& 400714.5 a_{12} x^{12.5} - 372392.7 a_{12} x^{11.0} - 1191625.009 a_{12} x^{21/2} + 699625.7 a_{10} x^{19/2} - \\
& 1437194.65 a_{12} x^{12} + 145099.94 a_9 x^{19/2} - 393606.860 a_{13} x^{11.0} - 172101.72 a_{11} x^{12} - \\
& 305561.4653 a_{11} x^{21/2} + 142578.73 a_9 x^8 + 189872.926 a_{13} x^{13.5} + 149029.40 a_{13} x^{29/2} + \\
& 209715.47 a_{12} x^{10.0} - 4977.3364 a_7 x^8 + 633.4361722 a_7 x^{15/2} - 914.4929615 a_9 x^{10} + \\
& 38060.54378 a_{13} x^{25/2} + 551.8604985 a_9 x^{17/2} + 80.685 a_6 x^{15/2} - 24563.656 a_{12} x^{23/2} + \\
& 8033.802 a_{12} x^{25/2} - 881.909142 a_2 x^{5/2} - 69.79037040 x^{3/2} a_1 + 881.3384858 a_{11} x^{25/2} - \\
& 42490.644 a_{12} x^{13} - 5038.957 a_{10} x^{10} - 27128.28225 a_{11} x^{23/2} + 1770.08270 a_8 x^{15/2} - \\
& 19143.439 a_{10} x^9 + 14.97466 a_{11} x^{13.0} - 4256.799124 a_{12} x^{10.5} + 1243.4343 a_{12} x^{15.0} + \\
& 5637632 t_3 x^5 + 33554432 t_1 x^{13} + 27.48895266 a_0 + 0.00543862 a_{13} x^{20} -
\end{aligned}$$

$$\begin{aligned}
& 3556.797824 a_{13}x^{13.0} + 89497.3936 a_{13}x^{12.0} + 4.0114 a_{12}x^{14.0} + 18157.87 a_{12}x^{13.5} - \\
& 12.54207032 a_3x^7 + 14.66076571 a_3x^6 + 29.530887 a_2x^6 - 2108.156440 a_{13}x^{11.5} - \\
& 356.7951 a_{13}x^{16.0} - 7.608209 a_{13}x^{15.5} - 2.257401740 a_6x^{10} + 3.0816477010 a_6x^9 + \\
& 4.938800772 a_5x^9 + 4.417864671 a_5x^8 + 1.7739803181 a_4x^8 - 18.84955592 a_4x^7 - \\
& 4370.074966 a_{12}x^{11} + 1153.687958 a_4x^{11/2} - 600.1340711 a_{10}x^{13} + 3.3724 a_9x^{14} - \\
& 6.819522590 a_9x^{13} + 119.3948 a_9x^{11} - 166.0990429 a_9x^{12} + 0.5337393220 a_8x^{13} - \\
& 48.84740760 a_8x^{12} + 254.01311 a_8x^{11} - 4.929079936 a_7x^{12} + 115.7345243 a_7x^{11} - \\
& 2.578727828 a_7x^{10} + 16.24260124 a_6x^{11} + 500.9498914 a_7x^9 - 10476.393 a_{11}x^{13} + \\
& 224.3481767 a_{11}x^{14} + 3.19836 a_{10}x^{16} - 1.06172448 a_{10}x^{15} + 16.80531539 a_{10}x^{14} - \\
& 8425.207662 a_{10}x^{12} - 6.73190 a_{11}x^{15} + 66.80796201 a_{11}x^{16} - 65.29307830 a_8x^{10} - \\
& 38.89122297 a_{12}x^{15} - 0.833303 a_{11}x^{17} + 4529.207236 a_{12}x^{14} + 24109.03484 a_8x^8 + \\
& 8476.716 a_9x^9 - 680.7214 a_{10}x^{21/2} + 11062.13450 a_8x^{19/2} - 138.51517300 a_{10}x^{11} + \\
& 557.897 a_{12}x^{16} + 1391.567109 a_{13}x^{27} - 14995.32898 a_6x^{11/2} + 6296.53824 a_4x^{9/2} - \\
& 1.14535190 a_{12}x^{37} - 148.6763 a_{13}x^{33} + 7957.75671 a_{13}x^{31} + 5.062957445 a_{13}x^{35} - \\
& 34.86310 a_{13}x^{37} + 0.2829423689 a_{13}x^{39} - 502.6548248 a_2x^3 + 1111.073781 a_3x^3 + \\
& 329.8672287 a_3x^4 + 165.5210964 a_4x^4 + 2701.777036 a_4x^5 + 7849.5210051 a_5x^5 - \\
& 1379.7657271 a_5x^6 - 63.43930947 a_6x^6 - 10787.95057 a_6x^7 - 1335.866378 a_6x^8 - \\
& 1103.409857 a_{11}x^{11} - 5.083277799 a_7x^{23/2} + 53.956772 a_7x^{19/2} - 7.4149 a_7x^{21/2} - \\
& 1.097263358 a_6x^{21/2} - 38.57227204 a_6x^{17/2} + 43.8999 a_6x^{19/2} + 9.123930 a_5x^{19/2} -
\end{aligned}$$

$$\begin{aligned}
& 77.48246 a_5 x^{17/2} - 26.56718 a_4 x^{17/2} + 1.4055911 a_4 x^{15/2} - \\
& 1.705295 a_3 x^{13/2} - 1.920158 a_2 x^{11/2} + 0.72766000 a_{13} x^{19} + \\
& 2.37101 a_{13} x^{18} - 221.7780 a_{13} x^{17} - 658.31258460 a_{13} x^{15} + \\
& 2238.401230 a_{13} x^{16} + 0.07868 a_{12} x^{18} - 21.2082111 a_{12} x^{17} - \\
& 11.898 a_8 x^{23/2} + 1033.480 a_8 x^{21/2} + 1.83752931111 a_8 x^{25/2} - \\
& 7.8413 a_8 x^{27/2} + 6.0288061 a_9 x^{25/2} - 352.5553477101 a_9 x^{23/2} - \\
& 102.8525 a_9 x^{27/2} + 2.18249 a_9 x^{29/2} - 495.9028000111 a_{10} x^{27/2} + \\
& 46.37839988 a_{10} x^{25/2} + 36.64489 a_{10} x^{29/2} - 0.2188601 a_{10} x^{31/2} - \\
& 953.8069 a_{11} x^{27/2} - 4.5015 a_{11} x^{31/2} - 1.71785568111 a_{11} x^{33/2} + \\
& 276.8451 a_{12} x^{31/2} + 5559.975039 a_{12} x^{29/2} - 363.88490 a_{12} x^{27/2} - \\
& 14.6532 a_{12} x^{33/2} + 0.4916108 a_{12} x^{35/2} = -115343360.0 t_3 x^{10} + \\
& 2097152 t_3 x^{11} - 1208064 t_3 x^4 + 151008 t_3 x^3 - 5.0000 x^4 + \\
& 8388608 t_2 x^{12} + 2471040 t_2 x^4 - 256256 t_2 x^3 + 24.0000 x + \\
& 190513152 t_2 x^8 - 120324096 t_2 x^7 + 50692096.0 t_2 x^6 - t_3 - \\
& 14057472 t_2 x^5 - t_1 + 13728 t_2 x^2 + 338 t_1 x - 288.000 t_2 x + \\
& 1133117440 t_1 x^9 + 132120576.0 t_2 x^{10} - 1049624576 t_1 x^{10} + \\
& 237.4 a_{11} x^{29/2} + 30638080 t_3 x^7 - 199229440 + 242 t_3 x t_2 x^9 +
\end{aligned}$$

$$\begin{aligned}
& 627048448 \tau_1 x^{11} - 218103808 \tau_1 x^{12} + 26 - 9680 \tau_3 x^2 + \\
& \frac{256}{315} x^{9/2} + 416416 \tau_1 x^3 - 825556992 \tau_1 x^8 + 27.00 x^{0.5} + \\
& 27394048 \tau_3 x^9 + 412778496 \tau_1 x^7 - 0.08132641 a_{10} x^{\frac{33}{2}} + \\
& 32361472 \tau_1 x^5 - 18928 \tau_1 x^2 + 0.019486200000 a_{11} x^{\frac{35}{2}} - \\
& 4759040 \tau_1 x^4 - 50331648 \tau_2 x^{11} - 36765696 \tau_3 x^8 + \tau_2 - \\
& 141213696 \tau_1 x^6 - 16400384 \tau_3 x^6
\end{aligned} \quad (3.265)$$

Collocating equation (3.265) at the $x = x_i$ gives

$$\begin{aligned}
& 2.03012 x_i^4 a_0 - 1.370330 x_i^5 a_1 - 10.63472310 x_i^{7/2} a_0 + 9.7484961800 x_i^{9/2} a_1 + \\
& 1.62226 a_4 x_i^9 - 0.481 a_5 x_i^{10} + 0.25330 a_8 x_i^{14} - 0.0639 a_9 x_i^{15} + 0.0236 a_{12} x_i^{19} - \\
& - 2.85669 a_2 x_i^{13/2} + 0.999842 a_3 x_i^{15/2} - 0.697645 a_6 x_i^{23/2} + 0.188364 a_7 x_i^{\frac{25}{2}} + \\
& 17.28142505 \sqrt[4]{\frac{a_0}{x}} - 111.3322575 \sqrt{\frac{a_0}{x_i a_1}} - 1515.708 a_2 x_i^2 - 779.1149778 a_3 x_i^2 + \\
& 2026.327262 a_4 x_i^3 - 839.39428 a_5 x_i^4 - 219.4819711 a_6 x_i^5 - 123.120360 a_7 x_i^6 - \\
& 84334.91416 a_9 x_i^7 + 1829.10450 a_9 x_i^{15/2} - 60171.78 a_{11} x_i^9 - 14035.42 a_{12} x_i^{10} - \\
& 7261.62713 a_{13} x_i^{11} + 66.55933473 a_2 x_i^{3/2} - 432.375 a_3 x_i^{5/2} + 1034.575 a_4 x_i^{7/2} + \\
& 18429.9461 a_5 x_i^{9/2} + 9398.692 a_6 x_i^{9/2} - 20479.485 a_7 x_i^{11/2} + 7563.683 a_8 x_i^{13/2} -
\end{aligned}$$

$$\begin{aligned}
& 6586.913387 a_6 x_i^{6.0} + 971.39417551 a_{10} x_i^{17/2} - 92.77533 x_i a_1 - 90.23422 a_6 x_i^{5.5} + \\
& 224.67665361 a_6 x_i^{7.0} - 3837.865 a_6 x_i^{7.5} + 4036.296909 a_6 x_i^{5.0} - 10.41161 a_6 x_i^{6.5} + \\
& 1519.6330 a_7 x_i^{7.0} - 27350.50053 a_7 x_i^{7.5} - 8088.391 a_7 x_i^{6.0} - 46.85227768 a_7 x_i^{6.5} + \\
& 30.27870089 a_7 x_i^{8.0} + 1523.859139 a_7 x_i^{8.5} + 2780.377 a_8 x_i^{7.0} + 45036.94 a_9 x_i^{10.0} - \\
& 60057.981 a_8 x_i^{7.5} + 266.36969 a_8 x_i^{8.0} + 15099.970 a_8 x_i^{8.5} + 10.119191531 a_8 x_i^{9.0} - \\
& 209.0711 a_8 x_i^{9.5} + 3433.358785 a_8 x_i^{10.0} - 1181.960 a_9 x_i^{11.0} - 2672.035711 a_9 x_i^{9.5} - \\
& 30013.306 a_9 x_i^{7.5} + 51332.67642 a_9 x_i^{8.5} - 24.2671 a_9 x_i^{10.5} + 631.4353400 a_9 x_i^{8.0} + \\
& 57457.067 a_{10} x_i^{8.5} + 110.4891955 a_9 x_i^{9.0} - 19844.74799 a_{10} x_i^{11.0} - 7.14 a_{10} x_i^{11.5} + \\
& 143.50 a_{10} x_i^{12.0} - 2297.06968 a_{10} x_i^{12.5} + 317.1281 a_{10} x_i^{9.0} - 12100.34325 a_{10} x_i^{9.5} - \\
& 382.14 a_{10} x_i^{10.5} - 19144.5323 a_{11} x_i^{9.5} + 2951.879 a_{11} x_i^{12.0} - 2149.28586 a_{11} x_i^{10.5} - \\
& 134.327 a_{11} x_i^{11.5} - 47982.80076 a_{11} x_i^{12.5} - 168758.3 a_8 x_i^7 + 171283.03551 a_{10} x_i^8 + \\
& 1351662.4 a_{11} x_i^{19/2} - 1327106.1 a_{13} x_i^{21/2} + 675495.4 a_{12} x_i^{19/2} + 217175 a_{10} x_i^{10.0} + \\
& 419639.13 a_{11} x_i^{10.0} - 128581.62 a_{11} x_i^{11.0} - 76870.091 a_7 x_i^7 - 13065.4414 a_{11} x_i^{10} - \\
& 1923.221 a_6 x_i^{13/2} + 686.5618 a_3 x_i^{7/2} - 126.3714 a_5 x_i^{11/2} + 10.233980000 a_7 x_i^{17/2} + \\
& 31.01794 \sqrt{x_i} a_0 + 121.0270 a_8 x_i^{17/2} - 526.92478 a_5 x_i^{13/2} + 9749.129863 a_9 x_i^{21/2} + \\
& 1021.834 a_7 x_i^{13/2} + 41110.86 a_{13} x_i^{14} - 5284.115910 a_{10} x_i^{23/2} + 4306.80966 a_8 x_i^9 + \\
& 264645.9 a_{13} x_i^{13} - 5771440.1 a_{13} x_i^{12} + 199852.7 a_{13} x_i^{23/2} - 1609300.01 a_{13} x_i^{12.5} -
\end{aligned}$$

$$\begin{aligned}
& 400714.5 a_{12} x_i^{12.5} - 372392.7 a_{12} x_i^{11.0} - 1191625.1 a_{12} x_i^{21/2} + 699625.7 a_{10} x_i^{19/2} - \\
& 1437194.652 a_{12} x_i^{12} + 145099.9 a_9 x_i^{19/2} - 393606.86 a_{13} x_i^{11.0} - 172101.7 a_{11} x_i^{12} - \\
& 305561.465 a_{11} x_i^{21/2} + 142578.7396 a_9 x_i^8 + 189872.9 a_{13} x_i^{13.5} + 149029.4 a_{13} x_i^{29/2} + \\
& 209715.4 a_{12} x_i^{10.0} - 4977.336 a_7 x_i^8 + 633.4361722 a_7 x_i^{15/2} - 914.4929615 a_9 x_i^{10} + \\
& 38060.54378 a_{13} x_i^{25/2} + 551.86 a_9 x_i^{17/2} + 80.685 a_6 x_i^{15/2} - 24563.656681 a_{12} x_i^{23/2} + \\
& 8033.802 a_{12} x_i^{25/2} - 881.9091 a_2 x_i^{5/2} - 69.79037040 x_i^{3/2} a_1 + 881.3384858 a_1 x_i^{25/2} - \\
& 42490.644 a_{12} x_i^{13} - 5038.957 a_{10} x_i^{10} - 27128.28 a_{11} x_i^{23/2} + 1770.082708 a_8 x_i^{15/2} - \\
& 19143.43929 a_{10} x_i^9 + 14.9746 a_{11} x_i^{13.0} - 4256.7991 a_{12} x_i^{10.5} + 1243.434 a_{12} x_i^{15.0} + \\
& 5637632 t_3 x_i^5 + 33554432 t_1 x_i^{13} + 27.488952660000 a_0 + 0.00543862000 a_{13} x_i^{20} - \\
& 3556.797824 a_{13} x_i^{13.0} + 89497.393 a_{13} x_i^{12.0} + 4.011 a_{12} x_i^{14.0} + 18157.877 a_{12} x_i^{13.5} - \\
& 12.54207032 a_3 x_i^7 + 14.66076571 a_3 x_i^6 + 29.53088 a_2 x_i^6 - 2108.156440 a_{13} x_i^{11.5} - \\
& 356.7951852 a_{13} x_i^{16.0} - 7.608209371 a_{13} x_i^{15.5} - 2.257401 a_6 x_i^{10} + 3.081647 a_6 x_i^9 + \\
& 4.938800772 a_5 x_i^9 + 4.417864671 a_5 x_i^8 + 1.773980318 a_4 x_i^8 - 18.84955592 a_4 x_i^7 - \\
& 4370.074966 a_{12} x_i^{11} + 1153.687958 a_4 x_i^{11/2} - 600.1340711 a_{10} x_i^{13} + 3.372 a_9 x_i^{14} + \\
& 6.8195 a_9 x_i^{13} + 119.3948240 a_9 x_i^{11} - 166.0990429 a_9 x_i^{12} + 0.53373932200 a_8 x_i^{13} - \\
& 38.8912 a_{12} x_i^{15} - 0.833303105 a_{11} x_i^{17} + 4529.20723 a_{12} x_i^{14} + 24109.03484 a_8 x_i^8 + \\
& 8476.716 a_9 x_i^9 - 680.72143 a_{10} x_i^{21/2} + 11062.134 a_8 x_i^{19/2} - 138.5151730 a_{10} x_i^{11} +
\end{aligned}$$

$$\begin{aligned}
& 38.89122297 a_{12} x_i^{15} - 0.8333031057 a_{11} x_i^{17} + 4529.20 a_{12} x_i^{14} + 24109.034 a_8 x_i^8 + \\
& 8476.7168 a_9 x_i^9 - 680.72 a_{10} x_i^{21/2} + 11062.13450 a_8 x_i^{19/2} - 138.5151730 a_{10} x_i^{11} + \\
& 557.8972838 a_{12} x_i^{16} + 1391.56710 a_{13} x_i^{27} - 14995.32 a_6 x_i^{11/2} + 6296.538 a_4 x_i^{9/2} - \\
& 1.1453519 a_{12} x_i^{37} - 148.676 a_{13} x_i^{33} + 7957.756712 a_{13} x_i^{31} + 5.062900000 a_{13} x_i^{35} - \\
& 34.86310727 a_{13} x_i^{37} + 0.282942 a_{13} x_i^{39} - 502.6548248 a_2 x_i^3 + 1111.073781 a_3 x_i^3 + \\
& 329.8672287 a_3 x_i^4 + 165.5210964 a_4 x_i^4 + 2701.777036 a_4 x_i^5 + 7849.521005 a_5 x_i^5 - \\
& 1379.765727 a_5 x_i^6 - 63.43930947 a_6 x_i^6 - 10787.95057 a_6 x_i^7 - 1335.866378 a_6 x_i^8 - \\
& 1103.409857 a_{11} x_i^{11} - 5.083277 a_7 x_i^{23/2} + 53.956772 a_7 x_i^{19/2} - 7.414927 a_7 x_i^{21/2} - \\
& 1.0972 a_6 x_i^{21/2} - 38.57227 a_6 x_i^{17/2} + 43.89991626 a_6 x_i^{19/2} + 9.12393002 a_5 x_i^{19/2} - \\
& 77.48246 a_5 x_i^{17/2} - 26.56718 a_4 x_i^{17/2} + 1.4055983 a_4 x_i^{15/2} - 1.70529591 a_3 x_i^{13/2} - \\
& 1.920158339 a_2 x_i^{11/2} + 0.72766012 a_{13} x_i^{19} + 2.3710108 a_{13} x_i^{18} - 221.778 a_{13} x_i^{17} - \\
& 658.312 a_{13} x_i^{15} + 2238.401 a_{13} x_i^{16} + 0.0786837935 a_{12} x_i^{18} - 21.20821334 a_{12} x_i^{17} - \\
& 11.8983 a_8 x_i^{23/2} + 1033.4807 a_8 x_i^{21/2} + 1.837529312 a_8 x_i^{25} - 7.841356343 a_8 x_i^{27} + \\
& 6.028806140 a_9 x_i^{25} - 352.5553477 a_9 x_i^{23/2} - 102.8525270 a_9 x_i^{27} + 2.1824 a_9 x_i^{29} - \\
& 495.902 a_{10} x_i^{27} + 46.378399 a_{10} x_i^{25} + 36.64480457 a_{10} x_i^{29} - 0.218861831 a_{10} x_i^{31} + \\
& 237.45620 a_{11} x_i^{29} - 953.80651 a_{11} x_i^{27} - 4.5015096 a_{11} x_i^{31} - 1.717855681 a_{11} x_i^{33} + \\
& 276.84511 a_{12} x_i^{31} + 5559.975039 a_{12} x_i^{29} - 363.8849 a_{12} x_i^{27} - 14.6532030 a_{12} x_i^{33} + \\
& 0.49161088341 a_{12} x_i^{35} = 26 - 11534336 t_3 x_i^{10} + 2097152 t_3 x_i^{11} - 1208064 t_3 x_i^4 + \\
& 151008 t_3 x_i^3 + 27.0811000 x_i^{0.5} + 8388608 t_2 x_i^{12} + 2471040 t_2 x_i^4 - 256256 t_2 x_i^3 +
\end{aligned}$$

$$\begin{aligned}
& 190513152 \, t_2 x_i^8 - 120324096 \, t_2 x_i^7 + 50692096 \, t_2 x_i^6 + \\
& 242 \, t_3 x_i + 13728 \, t_2 x_i^2 + 338 \, t_1 x_i + 1133117440 \, t_1 x_i^9 - \\
& 1049624576 \, t_1 x_i^{10} - 218103808 \, t_1 x_i^{12} - 288.000 \, t_2 x_i - \\
& 4759040 \, t_1 x_i^4 - 9680 \, t_3 x_i^2 - 5.0 \, x_i^4 + \frac{256}{315} x_i^{9/2} + t_2 + \\
& 416416 \, t_1 x_i^3 - 825556992 \, t_1 x_i^8 + 27394048.000 \, t_3 x_i^9 - \\
& t_3 + 412778496 \, t_1 x_i^7 - 0.0813264.0000000000 \, a_{10} x_i^{\frac{33}{2}} + \quad (3.266) \\
& 32361472 \, t_1 x_i^5 - 18928 \, t_1 x_i^2 - 199229440.000 \, t_2 x_i^9 + \\
& 132120576 \, t_2 x_i^{10} - 50331648 \, t_2 x_i^{11} - 36765696 \, t_3 x_i^8 + \\
& 30638080 \, t_3 x_i^7 - 16400384 \, t_3 x_i^6 + 0.0194862 \, a_{11} x_i^{\frac{35}{2}} - \\
& 14057472 \, t_2 x_i^5 - t_1 + 627048448 \, t_1 x_i^{11} + 24.0000 \, x_i - \\
& 141213696 \, t_1 x_i^6
\end{aligned}$$

where, $x_i = \frac{i}{15}$, $i = 1, 2, \dots, 14$.

Hence, for $i = 1$, $x_1 = \frac{1}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 102.4276489 a_0 - 36.13215945 a_1 - 6.751745197 a_2 - \\
& 3.570657066 a_3 + 0.718875126 a_4 + 0.0875770118 a_5 + \\
& 0.04713668404 a_6 + 0.00007721 a_{10} - 0.00814527 a_7 - \\
& 0.0008768266 a_8 - 0.00047381 a_9 + 0.000007934 a_{11} + \\
& 0.000004284 a_{12} - 0.0000006814 a_{13} = -0.874250 t_1 + \\
& 0.9998813151 t_2 - 0.858877532 t_3 + 34.59221
\end{aligned} \tag{3.267}$$

For $i = 2$, $x_2 = \frac{2}{19}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 86.13367232 a_0 - 56.41961529 a_1 - 30.62166659 a_2 - \\
& 13.32548195 a_3 + 6.609163713 a_4 + 2.181614247 a_5 + \\
& 0.9629600983 a_6 + 0.022703428 a_{10} - 0.42345064 a_7 - \\
& 0.1216629369 a_8 - 0.05380103 a_9 + 0.00616232 a_{11} + \\
& 0.002716390 a_{12} - 0.0011343483 a_{13} = 0.9571453 t_1 - \\
& 0.898802935 t_2 + 0.36109909 t_3 + 39.08713311
\end{aligned} \tag{3.268}$$

For $i = 3$, $x_3 = \frac{3}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
 &79.96823960 a_0 - 74.58006974 a_1 - 74.47084775 a_2 - \\
 &27.02572809 a_3 + 25.71170931 a_4 + 14.23466771 a_5 + \\
 &5.146531588 a_6 + 0.6743409573 a_{10} - 4.55647737 a_7 - \\
 &2.156772414 a_8 - 0.7728571122 a_9 + 0.29815141 a_{11} + \\
 &0.1054588357 a_{12} - 0.09301583305 a_{13} = -0.8719 t_1 + \\
 &0.1315857 t_2 + 0.714093 t_3 + 42.90361763
 \end{aligned} \tag{3.269}$$

For $i = 4$, $x_4 = \frac{4}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
 &76.87801103 a_0 - 91.81876934 a_1 - 140.5262611 a_2 - \\
 &41.81701088 a_3 + 70.28005162 a_4 + 53.78381295 a_5 + \\
 &15.20103377 a_6 + 7.816530768 a_{10} - 25.68373060 a_7 - \\
 &16.49854024 a_8 - 4.486654410 a_9 + 4.637159836 a_{11} + \\
 &1.212391463 a_{12} - 2.216843353 a_{13} = -0.028562 t_1 + \\
 &0.897396 t_2 - 0.809028 t_3 + 46.36145817
 \end{aligned} \tag{3.270}$$

For $i = 5$, $x_5 = \frac{5}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 75.12713354 a_0 - 108.5701377 a_1 - 230.7603094 a_2 - \\
& 54.38760944 a_3 + 157.9449203 a_4 + 150.6884068 a_5 + \\
& 30.17531175 a_6 + 53.89162906 a_{10} - 101.2480332 a_7 - \\
& 79.62954175 a_8 - 14.16027813 a_9 + 38.70871738 a_{11} + \\
& 6.12103750 a_{12} - 26.72597627 a_{13} = 0.956760000 t_1 - \\
& 0.59261 t_2 - 0.56186 t_3 + 49.57934472
\end{aligned} \tag{3.271}$$

For $i = 6$, $x_6 = \frac{6}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 74.05226185 a_0 - 125.0347541 a_1 - 346.9866695 a_2 - \\
& 61.03216057 a_3 + 312.8443504 a_4 + 349.4258816 a_5 + \\
& 39.22567568 a_6 + 266.5487474 a_{10} - 317.3417473 a_7 - \\
& 287.0972626 a_8 - 20.94025943 a_9 + 217.8055804 a_{11} + \\
& 8.6639860 a_{12} - 208.5607674 a_{13} = -0.500300000 t_1 - \\
& 0.74832 t_2 + 0.79962 t_3 + 52.61274980
\end{aligned} \tag{3.272}$$

For $i = 7$, $x_7 = \frac{7}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 73.33361101 a_0 - 141.3128086 a_1 - 490.8997053 a_2 - \\
& 57.6882647 a_3 + 566.7321704 a_4 + 711.05395060 a_5 + \\
& 12.25222559 a_6 + 1045.371509 a_{10} - 846.9184725 a_7 - \\
& 845.9935535 a_8 + 38.25137374 a_9 + 932.6442500 a_{11} - \\
& 90.7199157 a_{12} - 1201.894347 a_{13} = -0.7612000 t_1 + \\
& 0.6962 t_2 + 0.6699 t_3 + 55.48910798
\end{aligned} \tag{3.273}$$

For $i = 8$, $x_8 = \frac{8}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 72.79086767 a_0 - 157.4516456 a_1 - 664.0918899 a_2 - \\
& 39.9615076 a_3 + 960.0713257 a_4 + 1314.8844111 a_5 - \\
& 113.0311807 a_6 + 3452.186830 a_{10} - 2005.385827 a_7 - \\
& 2149.621761 a_8 + 384.8462191 a_9 + 3266.739429 a_{11} - \\
& 839.393619 a_{12} - 5535.405342 a_{13} = 58.220716100 + \\
& 0.7606667 t_1 + 0.69553 t_2 - 0.6697 t_3
\end{aligned} \tag{3.274}$$

For $i = 9$, $x_9 = \frac{9}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 72.30934873 a_0 - 173.4663575 a_1 - 868.0600410 a_2 - \\
& 3.1449071 a_3 + 1543.114308 a_4 + 2259.889023000 a_5 - \\
& 448.1950395 a_6 + 9981.554784 a_{10} - 4327.389127 a_7 - \\
& 4875.220904 a_8 + 1613.839137 a_9 + 9804.313484 a_{11} - \\
& 4351.206112 a_{12} - 21446.10092 a_{13} = 0.51000000 t_1 - \\
& 0.749 t_2 - 0.7995 t_3 + 60.81051482
\end{aligned} \tag{3.275}$$

For $i = 10$, $x_{10} = \frac{10}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 71.80854016 a_0 - 189.3498825 a_1 - 1104.204811 a_2 + \\
& 57.7642363 a_3 + 2376.970201 a_4 + 3665.8452200 a_5 - \\
& 1176.008717 a_6 + 25956.98615 a_{10} - 8668.715723 a_7 - \\
& 10103.28760 a_8 + 5092.022434 a_9 + 26019.71725 a_{11} - \\
& 17259.46440 a_{12} - 72405.19700 a_{13} = -0.940000 t_1 - \\
& 0.592 t_2 + 0.561 t_3 + 63.25504591
\end{aligned} \tag{3.276}$$

For $i = 11$, $x_{11} = \frac{11}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 71.22706517 a_0 - 205.0783994 a_1 - 1373.825577 a_2 + \\
& 148.0483520 a_3 + 3534.655961 a_4 + 5674.221877 a_5 - \\
& 2577.857754 a_6 + 61896.02000 a_{10} - 16337.33669 a_7 - \\
& 19454.14613 a_8 + 13639.36315 a_9 + 62436.19120 a_{11} - \\
& 57587.51402 a_{12} - 218478.2833 a_{13} = 0.0400000 t_1 + \\
& 0.89 t_2 + 0.810 t_3 + 65.54611699
\end{aligned} \tag{3.277}$$

For $i = 12$, $x_{12} = \frac{12}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 70.51486489 a_0 - 220.6143866 a_1 - 1678.111890 a_2 + \\
& 273.2533268 a_3 + 5102.127831 a_4 + 8448.800156 a_5 - \\
& 5067.382040 a_6 + 137303.7093 a_{10} - 29257.65732 a_7 - \\
& 35233.91582 a_8 + 32627.96492 a_9 + 137673.7252 a_{11} - \\
& 169283.9444 a_{12} - 600396.5993 a_{13} = 0.70000000 t_1 + \\
& 0.13 t_2 - 0.714 t_3 + 67.67181027
\end{aligned} \tag{3.278}$$

For $i = 13$, $x_{13} = \frac{13}{15}$ thus equation (3.266) after simplification gives

$$\begin{aligned}
& 69.62885618 a_0 - 235.9084328 a_1 - 2018.132166 a_2 + \\
& 439.1711183 a_3 + 7179.287595 a_4 + 12176.02091 a_5 - \\
& 9231.078751 a_6 + 286488.8088 a_{10} - 50172.06934 a_7 - \\
& 60577.86255 a_8 + 71674.05953 a_9 + 282333.5729 a_{11} - \\
& 450757.1421 a_{12} - 1524431.344 a_{13} = -0.800000 \tau_1 - \\
& 0.92 \tau_2 - 0.356 \tau_3 + 69.61713131
\end{aligned} \tag{3.279}$$

For $i = 14$, $x_{14} = \frac{14}{15}$, thus equation (3.266) after simplification gives

$$\begin{aligned}
& 68.53039258 a_0 - 250.9003387 a_1 - 2394.820025 a_2 + \\
& 651.8206011 a_3 + 9880.956911 a_4 + 17065.04708 a_5 - \\
& 15876.63410 a_6 + 567148.7593 a_{10} - 82883.88454 a_7 - \\
& 99574.24978 a_8 + 147146.6552 a_9 + 543340.7077 a_{11} - \\
& 1107597.579 a_{12} - 3616623.788 a_{13} = 0.9 \tau_1 + 1.1 \tau_2 + \\
& 0.85 \tau_3 + 71.36444769
\end{aligned} \tag{3.280}$$

For the perturbed collocation method with exponential fitted, an approximation to the conditions are fitted with exponentials to give

$$y_{13}(0) + \tau_4 e^0 = 0 \quad (3.281)$$

$$y'_{13}(0) + \tau_4 e^0 = 0 \quad (3.282)$$

$$y''_{13}(0) + \tau_4 e^0 = 0 \quad (3.283)$$

and,

$$y'''_{13}(0) + \tau_4 e^0 = 0 \quad (3.284)$$

Therefore, equation (3.280) after simplification gives

$$\begin{aligned} & 7.4 \times 10^{-39} a_0 - 5.60 \times 10^{-50} a_1 + 9.3412 \times 10^{-62} a_2 - \\ & 2.094 \times 10^{-66} a_3 + 2.35 \times 10^{-77} a_4 - 4.90 \times 10^{-89} a_5 + \\ & 1.28 \times 10^{-93} a_6 - 1.62 \times 10^{-104} a_7 + 3.72 \times 10^{-116} a_8 - \\ & 1.055 \times 10^{-120} a_9 + 1.4309 \times 10^{-131} a_{10} + \tau_4 e^{1.0 \times 10^{-11}} - \\ & 3.4 \times 10^{-143} a_{11} + 1.04 \times 10^{-147} a_{12} - 1.5 \times 10^{-158} a_{13} \\ & = 0 \end{aligned} \quad (3.285)$$

Also, equation (3.281) after simplification gives

$$\begin{aligned}
& 2.61 \times 10^{-27} a_0 - 2.52 \times 10^{-38} a_1 + 5.13 \times 10^{-50} a_2 - \\
& 1.25 \times 10^{-54} a_3 + 1.64 \times 10^{-65} a_4 - 3.92 \times 10^{-77} a_5 + \\
& 1.09 \times 10^{-81} a_6 - 1.543 \times 10^{-92} a_7 + 3.9 \times 10^{-104} a_8 - \quad (3.286) \\
& 1.2 \times 10^{-108} a_9 + 1.7 \times 10^{-119} a_{10} - 4.5 \times 10^{-131} a_{11} + \\
& 1.40 \times 10^{-135} a_{12} - 2.15 \times 10^{-146} a_{13} + \tau_4 = 0
\end{aligned}$$

Also, equation (3.282) after simplification gives

$$\begin{aligned}
& 6.539 \times 10^{-16} a_0 - 8.82 \times 10^{-27} a_1 + 2.31 \times 10^{-38} a_2 - \\
& 6.28 \times 10^{-43} a_3 + 9.89 \times 10^{-54} a_4 - 2.748 \times 10^{-65} a_5 + \\
& 8.18 \times 10^{-70} a_6 - 1.312 \times 10^{-80} a_7 + 3.71 \times 10^{-92} a_8 - \quad (3.287) \\
& 1.1 \times 10^{-96} a_9 + 1.8 \times 10^{-107} a_{10} - 5.43 \times 10^{-119} a_{11} + \\
& 1.759 \times 10^{-123} a_{12} - 2.905 \times 10^{-134} a_{13} + \tau_4 = 0
\end{aligned}$$

And, equation (3.283) after simplification gives

$$\begin{aligned}
& 0.0000981 a_0 - 2.206 \times 10^{-15} a_1 + 8.0917 \times 10^{-27} a_2 - \\
& 2.5 \times 10^{-31} a_3 + 4.948 \times 10^{-42} a_4 - 1.649 \times 10^{-53} a_5 + \\
& 5.32 \times 10^{-58} a_6 - 9.84 \times 10^{-69} a_7 + 3.150 \times 10^{-80} a_8 - \quad (3.288) \\
& 1.04 \times 10^{-84} a_9 + 1.8 \times 10^{-95} a_{10} - 5.97 \times 10^{-107} a_{11} + \\
& 2.022 \times 10^{-111} a_{12} - 3.6307300 \times 10^{-122} a_{13} + \tau_4 = 0
\end{aligned}$$

Hence, solving equations (3.267) – (3.287) to obtain

$$\begin{aligned}
 \alpha_0 &= -6.45424791710^{-27}, & \alpha_1 &= -2.23109808610^{-15}, \\
 \alpha_2 &= -0.001642899580, & \alpha_3 &= -35.82172252, \\
 \alpha_4 &= -212.3216120, & \alpha_5 &= -4396.689710, \\
 \alpha_6 &= 8554.191174, & \alpha_7 &= -6071.055231, \\
 \alpha_8 &= 1642.729475, & \alpha_9 &= 486.8496768, \\
 \alpha_{10} &= -1303.550297, & \alpha_{11} &= 768.8669727, \\
 \alpha_{12} &= -94.24492770, & \alpha_{13} &= -5.217623873, \\
 \tau_1 &= -7.1707 \times 10^{-6}, & \tau_2 &= -6.33317817 \times 10^{-10}, \\
 \tau_3 &= -1.8265176 \times 10^{-15}, & \tau_4 &= 1.624062844 \times 10^{-66}
 \end{aligned}$$

Thus, substituting the values of $\alpha_i (i = 0, 1, 2, \dots, 13)$ and τ_4 into equation (3.287), that is

$$y_{13}(x) = \sum_{j=0}^{13} \alpha_j Q_j(x) + \tau_4 e^x$$

after simplification gives

$$\begin{aligned}
 y_{13}(x) &= 1.6240621134 \times 10^{-66} e^x + 100.7510110000000000 x^{10} + \\
 &7012.65431 x^{11} + 165.3434629 x^{23/2} - 8555.481697 x^{12} - \\
 &72.76041719 x^{25/2} - 107.464619 x^{13} + 7440.63300000 x^{27/2} - \\
 &104.4193929 x^{14} - 4341.6104040 x^{29/2} + 0.196315913 x^{15} - \\
 &58.85661 x^{31/2} + 1002.6408 x^{16} + 3.9545183 \times 10^{-15} x^{9/2} - \\
 &1.52531 \times 10^{-26} x^{7/2} - 774.77602 x^{19/2} - 3.0224093 x^{33/2} -
 \end{aligned}$$

$$\begin{aligned}
& 139.80619 x^{17} - 0.05664133 x^{\frac{35}{2}} + 1.0401627250 x^{18} - \\
& 14.842073 x^{\frac{37}{2}} - 522.4325289 x^9 + 0.0016936722 x^{19} + \\
& 0.0720469 x^{\frac{39}{2}} - 0.0004853272 x^{11/2} + 75.0317219 x^6 - \quad (3.289) \\
& 1.3227572 x^{13/2} - 556.5396425 x^7 - 2.3520641 x^{15/2} + \\
& 2199.909266 x^8 - 1582.594 x^{17/2} - 1159.021422 x^{21/2}
\end{aligned}$$

Thus, equation (3.289) is the approximate solution of equation (3.251) together with conditions (3.252) for case $N = 13$

Problem 3.8.2: In order to test the effectiveness of the method on nonlinear problems, a second order nonlinear singular and multi-order fractional integro differential equation of the form:

$$\begin{aligned}
& \frac{d^{0.3}y}{dx^{0.3}} + 2\left(\frac{dy}{dx}\right)^2 + \frac{x^2 d^2y}{dx^2} + \int_0^x \frac{y(t)}{\sqrt[3]{x-t}} dt = 18x^4 + \\
& 20x^3 + 1.495438427 x^{8/3} + 0.55227272273 \quad (3.290) \\
& 0 \leq x \leq 1
\end{aligned}$$

subject to the conditions

$$y(0) = 1 \quad y'(0) = 1 \quad (3.291)$$

is considered.

The exact solution is given as $y(x) = x^3 + e^x$

Chapter 4

RESULTS AND DISCUSSIONS OF THE FINDINGS

4.1 Introduction

The results obtained from the solutions of the examples considered using the three methods with different cases of N are presented in both tabular and graphical forms.

4.2 Tables of Results

Here, the tables of results for problems 3.2.2.1, 3.2.2.2, 3.2.2.3 and 4.4.1 are presented

owo1

owo2

owo3

owo4

owo4

4.3 Graphical Representation of Results

Here, the graphs of the exact solution and approximation solutions for different values of N are presented

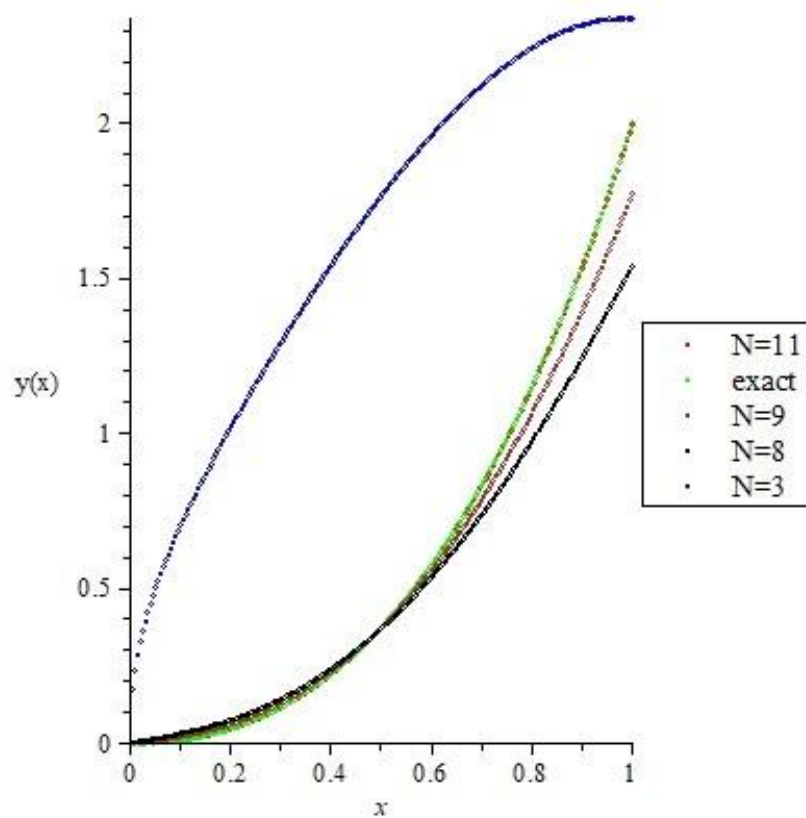


Figure 4.1: The Graph of the Approximate and Exact Solution for Problem 3.2.2.1

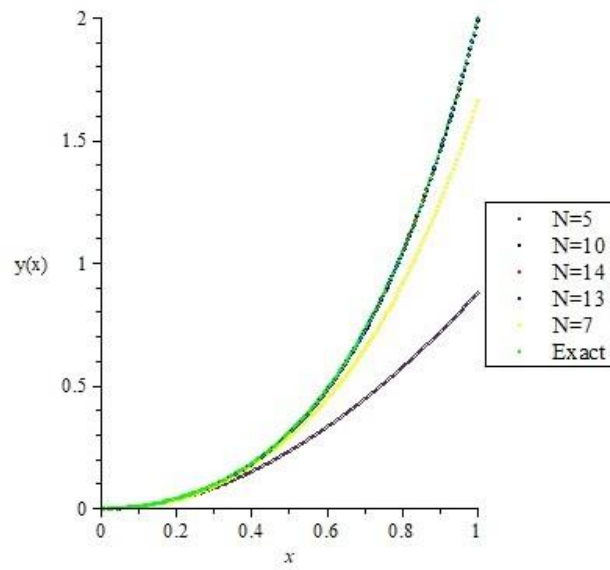


Figure 4.2: The Graph of the Approximate and Exact Solution for Problem 3.2.2.2

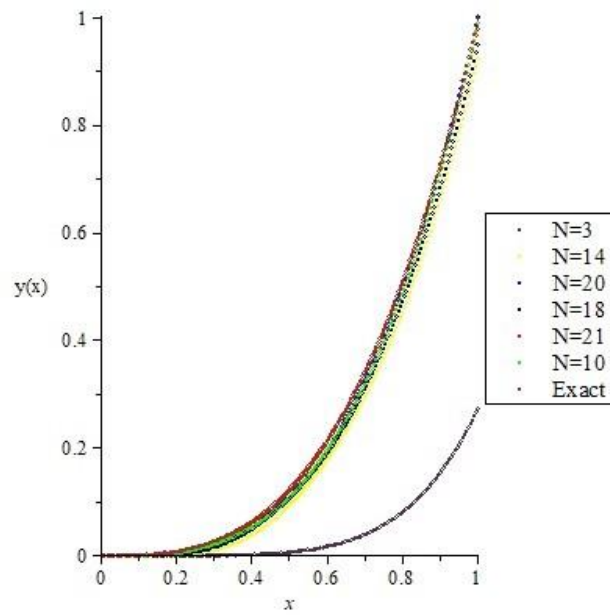


Figure 4.3: The Graph of the Approximate and Exact Solution for Problem 3.5.1

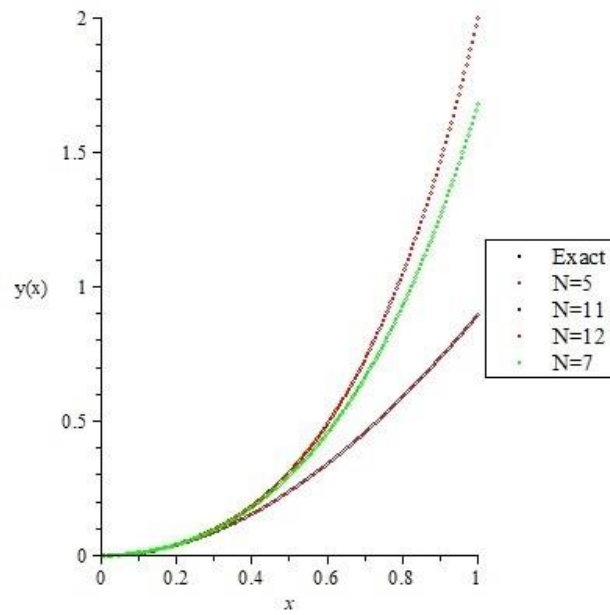


Figure 4.4: The Graph of the Approximate and Exact Solution for Problem 3.7.1

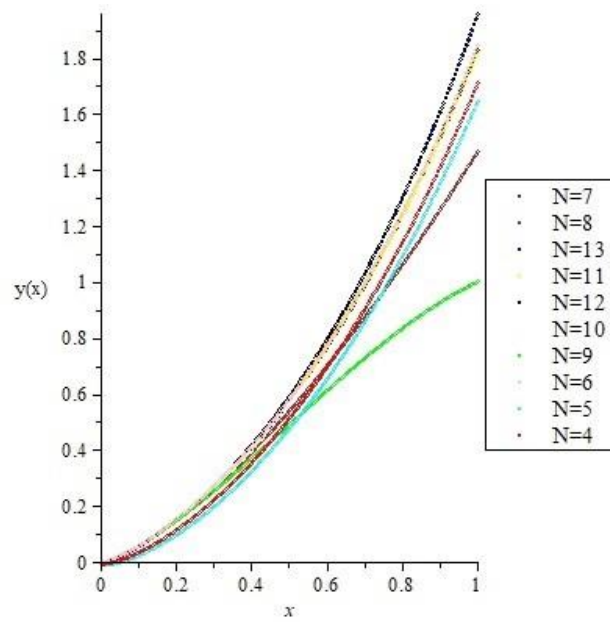


Figure 4.5: The Graph of the Approximate and Exact Solution for Problem 3.8.1

4.4 Discussion of Results

The problems presented in this thesis were solved using Maple 18. Table 1.1 shows the results of the exact solution, standard and perturbed collocation methods. For case $N = 10$, it is observed that the results obtained by Perturbed collocation method proved superior over that of the standard collocation method as the degree of the approximant (N) increases. However, the standard collocation method involved less computational cost than the perturbed collocation method. The extra workdone in perturbed collocation method is compensated for in term of accuracy achieved.

In tables 2.1 and 2.2, the results of the exact, standard, perturbed and exponential fitted collocation methods are displayed. It is observed that the results obtained by exponential fitting collocation method are better in terms of accuracy than the results obtained by standard and perturbed collocation methods without any extra work as the degree of the approximant (N) increases. In tables 3.1, 3.2, 4.1 and 4.2, the three collocation methods were used to solve problems 3.2.2.2 – 4.4.1 with the same constructed canonical polynomials as the new basis. However, it is clear that standard collocation method has the least accuracy among the three methods as the degree of the approximant (N) increases. It is obvious that the

methods performed equal well. For higher singular and multi-order fractional integro-differential equations whose solutions by numerical techniques usually requires their reduction to system of first order differential equations, direct application of the proposed method is recommended.

Finally, the constructed and adoption of the new basis function in this thesis yielded the best approximation solution to all the problems considered therein.

Furthermore, Figure 4.1 – 4.4 describes the graphs of the approximate solutions for different values N , and it is revealed that the behaviour of the graph becomes better when compared with that of the exact solution as the degree of the approximant N increases. Moreover, figure 4.5 describes the graph of the error of estimation, that is, the difference between two successive solutions, which reveals that as N increases the error of estimation becomes smaller. Thus, it asserts that the method is stable.

Chapter 5

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

The derivation of three methods for the numerical solution of singular and multi-order fractional integro-differential equations were carried out in this research work in which:

- (i) canonical polynomials are constructed and used as basis functions;
- (ii) the techniques using the canonical polynomials are demonstrated on some problems; and
- (iii) the graphical illustration and tables of the numerical results are presented, which indicate that the numerical results are well in

agreement with exact solutions.

5.2 Conclusion

Many alternative techniques with the Tau-method have been described for solving singular and multi-order fractional integro-differential equations.

In Chapter 3, it was described how canonical polynomial was easily constructed as new basis for collocation approximations. Three methods based on canonical polynomials are reported in chapter 3 and 4 for solving singular and multi-order fractional integro-differential equations. It is clear that the perturbed collocation method with exponentially fitted is more accurate for all the examples attempted in the work except for problem 3.2.2.1 in which the perturbed collocation method performed better than the standard collocation method since exponentially fitted could not be performed for first order singular and multi-order fractional integro-differential equation. Moreover, the three methods require the solution of system of algebraic equations for any specific value of N (say). The perturbed and the exponentially fitted collocation methods involve solving a large set of matrix equations than the standard collocation

method for the same value of N . It is interesting to compare the three methods for accuracy when the cost of computation is the same. It is observed that the standard collocation method will cost more than both the perturbed and exponentially fitted collocation methods if an accuracy is specified before embarking on computation. It is also seen from the tables of results displayed in chapter 4 that as N increases, the standard collocation method fails to converge compared with both the perturbed and exponentially fitted collocation methods without doing any extra work. Furthermore, an estimate of the errors in the collocation Tau method has been described using a practical approach when the exact solution is not known in closed form. A correction result has been reported for the error, in the sense of Zadunaisky's differential correction principle (see Zadunaisky (1979)).

5.3 Contribution to Knowledge

The contributions to the knowledge of this study are in line with the stated objectives. They are as follows:

- (a) construction of canonical polynomials for the general class of singular and multi-order fractional integro-differential

equations which are different from those mentioned in the literature were obtained;

- (b) formulation of three numerical methods were carried out based on the constructed canonical polynomials used as basis function. The formulation of the numerical methods are easily programmed;
- (c) demonstration of the three proposed methods using the constructed canonical polynomials as basis function on some singular and multi-order fractional integro-differential equations were successfully carried out;
- (d) comparisons of the results obtained and computations of the absolute errors with the available exact solutions in the literature were done.

5.4 Recommendation for Further Study

The findings of the work reported in this thesis has the limitation of being based on singular and multi-order fractional integro-differential equations. In furtherance of this work, it is recommended that the research be extended to singular partial differential equations, integral equations and, singular and multi-order fractional system of integro-differential equations.

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