*Journal of the Nigerian Association of Mathematical Physics*
*Volume 29, (March, 2015)*, pp 163 – 166


# On Existence of a Staged Progression HIV/AID Model with Control Measures

*S.T. Akinyemi[1], M.O Ibrahim[2], M.M. Dago[1], N.G. Bakare[2]*

[1]Department of Mathematics, Kebbi State University of Science and Technology Aliero, Nigeria
[2]Department of Mathematics, University of Ilorin, Kwara State, Nigeria.

*Abstract*

***In this paper we presented a new deterministic model of HIV/AIDS transmission dynamics with public health education program, condom usage and treatment by incorporating the staged progression nature of the disease and the rate of educating uncounseled AIDS individuals. We established that the model exist and has a unique solution using formulated theorems on existence and uniqueness of a solution.***

**Keywords:** HIV/AIDS, Mathematical Model, Control Measures, Existence and Uniqueness of Solution.

## 1.0 Introduction

Human Immunodeficiency Virus (HIV) is a retroviral infections that leads to Acquired Immuno-Deficiency Syndrome (AIDS), is still a global public health problem. HIV is a deadly infection that essentially weakens the immune system by gradually killing or destroying the $CD_4^+$ T cells, making it difficult for the body to fight against opportunistic infections such as cancer, tuberculosis, pneumonia, meningitis, e.t.c

Since it discovery in early 1980s, HIV has killed about 30million people in 2009 and with a global report 34 million people in at 2010 (UNAIDS[1]). In 2007, two-third of HIV infective reside in Sub-Saharan Africa [2] Nigeria ranks third in 2005 among countries with 10 percent of the worlds population of infected individuals [3].

Despite enormous breakthrough in the prevention and treatment of infectious diseases [4,5], HIV remains incurable with no perfect prophylactic vaccine [6-8]. Although Highly Active Anti-Retroviral Therapy (HAART) is used to treat HIV/AIDS with the aim of decreasing AIDS-induced death rate and prolonging the life span of infected individuals only. These pharmaceutical interventions are expensive and difficult to access, so that the effective control of HIV would primarily depend on decreasing behavioural risk through HIV-related public health campaign and condom usage [9].

## 2.0 Model Formulation

The total population of a sexually active population at time t, denoted by N(t) is sub divided into eight (8) mutually exclusive compartments of susceptible individuals S(t), non-counseled $E_u(t)$ and counseled $E_c(t)$ assymptotic individuals, non counseled $I_u(t)$ and counseled $I_c(t)$ symptomatic individuals, non-counseled $A_u(t)$ and counseled $I_c(t)$ individuals with AIDS symptoms and AIDS infected individuals receiving treatment $I_T(t)$, so that

$$N(t) = S(t) + E_u(t) + E_c(t) + I_u(t) + I_c(t) + A_u(t) + A_c(t) + I_T(t) \tag{1}$$

## 3.0 Basic Assumption of the Model

The following assumptions were made in the formulation of the model.

1. Individuals are only recruited into the susceptible class.
2. The studied population is homogeneous and varies with time.
3. Susceptible individuals acquire HIV infection through effective contact with infected individuals in the $E_u, E_c, I_u(t), I_c(t), A_u, A_c$ and $I_T$ classes at a rate $\Gamma$ where

$$\Gamma = \frac{\beta(1-E\alpha)[E_u + \phi_u I_u + \eta_u A_u + \theta_c(E_c + \phi_c I_c + \eta_c A_c) + \theta_T I_T]}{N} \tag{2}$$

is the force of infection.

The model takes the following deterministic system of non-linear equations.

Corresponding author: S.T. Akinyemi, E-mail: sammysalt047@gmail.com, Tel.: +23408169842337, 08036459002 (N.G.B.)

$$\begin{aligned}
\frac{dS}{dt} &= \pi-(\Gamma+\mu)S \\
\frac{dE_u}{dt} &= \Gamma S-(\gamma_1+w+\mu)E_u \\
\frac{dE_c}{dt} &= \gamma_1 E_u-(w+\mu)E_c \\
\frac{dI_u}{dt} &= wE_u-(\gamma_1+\sigma+\mu)I_u \\
\frac{dI_c}{dt} &= wE_c+\gamma_1 I_u-(\sigma+\mu)I_c \\
\frac{dA_u}{dt} &= \sigma I_u-(\gamma_2+\tau_u+\delta+\mu)A_u \\
\frac{dA_c}{dt} &= \sigma I_c+\gamma_2 A_u-(\tau_c+\delta+\mu)A_c \\
\frac{dI_T}{dt} &= \tau_c A_c+\tau_u A_u-(\mu+\psi\delta)I_T
\end{aligned} \tag{3}$$

**Table 1:** Description of the ModelParameters

| Parameters | Interpretation |
|---|---|
| $\pi$ | Recruitment rate of humans |
| $\mu$ | Natural death rate |
| $\beta$ | Effective contact rate |
| $w$ | Progressive rate from $E_u$ to $I_u$ and from $E_c$ to $I_c$ classes |
| $\sigma$ | Progressive rate from $I_u$ to $A_u$ and from $I_c$ to $A_c$ classes |
| $\delta$ | Disease induced death rate of AIDs individuals |
| $\tau_u$ | Treatment rate for individuals in $A_u$ class |
| $\tau$ | Treatment rate for individuals in $A_c$ class |
| $\gamma_1$ | Rate of counselling individuals in $E_u, I_u$ class |
| $\gamma_2$ | Rate of counselling individuals in $A_u$ class |
| $\eta_u, \eta_c$ | Modification parameter associated with infection by AIDs individuals |
| $\theta_T$ | Modification parameter associated with infection by treated individuals |
| $\phi_u, \phi_c$ | Modification parameters associated with infection by individuals in $I_u$ and $I_c$ class |
| $\theta_c$ | Modification parameter associated with infection by counselled individuals |
| $\psi$ | Modification parameter associated with reduced mortality of treated individuals. |
| $\in$ | Condom efficacy |
| $\alpha$ | Condom compliance |

## 4.0 Existence An I Uniqueness of Solution

We formulate theorem on existence of unique solution of system (3) and establish the proof. The system of equations are considered below

$$\left.\begin{aligned}
x_1' &= f_1(t,x_1,\cdots,x_n) \\
x_2' &= f_2(t,x_1,\cdots,x_n) \\
&\vdots \\
x_n' &= f_n(t,x_1,\cdots,x_n)
\end{aligned}\right\} \tag{4}$$

Equation (4) is express in a compact form as

$$x_i' = f_i(t,x), x(t_\circ) = x_\circ, i = 1,\cdots,n \tag{5}$$

**Theorem 1[10]**

Let D denote the region

$$|t-t|\leq a, \| x-x \| \leq b, x=(x_1,x_2,\cdots,x_n), x=(x_{10},x_{20},\cdots,x_n) \quad (6)$$

and suppose that $f(t,u)$ satisfies Lipschitz condition.

$$\| f(t,x_1)-f(t,x_2)\| \leq k \| x_1-x_2 \| \quad (7)$$

whenever the pairs $(t,x_1)$ and $(t,x_2)$ belong to D, where k is a positive constants. Then, there is a constant $\delta>0$ such that there exists a unique continuous vector solution $\underline{x}(t)$ of the system (3) in the interval $|t-t_0|\leq\delta$. It is essential to note that condition (7) is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}$ $i,j=1,2,\cdots,n$ are continuous and bounded in D.

The region of interest is

$$0\leq\xi\leq R$$

and a bounded solution of the form

$$0\leq R<\infty$$

is found in the region D, whose partial derivatives satisfy $\delta\leq\xi\leq 0$, where $\xi$ and $\delta$ are positive constants.

**Theorem 2**

Let D denote the region $0\leq\xi\leq R$. Then, the system (3) has a unique solution which is bounded and continuous in D.

Proof

Let

$$f_1=\pi-(\Gamma+\mu)S \quad (8)$$

$$f_2=\Gamma S-k_1E_u \quad (9)$$

$$f_3=\gamma_1E_u-k_2E_c \quad (10)$$

$$f_4=\omega_u-k_3I_u \quad (11)$$

$$f_5=\omega E_c+\gamma_1I_n-k_4I_c \quad (12)$$

$$f_6=\sigma I_u-k_5A_u \quad (13)$$

$$f_7=\sigma I_c+\gamma_2A_u-k_6A_c \quad (14)$$

$$f_8=\tau_cA_c+\tau_uA_u-k_7I_T \quad (15)$$

where $k_1=\gamma_1+\omega+\mu, k_2=\omega+\mu, k_3=\gamma_1+\sigma+\mu, k_4=\sigma+\mu, k_5=\gamma_2+\tau_u+\sigma+\mu, k_6=\tau_c+\sigma+\mu$ and $k_7=\mu+\psi\sigma$

Thus, the partial derivatives of equations (8)-(15) are given below.

$$\left|\frac{\partial f_1}{\partial s}\right|=|-(\Gamma+\mu)|<\infty, \left|\frac{\partial f_1}{\partial E_u}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)S}{N}\right|<\infty, \left|\frac{\partial f_1}{\partial E_c}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\theta_cS}{N}\right|<\infty$$

$$, \left|\frac{\partial f_1}{\partial I_n}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\varphi_uS}{N}\right|<\infty \left|\frac{\partial f_1}{\partial I_c}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\theta_c\varphi_cS}{N}\right|<\infty;$$

$$\left|\frac{\partial f_1}{\partial A_n}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\eta_uS}{N}\right|<\infty, \left|\frac{\partial f_1}{\partial A_c}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\theta_c\eta_cS}{N}\right|<\infty, \left|\frac{\partial f_1}{\partial I_T}\right|=\left|\frac{-\beta(1-\varepsilon\alpha)\theta_TS}{N}\right|<\infty$$

Similarly, $\left|\frac{\partial f_2}{\partial s}\right|=|\Gamma|<\infty, \left|\frac{\partial f_2}{\partial E_n}\right|=\left|\frac{\beta(1-\varepsilon\alpha)S}{N}-k\right|<\infty, \left|\frac{\partial f_2}{\partial E_c}\right|=\left|\frac{\beta(1-\varepsilon\alpha)\theta_cS}{N}\right|<\infty, \left|\frac{\partial f_2}{\partial I_u}\right|=$

$$\left|\frac{\beta(1-\varepsilon\alpha)\phi_uS}{N}\right|<\infty, \left|\frac{\partial f_2}{\partial I_c}\right|=\left|\frac{\beta(1-\varepsilon\alpha)\theta_c\phi_cS}{N}\right|<\infty, \left|\frac{\partial f_2}{\partial A_n}\right|=\left|\frac{\beta(1-\varepsilon\alpha)\eta_uS}{N}\right|<\infty, \left|\frac{\partial f_2}{A_c}\right|=$$

$$\left|\frac{\beta(1-\varepsilon\alpha)\theta_c\eta_cS}{N}\right|<\infty, \left|\frac{\partial f_2}{\partial I_T}\right|=\left|\frac{\beta(1-\varepsilon\alpha)\theta_TS}{N}\right|<\infty$$

$$\left|\frac{\partial f_3}{\partial s}\right|=\left|\frac{\partial f_3}{\partial I_u}\right|=\left|\frac{\partial f_3}{\partial I_c}\right|=\left|\frac{\partial f_3}{\partial A_u}\right|=\left|\frac{f_3}{\partial A_c}\right|=\left|\frac{\partial f_3}{\partial I_T}\right|=0, \left|\frac{\partial f_3}{\partial E_u}\right|=|\gamma_1|<\infty, \left|\frac{\partial f_3}{\partial E_c}\right|=|-k_2|<\infty$$

$$\left|\frac{\partial f_4}{\partial s}\right| = \left|\frac{\partial f_4}{\partial E_c}\right| = \left|\frac{\partial f_4}{\partial I_c}\right| = \left|\frac{\partial f_4}{\partial A_u}\right| = \left|\frac{f_4}{\partial A_c}\right| = \left|\frac{\partial f_4}{\partial I_T}\right| = 0 < \infty, \left|\frac{\partial f_4}{\partial E_u}\right| = |\omega| < \infty, \left|\frac{\partial f_4}{\partial I_n}\right| = |-k_3| < \infty$$

$$\left|\frac{\partial f_5}{\partial s}\right| = \left|\frac{\partial f_5}{\partial A_u}\right| = \left|\frac{\partial f_5}{\partial A_c}\right| = \left|\frac{\partial f_5}{\partial E_u}\right| = \left|\frac{f_5}{\partial I_T}\right| = 0 < \infty, \left|\frac{\partial f_5}{\partial E_c}\right| = |\omega| < \infty, \left|\frac{\partial f_5}{\partial I_u}\right| = |\gamma_1| < \infty, \left|\frac{\partial f_5}{\partial I_c}\right| = |-k_4| < \infty$$

$$\left|\frac{\partial f_6}{\partial s}\right| = \left|\frac{\partial f_6}{\partial E_u}\right| = \left|\frac{\partial f_6}{\partial E_c}\right| = \left|\frac{\partial f_6}{\partial I_c}\right| = \left|\frac{\partial f_6}{\partial A_c}\right| = \left|\frac{f_6}{\partial I_T}\right| = 0 < \infty, \left|\frac{\partial f_6}{\partial A_u}\right| = |-k_5| < \infty, \left|\frac{\partial f_6}{\partial I_u}\right| = |\sigma| < \infty$$

$$\left|\frac{\partial f_7}{\partial s}\right| = \left|\frac{\partial f_7}{\partial E_u}\right| = \left|\frac{\partial f_7}{\partial E_c}\right| = \left|\frac{\partial f_7}{\partial I_u}\right| = \left|\frac{\partial f_7}{\partial I_T}\right| = 0 < \infty, \left|\frac{\partial f_7}{\partial I_c}\right| < \infty, \left|\frac{\partial f_7}{\partial A_u}\right| = |\gamma_2| < \infty \left|\frac{\partial f_7}{\partial A_c}\right| = |-k_6| < \infty$$

$$\left|\frac{\partial f_8}{\partial s}\right| = \left|\frac{\partial f_8}{\partial E_c}\right| = \left|\frac{\partial f_8}{\partial E_u}\right| = \left|\frac{\partial f_8}{\partial I_u}\right| = \left|\frac{\partial f_8}{\partial I_c}\right| = 0 < \infty, \left|\frac{\partial f_8}{\partial A_c}\right| = |\tau_c| < \infty, \left|\frac{\partial f_8}{\partial A_u}\right| = |\tau_u| < \infty \left|\frac{\partial f_8}{\partial I_T}\right| = |-k_7| < \infty$$

Obviously, all the partial derivatives are continuous and bounded. Hence, Theorem 1 shows that in the region D there exists a unique solution of system (3).

## 5.0 Discussion and Conclusion

In this study, we present a new deterministic model for the spread of HIV, that includes the staged progression nature of HIV/AIDS, rate of counseling uncounseled AIDS individuals with public health education program, condom usage and treatment as control measures. Theorem on existence and uniqueness of solution was formulated and also proved to establish that the model exist and has a unique solution.The model presented in this paper can be used as study material for seminars, workshop or training programmes by public health workers to educate participants on the impact of aforementioned control measures to curb the menace of HIV epidemic.

## 6.0 References


[1]. UNAIDS (2010). Global report fact sheet. http://www.uniads.org/ documents/ 2010123 *Global* 1–10

[2]. Mukandavire, Z. and Garira, W.(2007). Effects of public health educational campaigns and the role of sex workers on the spread of HIV/AIDS among heterosexuals. Applied Mathematical Modelling, 33,2084-2095.

[3]. Eze, J.I.(2009). Modelling HIV/AIDS in Nigeria. Ph.D Dissertation, University of Glasgow.

[4]. Enagi, A.I and Ibrahim, M.O.(2011). Preventing mother to child transmission of Tuberculosis using Bacillus Calmette Geurrin Vaccine. A Deterministic Modelling Approach Research Journal of Mathematics and Statistics, 3(2), 67-71

[5]. Liao, S; and Yang, W.M.(2013). On the dynamics of a vaccination model with multiple transmission ways. Int.J.Appl. math.Comput.Sci, 23(4), 761-772

[6]. Smith, R.J. and Blower, S.M.(2004). Could disease-modifying HIV vaccines cause population level perversity? Lancet Infec. Dis.4, 636-639

[7]. Gummel, A.L Mc Cluskey, C.C; and Van den Driessche, P.(2006). Mathematical study of a staged-progression HIV model with imperfect vaccines. Bullettin of mathematical Biology, 68,2105-2128.

[8]. Elbasha, E.H and Gummel, A.B.(2006). Theoretical assessment of public health impact of imperfect prophylactic HIV-1 vaccines with therapentic benefits. Bullettin of mathematical biology, 68,577-614 doi:10:1007/511538-005-9057-5

[9] Hussaini,N., Winter, N., and Gummel, A.B.(2010). Qualitative assessment of the role of public health education program on HIV transmission dyanamics. IMA Journal of mathematical medicine and Biology. To appear dio:10.1093/imammb dqq2009.

[10]. Derrick, N.R., Grossman, S.L. (1976). Differential Equation with applications, Addison Wesley Publishing Company, Philipines, Inc.