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Approximation of Time - Fractional Partial Differential Equations with Caputo Derivatives by Laplace Sequential Iterative Decomposition Method

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ABSTRACT: *In this paper, iterative decomposition method is coupled with the Laplace transformation method and used to solve time fractional partial differential equations. The results obtained by the new proposed method are exact. Numerical examples are given to illustrate the reliability and applicability of the proposed method.*

Keywords: Iterative, Laplace transformation, Fractional Caputo Derivative, efficiency.

Introduction

Fractional-order partial differential equations which are generalizations of the integer-order partial differential equations are increasingly being used to model many physical phenomena. Examples of such physical phenomena occur in fluid mechanics, viscoelasticity, biology, physics, finance, engineering and other areas of science. Consequently, considerable attention has been given to the solution of various types of fractional differential equations, including the fractional-order partial differential equations (see Agarwal et. al 2010). Considerable attention has been paid to developing accurate and efficient methods of solving fractional partial differential equations (FPDEs). Recently, some approximate methods such as Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), Homotopy Analysis Method (HAM), Fractional Complex Transform (FCT) and Wavelets Method have been used to find an analytical approximation to partial differential equations. The Variational Iteration Method (VIM) which was first proposed by He, (1999) has been shown to be very efficient for handling a wide class of physical

problems. As early as 1998, the Variational Iteration Method was found to be an effective tool for fractional calculus (He, 2012), thereafter, the method has been routinely used to solve various fractional differential equations (He, 2012), for many years.

A modification called fractional VIM was carried by WU (2011) and called Fractional Variational Iteration Method (FVIM) to solve fractional differential equations. In the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomain Decomposition Method (Gejji and Jafari 2007; Momani and Noor 2006; Momani and Shawagfeh 2006; Ray et al 2006; Khan and Faraz 2011), He's Variational Iteration Method (Momani and Odibat 2006), Homotopy Perturbation Method (Mohyud-Din and Noor 2009); Laplace Transform Method (Jumaria 2009); (Duan and Xu 2004; Jumarie 2006). In this work, we coupled the Iterative Decomposition Method with Laplace transform to achieve our new proposed method termed Laplace Sequential Iterative Decomposition Method (LSIDM).

The LSIDM is applied to solve time fractional partial differential equations. The method does not require any linearization for nonlinear problems, unlike some existing iterative methods. Also, LSIDM does not require rigorous mathematical details. The solutions are presented as infinite series of easily compatible terms, with convergence guaranteed. We used the proposed method to solve some time-fractional partial differential equations and the results obtained are presented in the body of the paper.

Basic Definitions and Idea

In this section, some useful definitions and ideas are presented

Theorem 2.1 (Weilbeer, 2005) : (fundamental theorem of classical calculus)

Let $f : [a, b] \rightarrow R$ be a continuous function, and let $F : [a, b] \rightarrow R$ be defined as

$$U(x) = \int_a^x u(t)dt \quad (2.1)$$

Then, $U(x)$ is differentiable and $U' = u$. This shows that we have a very close relation between differential operators and integral operators

Definition 2.1:

- (a) Let D denote an operator that maps a differentiable function onto its derivatives, then

$$Du(x) = u'(x) \quad (2.2)$$

- (b) Let J_a denote the operator that maps a function u and assumed to be Riemann integrable on the compact interval $[a, b]$, onto its primitive centered at a , then

$$J_a u = \int_a^x u(t) dt \quad \text{for.} \\ a \leq x \leq b \quad (2.3)$$

(c) For $n \in \mathbb{N}$, we use the symbols D^n and J_a^n to denote the n -fold iterates of D and J_a , respectively, i.e. set

$$D^1 = D, J_a^1 = J_a, \quad \text{and } D^n = DD^{n-1} \quad \text{and } J_a^n = J_a J_a^{n-1} \quad \text{for} \\ n \geq 2 \quad (2.4)$$

Definition 2.2

The function: $\Gamma(0, \infty) \rightarrow \mathbb{R}$, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (2.5)$$

is called Euler's Gamma function (or Euler's integral of the second kind).

Definition 2.3

A real function $u(x)$, $x > 0$ is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $u(x) = x^p v(x)$ where $v(x) \in C[0, \infty]$ and it is said to be in the space C_μ^m iff

$$u^{(m)}(x) \in C_\mu, m \in \mathbb{N}. \quad (2.6)$$

Definition 2.4

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $u(x) \in C_\mu$, $\mu \geq -1$, is defined as:

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad \alpha > 0, x > 0 \quad (2.7)$$

where Γ is the Gamma function. Some of the most important properties of operator J^α are as follows

for $u(x) \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$

$$i) \quad J^\alpha J^\beta u(x) = J^{\alpha+\beta} u(x)$$

$$ii) \quad J^\alpha J^\beta u(x) = J^\beta J^\alpha u(x)$$

$$iii) J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \quad (2.8)$$

Definition 2.5

The fractional derivative of $u(x)$ in the Caputo's sense is defined

$$\text{as: } D_{*}^{\alpha} u(x) = J^{m-\alpha} D^m u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t) dt \quad (2.9)$$

where

$$m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0, u(x) \in C_{-1}^m$$

Caputo's Fractional Derivatives

Caputo's fractional derivative has a useful property given as

$$J^{\alpha} [D_{*}^{\alpha} f(x)] = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0), \quad m-1 < \alpha < m \quad (2.10)$$

Furthermore, for the Caputo derivatives, we have

$$D_{*}^{\alpha} (c) = 0 \quad (2.11)$$

where C is a constant

$$D_{*}^{\alpha} x^{\beta} = \begin{cases} 0, & \beta < \alpha-1 \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta > \alpha-1 \end{cases} \quad (2.12)$$

Definition 2.6

The Laplace transform of a function $u(t)$

$$L\{u(t)\} = \int_0^{\infty} e^{-st} u(t) dt = u(s) \quad (2.13)$$

where s is either real or complex

Discussion of the proposed Laplace sequential Iterative Decomposition Method (LSIDM)

We consider the Caputo form of two –Dimensional Time fractional differential equation of the form:

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = a(x, y, t) \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u(x, y, t)}{\partial y^\gamma} + f(x, y, t)$$

$$0 < x < L_x \quad 0 < y < L_y, \quad (3.1)$$

with the initial conditions:

$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y, 0) = \psi(x, y) \quad 0 \leq x \leq L_x \quad 0 \leq y \leq L_y, \quad (3.2)$$

where fractional order $0 < \alpha < 2, 1 < \beta \leq 2, 1 < \gamma \leq 2$.

The diffusion coefficients $a(x, y, t) > 0$, $b(x, y, t) > 0$ and the function $f(x, y, t)$ is representing source term.

Applying Laplace transform to both sides of equation (3.1), we obtain

$$L\left\{\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha}\right\} = L\left\{a(x, y, t) \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u(x, y, t)}{\partial y^\gamma} + f(x, y, t)\right\} \quad (3.3)$$

Where,

$$L\left\{\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha}\right\} = S^\alpha u(x, y, t) - S^{\alpha-1} u(x, y, 0) - S^{\alpha-2} u'(x, y, 0) - \dots - \quad (3.4a)$$

Substituting (3.2) into (3.4a), we obtain

$$L\left\{\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha}\right\} = S^\alpha u(x, y, s) - S^{\alpha-1} \{\phi(x, y) - S^{\alpha-2} \{\psi(x, y)\}\} \quad (3.4b)$$

Substituting (3.4b) into (3.3) to obtain

$$S^\alpha u(x, y, s) - S^{\alpha-1} \{\phi(x, y)\} - S^{\alpha-2} \{\psi(x, y)\} = \\ L\{a(x, y, t) \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u(x, y, t)}{\partial y^\gamma} + f(x, y, t)\} \quad (3.5)$$

Therefore, (3.5) becomes

$$u(x, y, s) = \\ \frac{1}{S} \{\phi(x, y)\} + \frac{1}{S^2} \{\psi(x, y)\} + \frac{L}{S^\alpha} \{a(x, y, t) \frac{\partial^\beta u(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u(x, y, t)}{\partial y^\gamma} + f(x, y, t)\} \quad (3.6)$$

Now, (3.6) is now decomposed into u_0 and u_n

Let,

$$u_0(x, y, s) = \frac{1}{S} \{\phi(x, y)\} + \frac{1}{S^2} \{\psi(x, y)\} + \frac{L}{S^\alpha} f(x, y, t) \quad (3.7)$$

and in general, the remaining terms are written as

$$u_n(x, y, s) = \frac{L}{S^\alpha} \{a(x, y, t) \frac{\partial^\beta u_{n-1}(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_{n-1}(x, y, t)}{\partial y^\gamma}\} \quad n \geq 1 \quad (3.8)$$

Now, taking the inverse Laplace of (3.8) we obtain

$$u_0(x, y, t) = L^{-1} \left\{ \frac{1}{S} \{\phi(x, y)\} + \frac{1}{S^2} \{\psi(x, y)\} + f(x, y, t) \right\} \quad (3.9)$$

Thus, for case $n=1$, (3.9) gives

$$u_1(x, y, s) = \frac{L}{S^\alpha} \{a(x, y, t) \frac{\partial^\beta u_0(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_0(x, y, t)}{\partial y^\gamma}\} \quad (3.10)$$

Taking Laplace transform of equation (3.10), we obtain

$$u_1(x, y, t) = L^{-1} \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta u_0(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_0(x, y, t)}{\partial y^\gamma} \right\} \right\} \quad (3.11)$$

After obtaining the solution for case $n=1$, the traditional iterative decomposition method

is then modified and given by the recurrence relation

$$\begin{aligned} a(x, y, t) \frac{\partial^\beta (u_0 + u_1 - u_0)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1 - u_0)}{\partial y^\gamma} = \\ a(x, y, t) \frac{\partial^\beta (u_0 + u_1)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1)}{\partial y^\gamma} - \\ (a(x, y, t) \frac{\partial^\beta u_0(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_0(x, y, t)}{\partial y^\gamma}) \end{aligned} \quad (3.12)$$

Thus,

$$\begin{aligned} u_2(x, y, t) = \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta (u_0 + u_1)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1)}{\partial y^\gamma} \right\} \right. \\ \left. - \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta u_0(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_0(x, y, t)}{\partial y^\gamma} \right\} \right\} \right\} \end{aligned} \quad (3.13)$$

Applying the inverse Laplace transform to both sides of (3.14) to obtain

$$\begin{aligned} u_2(x, y, t) = L^{-1} \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta (u_0 + u_1)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1)}{\partial y^\gamma} \right\} \right. \\ \left. - \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta u_0(x, y, t)}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma u_0(x, y, t)}{\partial y^\gamma} \right\} \right\} \right\} \end{aligned} \quad (3.14)$$

Then, it is generalized by the formula given as:

$$\begin{aligned} u_n(x, y, t) = L^{-1} \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta (u_0 + u_1 + \dots + u_{n-1})}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1 + \dots + u_{n-1})}{\partial y^\gamma} \right\} \right. \\ \left. - \left\{ \frac{L}{S^\alpha} \left\{ a(x, y, t) \frac{\partial^\beta (u_0 + u_1 + \dots + u_{n-2})}{\partial x^\beta} + b(x, y, t) \frac{\partial^\gamma (u_0 + u_1 + \dots + u_{n-2})}{\partial y^\gamma} \right\} \right\} \right\} \end{aligned} \quad (3.15)$$

Also, the solution is now written as solutions as

$$u = \sum_{n=0}^{n=\infty} u_n \quad (3.16)$$

Numerical Examples

Example 1

Consider the following fractional diffusion equation

$$\frac{\partial^{0.5} u}{\partial t^{0.5}} = a(x, y, t) D_x^{1.2} u + b(x, y, t) D_y^{1.8} u + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, 1],$$

$$u(x, y, 0) = x^3 y^3, \quad (x, y) \in \Omega$$

$$u(0, y, t) = 0, \quad u(1, y, t) = (t^2 + 1) y^3, \quad 0 \leq y \leq 1, \quad t > 0$$

$$u(x, 0, t) = 0, \quad u(x, 1, t) = (t^2 + 1) x^3, \quad 0 \leq x \leq 1, \quad t > 0$$

Where,

$$a(x, y, t) = \frac{\Gamma(2.8)}{\Gamma(4)} x^{1.2},$$

$$b(x, y, t) = \frac{\Gamma(2.2)}{\Gamma(4)} y^{1.8},$$

$$f(x, y, t) = x^3 y^3 \left(\frac{8}{3\Gamma(0.5)} t^{1.5} - 2t^2 - 2 \right)$$

And,

$$\Omega = \{(x, y) | 0 < x < 1, 0 < y < 1\},$$

The exact solution of the above equation is given as

$$u(x, y, t) = (t^2 + 1) x^3 y^3.$$

Solution:

$$\frac{\partial^{0.5} u}{\partial t^{0.5}} = \frac{\Gamma(2.8)}{\Gamma(4)} x^{1.2} D_x^{1.2} u + \frac{\Gamma(2.2)}{\Gamma(4)} y^{1.8} D_y^{1.8} u + \frac{x^3 y^3 8}{3\Gamma(0.5)} t^{1.5} - 2x^3 y^3 t^2 - 2x^3 y^3$$

Taking Laplace transform of both sides of the above equation, we obtain

$$L\left\{\frac{\partial^{0.5}u}{\partial t^{0.5}}\right\}=L\left\{\frac{\Gamma(2.8)}{\Gamma(2.4)}x^{1.2}D_x^{1.2}u+\frac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}D_y^{1.8}u+\frac{8x^3y^3}{3\Gamma(0.5)}t^{1.5}-2x^3y^3t^2-2x^3y^3\right\}$$

Where,

$$L\left\{\frac{\partial^{0.5}u}{\partial t^{0.5}}\right\}=S^{0.5}u(x,y,s)-S^{0.5-1}u(x,y,0)-\dots$$

Using the initial condition

$$L\left\{\frac{\partial^{0.5}u}{\partial t^{0.5}}\right\}=S^{0.5}u(x,y,s)-S^{-0.5}(x^3y^3)$$

Therefore,

$$S^{0.5}u(x,y,s)-\frac{x^3y^3}{s^{0.5}}=$$

$$L\left\{\frac{\Gamma(2.8)}{\Gamma(4)}x^{1.2}D_x^{1.2}u+\frac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}D_y^{1.8}u+\frac{8x^3y^3}{3\Gamma(0.5)}t^{1.5}-2x^3y^3t^2-2x^3y^3\right\}$$

$$U(x,y,s)=$$

$$\frac{x^3y^3}{s}+\frac{L}{s^{0.5}}\left\{\frac{\Gamma(2.8)}{\Gamma(4)}x^{1.2}D_x^{1.2}u+\frac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}D_y^{1.8}u+\frac{8x^3y^3}{3\Gamma(0.5)}t^{1.5}-2x^3y^3t^2-2x^3y^3\right\}$$

Taking

$$u_0(x,y,s)=\frac{x^3y^3}{s}+\frac{L}{s^{0.5}}\left\{\frac{8x^3y^3}{3\Gamma(0.5)}t^{1.5}\right\}$$

And,

$$u_n(x,y,s)=\frac{L}{s^{0.5}}\left\{\frac{\Gamma(2.8)}{\Gamma(4)}x^{1.2}D_x^{1.2}u_{n-1}+\frac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}D_y^{1.8}u_{n-1}-2x^3y^3t^2-2x^3y^3\right\}$$

$$\text{Now } u_0(x,y,s)=\frac{x^3y^3}{s}+\frac{L}{s^{0.5}}\left\{\frac{8x^3y^3}{3\Gamma(0.5)}t^{1.5}\right\}$$

$$u_0(x,y,s)=\frac{x^3y^3}{s}+\frac{1}{s^{0.5}}\left\{\frac{8x^3y^3\Gamma(2.5)}{3\Gamma(0.5)s^{2.5}}\right\}$$

$$u_0(x,y,s)=\frac{x^3y^3}{s}+\left\{\frac{8x^3y^3\Gamma(2.5)}{3\Gamma(0.5)s^3}\right\}$$

$$u_0(x,y,t)=x^3y^3+x^3y^3t^2$$

$$\begin{aligned}
 u_0(x, y, t) &= x^3 y^3 (1 + t^2) \\
 u_1(x, y, s) &= \frac{L}{s^{0.5}} \left\{ \frac{\Gamma(2.8)}{\Gamma(4)} x^{1.2} D_x^{1.2} u_0 + \frac{\Gamma(2.2)}{\Gamma(4)} y^{1.8} D_y^{1.8} u_0 - 2x^3 y^3 t^2 - 2x^3 y^3 \right\} \\
 u_1(x, y, s) &= \\
 \frac{L}{s^{0.5}} \left\{ \frac{\Gamma(2.8)}{\Gamma(4)} x^{1.2} D_x^{1.2} (x^3 y^3 (1 + t^2)) + \frac{\Gamma(2.2)}{\Gamma(4)} y^{1.8} D_y^{1.8} (x^3 y^3 (1 + t^2)) - 2x^3 y^3 t^2 - 2x^3 y^3 \right\} \\
 u_1(x, y, s) &= \frac{L}{s^{0.5}} \left\{ (x^3 y^3 + x^3 y^3 t^2 + x^3 y^3 + x^3 y^3 t^2 - 2x^3 y^3 (t^2 + 1)) \right\} \\
 u_1(x, y, s) &= \frac{L}{s^{0.5}} \left\{ (2x^3 y^3 (1 + t^2) - 2x^3 y^3 (t^2 + 1)) \right\} \\
 u_1(x, y, s) &= \frac{L}{s^{0.5}} \{ (0) \} \\
 u_1(x, y, t) &= 0 \\
 u &= u_0 + u_1 \\
 u &= x^3 y^3 (1 + t^2) \\
 u_2(x, y, t) &= 0 \\
 u &= u_0 + u_1 + u_2 \\
 u &= x^3 y^3 (1 + t^2)
 \end{aligned}$$

Example2

Consider the following fractional diffusion equation

$$\frac{\partial^{0.4} u}{\partial t^{0.4}} = \frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad (x, y, t) \in \Omega^* (0, 1]$$

$$u|_{\partial\Omega} = 0, \quad u(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Omega$$

Where,

$$f(x, y, t) = \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 2) \sin(\pi x) \sin(\pi y), \quad \Omega = \{(x, y) | 0 < x < 1, 0 < y < 1\}$$

And $\partial\Omega$ is the boundary of Ω . The exact solution of the above equation is $u(x, y, t) = (t^2 + 1) \sin(\pi x) \sin(\pi y)$

Solution:

$$\frac{\partial^{0.4} u}{\partial t^{0.4}} = \frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 2) \sin(\pi x) \sin(\pi y)$$

This is rewritten as

$$\begin{aligned} \frac{\partial^{0.4} u}{\partial t^{0.4}} &= \frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y) \\ &+ \frac{25t^{1.6}}{12\Gamma(0.6)} \sin(\pi x) \sin(\pi y) \end{aligned}$$

Taking Laplace transform of both sides of the equation above we obtain

$$\begin{aligned} L\left\{\frac{\partial^{0.4} u}{\partial t^{0.4}}\right\} &= L\left\{\frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y)\right\} \\ &+ L\left\{\frac{25t^{1.6}}{12\Gamma(0.6)} \sin(\pi x) \sin(\pi y)\right\} \end{aligned}$$

Considering the left side above,

$$L\left\{\frac{\partial^{0.4} u}{\partial t^{0.4}}\right\} = s^{0.4} u(x, y, t) - s^{0.4-1} u(x, y, 0)$$

Using the initial condition on the above equation, we obtain

$$L\left\{\frac{\partial^{0.4} u}{\partial t^{0.4}}\right\} = s^{0.4} u(x, y, t) - s^{0.4-1} \sin(\pi x) \sin(\pi y)$$

Therefore,

$$s^{0.4} u(x, y, s) - s^{0.4-1} \sin(\pi x) \sin(\pi y) = L\left\{\frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + \right.$$

$$\frac{25t^{1.6}}{12\Gamma(0.6)}(t^2 + 1)\sin(\pi x)\sin(\pi y)\} + L\left\{\frac{25t^{1.6}}{12\Gamma(0.6)}\sin(\pi x)\sin(\pi y)\right\}$$

which gives

$$u(x, y, s) = \frac{\sin(\pi x)\sin(\pi y)}{s} + \frac{L}{s^{0.4}} \left\{ \frac{2t^{1.6}}{\pi^2\Gamma(0.6)} \frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2\Gamma(0.6)} \frac{\partial^2 u}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)}(t^2 + 1)\sin(\pi x)\sin(\pi y)\right\} + \frac{L}{s^{0.4}} \left\{ \frac{25t^{1.6}}{12\Gamma(0.6)}\sin(\pi x)\sin(\pi y)\right\}$$

Now let,

$$u_0(x, y, s) = \frac{\sin(\pi x)\sin(\pi y)}{s} + \frac{L}{s^{0.4}} \left\{ \frac{25t^{1.6}}{12\Gamma(0.6)}\sin(\pi x)\sin(\pi y)\right\}$$

And

$$u_{n+1}(x, y, s) = \frac{L}{s^{0.4}} \left\{ \frac{2t^{1.6}}{\pi^2\Gamma(0.6)} \frac{\partial^2 u_n(x, y, t)}{\partial x^2} + \frac{t^{1.6}}{12\pi^2\Gamma(0.6)} \frac{\partial^2 u_n(x, y, t)}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)}(t^2 + 1)\sin(\pi x)\sin(\pi y)\right\}$$

Considering

$$u_0(x, y, s) = \frac{\sin(\pi x)\sin(\pi y)}{s} + \frac{1}{s^{0.4}} \left\{ \frac{25\Gamma(2.6)}{12\Gamma(0.6)s^{2.6}}\sin(\pi x)\sin(\pi y)\right\}$$

leads to

$$u_0(x, y, s) = \frac{\sin(\pi x)\sin(\pi y)}{s} + \frac{25\Gamma(2.6)}{12\Gamma(0.6)s^3}\sin(\pi x)\sin(\pi y)$$

Applying the inverse Laplace transform to both sides of the above equation ,we obtain

$$u_0(x, y, t) = L^{-1}\left\{\frac{\sin(\pi x)\sin(\pi y)}{s} + \frac{25\Gamma(2.6)}{12\Gamma(0.6)s^3}\sin(\pi x)\sin(\pi y)\right\}$$

leads to

$$u_0(x, y, t) = \sin(\pi x) \sin(\pi y) + \frac{25\Gamma(2.6)t^2}{12\Gamma(0.6)2} \sin(\pi x) \sin(\pi y)$$

And we have

$$u_0(x, y, t) = \sin(\pi x) \sin(\pi y) + t^2 \sin(\pi x) \sin(\pi y)$$

This is simplified as

$$u_0(x, y, t) = (t^2 + 1) \sin(\pi x) \sin(\pi y)$$

For case $n = 1$ and substituting u_0 into u_1 , we obtain

$$\begin{aligned} u_1(x, y, s) &= \frac{L}{s^{0.4}} \left\{ \frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} \frac{\partial^2 (t^2 + 1) \sin(\pi x) \sin(\pi y)}{\partial x^2} \right. \\ &\quad \left. + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} \frac{\partial^2 (t^2 + 1) \sin(\pi x) \sin(\pi y)}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y) \right\} \\ u_1(x, y, s) &= \frac{L}{s^{0.4}} \left\{ \frac{2t^{1.6}}{\pi^2 \Gamma(0.6)} (-\pi^2 (t^2 + 1) \sin(\pi x) \sin(\pi y)) \right. \\ &\quad \left. + \frac{t^{1.6}}{12\pi^2 \Gamma(0.6)} (-\pi^2 (t^2 + 1) \sin(\pi x) \sin(\pi y)) + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y) \right\} \\ u_1(x, y, s) &= \frac{L}{s^{0.4}} \left\{ \frac{-25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y) + \frac{25t^{1.6}}{12\Gamma(0.6)} (t^2 + 1) \sin(\pi x) \sin(\pi y) \right\} \\ u_1(x, y, s) &= \frac{L}{s^{0.4}} \{0\} \\ u_1(x, y, t) &= 0 \\ u &= u_0 + u_1 \\ u &= (t^2 + 1) \sin(\pi x) \sin(\pi y) + 0 \\ u_2 &= u_1 = 0 \\ u &= u_0 + u_1 + u_2 \\ u &= (t^2 + 1) \sin(\pi x) \sin(\pi y) \end{aligned}$$

TABLE 1 FOR EXAMPLE 1

T	x	$y=0.25$		$y=0.50$		$y = 0.75$		$y = 1.00$	
		exact LSIDM		exact LSIDM		LSIDM Exact		LSIDM Exact	
0.25	0.25	0.53125	0.53125	0.75130	0.75130	0.53125	0.53125	0.00000	0.00000
	0.50	0.75130	0.75130	1.06250	1.06250	0.75130	0.75130	0.00000	0.00000
	0.75	0.53125	0.53125	0.75130	0.75130	0.53125	0.53125	0.00000	0.00000
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.5	0.25	0.62500	0.62500	0.88388	0.88388	0.62500	0.62500	0.00000	0.00000
	0.50	0.88388	0.88388	1.25000	1.25000	0.88388	0.88388	0.00000	0.00000
	0.75	0.62500	0.62500	0.88388	0.88388	0.62500	0.62500	0.00000	0.00000
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.75	0.25	0.78125	0.78125	1.10485	1.10485	0.78125	0.78125	0.00000	0.00000
	0.50	1.10485	1.10485	1.56250	1.56250	1.01506	1.01506	0.00000	0.00000
	0.75	0.78125	0.78125	0.78125	0.78125	0.78125	0.78125	0.00000	0.00000
	1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 2 FOR EXAMPLE 2

t	X	$y=0.25$		$y=0.50$		$y = 0.75$		$y = 1.00$	
		exact	LSIDM	exact	LSIDM	LSIDM	Exact	LSIDM	Exact
0.25	0.25	0.000259399	0.000259399	0.0020751953	0.0020751953	0.007003784	0.007003784	0.01660156	0.01660156
	0.50	0.0020751950	0.0020751950	0.0166015600	0.0166015600	0.056030270	0.056030270	0.13281250	0.13281250
	0.75	0.0070037840	0.0070037840	0.0560302700	0.0560302700	0.189102170	0.189102170	0.44824200	0.44824200
	1.00	0.0166015600	0.0166015600	0.1328125000	0.1328125000	0.448242000	0.448242000	1.06250000	1.06250000
0.50	0.25	0.0003051758	0.0003051758	0.0024414060	0.0024414060	0.008239750	0.008239750	0.01953125	0.01953125
	0.50	0.0024414060	0.0024414060	0.0195312500	0.0195312500	0.065917970	0.065917970	0.15625000	0.15625000
	0.75	0.0082397460	0.0082397460	0.0659179700	0.0659179700	0.222473000	0.222473000	0.52734400	0.52734400
	1.00	0.0195312500	0.0195312500	0.1562500000	0.1562500000	0.527343000	0.527343000	1.25000000	1.25000000
0.75	0.25	0.0003814697	0.0003814697	0.0030517578	0.0030517578	0.010299680	0.010299680	0.02441406	0.02441406
	0.50	0.0030517578	0.0030517578	0.0244140600	0.0244140600	0.082397460	0.082397460	0.19531250	0.19531250
	0.75	0.0102996800	0.0102996800	0.0823974600	0.0823974600	0.278091400	0.278091400	0.65179690	0.65179690
	1.00	0.0244141000	0.0244141000	0.1953125000	0.1953125000	0.659179700	0.659179700	1.56250000	1.56250000

6. Conclusion

The LSIDM method has been constructed to solve 2-Dimensional Time fractional differential equations. The method produced the exact solutions for examples 1 and 2, as these are evident in Tables 1 and 2.

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