

Chapter 1

On the Coefficients of a Certain Class of Analytic Functions

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Abstract

In this paper we solve certain coefficient problems for functions of the class $T_n^\alpha(\beta)$ introduced in [7] by Opoola. In particular, for some index $\alpha > 0$, we obtain sharp bounds for the coefficients of these functions. Furthermore we give some rough estimate of the general coefficient bounds using the logarithmic coefficient approach. We also prove the analogue of the well-known Fekete-Szegő theorem for this class of functions.

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1. Introduction

Let A denote the class of functions:

$$f(z) = z + a_2 z^2 + \dots \quad (1.1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Also let P be the class of functions:

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \quad (1.2)$$

which are analytic in E and have positive real part.

In [7] Opoola introduced the subclass $T_n^\alpha(\beta)$ consisting of functions $f \in A$ which satisfy the inequality:

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta, \quad (1.3)$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, D^n ($n \in N_0 = \{0, 1, 2, \dots\}$) is the Salagean derivative operator defined as

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]' \quad (1.4)$$

with $D^0 f(z) = f(z)$ and powers in (1.3) meaning principal values only. The geometric condition (1.3) slightly modifies the one given originally in [7] (see [2]).

The coefficient problem for various families of analytic and univalent functions is basically about the search for precise bounds of the coefficient a_k . This problem has attracted much attention from researchers in the last century. In this work we attempt to solve this problem for the class $T_n^\alpha(\beta)$. Our main tool is the close association which exists between the class $T_n^\alpha(\beta)$ and the family P of Caratheodory functions. Though the problem is not completely solved for all values the index α could assume, yet considerable breakthrough has been recorded in the determination of the *best possible* estimate for certain $\alpha > 0$. Furthermore we give some rough estimates of the general coefficient bounds using the logarithmic coefficient approach.

Another closely related problem is the determination of a sharp bound for the Fekete-Szego functional $|a_3 - \lambda a_2^2|$ for both real and complex values of the parameter λ . The Fekete-Szego problem has its origin in a conjecture of Littlewood and Parley (1932) that the true bound on the coefficients of an odd univalent function is 1. This was disproved in 1933 by Fekete and Szego via the determination of the sharp bound on the functional (see [3]). Thus the functional has continued to receive attention of researchers in geometric function theory.

We state some preliminary lemmas in Section 2. while our main results are proved in Section 3..

2. Preliminaries

We shall need the following well-known lemmas:

Lemma 2.1 ([5]). *If $|z|$ is less than the radius of convergence of either of the power series $\sum_{k=0} a_k z^k$ and $\sum_{k=0} b_k z^k$, then the product of the two series is $\sum_{k=0} c_k z^k$, where*

$$c_k = \sum_{j=0}^k a_j b_{k-j}. \quad (2.1)$$

Inductively, the following corollary is true.

Corollary 2.2. *Let $\sum_{k=0} c_k z^k$ be a power series. Then the m th integer product is*

$$\sum_{k=0} c_k^{(m)} z^k = \left(\sum_{k=0} c_k z^k \right)^m$$

where $c_k^{(1)} = c_k$ and

$$c_k^{(m)} = \sum_{j=0}^k c_j c_{k-j}^{(m-1)}, \quad m = 2, 3, \dots \quad (2.2)$$

Lemma 2.3 ([4]). Let $p \in P$. Then $|p_k| \leq 2$, $k = 1, 2, 3, \dots$. Equality is attained for the Moebius function $L_0(z)$.

Lemma 2.4 ([8]). Let $p \in P$. Then

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2} \quad (2.3)$$

This result is sharp. Equality holds for the function

$$p(z) = \frac{1 + \frac{1}{2}(p_1 + \varepsilon \overline{p_1})z + \varepsilon z^2}{1 - \frac{1}{2}(p_1 - \varepsilon \overline{p_1})z - \varepsilon z^2}, \quad |\varepsilon| = 1. \quad (2.4)$$

Corollary 2.5. Let $p \in P$. Then we have the sharp inequalities

(1) for any real number σ ,

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leq \begin{cases} 2(1 - \sigma), & \text{if } \sigma \leq 0; \\ 2, & \text{if } 0 \leq \sigma \leq 2; \\ 2(\sigma - 1), & \text{if } \sigma \geq 2. \end{cases} \quad (2.5)$$

(2) for any complex number σ ,

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leq \begin{cases} 2, & \text{if } |1 - \sigma| \leq 1; \\ 2|1 - \sigma|, & \text{if } |1 - \sigma| \geq 1. \end{cases} \quad (2.6)$$

Proof. We write

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| = \left| p_2 - \frac{p_1^2}{2} + \frac{p_1^2}{2} - \sigma \frac{p_1^2}{2} \right|. \quad (2.7)$$

By the triangle inequality and an application of Lemma 2.3, (2.7) gives

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2} (1 - |1 - \sigma|). \quad (2.8)$$

Now if $1 - |1 - \sigma| \geq 0$ (that is $0 \leq \sigma \leq 2$) we have the second inequality of (2.5). Otherwise (that is $\sigma \leq 0$ or $\sigma \geq 2$) we have

$$\left| p_2 - \sigma \frac{p_1^2}{2} \right| \leq 2|1 - \sigma|. \quad (2.9)$$

The inequality (2.9) gives the first inequality of (2.5) for $\sigma \leq 0$ and the third if $\sigma \geq 2$.

The inequalities in (2.6) follow easily from (2.8) in the case σ is a complex number. This completes the proof of the lemma. $\square$

Lemma 2.6 ([3]). Let $\varphi(z) = \sum_{k=1}^{\infty} \eta_k z^k$ be an arbitrary power series with positive radius of convergence and normalized by $\varphi(0) = 0$. Suppose $\exp(\varphi(z)) = \sum_{k=0}^{\infty} \mu_k z^k$, $\mu_0 = 1$. Then

$$|\mu_k|^2 \leq \exp \left\{ \sum_{j=1}^k \left(j |\eta_j|^2 - \frac{1}{j} \right) \right\}.$$

Now the main results.

3. Main Results

Theorem 3.1. Let $f \in T_n^\alpha(\beta)$. Define

$$B_m = \frac{2^m(1-\beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1-j\alpha)}{m!}, \quad (3.1)$$

$$A_k(n, \alpha, \beta) = \sum_{m=1}^{k-1} B_m Q_{k-1}^{(m)}, \quad (3.2)$$

$$A_{k-1}(n, \alpha, \beta) = \sum_{m=1}^{k-2} B_m Q_{k-1}^{(m)}, \quad (3.3)$$

where for each $m = 1, 2, \dots$, $Q_{k-1}^{(m)}$ is defined by the power series

$$\left(\sum_{k=1}^{\infty} \frac{z^k}{(\alpha+k)^n} \right)^m = Q_m^{(m)} z^m + Q_{m+1}^{(m)} z^{m+1} + Q_{m+2}^{(m)} z^{m+2} + \dots \quad (3.4)$$

Also let

$$\begin{aligned} \Omega_1 &= \{\alpha | 0 < \alpha < (k-2)^{-1}, \quad k = 2, 3, \dots\}, \\ \Omega_2 &= \{\alpha | (k-2)^{-1} \leq \alpha \leq (k-3)^{-1}, \quad k = 4, 6, \dots\} \quad \text{and} \\ \Omega_3 &= \{\alpha | (k-2)^{-1} \leq \alpha < (k-3)^{-1}, \quad k = 3, 5, \dots\} \end{aligned}$$

Then

$$|a_k| \leq \begin{cases} A_k, & \text{if } \alpha \in \Omega_1 \cup \Omega_2; \\ A_{k-1}, & \text{if } \alpha \in \Omega_3. \end{cases} \quad (3.5)$$

The inequalities are sharp. Equalities are attained for $f(z)$ satisfying

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \begin{cases} \frac{1+(1-2\beta)z}{1-z}, & \text{if } \alpha \in \Omega_1 \cup \Omega_2, \\ \frac{1+(1-2\beta)z^2}{1-z^2}, & \text{if } \alpha \in \Omega_3. \end{cases} \quad (3.6)$$

Proof. Since $f \in T_n^\alpha(\beta)$, there exists an analytic function $p_n(z)$ such that

$$f(z)^\alpha = z^\alpha [\beta + (1-\beta)p_n(z)], \quad (3.7)$$

where

$$p_n(z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^n}{(\alpha+k)^n} p_k z^k$$

(see [2, Lemma 4.2]). Thus we have

$$f(z) = z \left(1 + (1-\beta)\alpha^n \sum_{k=1}^{\infty} \frac{p_k z^k}{(\alpha+k)^n} \right)^{\frac{1}{\alpha}}. \quad (3.8)$$

Equation (3.8) expands binomially as

$$\begin{aligned} \frac{f(z)}{z} = 1 + \frac{(1-\beta)\alpha^n}{\alpha} \sum_{k=1} \frac{p_k z^k}{(\alpha+k)^n} + \frac{(1-\beta)^2 \alpha^{2n} (1-\alpha)}{2! \alpha^2} \left(\sum_{k=1} \frac{p_k z^k}{(\alpha+k)^n} \right)^2 + \dots \\ + \frac{(1-\beta)^m \alpha^{mn} \prod_{j=0}^{m-1} (1-j\alpha)}{m! \alpha^m} \left(\sum_{k=1} \frac{p_k z^k}{(\alpha+k)^n} \right)^m + \dots \quad (3.9) \end{aligned}$$

Using Corollary 2.2 in (3.9), we get

$$\frac{f(z)}{z} = 1 + \sum_{k=1} \tilde{B}_1 C_k^{(1)} z^k + \dots + \sum_{k=m} \tilde{B}_m C_k^{(m)} z^k + \dots, \quad (3.10)$$

where

$$\tilde{B}_m = \frac{(1-\beta)^m \alpha^{m(n-1)} \prod_{j=0}^{m-1} (1-j\alpha)}{m!},$$

$$C_k^{(m)} = \sum_{j=1}^{k-1} q_j C_{k-j}^{(m-1)}, \quad m = 1, 2, 3, \dots; \quad k = m, m+1, \dots \quad (3.11)$$

and

$$q_k = \frac{p_k}{(\alpha+k)^n} \quad (3.12)$$

From (3.10) we have

$$f(z) = z + \sum_{k=2} \tilde{A}_k^{(m)} z^k, \quad (3.13)$$

where

$$\tilde{A}_k^{(m)} \equiv \tilde{A}_k^{(m)}(n, \alpha, \beta) = \sum_{m=1}^{k-1} \tilde{B}_m C_{k-1}^{(m)}. \quad (3.14)$$

Comparing coefficients in (1.1) and (3.13), we see that

$$a_k = \tilde{A}_k^{(m)}. \quad (3.15)$$

Now let $\alpha \in \Omega_1$, then the product $(1-\alpha)(1-2\alpha)\cdots(1-(m-1)\alpha)$ is nonnegative so that

$$|\tilde{A}_k^{(m)}| \leq \sum_{m=1}^{k-1} \tilde{B}_m |C_{k-1}^{(m)}|. \quad (3.16)$$

Observe from (3.11) and (3.12) that:

$$\begin{aligned} C_k^{(1)} &= q_k, \quad C_k^{(k)} = q_1^k, \quad k = 1, 2, 3, \dots \\ C_3^{(2)} &= 2q_1 q_2; \\ C_4^{(2)} &= q_2^2 + 2q_1 q_3, \quad C_4^{(3)} = 3q_1^2 q_2; \\ C_5^{(2)} &= 2q_1 q_4 + 2q_2 q_3, \quad C_5^{(3)} = 3q_1 q_2^2 + 3q_1^2 q_3, \quad C_5^{(4)} = 4q_1^3 q_2; \end{aligned} \quad (3.17)$$

Thus, in general, $C_k^{(m)}$ has the form

$$C_k^{(m)} = \sum_{j=1}^k G_j \prod_{l=1}^m q_l^{v_l} \quad (3.18)$$

for some nonnegative constants G_j , $j = 1, 2, \dots, k$ and indices v_l , $l = 1, 2, \dots, m$ taking values in the set $M = \{0, 1, 2, \dots, m\}$ such that the sum $v_1 + v_2 + \dots + v_m = m$. Hence

$$|C_k^{(m)}| \leq \sum_{j=1}^k G_j \prod_{l=1}^m |q_l^{v_l}| = \sum_{j=1}^k G_j \prod_{l=1}^m \left| \frac{p_l^{v_l}}{(\alpha + l)^{nv_l}} \right|. \quad (3.19)$$

Using Lemma 2.3, we obtain

$$|C_{k-1}^{(m)}| \leq 2^m Q_{k-1}^{(m)}, \quad m = 1, 2, \dots; \quad k = m+1, m+2, \dots, \quad (3.20)$$

where

$$Q_{k-1}^{(m)} = \sum_{j=1}^{k-1} \frac{G_j}{\prod_{l=1}^m (\alpha + l)^{nv_l}} \quad (3.21)$$

is the coefficient defined by the power series (3.2). Therefore, equation (3.21) is equivalent to (3.18) where $q_k = 2/(\alpha + k)^n$. Using (3.20) in (3.15) and (3.16) we have the first inequality of (3.4) for $\alpha \in \Omega_1$. Now, suppose $\alpha \in \Omega_2 \cup \Omega_3$. We write (3.14) as

$$\tilde{A}_k^{(m)} = \tilde{A}_k^{(k-3)} + \tilde{B}_{k-2} C_{k-1}^{(k-2)} + \tilde{B}_{k-1} C_{k-1}^{(k-1)}. \quad (3.22)$$

That is,

$$\tilde{A}_k^{(m)} = \tilde{A}_k^{(k-3)} + \tilde{B}_{k-2} \left\{ C_{k-1}^{(k-2)} + \frac{(1-\beta)\alpha^{(n-1)}(1-(k-2)\alpha)}{(k-1)} C_{k-1}^{(k-1)} \right\}. \quad (3.23)$$

Observe from (3.17) that the general form of $C_k^{(k-1)}$ and $C_k^{(k)}$ are respectively $C_k^{(k-1)} = (k-1)q_1^{k-2}q_2$ and $C_k^{(k)} = q_1^k$ so that, (3.23) becomes

$$\tilde{A}_k^{(m)} = \tilde{A}_k^{(k-3)} + (k-2)\tilde{B}_{k-2}q_1^{k-3} \left\{ q_2 - \frac{2(1-\beta)\alpha^{(n-1)}((k-2)\alpha-1)}{(k-1)(k-2)} \frac{q_1^2}{2} \right\}. \quad (3.24)$$

Using (3.11) in (3.24), we obtain

$$\tilde{A}_k^{(m)} = \tilde{A}_k^{(k-3)} + \frac{(k-2)\tilde{B}_{k-2}}{(\alpha+1)^{n(k-3)}(\alpha+2)^n} p_1^{k-3} \left\{ p_2 - \sigma \frac{p_1^2}{2} \right\}, \quad (3.25)$$

where

$$\sigma = \frac{2(1-\beta)\alpha^{(n-1)}((k-2)\alpha-1)(\alpha+2)^n}{(k-1)(k-2)(\alpha+1)^{2n}}$$

is a real number between 0 and 1. Hence by Corollary 2.5, the absolute value of the term in the Poisson brackets in (3.25) attains a maximum value 2. Therefore, from (3.25) we have

$$|\tilde{A}_k^{(m)}| \leq |\tilde{A}_k^{(k-3)}| + \frac{2^{k-2}(k-2)\tilde{B}_{k-2}}{(\alpha+1)^{n(k-3)}(\alpha+2)^n}. \quad (3.26)$$

That is

$$|\tilde{A}_k^{(m)}| \leq |\tilde{A}_k^{(k-3)}| + B_{k-2} Q_{k-1}^{(k-2)}, \quad (3.27)$$

where

$$Q_{k-1}^{(k-2)} = \frac{k-2}{(\alpha+1)^{n(k-3)}(\alpha+2)^n}.$$

By (3.15), (3.16) and (3.20), we have from (3.27)

$$|a_k| \leq A_{k-1} \quad (3.28)$$

for $\alpha \in \Omega_2 \cup \Omega_3$. From (2.4), we find that the boundary functions are given by

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 + (1-2\beta)z^2}{1-z^2}. \quad (3.29)$$

However, with regard to the even coefficients, we notice a disagreement between the upper bound in (3.28) and the coefficients in (3.29). This seeming contradiction suggests a possibility of refinement for the bound on the even coefficients. This is achieved thus: we write the inequality (2.3) of Lemma 2.4 as

$$p_2 = \frac{1}{2}p_1^2 + \varepsilon \left(2 - \frac{1}{2}|p_1|^2\right), \quad |\varepsilon| \leq 1. \quad (3.30)$$

By (3.30), we eliminate p_2 in (3.25) and we find that

$$\tilde{A}_k^{(m)} = \tilde{A}_k^{(k-3)} + \frac{(k-2)\tilde{B}_{k-2}}{(\alpha+1)^{n(k-3)}(\alpha+2)^n} p_1^{k-3} \left\{ (1-\sigma) \frac{p_1^2}{2} + \varepsilon \left(2 - \frac{1}{2}|p_1|^2\right) \right\}. \quad (3.31)$$

Since $(2 - \frac{1}{2}|p_1|^2) \geq 0$, the absolute value of the term in the Poisson brackets in (3.31) attains its maximum for $|p_1| = 2$. Hence, for $\alpha \in \Omega_2$ we have from (3.31)

$$|\tilde{A}_k^{(m)}| \leq |\tilde{A}_k^{(k-3)}| + \frac{2^{k-2}(1-\sigma)(k-2)\tilde{B}_{k-2}}{(\alpha+1)^{n(k-3)}(\alpha+2)^n}. \quad (3.32)$$

The inequality (3.32) together with (3.15), (3.16) and (3.20) and the fact that

$$Q_{k-1}^{(k-2)} = \frac{k-2}{(\alpha+1)^{n(k-3)}(\alpha+2)^n} \quad \text{and} \quad Q_{k-1}^{(k-1)} = \frac{1}{(\alpha+1)^{n(k-1)}}$$

gives

$$|a_k| \leq A_k \quad (3.33)$$

for $\alpha \in \Omega_2$. This completes the proof. $\square$

Remark 3.1.

- (1) If we set $n = \beta = 0$, $\alpha = 1$, we immediately obtain the sharp inequality $|a_k| \leq 2$, $k = 2, 3, \dots$, which was reported in [11].
- (2) Putting $n = \alpha = 1$, $\beta = 0$, we get a result of Macgregor [6, Theorem 1(1)].
- (3) By making $n = 1$ and $\beta = 0$, we obtain (in part) a result of Singh [10, Theorem 6].

(4) We observe from (3.4) that:

$$Q_1^{(1)} = \frac{1}{(\alpha+1)^n};$$

$$Q_2^{(1)} = \frac{1}{(\alpha+2)^n}, \quad Q_2^{(2)} = \frac{1}{(\alpha+1)^{2n}};$$

$$Q_3^{(1)} = \frac{1}{(\alpha+3)^n}, \quad Q_3^{(2)} = \frac{2}{(\alpha+1)^n(\alpha+2)^n}, \quad Q_3^{(3)} = \frac{1}{(\alpha+1)^{3n}};$$

$$Q_4^{(1)} = \frac{1}{(\alpha+4)^n},$$

$$Q_4^{(2)} = \frac{1}{(\alpha+2)^{2n}} + \frac{2}{(\alpha+1)^n(\alpha+3)^n},$$

$$Q_4^{(3)} = \frac{3}{(\alpha+1)^{2n}(\alpha+2)^n},$$

$$Q_4^{(4)} = \frac{1}{(\alpha+1)^{4n}};$$

$$Q_5^{(1)} = \frac{1}{(\alpha+5)^n},$$

$$Q_5^{(2)} = \frac{2}{(\alpha+1)^n(\alpha+4)^n} + \frac{2}{(\alpha+2)^n(\alpha+3)^n},$$

$$Q_5^{(3)} = \frac{3}{(\alpha+1)^n(\alpha+2)^{2n}} + \frac{3}{(\alpha+1)^{2n}(\alpha+3)^n},$$

$$Q_5^{(4)} = \frac{4}{(\alpha+1)^{3n}(\alpha+2)^n},$$

$$Q_5^{(5)} = \frac{1}{(\alpha+1)^{5n}}, \quad \dots$$

(5) Using (4) above in (3.1), (3.2) and (3.3), we can write out some of the bounds in explicit terms as follows:

$$|a_2| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+1)^n}, \quad \alpha > 0$$

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} + \frac{2(1-\beta)^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^{2n}}, & 0 < \alpha \leq 1; \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, & \alpha \geq 1, \end{cases}$$

$$|a_4| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+3)^n} + \frac{4(1-\beta)^2(1-\alpha)\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n}$$

$$+ \frac{4(1-\beta)^3(1-\alpha)(1-2\alpha)\alpha^{3n-3}}{3(\alpha+1)^{3n}}, \quad 0 < \alpha \leq 1;$$

$$|a_5| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} + \frac{2(1-\beta)^2(1-\alpha)[2(\alpha+2)^{2n} + (\alpha+1)^n(\alpha+3)^n]\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^{2n}(\alpha+3)^n} \\ + \frac{4(1-\beta)^3(1-\alpha)(1-2\alpha)\alpha^{3n-3}}{(\alpha+1)^{2n}(\alpha+2)^n} + \frac{2(1-\beta)^4(1-\alpha)(1-2\alpha)(1-3\alpha)\alpha^{4n-4}}{3(\alpha+1)^{4n}}$$

if $0 < \alpha \leq \frac{1}{3}$;

$$|a_5| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} + \frac{2(1-\beta)^2(1-\alpha)[2(\alpha+2)^{2n} + (\alpha+1)^n(\alpha+3)^n]\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^{2n}(\alpha+3)^n} \\ + \frac{4(1-\beta)^3(1-\alpha)(1-2\alpha)\alpha^{3n-3}}{(\alpha+1)^{2n}(\alpha+2)^n},$$

if $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$;

$$|a_6| \leq \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+5)^n} + \frac{4(1-\beta)^2(1-\alpha)[(\alpha+1)^n(\alpha+4)^n + (\alpha+2)^n(\alpha+3)^n]\alpha^{2n-2}}{(\alpha+1)^n(\alpha+2)^n(\alpha+3)^n(\alpha+4)^n} \\ + \frac{4(1-\beta)^3(1-\alpha)(1-2\alpha)[(\alpha+1)^n(\alpha+3)^n + (\alpha+2)^{2n}]\alpha^{3n-3}}{(\alpha+1)^{2n}(\alpha+2)^{2n}(\alpha+3)^n} \\ + \frac{8(1-\beta)^4(1-\alpha)(1-2\alpha)(1-3\alpha)\alpha^{4n-4}}{3(\alpha+1)^{3n}(\alpha+2)^n} \\ + \frac{4(1-\beta)^5(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)\alpha^{5n-5}}{15(\alpha+1)^{5n}}, \quad \left(0 < \alpha \leq \frac{1}{3}\right)$$

and so on.

Let S denote the subclass of A consisting only of univalent functions in E . Associated with each function $f \in S$ is a well-defined logarithmic function

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k, \quad z \in E.$$

The numbers γ_k are called the logarithmic coefficients of $f(z)$ (see [3]). The relevance of the logarithmic coefficients comes from the fact that, by the so called Lebedev-Milin inequality (Lemma 2.6), estimates of the logarithmic coefficients of $f(z)$ can be transferred to bounds on the coefficients of $f(z)$.

In the following theorem we shall associate with each $f \in T_n^\alpha(\beta)$ logarithmic coefficients:

$$\log \frac{f(z)^\alpha}{z^\alpha} = 2\alpha \sum_{k=1}^{\infty} \gamma_k z^k, \quad z \in E \quad (3.34)$$

and use it to approximate the general bounds on the coefficients of $f(z)$.

Theorem 3.2. Let $f \in T_n^\alpha(\beta)$, $n \in \mathbb{N}$. Suppose

$$\frac{f(z)^\alpha}{z^\alpha} = \sum_{k=0}^{\infty} A_{k+1}(\alpha) z^k, \quad A_1(\alpha) = 1, \quad (3.35)$$

then

$$|A_{k+1}(\alpha)| < \exp \left\{ 0.624\alpha^2 + \left(2\alpha^2 - \frac{1}{2}\right) \sum_{j=1}^k \frac{1}{j} \right\}. \quad (3.36)$$

Proof. From (3.34) and (3.35) we have

$$\log \left\{ \sum_{k=0}^{\infty} A_{k+1}(\alpha) z^k \right\} = \sum_{k=1}^{\infty} 2\alpha \gamma_k z^k,$$

that is,

$$\sum_{k=0}^{\infty} A_{k+1}(\alpha) z^k = \exp \left\{ \sum_{k=1}^{\infty} 2\alpha \gamma_k z^k \right\}. \quad (3.37)$$

By Lemma 2.6, we get

$$|A_{k+1}(\alpha)|^2 \leq \exp \left\{ \sum_{j=1}^k \left(4\alpha^2 j |\gamma_j|^2 - \frac{1}{j} \right) \right\}. \quad (3.38)$$

Hence by Milin's lemma (for each $f \in S$)

$$\sum_{j=1}^k j |\gamma_j|^2 \leq \sum_{j=1}^k \frac{1}{j} + \delta, \quad k = 1, 2, \dots$$

where $\delta < 0.312$ is called the *Milin's constant*, (see [3]), the inequality (3.38) yields

$$|A_{k+1}(\alpha)|^2 \leq \exp \left\{ 4\alpha^2 \delta + (4\alpha^2 - 1) \sum_{j=1}^k \frac{1}{j} \right\}, \quad (3.39)$$

which implies (3.36) and thus completes the proof. $\square$

Remark 3.2. If $\alpha = \frac{1}{2}$, then $|A_{k+1}(\frac{1}{2})| < 1.168826$, $k = 1, 2, \dots$. In particular $|A_2(\frac{1}{2})| = \frac{1}{2}|a_2|$ and $|A_3(\frac{1}{2})| = \left| \frac{1}{2}a_3 - \frac{1}{2} \left(\frac{1}{2}a_2 \right)^2 \right|$. Thus, we have $|a_2| < 2.337652$ and $|a_3| < 3.703806$. Obviously these estimates are not sharp. However this approach can conveniently lead to a rough estimate of the bounds on the coefficients of functions in the class.

Theorem 3.3. Let $f \in T_n^\alpha(\beta)$. Then

(i) for any real number λ ,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} - \frac{2(2\lambda+\alpha-1)(1-\beta)^2\alpha^{2n-2}}{(\alpha+1)^{2n}}, & \text{if } \lambda \leq \frac{1-\alpha}{2} \\ \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, & \text{if } \frac{1-\alpha}{2} \leq \lambda \leq \frac{1-\alpha}{2} + \frac{\alpha^{1-n}(\alpha+1)^{2n}}{(1-\beta)(\alpha+2)^n} \\ \frac{2(2\lambda+\alpha-1)(1-\beta)^2\alpha^{2n-2}}{(\alpha+1)^{2n}} - \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, & \text{if } \lambda \geq \frac{1-\alpha}{2} + \frac{\alpha^{1-n}(\alpha+1)^{2n}}{(1-\beta)(\alpha+2)^n} \end{cases}$$

(ii) for any complex number λ ,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{2(1-\beta)\alpha^{n-1}}{(\alpha+2)^n}, & \text{if } \left| 1 - \frac{(2\lambda+\alpha-1)(1-\beta)(\alpha+2)^n\alpha^{n-1}}{(\alpha+1)^{2n}} \right| \leq 1 \\ \frac{2(1-\beta)\alpha^{n-1}|(\alpha+1)^{2n} - (2\lambda+\alpha-1)(1-\beta)^2\alpha^{2n-2}|}{(\alpha+1)^{2n}(\alpha+2)^n}, & \text{if } \left| 1 - \frac{(2\lambda+\alpha-1)(1-\beta)(\alpha+2)^n\alpha^{n-1}}{(\alpha+1)^{2n}} \right| \geq 1. \end{cases}$$

The inequalities are sharp. For each λ , there is a function $f \in T_n^\alpha(\beta)$ for which equality holds.

Proof. From equations (3.14), (3.15) and (3.16) of Theorem 3.1 we get

$$a_2 = \frac{(1-\beta)\alpha^{n-1}}{(\alpha+1)^n} p_1$$

and

$$a_3 = \frac{(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} p_2 + \frac{(1-\alpha)(1-\beta)^2\alpha^{2n-2}}{2(\alpha+1)^{2n}} p_1^2.$$

Hence

$$|a_3 - \lambda a_2^2| = \frac{(1-\beta)\alpha^{n-1}}{(\alpha+2)^n} \left| p_2 - \frac{(2\lambda+\alpha-1)(1-\beta)(\alpha+2)^n\alpha^{n-1}}{(\alpha+1)^{2n}} \frac{p_1^2}{2} \right|.$$

By choosing

$$\sigma = \frac{(2\lambda+\alpha-1)(1-\beta)(\alpha+2)^n\alpha^{n-1}}{(\alpha+1)^{2n}}$$

in Corollary 2.5, the theorem follows. $\square$

Remark 3.3. Putting $n = 1$ and $\beta = 0$, we deduce a result of Singh [10, Theorem 7].

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