

SOME PROPERTIES OF SYMMETRIC INVERSE SEMIGROUP AND SOME OF ITS SUBSEMIGROUP

¹IBRAHIM, G. R. & ²BAKARE, G. N.

¹Department of Statistics and Mathematical Sciences, Kwara State University, Malete

²Department of Mathematics, University of Ilorin

ABSTRACT

In this paper, we studied the number of Nilpotent element of order preserving one-one partial transformation semi groups and number of nilpotent elements of order decreasing + order preserving one-one partial transformation semi group. Also, we showed that every Idempotent element of one-one partial transformation semigroup is a contraction mapping.

1. INTRODUCTION

Consider the finite chain $X_n = \{1, 2, \dots, n\}$ under the natural ordering and let T_n, P_n , and I_n be the full, partial and one-one partial transformation semigroups on X_n respectively. We define $Ln = On \cap Dn$, $PLn = POn \cap PDn$ and $ILn = IO_n \cap IDn$ as the semigroups of order decreasing + order preserving full, partial, one – one transformation respectively.

Higgins (1994) showed in particular that $|L_n|$ is the nth Catalan number given by

$$|(L_n)| = 1/n + \binom{2n}{n} \text{ and } E(Ln) = 2^{n-1} \text{ furthermore, combinatorial properties}$$

of L_n were investigated by Laradji and Umar (2007), they showed that the number of maps in L_n such that $|Im \alpha| = r$ is the triangle of Mahayana =(A001263) numbers given by

$$|\{\alpha \in L_n : |im \alpha| = r\}| = 1/n - r + 1 \binom{n-1}{r-1} \binom{n}{r}$$

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The summary of symbols used in this work are given on this table

Transformation	Full	Partial	Partial one - one
Transformation	T_n	P_n	I_n
Order - Preserving	O_n	PO_n	IO_n
Order - decreasing	D_n	PD_n	ID_n
Order-Preserving+Order-decreasing	L_n	PL_n	IL_n

Table 1

2 METHODOLOGY

Number of Nilpotent Elements for order preserving one -one partial transformation semigroup

When $n=1$ we have the following partial transformation $|P_1|=2$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ - \end{pmatrix}$$

The injective order preserving elements are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ - \end{pmatrix}$$

and the Nilpotent element for IO_1 is :

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $n=2$, we have the following partial transformation semi group

$$\begin{pmatrix} 12 \\ 11 \end{pmatrix} \begin{pmatrix} 12 \\ 22 \end{pmatrix} \begin{pmatrix} 12 \\ 12 \end{pmatrix} \begin{pmatrix} 12 \\ 21 \end{pmatrix} \begin{pmatrix} 12 \\ 1- \end{pmatrix} \begin{pmatrix} 12 \\ 2- \end{pmatrix} \begin{pmatrix} 12 \\ -1 \end{pmatrix} \begin{pmatrix} 12 \\ -2 \end{pmatrix} \begin{pmatrix} 12 \\ -- \end{pmatrix}$$

the injective element and order preserving

$$\begin{pmatrix} 12 \\ 12 \end{pmatrix} \begin{pmatrix} 12 \\ 1- \end{pmatrix} \begin{pmatrix} 12 \\ 2- \end{pmatrix} \begin{pmatrix} 12 \\ -1 \end{pmatrix} \begin{pmatrix} 12 \\ -2 \end{pmatrix} \begin{pmatrix} 12 \\ -- \end{pmatrix}$$

= 6 elements

And the Nilpotent elements for IO_2 are:

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$$\begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}$$

= 3 elements

3.0 MAIN RESULT

3.1 Number of Nilpotent Element in Partial One-One Order Preserving Transformation Semigroup

Thus we repeated for $n = 3, n = 4$ and $n = 5$ and the result gives the following

Table 2

	$ P_n $	$ I_n $	$ IO_n $	$ E(IO_n) $	$ N(IO_n) $
$n = 1$	2	2	2	2	1
$n = 2$	9	7	6	4	3
$n = 3$	64	34	20	8	9
$n = 4$	625	209	70	16	29
$n = 5$	7776	1546	252	32	97

Here, for $N(IO_n)$ we obtained up to $n = 5$ the sequence 1, 3, 9, 29, 97 for the number of Nilpotent element in partial one-one order preserving transformation but the general expression for these sequences is yet to be derived.

3.2 Number of Nilpotent element in order decreasing + order preserving partial one-one transformation semi group.

Using the method in (2) above the result for this class of semi group as in the following table

Table 3

	$ P_n $	$ I_n $	$ IL_n $	$ E(IL_n) $	$ N(IL_n) $
$n = 1$	2	2	2	2	1
$n = 2$	9	7	5	4	2
$n = 3$	64	34	14	8	5
$n = 4$	625	209	42	16	14
$n = 5$	7776	1546	132	32	42

Here, we got the sequence 1, 2, 5, 14 and 42 for the set of Nilpotent element of partial one-one order decreasing + order preserving transformation semi group up to $n = 5$ and the general expression for these sequence.

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3.2.1 Proposition

The general expression for the number of Nilpotent elements in order decreasing + order preserving one-one transformation semi group is

$$N(IL_n) = \binom{2n}{n-1}$$

Umar did this work and got the same result with me (independent) which I was unable to prove it in his own work [5]

4. CONTRACTION MAPPING

Umar A. [8], A transformation α in P_n for which $|x\alpha - y\alpha| \leq |x - y| \forall x, y \in X_n$ is said to be a contraction. That is, a contraction mapping is one in which the difference between any two elements (in the base set X) is greater than or equal to the difference their images. Examples of contraction mapping $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & - & 2 \end{pmatrix}$

The set of all contraction mapping I_n forms a semi group denoted by $ICONT_n$ (one-one contraction mapping).

Since if

$\alpha, \beta \in ICONT_n, |x\alpha - y\alpha| \leq |x - y| \forall x, y \in X_n$ and $|x\beta - y\beta| \leq |x - y| \forall x, y \in X_n$. Suppose $t = x\alpha$ and $s = y\alpha$, then $t\beta = x\alpha\beta$ and $s\beta = y\alpha\beta$.

Therefore,

$$|x\alpha\beta - y\alpha\beta| \leq |t\beta - s\beta| \leq |t - s| \quad (1)$$

But

$$|t - s| \leq |x\alpha - y\alpha| \leq |x - y| \quad (2)$$

$|x\alpha\beta - y\alpha\beta| \leq |x - y|$ It is a subsemigroup.

It was shown by Kehinde and Makanjuola in [3] that the number of idempotent elements $|E(ICONT_n)|$ forms a sequence 2, 4, 8, 32, that was generalized to $2^n, n = 1, 2, \dots, 2^n$. We shall use the following result to support these findings

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Theorem

Every idempotent element of one – one partial transformation semi group is a contraction mapping.

Proof

For any Idempotent α , $|F(\alpha)| = |\text{Im}(\alpha)|$. In other word, all the elements in $\text{DOM} \alpha$ is fixed. Where $|F(\alpha)|$ is the number of fixed elements in and $|\text{Im}(\alpha)|$ is the number of image in α . Hence if α is idempotent then $|x\alpha - y\alpha| = |x - y|$ for all $x, y \in d(\alpha)$ since $x\alpha = x$ and $y\alpha = y$. So all the idempotent element in I_n are contraction mapping. It is a well known fact that the number of idempotent elements in I_n is 2^n . Hence, $|E(\text{ICONT}_n)| = |E(I_n)| = 2^n$ as got earlier.

CONCLUSION

We obtained the sequence for set of Nilpotent element of partial one –one order preserving transformation up to $n=5$ as 1,6,9,29 and 97 but the general expression for these figures has not yet derived .

Also the sequence for the set of Nilpotent elements of partial one –one order decreasing + order preserving were also obtained up to $n=5$ and the general expression for these 1,2,5,14 and 42 has been derived as

$$N(IL_n) = 1/n \binom{2n}{n-1}$$

Furthermore, we showed that every idempotent of one-one partial transformation semi group is a contraction mapping.

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