SOME MODIFIED FACTOR-TYPE ESTIMATORS FOR POPULATION MEAN IN SAMPLE SURVEY

BY

AUDU, AHMED

14/68DV005

(B.Sc.; M.Sc. (UDU SOKOTO))

THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (Ph.D) IN THE DEPARTMENT OF STATISTICS, UNIVERSITY OF ILORIN, ILORIN, KWARA STATE, NIGERIA

CERTIFICATION

This is to certify that this doctoral thesis was carried out by AUDU, Ahmed, (14/68DV005) in the Department of Statistics, in partial fulfillment of the requirement for the award of Doctor of Philosophy in Statistics of the University of Ilorin, Ilorin, Kwara State, Nigeria and has not been presented elsewhere for the award of a degree, or any other purpose.

……. ...……..

Dr. A. A. Adewara Date

Supervisor

……. …….....

Prof. B. L. Adeleke Date

H.O.D. & Chief Examiner

……. …….

External Examiner Date

DEDICATION

This research work is dedicated to ALMIGHTY ALLAH, the Supreme God (Subuhanahu Wata'ala) for His infinite mercy, guidance and protection on me and my family.

ACKNOWLEDGEMENT

In the name of Allah, Most Gracious, Most Merciful. I wish to express my sincere gratitude to Almighty Allah (SWT) for granting me opportunity to carry out this research work successfully.

My special thanks go to my major supervisor, Dr. A. A. Adewara for his constructive criticism, assistance, guidance and fatherly advice and for the many hours he spent guiding me in my thesis as well as in my career. I sincerely thank Dr. R.V.K. Singh and Dr. A. Danbaba of the Department of Mathematics, Kebbi State University of Science and Technology and Department of Mathematics, Usmanu Danfodiyo University, Sokoto respectively for their support and priceless advices. I would also like to express my gratitude to all lecturers in the Department of Statistics, starting Prof. B.L. Adeleke, Prof. B. A. Oyejola, Prof. E. T. Jolayemi, Prof. P.A. Osanaiye, Prof. R. A. Ipinyomi, Prof. I.O. Oshungade, Dr. G.M. Oyeyemi, Dr. W. B. Yahya, Dr. O.O. Sanni, Dr. A.O. Adejumo, Dr. A.O. Abidoye, among others, including non-academic staff in the Department, May God bless you all.

My profound gratitude goes to my Mother, Mallama Hawawu Muhammed, my aunts, my sisters, my brothers and my entire family for their spiritual support. In addition, my profound gratitude goes to my beloved wife Tawakaltu Abdulsalam and children, Princess Raddiyah Ahmad, Prince Abdullahi Ahmad and Princess Aisha Ahmad for their patient and support throughout my period of study in the University of Ilorin. I would not forget with my close friends like I. Suleiman, S. Suleiman, A. A. Sanusi, H. Lawal, N. S. Dauran, Aremu O. Kazeem, M. M. Hamza, M. S. Magami, A. Dogondaji and the rest of them for

their perseverance, encouragement and constant prayers for me while undergoing the program.

Finally, I would like to acknowledge the management of Usmanu Danfodiyo University, Sokoto for their financial support throughout the program, I pray Almighty Allah to bless you all (Ameen).

ABSTRACT

Factor-type estimators of population mean are applicable to where correlations between the study and auxiliary variables are positive or negative. Their efficiency depends on the optimum estimate of the value of positive real number $d\left(d \in \Re^{+}\right)$ which is usually a function of the study and auxiliary variables. However, existing factor-type estimators are found not to be defined for some values of d and some have lower precision for positive correlation coefficient $(\rho_{xy} > 0)$. This study aimed at modifying some conventional factor-type estimators under single-phase and two-phase sampling schemes by incorporating more information on auxiliary variables like coefficients of skewness ($\beta_1(x)$), kurtosis ($\beta_2(x)$), variation (C_x) and standard deviation (σ_x) to produce more efficient ones. The specific objectives of this study were to: (i) obtain four modified factor-type estimators under single and two-phase sampling; (ii) derive the Biases and Mean Square Errors (MSEs) of these modified factor-type estimators; (iii) derive the conditions under which these modified estimators will be more efficient than the conventional ones; (iv) determine the relative efficiency of these modified estimators using both real life and simulated data; and (v) determine the robustness of these modified factor-type estimators under super-population and non-response model.

The four modified estimators under single-phase and two-phase sampling schemes considered are: $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$. These are done by incorporating more information on auxiliary variables like coefficients of skewness ($\beta_1(x)$), kurtosis ($\beta_2(x)$), variation (C_x) and standard deviation (σ_x) to produce more efficient ones.

The findings were:

i. four modified factor-type estimators for population mean were obtained;

ii. Biases and MSEs of these modified factor-type estimators were theoretically and empirically derived;

iii. conditions under which these modified estimators were more efficient were derived;

iv. using both real life and simulated data, the modified factor-type estimators were found to have lower Mean Square Errors and highest relative efficiency, hence more efficient than the conventional ones; and

v. the four modified estimators demonstrated high level of robustness under super-population and non-response models in comparison to the conventional ones.

This study concluded that when: (i). $-\frac{\rho_{xy}C_y}{\delta_x C_x} < \frac{2\rho_{xy}C_y}{[\delta_x + 1/(d-1)]C_x}, (d>1)$ (ii). $-\frac{C_{xy}}{\delta_x} < 2\rho_{xy}C_y/[\delta_x+1]C_x$ (iii). $C_{yx} < \frac{\rho_{xy}C_y}{C_x}$ and (iv). $\frac{\rho_{xy}C_y + \rho_{yz}C_zC_{yz}}{C_x - \rho_{xz}C_zC_{xz}} < \frac{\rho_{xy}C_y[2(1-\rho^2{}_{xy}) - \theta_2(1-2\rho^2{}_{xy})]}{(2C_y-1)\theta_2\rho^2{}_{xz}}$, these modified factor-type estimators are better than the existing ones where, $\theta_2 = \frac{(1-\frac{n}{N})}{n_2}$ while ρ_{xy}, ρ_{xz} and ρ_{yz} are the correlation coefficients between x and y, x and z and, y and z respectively . Therefore, these modified estimators are recommended for use in estimating finite population mean in sample survey.

TABLE OF CONTENTS

TITLE PAGE	i
CERTIFICATION	ii
DEDICATION	iii
ACKNOWLEDGEMENT	iv
ABSTRACT	vi
TABLE OF CONTENTS	viii
LIST OF TABLES	xiv
LIST OF FIGURES	xxi
CHAPTER ONE (INTRODUCTION)	
1.0 BACKGROUND OF THE STUDY	1
1.1 STATEMENT OF THE PROBLEM	4
1.2 AIM AND OBJECTIVES	4
1.3 SIGNIFICANCE OF THE STUDY	5
1.4 SCOPE AND LIMITATION OF THE STUDY	5
1.5 DEFINITION OF SOME RELATED TERMS	6
1.5.1 Sample	6
1.5.2 Parameter	6
1.5.3 Statistic	6
1.5.4 Estimator	6
1.5.5 Bias	7
1.5.6 Mean Square Error (MSE)	8
1.5.7 Kurtosis	8

1.5.8 Coefficient of variation (CV) 9

CHAPTER TWO (LITERATURE REVIEW)

2.0 INTRODUCTION 10

2.1 RATIO-PRODUCT ESTIMATORS 10

2.2 FACTOR-TYPE ESTIMATORS 26

2.3 ESTIMATION UNDER SUPER-POPULATION MODEL 29

CHAPTER THREE (METHODOLOGY)

3.0 INTRODUCTION 31

3.1 NON-RESPONSE MODEL 31

3.2 SUPER-POPULATION MODEL 31

3.3 MODIFIED ESTIMATORS 32

3.3.1 Under single-phase sampling 32

3.3.2 Under two-phase sampling 33

3.3.3 Assumptions of the Modified Factor-Type Estimators 33

3.3.4 Members of the Family of Modified Estimators 34

3.4 PROPERTIES OF THE MODIFIED FACTOR-TYPE ESTIMATORS 37

3.4.1 Bias and MSE of the estimator $\bar{y}_{FTAA}$ 38

3.4.2 Bias and MSE of the estimator $\bar{y}_{FTAA}^{(d)}$ 41

3.4.3 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ 46

3.4.4 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ 52

3.5 EFFICIENCY COMPARISONS 58

3.5.1 Efficiency of $\bar{y}_{FTAA}$ over some related Estimators 58

3.5.2 Efficiency of $\bar{y}_{FTAA}^{(d)}$ over some related Estimators 66

3.5.3 Efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}$ over some related Estimators 70

3.5.4 Efficiency of $\bar{y}_{FTAA}^{\beta_2(d)}$ over some related Estimators 76

3.6 PROPERTIES OF THE MODIFIED FACTOR-TYPE ESTIMATORS UNDER NON-RESPONSE MODEL 77

3.7 OPTIMUM SAMPLE SIZE 78

3.7.1 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{(d)}$ for the Fixed Cost 78

3.7.2 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{\beta_1(d)}$ for the Fixed Cost 80

3.7.3 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{\beta_2(d)}$ for the Fixed Cost 81

3.7.4 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{(d)}$ for Specified Precision 83

3.7.5 Determination of n_1 and n_2 for $\bar{y}_{FTAA}^{\beta_1(d)}$ for Specified Precision 85

3.7.6 Determination of n_1 and n_2 for $\bar{y}_{FTAA}^{\beta_2(d)}$ for Specified Precision 86

CHAPTER FOUR (EMPIRICAL STUDY AND DISCUSSION)

4.0 INTRODUCTION 89

4.1 EMPIRICAL STUDY USING REAL LIFE DATA 89

4.1.1 Efficiency Comparison of $\bar{y}_{FTAA}$ with some existing related Estimators 92

4.1.1.1 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio Estimators 92

4.1.1.2 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product Estimators 93

4.1.1.3 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators 95

4.1.2 Efficiency Comparison of $\bar{y}_{FTAA}^{(d)}$ with some existing related Estimators 105

4.1.2.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some existing related Ratio Estimators 105

4.1.2.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some Related Product Estimators 107

4.1.2.3 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ 108

4.1.3 Efficiency Comparison of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ with some existing related Estimators 124

4.1.3.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Ratio Estimators 124

4.1.3.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Product Estimators 126

4.1.4 Cost Analysis 128

4.1.5 Robustness under Non-Response Model 129

4.1.6 Robustness under Super-Population Model 133

4.2 EMPIRICAL STUDY USING SIMULATED DATA 141

4.2.1 Efficiency Comparisons of $\bar{y}_{FTAA}$ with some related Estimators 142

4.2.1.1 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio Estimators 142

4.2.1.2 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product Estimators 145

4.2.1.3 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators 148

4.2.2 Efficiency Comparison of $\bar{y}_{FTAA}^{(d)}$ with some existing related Estimators 162

4.2.2.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some existing related Ratio Estimators 162

4.2.2.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some Related Product Estimators 165

4.2.2.3 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and Shukla (2002) Estimator 168

4.2.3. Efficiency Comparison of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ with some existing related Estimators 189

4.2.3.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Ratio Estimators 189

4.2.3.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Product Estimators 192

4.2.4 Robustness under Non-Response Model 195

4.2.5 Robustness under Super-Population Model 204

CHAPTER FIVE (SUMMARY, CONCLUSION AND RECOMMENDATION)

5.0 INTRODUCTION 237

5.1 SUMMARY OF RESULTS 237

5.2 CONCLUSION 237

5.3 GENERAL FINDINGS OF THIS RESEARCH WORK 238

5.4 CONTRIBUTION TO KNOWLEDGE 238

5.5 SUGGESTION FOR FURTHER STUDY 238

REFERENCE 240

APPENDIX 246

LIST OF TABLES

Table 3.1: Some Members of the Family of Estimator $\bar{y}_{FTAA}$ 35

Table 3.2: Some Members of the Family of Estimator $\bar{y}_{FTAA}^{(d)}$ 36

Table 3.3: Some Members of the Family of Estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ 36

Table 3.4: Some Members of the Family of Estimator $\bar{y}_{FTAA}^{\beta_2(d)}$. 37

Table 4.1A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators when $\rho_{xy} > 0$ 92

Table 4.1B: Efficiency conditions of $\bar{y}_{FTAA}$ over Ratio estimators in Table 4.1A 93

Table 4.2: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some related Product estimators when $\rho_{xy} < 0$ 94

Table 4.2B: Efficiency conditions of $\bar{y}_{FTAA}$ over Product estimators in Table 4.2A 95

Table 4.3A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} > 0$ 96

Table 4.3B: Efficiency conditions of $\bar{y}_{FTAA}$ over Factor-type estimators in Table 4.3A 97

Table 4.4A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} < 0$ 101

Table 4.4B: Efficiency conditions of $\bar{y}_{FTAA}$ over Factor-type estimators in Table 4.4A 102

Table 4.5A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related Ratio estimators when $\rho_{xy} > 0$ 106

Table 4.5B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Ratio estimators in Table 5A 106

Table 4.6A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related Ratio estimators when $\rho_{xy} < 0$ 107

Table 4.6B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Product estimators in Table 6A 108

Table 4.7A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} > 0$ 109

Table 4.7B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimator in Table 4.7A 109

Table 4.8A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} < 0$ 117

Table 4.8B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimator in Table 4.8A 117

Table 4.9A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related ratio estimators when $\rho_{xy} > 0$ 124

Table 4.9B: Efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over Estimators in Table 4.9A 125

Table 4.10A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} < 0$ 126

Table 4.10B: Efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over Estimators in Table 4.10A 127

Table 4.11: Expected cost of $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and some existing related Estimators 128

Table 4.12: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C1 129

Table 4.13: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C2 130

Table 4.14: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C3 131

Table 4.15: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C4 132

Table 4.16A: Bias and MSE of suggested Estimators and some related Estimators under some Super-population models I using Data D1 $\left(\rho_{xy} > 0\right)$ 134

Table 4.16B: Bias and MSE of suggested Estimators and some related Estimators under some Super-population model II using Data D1 $\left(\rho_{xy} > 0\right)$ 135

Table 4.16C: Bias and MSE of suggested Estimators and some related Estimators under some Super-population models III using Data D1 $\left(\rho_{xy} > 0\right)$ 136

Table 4.17A: Bias and MSE of suggested Estimators and some related Estimators under some Super-population models I using Data D2 $\left(\rho_{xy} < 0\right)$ 138

Table 4.17B: Bias and MSE of suggested Estimators and some related Estimators under some Super-population model II using Data D2 $\left(\rho_{xy} < 0\right)$ 139

Table 4.17C: Bias and MSE of suggested Estimators and some related Estimators under some Super-population models III using Data D2 $\left(\rho_{xy} < 0\right)$ 140

Table 4.18A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 143

Table 4.18B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 144

Table 4.19A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 143

Table 4.19B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 144

Table 4.20A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 149

Table 4.20B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 150

Table 4.21A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$ 156

Table 4.21B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$ 157

Table 4.22A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 163

Table 4.22B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 164

Table 4.23A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$ 166

Table 4.23B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$ 167

Table 4.24A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 169

Table 4.24A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 170

Table 4.25A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$ 179

Table 4.25B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$ 180

Table 4.26A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$ 187

Table 4.26B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$ 188

Table 4.27A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$ 193

Table 4.27B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$ 194

Table 4.28: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.2$ 196

Table 4.29: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.5$ 197

Table 4.30: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.8$ 198

Table 4.31: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.99$ 199

Table 4.32: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.2$ 200

Table 4.33: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.5$ 201

Table 4.34: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.8$ 202

Table 4.35: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.99$ 203

Table 4.36A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.2$ 205

Table 4.36B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.2$ 206

Table 4.36C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.2$ 207

Table 4.37A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.5$ 209

Table 4.37B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.5$ 210

Table 4.37C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.5$ 211

Table 4.38A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = 0.8$ 213

Table 4.38B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = 0.8$ 214

Table 4.38C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = 0.8$ 215

Table 4.39A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.99$ 217

Table 4.39B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.99$ 218

Table 4.39C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.99$ 219

Table 4.40A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = -0.2$ 221

Table 4.40B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = -0.2$ 222

Table 4.40C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = -0.2$ 223

Table 4.41A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = -0.5$ 225

Table 4.41B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = -0.5$ 226

Table 4.41C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = -0.5$ 227

Table 4.42A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = -0.8$ 229

Table 4.42B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = -0.8$ 230

Table 4.42C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = -0.8$ 231

Table 4.43A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = -0.99$ 233

Table 4.43B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = -0.99$ 234

Table 4.43C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = -0.99$ 235

LIST OF FIGURES

Figure 4.1A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 1 97

Figure 4.2A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 2 98

Figure 4.3A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 3 98

Figure 4.1B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 1 99

Figure 4.2B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 2 99

Figure 4.3B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 3 100

Figure 4.4A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 4 102

Figure 4.5A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 5 103

Figure 4.6A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 6 103

Figure 4.4B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 4 104

Figure 4.5B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 5 104

Figure 4.6B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data 6 105

Figure 4.7A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 1 110

Figure 4.8A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 1 111

Figure 4.9A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 2 111

Figure 4.10: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 2 112

Figure 4.11A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 3 112

Figure 4.12A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 3 113

Figure 4.7B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 1 113

Figure 4.8B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 1 114

Figure 4.9B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 2 114

Figure 4.10B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 2 115

Figure 4.11: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 3 115

Figure 4.12B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II using Data 3 116

Figure 4.1.13A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I using Data 4 118

Figure 4.1.13B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 4 118

Figure 4.1.14A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 5 119

Figure 4.1.14B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 5 case 119

Figure 4.1.15A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 6 120

Figure 4.1.15B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 6 120

Figure 4.1.16A: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 4 121

Figure 4.1.16B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 4 121

Figure 4.1.17A: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 5 case I 122

Figure 4.1.17B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 5 122

Figure 4.1.18A: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 6 123

Fig. 4.1.18B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 6 123

Figure 4.2.1A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.2$ 151

Figure 4.2.2A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.5$ 152

Figure 4.2.3A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.8$ 152

Figure 4.2.4A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.99$ 153

Figure 4.2.1B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.2$ 153

Figure 4.2.2B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.5$ 154

Figure 4.2.3B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.8$ 154

Figure 4.2.4B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.99$ 155

Figure 4.2.5A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.2$ 158

Figure 4.2.6A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.5$ 158

Figure 4.2.7A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.8$ 159

Figure 4.2.8A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.99$ 159

Figure 4.2.5B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.2$ 160

Figure 4.2.6B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.5$ 160

Figure 4.2.7B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.8$ 161

Figure 4.2.8B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.99$ 161

Figure 4.2.9A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.2$ 171

Figure 4.2.10A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.2$ 171

Figure 4.2.11A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.5$ 172

Figure 4.2.12A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.5$ 172

Figure 4.2.13A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.8$ 173

Figure 4.2.14A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.8$ 173

Figure 4.2.15A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.99$ 174

Figure 4.2.16A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.99$ 174

Figure 4.2.9B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.2$ 175

Figure 4.2.10B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.2$ 175

Figure 4.2.11B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.5$ 176

Figure 4.2.12B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.5$ 176

Figure 4.2.13B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.8$ 177

Figure 4.2.14B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.8$ 177

Figure 4.2.15B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.99$ 178

Figure 4.2.16B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.99$ 178

Figure 4.2.17A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.2$ 181

Figure 4.2.18A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.2$ 181

Figure 4.2.19A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.5$ 182

Figure 4.2.20A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.5$ 182

Figure 4.2.21A: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.8$ 183

Figure 4.2.22A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.8$ 183

Figure 4.2.23A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I when $\rho_{xy} = -0.99$ 184

Figure 4.2.24A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II when $\rho_{xy} = -0.99$ 184

Figure 4.2.17B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I when $\rho_{xy} = -0.2$ 185

Figure 4.2.18B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II when $\rho_{xy} = -0.2$ 185

Figure 4.2.19B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I when $\rho_{xy} = -0.5$ 186

Figure 4.2.20B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II when $\rho_{xy} = -0.5$ 186

Figure 4.2.21B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I when $\rho_{xy} = -0.8$ 187

Figure 4.2.22B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II when $\rho_{xy} = -0.8$ 187

Figure 4.2.23B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case I when $\rho_{xy} = -0.99$ 188

Figure 4.2.24B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}^{(d)}_{FTAA}$ Estimators under case II when $\rho_{xy} = -0.99$ 188

CHAPTER ONE

INTRODUCTION

1.0 BACKGROUND OF THE STUDY

In sample surveys, supplementary population information is often used at the estimation stage to increase the precision of estimators of a population mean or total. It is common practice to use auxiliary information on a character X in the estimation of the finite population parameters (mean, total, variance, proportion) of a character under study. The assumption is that the information on auxiliary variable are known either by past data, past experience or guess of the investigator. Prior knowledge about population mean along with coefficient of variation, kurtosis and correlation coefficient of the population of an auxiliary variable are known to be very useful particularly when the ratio, product and regression estimators are used for estimation of population mean of a variable of interest. The efficiency of ratio and product estimators solely depends on strong-positive and strong-negative correlations respectively between the study and auxiliary variables. Several authors like Cochran (1942), Sukhatme (1962), Murthy (1964), Chand (1975),Srivenkataramana (1980), Pandey and Dubey (1988), Singh and Upadhyaya (1999), Singh (2001), Singh (2003), Singh and Tailor (2003), Singh et. al. (2004), Kadilar and Cingi (2004), Kadilar and Cingi (2006), Singh et. al. (2007), Samiuddin and Hanif (2007), Choudhury and Singh (2011), Bijoy (2012), Choudhury and Singh (2012), Malik and Tailor (2013), Subramani and Prabavathy (2014), Singh (2015), Khan and Hussain (2015), Khare and Rehman (2015), Audu et. al. (2016) e.t.c. had contributed significantly in this direction. In 1993, Singh and Shukla modified conventional factor-type estimator which is applicable when correlations between the study and auxiliary variables are either positive or negative.

The work of Singh and Shukla (1993) has been extended by other researchers like Shukla (2002), Shukla et al. (2013), Jain and Shukla (2013) and Thakur and Gupta (2013). Extension and modification of estimators are being aimed at improving their performance (efficiency); that is, their closeness to the corresponding population parameters which is being measured using their mean squared error (MSE).

In the comparison of various statistical procedures, efficiency is a measure of quality of an estimator, of an experimental design, or of a hypothesis testing procedure *(*Everitt, 2002*)*. Essentially, a more efficient estimator, experiment, or test needs fewer observations than a less efficient one to achieve a given performance. Efficiency of an estimator may change significantly if the distribution changes, often dropping. This is one of the motivations of robust statistics. An estimator such as the sample mean is an efficient estimator of the population means of a normal distribution, for example, but can be an inefficient estimator of a mixture distribution of two normal distributions with the same mean and different variances. By contrast, the trimmed mean is less efficient for a normal distribution, but is more robust (less affected) by changes in distribution, and thus, may be more efficient for a mixture distribution. Similarly, the shape of a distribution, such as skewness or heavy tails, can significantly reduce the efficiency of estimators that assume a symmetric distribution or thin tails. While efficiency is a desirable quality of an estimator, it must be weighed against other considerations, and an estimator that is efficient for certain distributions may well be inefficient for other distributions. Most significantly, estimators that are efficient for clean data from a simple distribution, such as the normal distribution (which is symmetric, unimodal, and has thin tails) may not be robust to contamination by outliers, and may be inefficient for more complicated distributions. In robust statistics, more importance is

placed on robustness and applicability to a wide variety of distributions, rather than efficiency on a single distribution (Nikulin, 2001). The efficiency of estimators based on simple random sampling is not only influenced by the underlying distributions, but also the sample size of the survey. Their consistency improved as the sample size increases and decreases due to incomplete information caused by missing observations or non-response by respondents, experiments e.t.c. Determination of efficiency of ratio-type, product-type and factor-type estimators in the presence of non-response are general classes of solutions motivated by these concerns, yielding both robustness and high relative efficiency which can be measured using nonresponse model; though possibly lower efficiency than traditional estimators for some cases.

It is well known especially in human surveys that information is generally not obtained from all the sample units even after callbacks. This is called non-response. The problem of estimating the parameters such as ratio of two means, population mean and variance when some observations are missing due to random non response has been discussed by Toutenberg and Srivastava (1998), Singh and Joarder (1998), Singh et al. (2000), Singh and Tracy (2001) and Singh and Tailor (2005). In case of non-random non-response, the problem of estimation of population mean using information on single auxiliary character has been considered by different authors such as Tabasum and Khan (2004, 2006), Khare and Sinha (2007), Singh and Kumar (2009, 2011), Kumar et al. (2011).

The direction and quantity of correlation (relationship) between the study and auxiliary variables are not only the determinants of the efficiency of estimators, but also the nature of the relationship; linear or non-linear and the robustness and efficiency of estimators for the population mean and total can be investigated by a variety of approaches which include

design based and model based methods. The model-based approach is based on super population models, which assume that the population under study is a realization of super-population random variables having a super population model ξ. This super population model ξ formalizes our prior knowledge about the population and is used to predict the non-sampled values of the population, and hence finite population quantities or total.

1.1 STATEMENT OF THE PROBLEM

The efficiency of factor-type estimators depends on the optimum estimate of the value of positive real number $d\left(d \in \Re^{+}\right)$ and they are expected to be applicable and robust irrespective of the direction of correlation $\left(\rho_{xy}\right)$ between the study and auxiliary variables. However, some existing factor-type estimators are found to be challenged either for some values of $d \in \Re^{+}$ or correlation $\left(\rho_{xy}\right)$. For example, Shukla et. al. (2013) factor estimator $\bar{y}_{TFT}$ is zero at d=1, Jain and Shukla (2013) factor estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ are independent of d and Thakur and Gupta (2013) factor estimator is less efficient when $\rho_{xy} > 0$ and also less robust for both $\rho_{xy} > 0$ and $\rho_{xy} < 0$ under super-population models. These limitations identified above prompt the present study.

1.2 AIM AND OBJECTIVES

This study aimed at modifying some conventional factor-type estimators under single-phase and two-phase sampling schemes by incorporation of more information on auxiliary variables like coefficient of skewness ($\beta_1(x)$), coefficient of kurtosis ($\beta_2(x)$), variation (C_x) and standard deviation (σ_x) to produce more efficient ones.

The objectives are to:

i. obtain four modified factor-type estimators under single and two-phase sampling;

ii. derive the Biases and Mean Square Errors (MSEs) of these modified factor-type estimators;

iii. derive the conditions under which these modified estimators will be more efficient than those conventional ones.

iv. determine the relative efficiency of these modified estimators using both real life and simulated data; and

v. determine the robustness of these modified factor-type estimators under super-population and non-response model.

1.3 SIGNIFICANCE OF THE STUDY

The applications of factor-type estimators are recently receiving greater attention, especially in the areas of estimation of ready queue processing time in multiprocessor environment, lottery scheduling, thread scheduling (Jain and Shukla (2013) and Shukla et. al. (2013)). The applicability of these methods calls for the development of robust estimators with high relative efficiencies of which is the bases for this research. Also the present ongoing research can be utilized in any field in area of estimation of study variable. The modified estimators in the present research work provide a various estimators as their particular cases which can be used in sampling theory at estimation stage.

1.4 SCOPE AND LIMITATION OF THE STUDY

The present study will focus on the use of auxiliary information like Coefficients of Kurtosis, Skewness, Variation, Correlation and Standard Deviation to develop robust factor type estimators under one-phase and two-phase sampling strategies and the robustness properties of the modified estimators will be investigated under:

1. Super-population (Linear, Quadratic and Cubic) models.
2. Non-response model.

1.5 DEFINITION OF RELATED TERMS

1.5.1 Sample

A sample is a group of units selected from larger group (population). By studying the sample, it is hoped to draw valid conclusions about the larger group. A sample is generally selected for study because the population is too large to study in its entirety. The sample should be representative of general population. This is often best achieved by random sampling. Also, before collecting the sample, it is important that the researcher carefully and completely defines the population, including a description of the members to be included (Cochran, 1977).

1.5.2 Parameter

A parameter is a value usually unknown (and which therefore has to be estimated), used to represent a certain population characteristic. Within a population, a parameter is fixed value which does not vary. They are often denoted by Greek letters (Cochran, 1977).

1.5.3 Statistic

A statistic is a quantity that is calculated from a sample data. It is used to give information about unknown values in the corresponding population. it is possible to draw more than one sample from the same population and the value of a statistic will in general vary from sample to sample. Therefore, statistic is a random variable (Cochran, 1977).

1.5.4 Estimator

An estimator is a rule for calculating an estimate of a given quantity based on observed data. There are point and interval estimators. The point estimator yields single-valued

results, although this includes the possibility of single vector-valued results and results that can be expressed as a single function. This is in contrast to an interval estimator, where the results would be a range of plausible values (or vectors or functions). An estimator is a statistic, (that is, a function of data) that is used to infer the value of an unknown parameter in statistical model. The parameter being estimated is sometimes called estimand. It can be either finite-dimensional (in parametric and semi-parametric) or finite-dimensional (in nonparametric and semi-nonparametric models). If the parameter is denoted by θ, then the estimator is typically written as $\hat{\theta}$. Being a function of data, the estimator is a random variable (Cochran, 1977).

1.5.5 Bias

The bias of an estimator is the difference between this estimator's expected value and the true value of the parameter being estimated. An estimator with zero bias is called unbiased, otherwise the estimator is said to be biased (Cochran, 1977). Suppose we have a statistical model parameterized by θ giving rise to a probability distribution for observed data $p(x \backslash \theta)$ and a statistic $\hat{\theta}$ which serves as an estimator based on any observed data x. That is, we assume that our data follows some unknown distribution $p(x \backslash \theta)$ (where θ is a fixed constant that is part of this distribution, but is unknown), and then, we construct some estimators $\hat{\theta}$ that maps observed data to values that we hope are close to. Then, the bias of this estimator is defined to be;

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \qquad (1.1)$$

1.5.6 Mean Square Error (MSE)

The MSE of an estimator is one of many ways to quantify the difference between values implies by an estimator and the true values of the quantity being estimated. MSE is a risk function, corresponding to the expected value of the squared error loss or quadratic loss. MSE measures the average of the squares of the errors. The error is the amount by which the value implied by the estimator differs from the quantity to be estimated. The difference occurs because of randomness or because the estimator doesn't account for information that could produce a more accurate estimate. The MSE is the second moment (about the origin) of the error, and thus incorporates both the variance of the estimator and its bias. For an unbiased estimator, the MSE is the variance. The MSE of an estimator $\hat{\theta}$ with respect to the estimated θ is defined mathematically as;

$$MSE\left(\hat{\theta}\right) = E\left[\hat{\theta} - \theta\right]^2 = \text{var}\left(\hat{\theta}\right) + \left(Bias\left(\hat{\theta}\right)\right)^2 \qquad (1.2)$$

The MSE thus assess the quality of an estimator in terms of its variation and unbiasedness (Cochran, 1977).

1.5.7 Kurtosis

Kurtosis is any measure of the peakedness of the probability distribution of a real-valued random variable. It is descriptor of the shape of probability distributions. One common measure of kurtosis originated by Pearson, is based on a scaled version of the fourth moment of the data or population. For this measure, higher kurtosis means more of the variance is the result of infrequent extreme deviations as opposed to frequent modestly sized deviations. Distributions with negative or positive excess are called platykurtic or leptokurtic respectively. The fourth standardized moment is defined as;

$$\beta_2 = \frac{\mu_4}{\sigma^4}, \tag{1.3}$$

where μ_4 is the fourth moment about the mean and σ is the standard deviation (Cochran, 1977).

1.5.8 Coefficient of Variation (CV)

CV is a normalized measure of dispersion of a probability distribution. It is known as unitized risk or the variation coefficient. The absolute value of CV is sometimes known as relative standard deviation (RSD), which is express as a percentage. The CV is defined as the ratio of the standard deviation to the mean;

$$CV = \frac{\sigma}{\mu} \tag{1.4}$$

where μ is the population mean.

CV is the inverse of the signal-to-noise ratio. It shows the extent of variability in relation to mean of the population (Cochran, 1977).

CHAPTER TWO

LITERATURE REVIEWS

2.0 INTRODCTION

In the decade of 1940-1950, the contributions of Cochran (1942), Deming (1956), Hurvitz (1952) and others helped in laying the foundation of modern sampling theory. The work done during this period related mainly with the development of sampling and estimation procedures under different sets of assumptions about the population values.

In 1942, Cochran made a particular important contribution to the modern sampling theory by suggesting methods of utilizing the auxiliary information for the purpose of estimation of the population mean in order to increase the precision of the estimates (Cochran 1942). It is a common experience that use of auxiliary information gives rise to some estimators that are known today, and in under certain conditions, these estimators are more efficient than the estimators based on simple random sampling. The use of auxiliary information in the estimation of population values of the study variable has been a common phenomenon in the sampling theory of surveys.

2.1 RATIO-PRODUCT ESTIMATORS

Let $\Omega = (1\ 2, 3\ \ldots N)$be a population of size N andY, X be two real valued functions having values$(Y_i,\ X_i) \in \mathbb{R}^+ > 0$ on the i^{th} unit of $U(1 \leq i \leq N)$.. Let $\bar{Y}$ and $\bar{X}$ be the population means of Y and X respectively with C_y and C_x as coefficients of variation of Y and X.

Let a pair (y_i, x_i) of simple random sample of size n be drawn without replacement from the population $\Omega = (1\ 2, 3\ \ldots N)$ and $\bar{y}$, $\bar{x}$ be sample means based on the sample drawn. The usual sample mean $\bar{y}$, traditional ratio estimator $\bar{y}_r$ (Cochran (1942)), product

estimator $\bar{y}_p$ (Murthy (1964)) and Dual to ratio estimator $\bar{y}_{st}$ (Strivenkataramana (1980)) with their respective bias and variance/mean square error are defined as

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{2.1}$$

$$\text{var}\left(\bar{y}\right) = \frac{1-f}{n}\bar{Y}^2 C_y^2 \tag{2.2}$$

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\bar{X} \tag{2.3}$$

$$Bias\left(\bar{y}_r\right) = \bar{Y}\frac{1-f}{n}\left[C_x^2 - \rho_{xy}C_xC_y\right] \tag{2.4}$$

$$MSE\left(\bar{y}_r\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + C_x^2 - 2\rho_{xy}C_xC_y\right] \tag{2.5}$$

$$\bar{y}_p = \frac{\bar{y}}{\bar{X}}\bar{x} \tag{2.6}$$

$$Bias\left(\bar{y}_p\right) = \bar{Y}\frac{1-f}{n}\left[C_x^2 + \rho_{xy}C_xC_y\right] \tag{2.7}$$

$$MSE\left(\bar{y}_p\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + C_x^2 + 2\rho_{xy}C_xC_y\right] \tag{2.8}$$

$$\bar{y}_{st} = \bar{y}\frac{N\bar{X} - n\bar{x}}{\left(N-n\right)\bar{X}} \tag{2.9}$$

$$Bias\left(\bar{y}_{st}\right) = \bar{Y}\frac{1-f}{n}\left[\left(\frac{n}{N-n}\right)^2 C_x^2 - \frac{n}{N-n}\rho_{xy}C_xC_y\right] \tag{2.10}$$

$$MSE\left(\bar{y}_{st}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{n}{N-n}\right)^2 C_x^2 - 2\frac{n}{N-n}\rho_{xy}C_xC_y\right] \tag{2.11}$$

The traditional ratio estimator $\bar{y}_r$ and Dual to ratio estimator $\bar{y}_{st}$ have higher efficiencies when the correlation between the study and auxiliary variables is strong and positive while

product estimator $\bar{y}_p$ has higher efficiency when the correlation between the study and auxiliary variables is strong and negative.

Sarndal (1972) in his study of estimation of population mean under sample survey and general statistical theories discovered that in many practical situations, when we want to estimate the unknown characteristic of the population there exists some large $(y_{\max})$ or small values $(y_{\min})$ and to estimate the population quantities without considering these information is very sensitive in either case the result will be either over estimated or under estimated. In order to tackle this situation, he modified the following unbiased estimator for finite population mean.

$$\bar{y}_{Sa} = \begin{cases} \bar{y} + c \;\; if \;\; sample\, contains\, y_{\min}\, but\, not\, y_{\max} \\ \bar{y} - c \;\; if \;\; sample\, contains\, y_{\max}\, but\, not\, y_{\min} \\ \bar{y} \qquad for\, other\, samples \end{cases} \tag{2.12}$$

The variance of the modified estimator is

$$\operatorname{var}\left(\bar{y}_{Sa}\right) = \operatorname{var}\left(\bar{y}\right) - \frac{1-f}{n}\frac{\left[y_{\max} - y_{\min}\right]^2}{2\left(N-1\right)} \tag{2.13}$$

The result of the study showed that the modified estimator is at optimum when $c = \left(y_{\max} - y_{\min}\right)/2n$.

Chand (1975) modified chain ratio estimator with two auxiliary variables under double sampling scheme. The estimator and its mean square error are given as follow:

$$\bar{\Omega}_3 = \bar{y}_2 \frac{\bar{x}_2}{\bar{x}_1}\frac{\bar{z}_1}{\bar{Z}} \tag{2.14}$$

$$MSE\left(\bar{\Omega}_3\right) = \bar{Y}^2\left[\delta_2 C_y^2 + \delta_3\left(\left(C_x + \rho_{xy}C_y\right)^2 - C_y^2\rho_{xy}^2\right) + \delta_1\left(\left(C_z + \rho_{yz}C_z\right)^2 - C_z^2\rho_{yz}^2\right)\right] \tag{2.15}$$

Srivastava et al. (1990) modified ratio estimator with two auxiliary variables and two unknown as power transformation under double sampling scheme. The estimator and its mean square error are given as follow:

$$\bar{\Omega}_6 = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2} \right]^{\alpha_1} \left[\frac{\bar{Z}}{\bar{z}_1} \right]^{\alpha_2} \tag{2.16}$$

$$MSE\left(\bar{\Omega}_6\right) = \bar{Y}^2 C_y^2 \left[\delta_2 - \delta_3 \rho_{xy}^2 - \delta_1 \rho_{yz}^2 \right] \tag{2.17}$$

Singh (2001) also modified chain ratio type estimator with two auxiliary variables under double sampling scheme. The second auxiliary variable of the estimator is subjected to both linear and power transformation. The estimator and its minimum mean square error is given as follow:

$$\bar{\Omega}_8 = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2} \right] \left[\frac{\alpha \bar{Z} + \sigma_z}{\alpha \bar{z}_1 + \sigma_z} \right]^g \tag{2.18}$$

$$MSE\left(\bar{\Omega}_8\right) = \bar{Y}^2 \left[\delta_2 C_y^2 + \delta_3 \left(C_x^2 - 2 C_x C_y \rho_{xy} \right) - \delta_1 C_y^2 \rho_{yz}^2 \right] \tag{2.19}$$

Singh et al. (2004) modified ratio estimator with two auxiliary variables, unknown powers and known functions of auxiliary variable Z under two-phase sampling. Their modified estimator and minimum MSE are given below;

$$\bar{\Omega}_{14} = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2} \right]^{\alpha_1} \left[\frac{a\bar{Z} + b}{a\bar{z}_1 + b} \right]^{\alpha_2} \left[\frac{a\bar{Z} + b}{a\bar{z}_2 + b} \right]^{\alpha_3} \tag{2.20}$$

$$MSE\left(\bar{\Omega}_{14}\right) = \bar{Y}^2 C_y^2 \left[\delta_2 - \delta_1 \rho_{yz}^2 - \delta_3 \rho_{y.xz}^2 \right] \tag{2.21}$$

Samiuddin and Hanif (2007) also modified ratio estimator with two auxiliary variables and unknown powers under two-phase sampling. The estimator utilized both the sample and population means of auxiliary variable Z. Their modified estimator and minimum MSE are given below;

$$\bar{\Omega}_{10} = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2}\right]^{\alpha_1} \left[\frac{\bar{z}_1}{\bar{z}_2}\right]^{\alpha_2} \left[\frac{\bar{Z}}{\bar{z}_2}\right]^{\alpha_3} \tag{2.22}$$

$$MSE\left(\bar{\Omega}_{10}\right) = \bar{Y}^2 C_y^2 \left[\delta_2\left(1-\rho_{y.xz}^2\right) + \delta_1\left(1-\rho_{yz}^2\right)\rho_{yx.z}^2\right] \tag{2.23}$$

Singh *et al*. (2007) modified a transformed chain ratio type estimator for the population mean *Y* utilizing the known correlation coefficient (of the second auxiliary character through a simple transformation for estimating the population mean of auxiliary character more precisely in the first phase (preliminary) sample as

$$\bar{y}_S = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2}\right]\left[\frac{\bar{Z}+\rho_{xz}}{\bar{z}_1+\rho_{xz}}\right] \tag{2.24}$$

Singh et al. (2007) extended their work by using power transformation to generalize the power of auxiliary variables. The modified estimator and its minimum MSE are given below;

$$\bar{\Omega}_{13} = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2}\right]^{\alpha_1} \left[\frac{\bar{Z}+\rho_{xz}}{\bar{z}_1+\rho_{xz}}\right]^{\alpha_2} \tag{2.25}$$

$$MSE\left(\bar{\Omega}_{13}\right) = \bar{Y}^2 C_y^2 \left[\delta_2 - \delta_3\rho_{xy}^2 - \delta_1\rho_{yz}^2\right] \tag{2.26}$$

For estimating the population mean $\bar{Y}$ of the study variate y, Singh and Espejo (2003) considered a linear combination of ratio and product estimators' type given by

$$\bar{y}_{RP} = \bar{y}\left[k\frac{\bar{X}}{\bar{x}} + (1-k)\frac{\bar{x}}{\bar{X}}\right] \tag{2.27}$$

and obtained that estimator attained optimality when $k = \frac{(1+C)}{2}$ with $C = \rho C_y/C_x$, where *Cy* is the coefficient of variation of y and *Cx* is the coefficient of variation of *x*.

Singh and Espejo (2007), modified that when the population mean $\bar{X}$ of x is not known, a first-phase sample of size n_1 should be drawn from the population on which only the x-

characteristic is measured in order to furnish a good estimate of $\bar{X}$. Then a second-phase sample of size n is drawn on which both the variables y and x are measured. Let $\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ denote the sample mean of x based on the first-phase sample of size n_1, Singh and Espejo(2007) considered a ratio–product type estimator in the two-phase sampling given by

$$\bar{y}_{RP}^{(d)} = \bar{y}\left[k\frac{\bar{x}_1}{\bar{x}} + (1-k)\frac{\bar{x}}{\bar{x}_1}\right] \tag{2.28}$$

Choudhury and Singh (2012) consider the work of Singh and Espejo (2007) and modified a class chain ratio-product type estimator $\bar{y}_{RP}^{(dc)}$ for estimating population mean $\bar{Y}$using two auxiliary characters under two conditions in Singh and Espejo (2007). The modified estimator, its bias and MSE are respectively given as

$$\bar{y}_{RP}^{(dc)} = \bar{y}\left[k\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}}{\bar{z}_1} + (1-k)\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1}{\bar{Z}}\right] \tag{2.29}$$

$$Bias\left(\bar{y}_{RP}^{(dc)}\right)_{\mathrm{I}} = \bar{Y}\left[\frac{1-f^*}{n}C_x^2\left(k+(1-2k)C_{yx}\right) + \frac{1-f_1}{n_1}C_z^2\left(k+(1-2k)C_{yz}\right)\right] \tag{2.30}$$

$$Bias\left(\bar{y}_{RP}^{(dc)}\right)_{II} = \bar{Y}\left[\frac{1-f_1}{n_1}(1-k)C_x^2 + \frac{1-f}{n}C_x^2\left(k+(1-2k)C_{yx}\right) + \frac{1-f_1}{n_1}C_z^2\left(k-C_{xz}\right)\right] \tag{2.31}$$

$$MSE\left(\bar{y}_{RP}^{(dc)}\right)_{\mathrm{Im}in} = \bar{Y}^2\left[\frac{1-f}{n}C_y^2 - \left\{\frac{1-f^*}{n}C_x^2C_{yx} + \frac{1-f_1}{n_1}C_z^2C_{yz}\right\} / \left\{\frac{1-f^*}{n}C_x^2 + \frac{1-f_1}{n_1}C_z^2\right\}\right] \tag{2.32}$$

$$MSE\left(\bar{y}_{RP}^{(dc)}\right) = \bar{Y}^2\left[\frac{1-f}{n}C_y^2 - \left\{\left(\frac{1-f}{n}\right)^2 C_{yx}^2C_x^4\right\} / \left\{f^{**}C_x^2 + \frac{1-f_1}{n_1}C_z^2\left(1-2C_{xz}\right)\right\}\right] \tag{2.33}$$

Where $f = \frac{n}{N}$, $f^* = \frac{n}{n_1}$, $f_1 = \frac{n_1}{N}$, $f^{**} = \frac{1-f}{n} + \frac{1-f_1}{n_1}$, $C_{yx} = \rho_{xy} C_y / C_x$, $C_{yz} = \rho_{yz} C_y / C_z$, $C_{xz} = \rho_{xz} C_x / C_z$.

In their study, it was discovered that if another auxiliary variable Z is closely related to auxiliary variable X, high efficient estimate can be obtained by estimating $\bar{X}$ by $\bar{X} = \frac{\bar{x}_1 \bar{Z}}{\bar{z}_1}$ especially when $\frac{\rho_{xz} C_x}{C_z} > \frac{1}{2}$.

Khan et.al (2012) modified a class of improved estimators for estimating population mean regarding partial information in double sampling. In their work, functions of some known parameters of auxiliary variables like standard deviation, coefficient of correlation, skewness and kurtosis e.t.c. and power transformation of auxiliary variable were incorporated in the work of Singh (2001) and Singh and Agnihortie (2008). This transformation increased the efficiencies of latter's. The modified estimators and their MSEs are given below;

$$\hat{\bar{Y}}_1 = \bar{y}_2 \left[\frac{\bar{x}_1}{\bar{x}_2} \right]^{\alpha} \left(\frac{a_z \bar{Z} + b_z}{a_z \bar{z}_1 + b_z} \right) \tag{2.34}$$

$$Bias\left(\hat{\bar{Y}} \right) = \bar{Y} \left[-\frac{\theta_3}{2} C_y^2 \rho_{xy}^2 + \frac{\theta_3}{2} C_x \left(C_y \rho_{xy} - \frac{C_x}{4} \right) - \frac{\theta_1}{4} C_y^2 \rho_{yz}^2 \right] \tag{2.35}$$

$$MSE\left(\hat{\bar{Y}} \right)_{\min} = \bar{Y}^2 C_y^2 \left[\theta_2 - \theta_3 \rho_{xy}^2 - \theta_1 \rho_{yz}^2 \right] \tag{2.36}$$

$$\hat{\bar{Y}}_2 = \bar{y}_2 \left[\frac{\bar{z}_1}{\bar{z}_2} \right]^{\alpha} \left[\delta \left(\frac{a_x \bar{X} + b_x}{a_x \bar{x} + b_x} \right) + (1-\delta) \left(\frac{a_x \bar{x}_1 + b_x}{a_x \bar{X} + b_x} \right) \right] \tag{2.37}$$

$$Bias\left(\hat{\bar{Y}}_2 \right) = \bar{Y} \left[\frac{\theta_3}{2} \left(C_y \rho_{yz} - \frac{C_z}{2} \right)^2 \right] \tag{2.38}$$

$$MSE\left(\hat{\bar{Y}}_2\right)_{\min} = \bar{Y}^2 C_y^2 \left[\theta_2 - \theta_1 \rho_{xy}^2 - \theta_3 \rho_{yz}^2\right] \tag{2.39}$$

where $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$ and $\theta_3 = \theta_2 - \theta_1$

Bijoy (2012) modified a class of regression-cum-ratio estimators in two-phase sampling for estimating average volume (in cubic meter) of forest timber by utilizing information from high resolution satellite data. The modified estimator is an improved work of Mohanty (1967) by using the transformation modified by Singh et. al. (2007). The modified estimator and its MSE are given below;

$$\bar{Y}_{BH} = \left[\bar{y}_2 + b_{yx}\left(\bar{x}_1 - \bar{x}_2\right)\right]\left[\frac{\bar{Z} + \rho_{xz}}{\bar{z}_1 + \rho_{xz}}\right] \tag{2.40}$$

$$MSE\left(\bar{Y}_{BH}\right) = \bar{Y}^2\left[\theta_2 C_y^2 - \theta_3 C_y^2 \rho_{xy}^2 + \theta_1\left\{\left(\bar{Z}C_z / \left[\bar{Z} + \rho_{xz}\right]\right)^2 - C_y^2 \rho_{yz}^2\right\}\right] \tag{2.41}$$

In their study, the efficiency of the modified estimator $\bar{Y}_{BH}$ was compared to that of the estimators modified by Mohanty (1967), Chand (1975) and Singh et. al. (2007) and results of their study revealed that the estimator modified is more efficient if $\rho_{xz} > 0$, $\left(C_x - C_y \rho_{xy}\right)^2 > 0$ and $C_x^2 > 0$

Malik and Tailor (2013) modified double sampling version of (Singh and Tailor, 2003) ratio estimator. Both sample means of preliminary and secondary sampling were modified using correlation coefficient between the study and auxiliary variables. The modified estimator, its bias and MSE are given below;

$$\hat{\bar{Y}}_{ST} = \bar{y}_2\left[\frac{\bar{x}_1 + \rho_{xy}}{\bar{x}_2 + \rho_{xy}}\right] \tag{2.42}$$

$$Bias\left(\hat{\bar{Y}}_{ST}\right)=\bar{Y}\theta_3\theta C_x^2\left[\theta-C_{yx}\right] \tag{2.43}$$

$$MSE\left(\hat{\bar{Y}}_{ST}\right)=\bar{Y}^2\left[\theta_2 C_y^2+\theta_3\theta C_x^2\left(\theta-2C_{yx}\right)\right] \tag{2.44}$$

where $\theta_1=\frac{1}{n_1}-\frac{1}{N}$, $\theta_2=\frac{1}{n_2}-\frac{1}{N}$, $\theta_3=\theta_2-\theta_1$, $\theta=\frac{\bar{X}}{\bar{X}+\rho_{xy}}$, $C_{yx}=\rho_{xy}C_y/C_x$.

The results of their study showed that the modified estimator performed better than sample mean and Singh and Tailor (2003) estimators if $0<\theta<2C_{yx}$ and $2C_{yx}<\theta<1$ respectively.

Chanu and Singh (2014) modified improved class of ratio-cum-product estimators of finite population mean in two phase sampling. The modified estimator is an improvement over the work of Choudhury and Singh (2011) using power transformation. The modified estimator and its properties under the two cases of two-phase sampling are given below as;

$$\bar{Y}_{RP}^{w(d)}=\bar{y}_2\left[\frac{\bar{x}_1}{\bar{x}_2}.\frac{\bar{z}_2}{\bar{z}_1}\right]^{\alpha} \tag{2.45}$$

$$Bias\left(\bar{Y}_{RP}^{w(d)}\right)_{I\min}=\frac{1}{2}\bar{Y}\theta_3\frac{k_1\left[C_x^2\left(1-C_{yx}\right)-C_z^2\left(1-C_{yz}\right)\right]}{\left[C_x^2+C_z^2\left(1-2C_{xz}\right)\right]} \tag{2.46}$$

$$MSE\left(\bar{Y}_{RP}^{w(d)}\right)_{I\min}=\bar{Y}^2\left[\theta_2 C_y^2-\theta_3 k_1^2/\left(C_x^2+C_z^2\left\{1-2C_{xz}\right\}\right)\right] \tag{2.47}$$

$$Bias\left(\bar{Y}_{RP}^{w(d)}\right)_{II\,min}=\bar{Y}\frac{\theta_2 k_1}{N_1 k_2}\left[\frac{1}{2}\theta_3\left(C_x^2-C_z^2\right)-\theta_2\frac{k_1}{2N_1C_x^2\left(1-2C_{yx}\right)}\right] \tag{2.48}$$

$$MSE\left(\bar{Y}_{RP}^{w(d)}\right)_{II\min}=\bar{Y}^2\theta_2\left[C_y^2-\theta_2 k_1^2/2N_1k_2\right] \tag{2.49}$$

where $k_1=C_y\left[\rho_{xy}C_x-\rho_{yz}C_z\right]$, $k_2=C_x^2+C_z^2\left(1-2C_{xz}\right)$ and $N_1=\frac{1}{2}\left(\frac{1}{n_2}+\frac{1}{n_1}-\frac{2}{N}\right)$

The results of their study showed that the modified estimator attained optimality if $\alpha = k_1 / k_2$ and $\alpha = \theta_2 k_1 / 2N_1 k_2$ when secondary sample is a subset of preliminary sample and population respectively. The results of the empirical study revealed that to the first degree of approximation, the modified class of estimators under the optimality condition is consider to be more efficient than two-phase traditional product estimator, ratio estimator, Singh (1967) estimator and sample mean if $k_2 > 0, k_3 > 0, k_4 > 0$ and $k_5 > 0$ where $k_2 = C_x^2\left(1-2C_{xy}\right)$, $k_3 = C_x^2 + C_z^2\left(1-2C_{xz}\right)$, $k_4 = C_z^2\left(1-2C_{xz}+2C_{yz}\right)$

Shirley et al. (2014) in their work titled "On improving ratio and product estimators by ratio and product-cum-mean-per-unit estimator targeting more efficient use of auxiliary information" used single parameter linear combinations of $\bar{y}_r$ and $-\bar{y}$ as well as $\bar{y}_p$ and $-\bar{y}$ to modified the following estimators;

$$\hat{\bar{Y}}_{SSDR} = \left(1+\theta\right)\bar{y}_r - \theta\bar{y} \tag{2.50}$$

$$\hat{\bar{Y}}_{SSDP} = \left(1+\theta\right)\bar{y}_p - \theta\bar{y} \tag{2.51}$$

where θ is the designed parameter for the modified estimators to be assigned an optimal value which minimizes the MSE of the modified estimators.

The biases and MSEs of $\hat{\bar{Y}}_{SSDR}$ and $\hat{\bar{Y}}_{SSDP}$ are respectively given below;

$$Bias\left(\hat{\bar{Y}}_{SSDR}\right) = B_0\left[a\left(2-F\right) - \rho_{xy}F\right] \tag{2.52}$$

$$Bias\left(\hat{\bar{Y}}_{SSDP}\right) = B_0\rho_{xy}F \tag{2.53}$$

$$MSE\left(\hat{\bar{Y}}_{SSDR}\right) = V_0\left[1 + a^2F^2 - 2aF\rho_{xy}\right] \tag{2.54}$$

$$MSE\left(\hat{\bar{Y}}_{SSDP}\right)=V_0\left[1+a^2F^2+2aF\rho_{xy}\right] \tag{2.55}$$

where $a=C_x/C_y$, $F=1+\theta$, $B_0=a(1-f)\bar{Y}C_y^2/n$, $V_0=\text{var}(\bar{y})$.

The theoretical result of their study showed that the performances of modified estimators are at optimum when $F=\rho_{xy}/a$. An empirical study to compare the efficiency of the modified estimators to traditional ratio, product and regression estimators was conducted under the assumptions that the parent population came from a super-population which bivariate normal with parameters $\bar{Y}=20.0, \bar{X}=15.0, S_y=3.0, S_x=1.0$ and a guess value of F defined by $F^*=F(1+R)$, where $R=\pm0.02,\pm0.04,\pm0.08,\pm0.10$ is the quantum of relative under guess or over guess and the results showed that the modified estimators are not only more efficient but also robust against possible under guess or over guess of the key-population parameter F.

Enang et al. (2014) modified ratio estimator obtained by forming linear combination of Singh and Tailor (2003) and Kadilar and Cingi (2004) estimators as shown below;

$$\bar{y}_{EAE}=\frac{\alpha\bar{y}\left(\bar{X}+\rho_{xy}\right)}{\bar{x}+\rho_{xy}}+\frac{\beta\left(\bar{y}+b_{yx}\left[\bar{X}-\bar{x}\right]\right)\bar{X}}{\bar{x}} \tag{2.56}$$

such that $\alpha+\beta=1$.

The bias and MSE of their estimator are given below as;

$$Bias\left(\bar{y}_{EAE}\right)=\frac{1-f}{n}\bar{Y}\left[\left(\alpha-\alpha w-1\right)\rho_{xy}C_xC_y+\left(\alpha\beta_{yx}K-\beta_{yx}K-\alpha-\alpha w\right)^2C_x^2\right] \tag{2.57}$$

$$\begin{aligned}MSE\left(\bar{y}_{EAE}\right)=&\frac{1-f}{n}\bar{Y}^2\left[C_y^2+2\left(\alpha\beta_{yx}K-\beta_{yx}K-\alpha-\alpha w\right)\rho_{xy}C_xC_y\right.\\&\left.+\left(\alpha\beta_{yx}K-\beta_{yx}K-\alpha-\alpha w\right)^2C_x^2\right]\end{aligned} \tag{2.58}$$

The results of their work showed that the performance of their modified estimator is at optimal when $\alpha = \frac{\left(\beta_{yx}K+1\right)C_x^2 - \rho_{xy}C_xC_y}{\left(\beta_{yx}K+1-w\right)C_x^2}$ and that the modified estimator performed better than the estimators of Cochran (1942), Singh and Tailor (2003) and Kadiliar and Cingi (2004) when $\left(C_y\rho_{xy} - C_x\right)^2 > 0$, $C_y^2\rho_{xy}^2 + C_x^2 w\left(w - 2C_{yx}\right) > 0$ and $C_x^2 > 0$ respectively.

Subramani and Prabavathy (2014) modified two estimators of population mean based on median of study variable using auxiliary information as:

$$\bar{y}_{SP1} = \bar{y}\left[\frac{M_d M + \bar{X}}{M_d m + \bar{X}}\right] \tag{2.59}$$

`

$$\bar{y}_{SP2} = \bar{y}\left[\frac{\bar{X}M + M_d}{\bar{X}m + M_d}\right] \tag{2.60}$$

where M, M_d and m are Median of the study variable, Median of the auxiliary variable and Sample median of study variable respectively.

The MSEs of modified estimators are given below;

$$MSE\left(\bar{y}_{SPi}\right) = \text{var}\left(\bar{y}\right) + R_m^2\phi_i^2\,\text{var}\left(m\right) - 2R_m\phi_i\,\text{cov}\left(\bar{y}, m\right) \quad i = 1, 2 \tag{2.61}$$

where $R_m = \bar{Y} / M$ $\phi_1 = MM_d / \left(MM_d + \bar{X}\right)$ and $\phi_2 = \bar{X}M / \left(\bar{X}M + M_d\right)$.

Yadav et al. (2014) modified two efficient estimators of population mean using linear combination of population mean and median of auxiliary variable. They improved the work of Subramani and Prabavathy (2014) using unknown weights k_1 and k_2. The modified estimators and their MSEs are given below;

$$\bar{y}_{YMS1} = k_1\bar{y}\left[\frac{M_d M + \bar{X}}{M_d m + \bar{X}}\right] \tag{2.62}$$

` $$\bar{y}_{YMS2} = k_2 \bar{y}\left[\frac{\bar{X}M + M_d}{\bar{X}m + M_d}\right] \tag{2.63}$$

$$MSE\left(\bar{y}_{YMS1}\right) = \bar{Y}^2\left[1 - A_1^2 / B_1\right] \tag{2.64}$$

$$MSE\left(\bar{y}_{YMS2}\right) = \bar{Y}^2\left[1 - A_2^2 / B_2\right] \tag{2.65}$$

where $A_1 = 1 + \phi_1^2 \operatorname{var}(m)/M^2 - \phi_1 c\operatorname{ov}(\bar{y}, m)/\bar{Y}M$,

$A_2 = 1 + \phi_2^2 \operatorname{var}(m)/M^2 - \phi_2 c\operatorname{ov}(\bar{y}, m)/\bar{Y}M$

$B_1 = 1 + \operatorname{var}(\bar{y})/\bar{Y}^2 + 3\phi_1^2 \operatorname{var}(m)/M^2 - 4\phi_1 \operatorname{cov}(\bar{y}, m)/\bar{Y}M$,

$B_2 = 1 + \operatorname{var}(\bar{y})/\bar{Y}^2 + 3\phi_2^2 \operatorname{var}(m)/M^2 - 4\phi_2 \operatorname{cov}(\bar{y}, m)/\bar{Y}M$,

The results of their study showed that the estimators $\bar{y}_{YMS1}$ and $\bar{y}_{YMS2}$ are optimized when $k_1 = A_1 / B_1$ and $k_2 = A_2 / B_2$ respectively.

Singh (2015) modified an improved class of ratio type estimator for finite population mean using unknown weight and power transformation strategies. The particular cases of this estimator are Walsh (1970) estimator at $k = 1$ and $\alpha = w$, Ray et al. (1979) estimator at $k = -1$ and $\alpha = -\gamma$, Srivenkataramana and Tracy (1979) at $k = -1$ and $\alpha = 1 - f$, Srivenkataramana and Tracy (1979) at $k = -1$ and $\alpha = -n/(N - n)$ and Srivenkataramana (1980) at $k = -1$ and $\alpha = 1 - w$. The modified estimator and its properties are given below;

$$\bar{y}_{RVK} = \bar{y}\left[\frac{\sqrt{\bar{X}}}{\alpha\sqrt{\bar{x}} + (1-\alpha)\sqrt{X}}\right]^K \tag{2.66}$$

$$Bias\left(\bar{y}_{RVK}\right) = \bar{Y}\frac{1-f}{n}\left[\left(\frac{k(k+1)}{4}\alpha^2 + \frac{1}{8}k\alpha\right)C_x^2 - \frac{1}{2}k\alpha\rho_{xy}C_xC_y\right] \tag{2.67}$$

$$MSE\left(\bar{y}_{RVK}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \frac{1}{4}k^2\alpha^2C_x^2 - k\alpha\rho_{xy}C_xC_y\right] \tag{2.68}$$

In his study, the modified estimators attain the optimality when $\alpha k = 2\rho_{xy}C_y / C_x$. The empirical study of in his work revealed that the modified estimator is more efficient than the estimators of Walsh (1979), Ray et al. (1979), Srivenkataramana and Tracy (1979), Srivenkataramana (1980) when

$$\min\left(2, \frac{4\rho_{xy}C_y}{wC_x} - 2\right) \le k \le \max\left(2, \frac{4\rho_{xy}C_y}{wC_x} - 2\right),$$

$$\min\left(-2, 2 - \frac{4\rho_{xy}C_y}{\gamma C_x}\right) \le k \le \max\left(-2, 2 - \frac{4\rho_{xy}C_y}{\gamma C_x}\right),$$

$$\min\left(-2, \frac{4\rho_{xy}C_y}{(1-f)C_x} + 2\right) \le k \le \max\left(-2, \frac{4\rho_{xy}C_y}{(1-f)C_x} + 2\right),$$

$$\min\left(-2, 2 - \frac{4\rho_{xy}C_y}{nC_x/(N-n)}\right) \le k \le \max\left(-2, 2 - \frac{4\rho_{xy}C_y}{nC_x/(N-n)}\right) \quad \text{and}$$

$$\min\left(-2, \frac{4\rho_{xy}C_y}{(1-w)C_x} + 2\right) \le k \le \max\left(-2, \frac{4\rho_{xy}C_y}{(1-w)C_x} + 2\right) \text{ respectively.}$$

Khan and Hussain (2015) modified a class of ratio estimators for the estimation of finite population mean using simple random sampling scheme when there are maximum and minimum values on both the study and the auxiliary variables. The modified class of estimators is modification of the work of Kadilar and Cingi (2006) using the strategy modified by Sarndal (1972) and they are defined as

$$\bar{y}_{M1} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\bar{x}_{c_{21}}}\bar{X} \tag{2.69}$$

$$\bar{y}_{M2} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}} + C_x\right)}\left(\bar{X} + C_x\right) \tag{2.70}$$

$$\bar{y}_{M3} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}} + \beta_2(x)\right)}\left(\bar{X} + \beta_2(x)\right) \tag{2.71}$$

$$\bar{y}_{M4} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}\beta_2(x) + C_x\right)}\left(\bar{X}\beta_2(x) + C_x\right) \tag{2.72}$$

$$\bar{y}_{M5} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}C_x + \beta_2(x)\right)}\left(\bar{X}C_x + \beta_2(x)\right) \tag{2.73}$$

$$\bar{y}_{M6} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}} + M_x\right)}\left(\bar{X} + M_x\right) \tag{2.74}$$

$$\bar{y}_{M7} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}M_x + \rho_{xy}\right)}\left(\bar{X}M_x + \rho_{xy}\right) \tag{2.75}$$

$$\bar{y}_{M8} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}\rho_{xy} + M_x\right)}\left(\bar{X}\rho_{xy} + M_x\right) \tag{2.76}$$

$$\bar{y}_{M9} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}C_x + M_x\right)}\left(\bar{X}C_x + M_x\right) \tag{2.77}$$

$$\bar{y}_{M10} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}\beta_2(x) + M_x\right)}\left(\bar{X}\beta_2(x) + M_x\right) \tag{2.78}$$

$$\bar{y}_{M11} = \frac{\bar{y}_{c_{11}} + b_{yx}\left(\bar{X} - \bar{x}_{c_{21}}\right)}{\left(\bar{x}_{c_{21}}M_x + \beta_2(x)\right)}\left(\bar{X}M_x + \beta_2(x)\right) \tag{2.79}$$

where M_x and $\beta_2(x)$ are the known values of the population median and population coefficient of kurtosis of the auxiliary variable respectively and $\left(\bar{y}_{c_{11}} = \bar{y} + c_1\ ,\ \bar{x}_{c_{21}} = \bar{x} + c_2\right)$ where c_1 and c_2 are unknown constants whose values are to be find for optimality conditions.

The Biases and MSEs of their estimators are given by;

$$Bias\left(\bar{y}_{Mj}\right)=\frac{\alpha_{Mj}}{\bar{X}}\frac{1-f}{n}\left[\left(\frac{\alpha_{Mj}}{2}+\beta_{yx}\right)\left(S_x^2-\frac{2nc_2}{N-1}\left\{x_{\max}-x_{\min}-nc_2\right\}\right)\right.$$
$$\left.-S_{yx}+\frac{n}{N-1}\left(c_2\left\{y_{\max}-y_{\min}\right\}+c_1\left\{x_{\max}-x_{\min}\right\}-2nc_1c_2\right)\right] \quad (2.80)$$

$$MSE\left(\bar{y}_{Mj}\right)_{min}=\frac{1-f}{n}\left[\alpha_{Mj}^2S_x^2+S_y^2\left(1-\rho_{xy}^2\right)-\frac{1}{2\left(N-1\right)}\right.$$
$$\left.\left(\left\{y_{\max}-y_{\min}\right\}-\left(\alpha_{Mj}+\beta_{yx}\right)\left\{x_{max}-x_{\min}\right\}\right)^2\right] \quad (2.81)$$

for $j=1,2,3,...,11$ where $\alpha_{M1}=\bar{Y}/\bar{X}$, $\alpha_{M2}=\bar{Y}/\left(\bar{X}+C_x\right)$, $\alpha_{M3}=\bar{Y}/\left(\bar{X}+\beta_2(x)\right)$

$\alpha_{M4}=\bar{Y}\beta_2(x)/\left(\bar{X}\beta_2(x)+C_x\right)$, $\alpha_{M5}=\bar{Y}C_x/\left(\bar{X}C_x+\beta_2(x)\right)$, $\alpha_{M6}=\bar{Y}/\left(\bar{X}+M_x\right)$,

$\alpha_{M7}=\bar{Y}M_x/\left(\bar{X}M_x+\rho_{xy}\right)$, $\alpha_{M8}=\bar{Y}\rho_{xy}/\left(\bar{X}\rho_{xy}+M_x\right)$, $\alpha_{M9}=\bar{Y}C_x/\left(\bar{X}C_x+M_x\right)$

$\alpha_{M10}=\bar{Y}\beta_2(x)/\left(\bar{X}\beta_2(x)+M_x\right)$, $\alpha_{M11}=\bar{Y}M_x/\left(\bar{X}M_x+\beta_2(x)\right)$

The results of their study showed that the performances of the modified estimators attained optimal level when $c_1=\left(y_{\max}-y_{\min}\right)/2n$ and $c_2=\left(x_{\max}-x_{min}\right)/2n$.

Akkus (2016) in his work titled "A review on the theoretical and empirical efficiency comparison of some ratio and product type mean estimators in two phase sampling scheme compared the efficiency of some estimators for population mean in the estimation of average olive production. The estimators considered in the study include that of Srivastava (1970), Chand (1975), Kiregyera (1980), Upadhyaya et al. (1990), Singh and Upadhyaya (1995), Singh and Upadhyaya (2001), Singh (2001), Singh et al.(2004), Singh et al. (2007), Singh and Espejo (2007), Singh, et. al. (2008), Choudhury and Singh (2012) and classical ratio and product estimators,

2.2 FACTOR-TYPE ESTIMATORS

Definition: Let $A=(d-1)(d-2)$, $B=(d-1)(d-4)$, $C=(d-2)(d-3)(d-4)$,

$$\psi_1=\frac{A+C}{A+fB+C},\ \psi_2=\frac{fB}{A+fB+C},\ \psi_3=\frac{A+fB}{A+fB+C},\ \psi_4=\frac{C}{A+fB+C},$$

$$P=\psi_3-\psi_1=\psi_2-\psi_4$$

d is an unknown positive real number to be estimated i.e $d\in\Re^+$

Singh and Shukla (1987, 1993) modified a traditional factor-type estimator for population mean. This estimator is a class of traditional estimators for some values of d and it is defined as

$$\bar{y}_{FT}=\bar{y}\left[\frac{(A+C)\bar{X}+fB\bar{x}}{(A+fB)\bar{X}+C\bar{x}}\right] \tag{2.82}$$

For $d=1$, the estimator $\bar{y}_{FT}$ becomes $\bar{y}_r$, for $d=2$, $\bar{y}_{FT}$ becomes $\bar{y}_p$, for $d=3$, $\bar{y}_{FT}$ becomes $\bar{y}_{st}$ and for $d=4$, $\bar{y}_{FT}$ becomes $\bar{y}$

$$Bias(\bar{y}_{FT})=\bar{Y}\frac{1-f}{n}P(\psi_4C_x^2+\rho_{xy}C_xC_y) \tag{2.83}$$

$$MSE(\bar{y}_{FT})=\bar{Y}^2\frac{1-f}{n}\left[C_y^2+P^2C_x^2+2P\rho_{xy}C_xC_y\right] \tag{2.84}$$

In their work, it was observed that factor-type estimator $\bar{y}_{FT}$ was more efficient than classical ratio estimator $\bar{y}_r$ if $P>2C_{yx}-1$.

Shukla (2002) modified a factor-type estimator for population mean under two-phase sampling as

$$\bar{y}_{FTd}=\bar{y}_2\frac{(A+C)\bar{x}_1+fB\bar{x}_2}{(A+fB)\bar{x}_1+C\bar{x}_2} \tag{2.85}$$

$$Bias\left(\bar{y}_{FTd}\right)_I = \bar{Y}P\theta_3\left[\rho_{xy}C_xC_y - \psi_4 C_x^2\right] \tag{2.86}$$

$$Bias\left(\bar{y}_{FTd}\right)_{II} = \bar{Y}P\left[\left(\theta_1\psi_3 - \theta_2\psi_4\right)C_x^2 + \theta_2\rho_{xy}C_xC_y\right] \tag{2.87}$$

$$MSE\left(\bar{y}_{FTd}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P^2 C_x^2 + 2\theta_3 P\rho_{xy}C_xC_y\right] \tag{2.88}$$

$$MSE\left(\bar{y}_{FTd}\right)_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_4 P^2 C_x^2 + 2\theta_2 P\rho_{xy}C_xC_y\right] \tag{2.89}$$

where $\theta_1 = \frac{1}{n_1} - \frac{1}{N}$, $\theta_2 = \frac{1}{n_2} - \frac{1}{N}$, $\theta_3 = \frac{1}{n_2} - \frac{1}{n_1}$, $\theta_4 = \theta_1 + \theta_2$

In their work, it was observed that factor-type estimator $\bar{y}_{FTd}$ was more efficient than classical ratio estimator $\bar{y}_r^d$, if $-2C_{yx} < P < 0$ under case I and if $-2C_{yx}\left(1+\delta\right)^{-1} < P < 0$ under case II where $\delta = \theta_1\theta_2^{-1}$.

Thakur and Gupta (2013) modified a linear combination based factor-type estimator $\bar{y}_{FTRP}$ for estimating population mean in sample surveys. It was discovered that when the correlation between the study and auxiliary variables is negative, the modified estimator based on linear combination produced better estimate than some existing estimators. The modified estimator, its bias and MSE are given below;

$$\bar{y}_{FTRP} = f\bar{y}\frac{\left(A+C\right)\bar{X} + fB\bar{x}}{\left(A+fB\right)\bar{X} + C\bar{x}} + \left(1-f\right)\bar{y}\frac{\left(A+C\right)\bar{x} + fB\bar{X}}{\left(A+fB\right)\bar{x} + C\bar{X}} \tag{2.90}$$

$$Bias\left(\bar{y}_{FTPR}\right) = \bar{Y}\frac{1-f}{n}P\left[\left(\psi_3 - f\right)C_x^2 + \left(2f-1\right)\rho_{xy}C_xC_y\right] \tag{2.91}$$

$$MSE\left(\bar{y}_{FTRP}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(2f-1\right)^2 P^2C_x^2 + 2\left(2f-1\right)P\rho_{xy}C_xC_y\right] \tag{2.92}$$

Jain and Shukla (2013) modified factor-type estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ in multiprocessor environment for estimation of ready queue processing time. The modified estimators were compared with ratio estimator in terms of total processing time and the modified factor-type

estimators were found to be more accurate, precise and efficient. The modified estimators, their biases and MSEs are given below;

$$\bar{y}_{FT1} = \bar{y}\frac{9\bar{X} + 2f\bar{x}}{(6+2f)\bar{X} + 3\bar{x}} \tag{2.93}$$

$$Bias(\bar{y}_{FT1}) = \frac{(3-2f)}{(9+2f)}\frac{1-f}{n}\bar{Y}\left[\frac{3}{9+2f}C_x^2 - \rho_{xy}C_xC_y\right] \tag{2.94}$$

$$MSE(\bar{y}_{FT1}) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \frac{4f^2-12f+9}{4f^2+36f+81}C_x^2 + \frac{4f-6}{4f+18}\rho_{xy}C_xC_y\right] \tag{2.95}$$

$$\bar{y}_{FT2} = \bar{y}\frac{22\bar{X} + 5f\bar{x}}{(10+5f)\bar{X} + 12\bar{x}} \tag{2.96}$$

$$Bias(\bar{y}_{FT2}) = \frac{(12-5f)}{(22+5f)}\frac{1-f}{n}\bar{Y}\left[\frac{12}{22+5f}C_x^2 - \rho_{xy}C_xC_y\right] \tag{2.97}$$

$$MSE(\bar{y}_{FT2}) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \frac{25f^2-120f+144}{25f^2+220f+448}C_x^2 + \frac{10f-24}{10f+44}\rho_{xy}C_xC_y\right] \tag{2.98}$$

Shukla et. al. (2013) modified a transformed factor-type estimator for population mean in multiprocessor environment for estimation of ready queue processing time. The modified estimators were used to predict the remaining total processing time required to process completely the ready queue provided sources of auxiliary information are negatively correlated. The modified estimator is given below as;

$$\bar{y}_{TFT} = \bar{y}\frac{(A+C)(d-1)\bar{X} + fB(d\bar{X} - \bar{x})}{(A+fB)(d-1)\bar{X} + C(d\bar{X} - \bar{x})} \tag{2.99}$$

The function $\bar{y}_{TFT}$ equals zero at $d = 1$ and equals sample mean at $d = 4$, so the properties of the modified estimator was studied at $d = 2$, $d = 3$, $d = 5$, and $d = 6$.

The bias and MSE of the modified estimator at $d = 2$, $d = 3$, $d = 5$, and $d = 6$ respectively as follows;

$$Bias\left(\bar{y}_{TFT2}\right) = -\frac{1-f}{n}\bar{Y}\rho_{xy}C_xC_y \tag{2.100}$$

$$MSE\left(\bar{y}_{TFT2}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + C_x^2 - 2\rho_{xy}C_xC_y\right] \tag{2.101}$$

$$Bias\left(\bar{y}_{TFT3}\right) = -\frac{1}{2}\frac{1-f}{n}\bar{Y}\left[\frac{n}{N-n}\rho_{xy}C_xC_y\right] \tag{2.102}$$

$$MSE\left(\bar{y}_{TFT3}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{f}{2-2f}\right)^2 C_x^2 - \left(\frac{f}{1-f}\right)\rho_{xy}C_xC_y\right] \tag{2.103}$$

$$Bias\left(\bar{y}_{TFT5}\right) = \frac{(3-2f)}{4(2f+9)}\frac{1-f}{n}\bar{Y}\left[\frac{3}{4(2f+9)}C_x^2 - \rho_{xy}C_xC_y\right] \tag{2.104}$$

$$MSE\left(\bar{y}_{TFT5}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{3-2f}{36+8f}\right)^2 C_x^2 + 2\left(\frac{3-2f}{36+8f}\right)\rho_{xy}C_xC_y\right] \tag{2.105}$$

$$Bias\left(\bar{y}_{TFT6}\right) = \frac{(12-5f)}{5(22+5f)}\frac{1-f}{n}\bar{Y}\left[\frac{12}{5(22+5f)}C_x^2 - \rho_{xy}C_xC_y\right] \tag{2.106}$$

$$MSE\left(\bar{y}_{TFT6}\right) = \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{12-5f}{110+25f}\right)^2 C_x^2 + 2\left(\frac{12-5f}{110+25f}\right)\rho_{xy}C_xC_y\right] \tag{2.107}$$

The properties of modified estimator obtained at $d = 2$, $d = 3$, $d = 5$, and $d = 6$ were compared through empirical study and the result of the empirical analysis showed that the modified estimator $\bar{y}_{TFT}$ performed better when $d = 6$.

2.3 ESTIMATION UNDER SUPER-POPULATION MODEL

Singh and Singh (2011) studied factor-type estimator and some modified product estimators which are its special cases under polynomial Regression model when there is

misspecification in the model with the aim that a better strategy can be selected. The robustness of the factor-type estimator is studied in order to observe the effect of misspecification of the model on the performance of the estimator. The stabilities of the efficiencies of the estimators were considered under the following four models when $\bar{x}_s > \bar{X}$ and $\bar{x}_s < \bar{X}$;

$$Model\ I: y_k = 2x_k + \varepsilon_k x_k^{g/2} \tag{2.108}$$

$$Model\ II: y_k = 4 + 2x_k + \varepsilon_k x_k^{g/2} \tag{2.109}$$

$$Model\ III: y_k = 4 + 2x_k - 1.5x_k^2 + \varepsilon_k x_k^{g/2} \tag{2.110}$$

$$Model\ I: y_k = 4 + 2x_k - 1.5x_k^2 + 2.4x_k^3 + \varepsilon_k x_k^{g/2} \tag{2.111}$$

Their results of the empirical study showed that factor-type estimators is robust under model III when $\bar{x}_s > \bar{X}$ and robust under Models I, II, and IV when $\bar{x}_s < \bar{X}$.

Sharma and Singh (1998) studied empirical relative efficiencies of ratio, product, regression estimators, super population model based generalized least square (GLS) linear regression estimator, GLS with zero intercept linear estimator, GLS zero intercept reciprocal estimator and other estimators in respect of their bias, variance and mean square error. The populations used for the study were simulated under different model assumptions and probability distributions using super-population model defined as;

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \tag{2.112}$$

with assumption that $E(y_i \setminus x_i) = 0,\ \mathrm{var}(y_i \setminus x_i) = \gamma x_i^g$ where $e_i's$ are uncorrelated with $0 < \gamma < \infty$ and $0 \le g \le 2$.

The results of their study revealed that no estimator is uniformly superior over other estimators and the regression estimator is the most stable estimator among the different model situations studied.

CHAPTER THREE

METHODOLOGY

5.0 INTRODUCTION

In this chapter, the modified estimators and their theoretical properties (Bias and MSE) as well as the superpopulation and nonresponse models used to study their robustness are presented. Also, efficiency conditions of the modified estimators over some existing estimators as well as their optimum sample size and cost functions under minimum cost for a desired precision or optimum presision for a fixed cost were derived and established.

3.1 NON-RESPONSE MODEL

Hansen and Hurwitz (1946) gave a procedure to deal with non-response: Select a sample of size n from the population (denote this sample by s). Let n_1 units respond (sample s_1) and n_2 not respond (sample S_2), for $n_1 + n_2 = n$. Let out of n_2 , h_2 were re-interviewed and they responded (sample s_{h_2}). The modified unbiased estimator is given as;

$$\bar{y}_w = \frac{n_1 \bar{y}_{s_1} + n_2 \bar{y}_{s_{h_2}}}{n} \tag{3.1}$$

where $\bar{y}_{s_1} = \frac{1}{n_1} \sum_{i \in s_1} y_i$, $\bar{y}_{s_{h_2}} = \frac{1}{h_2} \sum_{i \in s_{h_2}} y_i$

3.2 SUPER-POPULATION MODEL

Royal and Herson (1973) found that the efficiency and robustness can be combined by choosing an optimal estimator of population total under super-population model and a selection procedure so that the resultant estimator is a BLU estimator under a more general family of polynomial models. They used the notation $\xi\left\lfloor \delta_0, \delta_1, ..., \delta_p : v(x) \right\rfloor$ to represent the super-population model given by

$$y_i = \sum_{j=1}^{p} \delta_j B_j x_i^j + \varepsilon_i \left[\mathrm{v}\left(x_i\right) \right]^{1/2} , \quad i = 1, 2, ..., N \tag{3.2}$$

where

$$\delta_j = \begin{cases} 1, & \text{if the term } B_j x^j \text{ appears in the equation} \\ 0, & \text{otherwise} \end{cases}$$

The bias and MSE of any estimator $\hat{\bar{Y}}$ of population mean under model $\xi\left[\delta_0, \delta_1, ..., \delta_p : v\left(x\right) \right]$ are given as;

$$Bias\left(\hat{\bar{Y}}\right)_{\xi} = \sum_{j=1}^{p} \delta_j B_j \left[f\left(\bar{x}\right) \bar{x}_s^{(j)} - \bar{X}_s^{(j)} \right] \tag{3.3}$$

$$MSE\left(\hat{\bar{Y}}\right)_{\xi} = \left[\left(\left\{ \sum_{j=1}^{p} \delta_j B_j \right\} \left(f\left(\bar{x}\right) \bar{x}_s^{(j)} - \bar{X}^{(j)} \right) \right)^2 + \frac{1}{N^2} \sigma^2 \left(N - n\right) \bar{x}_{\bar{s}} + \sigma^2 n \bar{x}_s \left\{ f\left(\bar{x}\right) \frac{1}{n} - \frac{1}{N} \right\}^2 \right] \tag{3.4}$$

where $\bar{x}_s^{(j)} = \frac{1}{n} \sum_{i \in s}^{n} x_i^{(j)}$ and $\bar{X}^{(j)} = \frac{1}{N} \sum_{i=1}^{N} x_i^j$

3.3 MODIFIED ESTIMATORS

Motivated by Singh and Shukla (1987, 1993), Shukla (2002) Choudhury and Singh (2012), Khan, et.al (2012) and Malik and Tailor (2013), the following modified estimators are suggested.

3.3.3 Under single-phase sampling

Let the sample of size n be drawn by SRSWOR from population of size N, then the modified estimator and its assumptions under single-phase sampling is;

$$\bar{y}_{FTAA} = \bar{y} \frac{\left[\left(A + C\right) \bar{X} + fB\bar{x} \right] a_x + \left[A + C + fB \right] b_x}{\left[\left(A + fB\right) \bar{X} + C\bar{x} \right] a_x + \left[A + C + fB \right] b_x} \tag{3.5}$$

where $a_x > 0$ and $b_x > 0$ are assumed to be known as either real numbers or (Linear or Non-linear) functions.

3.3.4 Under two-phase sampling

Consider a preliminary large sample S_1 of size n_1 drawn from population Ω of size N by SRSWOR and secondary sample S_2 of size $n(n < n_1)$ drawn either of the following manners:

Case I: as a subset from the preliminary sample i.e. $S_2 \subset S_1$.

Case II: as an independent sample from population i.e. $S_2 \subset \Omega$.

The modified estimators and their assumptions under two-phase sampling are;

$$\bar{y}_{FTAA}^{(d)} = \bar{y}_2 \frac{\left[(A+C)\bar{x}_1 + fB\bar{x}_2\right]a_x + \left[A+C+fB\right]b_x}{\left[(A+fB)\bar{x}_1 + C\bar{x}_2\right]a_x + \left[A+C+fB\right]b_x} \tag{3.6}$$

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{y}_2 \left[\frac{(A+C)\bar{x}_1 + fB\bar{x}_2}{(A+fB)\bar{x}_1 + C\bar{x}_2}\right]\left[\frac{a_z\bar{Z} + b_z}{a_z\bar{z}_1 + b_z}\right]^{\beta_1} \tag{3.7}$$

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{y}_2 \left[\frac{(A+C)\bar{x}_1 + fB\bar{x}_2}{(A+fB)\bar{x}_1 + C\bar{x}_2}\right]\left[\frac{a_z\bar{z}_1 + b_z}{a_z\bar{z}_2 + b_z}\right]^{\beta_2} \tag{3.8}$$

where $a_x > 0$, $b_x > 0$, $a_z > 0$ and $b_z > 0$ are assumed to be known as either real numbers or (Linear or Non-linear) functions, $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$

3.3.3 Assumptions of the Modified Factor-Type Estimators

The modified estimators suggested in this study are based on the following assumptions;

(i) Correlation between X and Y, (ρ_{xy}), $-1 \leq \rho_{xy} < 0$ (product estimators) and $0 < \rho_{xy} \leq 1$ (ratio estimators)

(ii) Population mean $\bar{X}$ is known or can be estimated by large sample S_1 of size n_1.

(iii) Population mean $\bar{Z}$ is known or can be estimated by large sample S_1 of size n_1.

(iv) The values of N , n_1 and n are known

(v) The information about X, Y and Z variables in sample is completely available

(vii) a_x and b_x are some known parameters of auxiliary variable X such as standard deviation σ_x, coefficient of variation C_x, skewness $\beta_{1(x)}$, kurtosis $\beta_{2(x)}$, correlation coefficient ρ_{xy}.

(viii) $a_z > 0$ and $b_z > 0$ are assumed to be known as either real numbers or (Linear or Non-linear) functions

(ix) a_z and b_z are some known parameters of auxiliary variable Z such as standard deviation σ_z, coefficient of variation C_z, skewness $\beta_{1(z)}$, kurtosis $\beta_{2(z)}$, correlation coefficient ρ_{xz}.

3.3.5 Members of the Family of Modified Estimators

Some existing estimators can be obtained from the modified estimators modified by varying the values of $d\left(d \in \Re^{+}\right)$ and taking pairs of σ_x, C_x, $\beta_{1(x)}$, $\beta_{2(x)}$, ρ_{xy} or σ_z, C_z, $\beta_{1(z)}$, $\beta_{2(z)}$, ρ_{xz}. Some of these members of $\bar{y}_{FTAA,}$ $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ are presented in Tables 3.1 – 3.4 below respectively and these indicate the validity of the modified estimators.

Table 3.1: Some Members of the Family of Estimator $\bar{y}_{FTAA}$

S/N	a	b	d	DEDUCED ESTIMATOR
1	$a_0 \neq 0$	0	1	$\bar{y}\frac{\bar{X}}{\bar{x}}$ Usual Ratio type
2	1	C_x	1	$\bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x}$ Sisodia and Diwivedi (1981)
3	1	ρ_{xy}	1	$\bar{y}\frac{\bar{X}+\rho}{\bar{x}+\rho}$ Singh and Tailor (2003)
4	1	$\beta_{2(x)}$	1	$\bar{y}\frac{\bar{X}+\beta_{2(x)}}{\bar{x}+\beta_{2(x)}}$ Singh et. al. (2004)
5	C_x	$\beta_{2(x)}$	1	$\bar{y}\frac{C_x\bar{X}+\beta_{2(x)}}{C_x\bar{x}+\beta_{2(x)}}$ Upadhyaya and Singh (1999)
6	$\beta_{2(x)}$	C_x	1	$\bar{y}\frac{\beta_{2(x)}\bar{X}+C_x}{\beta_{2(x)}\bar{x}+C_x}$ Upadhyaya and Singh (1999)
7	$a_0 \neq 0$	0	2	$\bar{y}\frac{\bar{x}}{\bar{X}}$ Usual product estimator
8	1	C_x	2	$\bar{y}\frac{\bar{x}+C_x}{\bar{X}+C_x}$ Pandey and Dubey (1988)
9	1	ρ_{xy}	2	$\bar{y}\frac{\bar{x}+\rho_{xy}}{\bar{X}+\rho_{xy}}$ Singh and Tailor (2003)
10	1	σ_x	2	$\bar{y}\frac{\bar{x}+\sigma_x}{\bar{X}+\sigma_x}$ Singh (2003)
11	1	$\beta_{2(x)}$	2	$\bar{y}\frac{\bar{x}+\beta_{2(x)}}{\bar{X}+\beta_{2(x)}}$ Singh et. al. (2004)
12	C_x	$\beta_{2(x)}$	2	$\bar{y}\frac{C_x\bar{x}+\beta_{2(x)}}{C_x\bar{X}+\beta_{2(x)}}$ Upadhyaya and Singh (1999)
13	$\beta_{1(x)}$	C_x	2	$\bar{y}\frac{\beta_{1(x)}\bar{x}+C_x}{\beta_{1(x)}\bar{X}+C_x}$ Singh(2003)
14	$\beta_{1(x)}$	σ_x	2	$\bar{y}\frac{\beta_{1(x)}\bar{x}+\sigma_x}{\beta_{1(x)}\bar{X}+\sigma_x}$ Singh (2003)
15	$\beta_{2(x)}$	C_x	2	$\bar{y}\frac{\beta_{2(x)}\bar{x}+C_x}{\beta_{2(x)}\bar{X}+C_x}$ Upadhyaya and Singh (1999)
16	$\beta_{2(x)}$	σ_x	2	$\bar{y}\frac{\beta_{2(x)}\bar{x}+\sigma_x}{\beta_{2(x)}\bar{X}+\sigma_x}$ Singh (2003)
17	$a_0 \neq 0$	0	3	$\bar{y}\frac{N\bar{X}-n\bar{x}}{(N-n)\bar{X}}$ Strivenkataramana (1980)
18	$a_0 \neq 0$	0	4	$\bar{y}$ Usual unbiased estimator
19	$a_0 \neq 0$	0	5	$\bar{y}\frac{9\bar{X}+2f\bar{x}}{(6+2f)\bar{X}+3\bar{x}}$ Jain and Shukla (2013)
20	$a_0 \neq 0$	0	6	$\bar{y}\frac{22\bar{X}+5f\bar{x}}{(10+5f)\bar{X}+12\bar{x}}$ Jain and Shukla (2013)
21	1	0	d	$\bar{y}\frac{(A+C)\bar{X}+fB\bar{x}}{(A+fB)\bar{X}+C\bar{x}}$ Singh and Shukla (1987)

Table 3.2**:** Some Members of the Family of Estimator $\bar{y}_{FTAA}^{(d)}$

S/N	a	b	d	DEDUCED ESTIMATOR
1	$a_0 \neq 0$	0	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}$ Sukhatme (1962)
2	1	C_x	1	$\bar{y}\frac{\bar{x}_1 + C_x}{\bar{x} + C_x}$
3	1	ρ_{xy}	1	$\bar{y}\frac{\bar{x}_1+\rho_{xy}}{\bar{x}+\rho_{xy}}$ Malik and Tailor (2013)
4	1	σ_x	1	$\bar{y}\frac{\bar{x}_1 + \sigma_x}{\bar{x} + \sigma_x}$
5	1	$\beta_{1(x)}$	1	$\bar{y}\frac{\bar{x}_1 + \beta_{1(x)}}{\bar{x} + \beta_{1(x)}}$
6	$a_0 \neq 0$	0	2	$\bar{y}\frac{\bar{x}}{\bar{x}_1}$ Two phase product estimator
7	1	C_x	2	$\bar{y}\frac{\bar{x} + C_x}{\bar{x}_1 + C_x}$
8	1	ρ_{xy}	2	$\bar{y}\frac{\bar{x} + \rho_{xy}}{\bar{x}_1 + \rho_{xy}}$
9	1	σ_x	2	$\bar{y}\frac{\bar{x} + \sigma_x}{\bar{x}_1 + \sigma_x}$
10	1	$\beta_{2(x)}$	2	$\bar{y}\frac{\bar{x} + \beta_{2(x)}}{\bar{x}_1 + \beta_{2(x)}}$
11	$a_0 \neq 0$	0	3	$\bar{y}\frac{N\bar{x}_1 - n\bar{x}}{(N-n)\bar{x}_1}$
12	$a_0 \neq 0$	0	4	$\bar{y}$

Table 3.3: Some Members of the Family of Estimator $\bar{y}_{FTAA}^{\beta_1(d)}$

S/N	a	b	d	α_{21}	DEDUCED ESTIMATOR
1	$a_0 \neq 0$	0	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}}{\bar{z}_1}$ Chand (1975)
2	1	C_z	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}+C_z}{\bar{z}_1+C_z}$ Upadhayaya and Singh (2001)
3	1	ρ_{xz}	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{Z}+\rho_{xz}}{\bar{z}_1+\rho_{xz}}$ Singh et.al. (2007)
4	C_z	$\beta_{2(z)}$	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{C_z\bar{Z} + \beta_{2(z)}}{C_z\bar{z}_1 + \beta_{2(z)}}$ Upadhayaya and Singh (2001)

5	$\beta_{1(z)}$	σ_z	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\beta_{1(z)}\bar{Z}+\sigma_z}{\beta_{1(z)}\bar{z}_1+\sigma_z}$ Singh (2001)
6	$a_0 \neq 0$	0	4	0	$\bar{y}$

Table 3.4: Some Members of the Family of Estimator $\bar{y}_{FTAA}^{\beta_2(d)}$

S/N	a_z	b_z	d	α_{22}	DEDUCED ESTIMATOR
1	$a_0 \neq 0$	0	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1}{\bar{z}}$
2	1	C_x	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1+C_z}{\bar{z}+C_z}$
3	1	ρ	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1+\rho_{xz}}{\bar{z}+\rho_{xz}}$
4	1	σ_x	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1+\sigma_z}{\bar{z}+\sigma_z}$
5	1	$\beta_{1(x)}$	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1+\beta_{1(z)}}{\bar{z}+\beta_{1(z)}}$
6	1	$\beta_{2(x)}$	1	1	$\bar{y}\frac{\bar{x}_1}{\bar{x}}\frac{\bar{z}_1+\beta_{2(z)}}{\bar{z}+\beta_{2(z)}}$
7	$a_0 \neq 0$	0	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1}{\bar{z}}$
8	1	C_x	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1+C_z}{\bar{z}+C_z}$
9	1	ρ	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1+\rho_{xz}}{\bar{z}+\rho_{xz}}$
10	1	σ_x	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1+\sigma_z}{\bar{z}+\sigma_z}$
11	1	$\beta_{1(x)}$	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1+\beta_{1(z)}}{\bar{z}+\beta_{1(z)}}$
12	1	$\beta_{2(x)}$	2	1	$\bar{y}\frac{\bar{x}}{\bar{x}_1}\frac{\bar{z}_1+\beta_{2(z)}}{\bar{z}+\beta_{2(z)}}$
13	$a_0 \neq 0$	0	3	1	$\bar{y}\frac{N\bar{x}_1-n\bar{x}}{(N-n)\bar{x}_1}\frac{\bar{z}_1}{\bar{z}}$
14	$a_0 \neq 0$	0	4	0	$\bar{y}$

3.4 PROPERTIES OF THE MODIFIED FACTOR-TYPE ESTIMATORS

In this section, the theoretical biases and MSEs of the modified estimators are derived up to first order approximation using Taylor's series expansion.

3.4.1 Bias and MSE of the estimator $\bar{y}_{FTAA}$

In order to obtained the Bias and Mean Square Error (MSE) of the modified estimator $\bar{y}_{FTAA}$, we defined the error terms $\Delta_{\bar{y}} = \left(\bar{y} - \bar{Y}\right)/\bar{Y}$ and $\Delta_{\bar{x}} = \left(\bar{x} - \bar{X}\right)/\bar{X}$ such that $\left|\Delta_{\bar{y}}\right| < 1, \left|\Delta_{\bar{x}}\right| < 1$

$$\left.\begin{aligned} &E\left(\Delta_{\bar{y}}\right) = E\left(\Delta_{\bar{x}}\right) = 0,\ E\left(\Delta_{\bar{y}}^2\right) = \gamma C_y^2 \\ &E\left(\Delta_{\bar{x}}^2\right) = \gamma C_x^2,\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right) = \gamma \rho_{xy} C_x C_y, \gamma = \frac{1}{n} - \frac{1}{N} \end{aligned}\right\} \tag{3.9}$$

Theorem 3.4.1: To the first degree of approximation, the modified factor-type estimator $\bar{y}_{FTAA}$ can be expressed in terms of error terms $\Delta_{\bar{x}}$ and $\Delta_{\bar{y}}$ as

$$\bar{y}_{FTAA} = \bar{Y}\left[1 + \Delta_{\bar{y}}\right]\left[1 + \delta_x \psi_2 \Delta_{\bar{x}}\right]\left[1 + \delta_x \psi_4 \Delta_{\bar{x}}\right]^{-1} \tag{3.10}$$

where $\delta_x = \dfrac{a_x \bar{X}}{a_x \bar{X} + b_x}$

Proof:

Equation (3.10) can be expressed in terms of $\Delta_{\bar{x}}$ and $\Delta_{\bar{y}}$ as

$$\bar{y}_{FTAA} = \bar{Y}\left(1+\Delta_{\bar{y}}\right)\left[\frac{\left\{\left(A+C\right)\bar{X} + fB\bar{X}\left(1+\Delta_{\bar{x}}\right)\right\}a_x + \left\{A+ fB+C\right\}b_x}{\left\{\left(A+ fB\right)\bar{X} + C\bar{X}\left(1+\Delta_{\bar{x}}\right)\right\}a_x + \left\{A+ fB+C\right\}b_x}\right] \tag{3.11}$$

Factorize $\left(A+ fB+C\right)$ from the numerator and denominator of the right hand side of equation (3.11), we have

$$\bar{y}_{FTAA} = \bar{Y}\left(1+\Delta_{\bar{y}}\right)\left[\frac{\left(1+\psi_2\Delta_{\bar{x}}\right)\bar{X}a_x + b_x}{\left(1+\psi_4\Delta_{\bar{x}}\right)\bar{X}a_x + b_x}\right] \tag{3.12}$$

Factorize $\left(a_x\bar{X}+b_x\right)$ from the numerator and denominator of the right hand side of equation (3.12), we have

$$\bar{y}_{FTAA} = \bar{Y}\left(1+\Delta_{\bar{y}}\right)\left[\frac{1+\delta_x\psi_2\Delta_{\bar{x}}}{1+\delta_x\psi_4\Delta_{\bar{x}}}\right] \tag{3.13}$$

which can be written equivalently as

$$\bar{y}_{FTAA} = \bar{Y}\left(1+\Delta_{\bar{y}}\right)\left[1+\delta_x\psi_2\Delta_{\bar{x}}\right]\left[1+\delta_x\psi_4\Delta_{\bar{x}}\right]^{-1} \tag{3.14}$$

where $\delta_x = \dfrac{a_x\bar{X}}{a_x\bar{X}+b_x}$

Theorem 3.4.2: To the first degree of approximation, the bias denoted by $Bias\left(\bar{y}_{FTAA}\right)$, of the modified estimator $\bar{y}_{FTAA}$ is

$$Bias\left(\bar{y}_{FTAA}\right) = \bar{Y}P\gamma\left[\delta_x\rho_{xy}C_xC_y - \delta_x^2\psi_2C_x^2\right] \tag{3.15}$$

Proof:

Here the assumption is that in (3.14), $\left|\delta_x\theta_2\Delta_{\bar{x}}\right|<1$ so that $\left(1+\delta_x\theta_2\Delta_{\bar{x}}\right)^{-1}$ is expandable.

Subtract $\bar{Y}$ for both side of equation (3.14) and using power series expansion; the simplification of (3.19) up to first order approximation $O(n^{-1})$ is given by

$$\bar{y}_{FTAA} - \bar{Y} = \bar{Y}\left[\Delta_{\bar{y}} + \delta_xP\Delta_{\bar{x}} + \delta_xP\Delta_{\bar{y}}\Delta_{\bar{x}} - \delta_x^2\psi_4P\Delta_{\bar{x}}^2\right] \tag{3.16}$$

Taking expectation of $\bar{y}_{FTAA} - \bar{Y}$ and using the results in equation (3.16), the bias of the modified estimator is obtained as

$$Bias\left(\bar{y}_{FTAA}\right) = \bar{Y}P\gamma\left[\delta_x\rho_{xy}C_xC_y - \delta_x^2\psi_2C_x^2\right] \tag{3.17}$$

Theorem 3.4.3: To the first degree of approximation, the MSE of modified estimator $\bar{y}_{FTAA}$ denoted by $MSE\left(\bar{y}_{FTAA}\right)$ is

$$MSE\left(\bar{y}_{FTAA}\right) = \bar{Y}^2\gamma\left[C_y^2 + \delta_x^2P^2C_x^2 + 2\delta_xP\rho_{xy}C_xC_y\right] \tag{3.18}$$

Proof:

Square both sides of $\bar{y}_{FTAA} - \bar{Y}$, taking expectation and using the results in equation (3.16), we obtain the MSE of the modified estimators as:

$$MSE\left(\bar{y}_{FTAA}\right) = \bar{Y}^2\gamma\left[C_y^2 + \delta_x^2P^2C_x^2 + 2\delta_xP\rho_{xy}C_xC_y\right] \tag{3.19}$$

Theorem 3.4.4: The $MSE\left(\bar{y}_{FTAA}\right)$ is at minimum under the optimal condition that $P = -\rho_{xy}C_y / \delta_xC_x$.

Proof:

Differentiate (3.19) partially with respect to P and equate the result to zero as

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}\right) = \bar{Y}^2\gamma\left[2P\delta_xC_x^2 + 2\delta_x\rho_{xy}C_xC_y\right] \tag{3.20}$$

$$\bar{Y}^2\gamma\left[2P\delta_xC_x^2 + 2\delta_x\rho_{xy}C_xC_y\right] = 0 \tag{3.21}$$

$$P = -\rho_{xy}C_y / \delta_xC_x \tag{3.22}$$

Substitute (3.22) in (3.19), the minimum $MSE\left(\bar{y}_{FTAA}\right)$ written as $MSE\left(\bar{y}_{FTAA}\right)_{\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}\right)_{\min} = \bar{Y}^2 \gamma C_y^2 \left[1-\rho_{xy}^2\right] \tag{3.23}$$

Remark 3.4.1: The estimator $\bar{y}_{FTAA}$ under optimum condition is equally efficient as regression estimator, so therefore $\bar{y}_{FTAA}$ is an alternative to regression estimator when the slope is unknown.

In order to estimate unknown constant $d \in \Re^+$, $P = -C_{yx} / \delta_x$, where $C_{yx} = \rho_{xy} C_y / C_x$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = -C_{yx} / \delta_x \tag{3.24}$$

$$\frac{fB - C}{A + C + fB} = -C_{yx} / \delta_x \tag{3.25}$$

$$\begin{aligned}(w-1)d^3 + (fw + f - 8w + 9)d^2 - (5fw + 5f - 23w + 26)d \\ + (4fw + 4f - 22w + 24) = 0\end{aligned} \tag{3.26}$$

where $w = -C_{yx} / \delta_x$

By solving (3.26), at most 3 zeros d_1 , d_2 and d_3 of the polynomials for which (3.5) is optimal will be obtained.

3.4.2 Bias and MSE of the estimator $\bar{y}_{FTAA}^{(d)}$

In order to study the properties of the modified estimator, we define the following error terms $\Delta_{\bar{y}_2} = \left(\bar{y}_2 - \bar{Y}\right) / \bar{Y}$, $\Delta_{\bar{x}_2} = \left(\bar{x}_2 - \bar{X}\right) / \bar{X}$ and $\Delta_{\bar{x}_1} = \left(\bar{x}_1 - \bar{X}\right) / \bar{X}$ such that $\left|\Delta_{\bar{y}_2}\right| < 1, \left|\Delta_{\bar{x}_2}\right| < 1, \left|\Delta_{\bar{x}_2}\right|$

Under Case I

In case I, we have

$$\left.\begin{array}{c} E\left(\Delta_{\bar{y}}\right)=E(\Delta_{\bar{x}})=E\left(\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}_1}\right)^2=\theta_1 C_x^2 \\ E\left(\Delta_{\bar{y}}\right)^2=\theta_2 C_y^2, E(\Delta_{\bar{x}})^2=\theta_2 C_x^2, E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right)=\theta_2\rho_{xy}C_yC_x \\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}_1}\right)=\theta_1\rho_{xy}C_yC_x, E\left(\Delta_{\bar{x}}\Delta_{\bar{x}_1}\right)=\theta_1 C_x^2, \\ \theta_1=\frac{1}{n_1}-\frac{1}{N},\quad \theta_2=\frac{1}{n_2}-\frac{1}{N},\quad \theta_3=\theta_2-\theta_1,\ \theta_4=\theta_2+\theta_1 \end{array}\right\} \tag{3.27}$$

Under Case II

In case II, we have

$$\left.\begin{array}{c} E\left(\Delta_{\bar{y}}\right)=E(\Delta_{\bar{x}})=E\left(\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}_1}\right)^2=\theta_1 C_x^2 \\ E\left(\Delta_{\bar{y}}\right)^2=\theta_2 C_y^2, E(\Delta_{\bar{x}})^2=\theta_2 C_x^2, E\left(\Delta_{\bar{y}}\Delta_{\bar{x}}\right)=\theta_2\rho_{xy}C_yC_x \\ E\left(\Delta_{\bar{y}}\Delta_{\bar{x}_1}\right)=0, E\left(\Delta_{\bar{x}}\Delta_{\bar{x}_1}\right)=0, \\ \theta_1=\frac{1}{n_1}-\frac{1}{N},\quad \theta_2=\frac{1}{n_2}-\frac{1}{N},\quad \theta_3=\theta_2-\theta_1,\ \theta_4=\theta_2+\theta_1 \end{array}\right\} \tag{3.28}$$

Theorem 3.4.5: To the first degree of approximation, the modified factor-type estimator $\bar{y}_{FTAA}^{(d)}$ can be expressed in terms of error terms $\Delta_{\bar{x}_1}, \Delta_{\bar{x}_2}$ and $\Delta_{\bar{y}_2}$ as

$$\bar{y}_{FTAA}^{(d)}=\bar{Y}\left[1+\Delta_{\bar{y}_2}\right]\left[1+\delta_x\psi_1\Delta_{\bar{x}_1}+\delta_x\psi_2\Delta_{\bar{x}_2}\right]\left[1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right]^{-1} \tag{3.29}$$

Proof:

Equation (3.6) can be expressed in terms of $\Delta_{\bar{x}}$ and $\Delta_{\bar{y}}$ as

$$\bar{y}_{FTAA}^{(d)}=\bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[\frac{\left\{(A+C)\bar{X}\left(1+\Delta_{\bar{x}_1}\right)+fB\bar{X}\left(1+\Delta_{\bar{x}_2}\right)\right\}a_x+\{A+fB+C\}b_x}{\left\{(A+fB)\bar{X}\left(1+\Delta_{\bar{x}_1}\right)+C\bar{X}\left(1+\Delta_{\bar{x}_2}\right)\right\}a_x+\{A+fB+C\}b_x}\right] \tag{3.30}$$

Factorize $(A+fB+C)$ from the numerator and denominator of the right hand side of equation (3.30), we have

$$\bar{y}_{FTAA}^{(d)}=\bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[\frac{\left(1+\psi_1\Delta_{\bar{x}_1}+\psi_2\Delta_{\bar{x}_2}\right)\bar{X}a_x+b_x}{\left(1+\psi_3\Delta_{\bar{x}_1}+\psi_4\Delta_{\bar{x}_2}\right)\bar{X}a_x+b_x}\right] \tag{3.31}$$

Factorize $\left(a_x\bar{X}+b_x\right)$ from the numerator and denominator of the right hand side of equation (3.31), we have

$$\bar{y}_{FTAA}^{(d)} = \bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[\frac{1+\delta_x\psi_1\Delta_{\bar{x}_1}+\delta_x\psi_2\Delta_{\bar{x}_2}}{1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}}\right] \tag{3.32}$$

which can be written equivalently as

$$\bar{y}_{FTAA}^{(d)} = \bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[1+\delta_x\psi_1\Delta_{\bar{x}_1}+\delta_x\psi_2\Delta_{\bar{x}_2}\right]\left[1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right]^{-1} \tag{3.33}$$

Theorem 3.4.6: To the first degree of approximation, the biases denoted by $Bias\left(\bar{y}_{FTAA}^{(d)}\right)_I$ and $Bias\left(\bar{y}_{FTAA}^{(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{(d)}$ under Cases I and II respectively are;

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_I = \bar{Y}P\theta_3\left[\delta_x\rho_{xy}C_xC_y-\delta_x^2\psi_4C_x^2\right] \tag{3.34}$$

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_{II} = \bar{Y}P\left[\theta_2\delta_x\rho_{xy}C_xC_y+\left(\theta_1\psi_3-\theta_2\psi_4\right)\delta_x^2C_x^2\right] \tag{3.35}$$

Proof:

Here the assumption is that in (3.33) $\left|\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right|<1$ so that $\left(1+\delta_x\psi_3\Delta_{\bar{x}_1}+\delta_x\psi_4\Delta_{\bar{x}_2}\right)^{-1}$ is expandable.

Subtract $\bar{Y}$ from both sides of equation (3.33) and using power series expansion, the simplification of equation (3.33) up to first order approximation $O(n^{-1})$ is given by

$$\begin{aligned}\bar{y}_{FTAA}^{(d)}-\bar{Y} = \bar{Y}\Big[&\Delta_{\bar{y}_2}-\delta_xP\Delta_{\bar{x}_1}+\delta_xP\Delta_{\bar{x}_2}+\delta_x^2\psi_3P\Delta_{\bar{x}_1}^2-\delta_x^2\psi_4P\Delta_{\bar{x}_1}^2\\&+\delta_x^2\left(\psi_4-\psi_3\right)P\Delta_{\bar{x}_1}\Delta_{\bar{x}_2}-\delta_xP\Delta_{\bar{y}_2}\Delta_{\bar{x}_1}+\delta_xP\Delta_{\bar{y}_2}\Delta_{\bar{x}_2}\Big]\end{aligned} \tag{3.36}$$

Taking expectation of (3.36) and using results of (3.27), the bias of the modified estimator to terms of order n^{-1}, when $S_2 \subset S_1$ is obtained as:

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_I = \bar{Y}P\theta_3\left[\delta_x\rho_{xy}C_xC_y-\delta_x^2\psi_4C_x^2\right] \tag{3.37}$$

Also, taking expectation of (3.36) and using results of (3.28), the bias of the modified estimator to terms of order n^{-1}, when $S_2 \subset \Omega_N$ is obtained as

$$Bias\left(\bar{y}_{FTAA}^{(d)}\right)_{II} = \bar{Y}P\left[\theta_2\delta_x\rho_{xy}C_xC_y + \left(\theta_1\psi_3 - \theta_2\psi_4\right)\delta_x^2C_x^2\right] \tag{3.38}$$

Theorem 3.4.7: To the first degree of approximation, the biases denoted by $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{(d)}$ under Cases I and II respectively are;

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I = \bar{Y}^2\left[\theta_2C_y^2 + \theta_3\delta_x^2P^2C_x^2 + 2\theta_3\delta_xP\rho_{xy}C_xC_y\right] \tag{3.39}$$

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} = \bar{Y}^2\left[\theta_2C_y^2 + \theta_4\delta_x^2P^2C_x^2 + 2\theta_2\delta_xP\rho_{xy}C_xC_y\right] \tag{3.40}$$

Proof:

Squaring both sides of (3.36), then taking expectation and using results in (3.27), we obtain the MSE of the modified estimator to terms of order n^{-1}, when $S_2 \subset S_1$ as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I = \bar{Y}^2\left[\theta_2C_y^2 + \theta_3\delta_x^2P^2C_x^2 + 2\theta_3\delta_xP\rho_{xy}C_xC_y\right] \tag{3.41}$$

Squaring both sides of (3.36), then taking expectation and using results in (3.28), we obtain the MSE of the modified estimator to terms of order n^{-1}, when $S_2 \subset \Omega_N$ as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} = \bar{Y}^2\left[\theta_2C_y^2 + \theta_4\delta_x^2P^2C_x^2 + 2\theta_2\delta_xP\rho_{xy}C_xC_y\right] \tag{3.42}$$

Theorem 3.4.8: The $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}$ are at minimum under the optimal conditions that $P = -C_{yx}/\delta_x$ and $P = -\theta_2C_{yx}/\delta_x\theta_4$ respectively.

Proof:

Differentiate (3.41) partially with respect to P and equate the result to zero as

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I = \bar{Y}^2\theta_3\left[2P\delta_x C_x^2 + 2\delta_x\rho_{xy}C_xC_y\right] \tag{3.43}$$

$$\bar{Y}^2\theta_3\left[2P\delta_x C_x^2 + 2\delta_x\rho_{xy}C_xC_y\right] = 0 \tag{3.44}$$

$$P = -C_{yx}/\delta_x \tag{3.45}$$

Substitute (3.45) in (3.41), the minimum $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I$ written as $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{I\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{I\min} = \bar{Y}^2C_y^2\left[\theta_2 - \theta_3\rho_{xy}^2\right] \tag{3.46}$$

Lemma 3.4.1: The values of unknown constant $d \in \Re^+$ that optimize the estimator $\bar{y}_{FTAA}^{(d)}$ when $S_2 \subset S_1$ are the same as that of the estimator $\bar{y}_{FTAA}$

Also, differentiate (3.42) partially with respect to P and equate the result to zero as

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} = \bar{Y}^2\left[2\theta_4 P\delta_x^2 C_x^2 + 2\theta_2\delta_x\rho_{xy}C_xC_y\right] \tag{3.47}$$

$$\bar{Y}^2\left[2\theta_4 P\delta_x^2 C_x^2 + 2\theta_2\delta_x\rho_{xy}C_xC_y\right] = 0 \tag{3.48}$$

$$P = -\theta_2 C_{yx}/\delta_x\theta_4 \tag{3.49}$$

Substitute (3.49) in (3.42), the minimum $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}$ written as $MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min}$ is obtained as

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min} = \bar{Y}^2C_y^2\left[\theta_2 - \frac{\theta_2^2}{\theta_4}\rho_{xy}^2\right] \tag{3.50}$$

In order to estimate unknown constant $d \in \Re^+$, $P = -\theta_2 C_{yx}/\delta_x\theta_4$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = -\theta_2 C_{yx}/\delta_x\theta_4 \tag{3.51}$$

$$\frac{fB-C}{A+C+fB}=\frac{-\theta_2 C_{yx}}{\delta_x\theta_4} \tag{3.52}$$

$$(v-1)d^3+(fv+f-8v+9)d^2-(5fv+5f-23v+26)d+(4fv+4f-22v+24)=0 \tag{3.53}$$

where $v=\dfrac{-\theta_2 C_{yx}}{\delta_x\theta_4}$

By solving (3.53), at most 3 zeros d_1 , d_2 and d_3 of the polynomials for which (3.6) is optimal will be obtained.

3.4.3 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$

In order to study the properties of the modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$, we define the following error terms

$\in_{\bar{y}_2}=(\bar{y}_2-\bar{Y})/\bar{Y}$, $\in_{\bar{x}_1}=(\bar{x}_1-\bar{X})/\bar{X}$, $\in_{\bar{x}_2}=(\bar{x}_2-\bar{X})/\bar{X}$, $\in_{\bar{z}_1}=(\bar{z}_1-\bar{Z})/\bar{Z}$, $\in_{\bar{z}_2}=(\bar{z}_2-\bar{Z})/\bar{Z}$,

such that $|\in_{\bar{y}_2}|<1$, $|\in_{\bar{x}_1}|<1$, $|\in_{\bar{x}_2}|<1$, $|\in_{\bar{z}_1}|<1$, $|\in_{\bar{z}_2}|<1$

Under case I

$$\left.\begin{aligned}
&\bar{y}_1=\bar{Y}(1+\in_{\bar{y}_1}),\qquad \bar{x}_1=\bar{X}(1+\in_{\bar{x}_1}),\qquad \bar{z}_1=\bar{Z}(1+\in_{\bar{z}_1})\\
&\bar{y}_2=\bar{Y}(1+\in_{\bar{y}_2}),\qquad \bar{x}_2=\bar{X}(1+\in_{\bar{x}_2}),\qquad \bar{z}_2=\bar{Z}(1+\in_{\bar{z}_2})\\
&E(\in_{\bar{x}_1})=E(\in_{\bar{z}_1})=E(\in_{\bar{y}_2})=E(\in_{\bar{x}_2})=E(\in_{\bar{z}_2})=0\\
&E(\in_{\bar{y}_2}^2)=\theta_2 C_y^2,\quad E(\in_{\bar{x}_1}^2)=\theta_1 C_x^2,\quad E(\in_{\bar{z}_1}^2)=\theta_1 C_z^2,\ E(\in_{\bar{x}_2}^2)=\theta_2 C_x^2\\
&E(\in_{\bar{z}_2}^2)=\theta_2 C_z^2,\quad E(\in_{\bar{y}_2}\in_{\bar{x}_1})=\theta_1\rho_{xy}C_yC_x,\ E(\in_{\bar{y}_2}\in_{\bar{z}_1})=\theta_1\rho_{yz}C_yC_z\\
&E(\in_{\bar{y}_2}\in_{\bar{x}_2})=\theta_2\rho_{xy}C_yC_x,\ E(\in_{\bar{y}_2}\in_{\bar{z}_2})=\theta_2\rho_{yz}C_yC_z,\ E(\in_{\bar{x}_1}\in_{\bar{z}_1})=\theta_1\rho_{xz}C_xC_z\\
&E(\in_{\bar{x}_1}\in_{\bar{z}_2})=\theta_1\rho_{xz}C_xC_z,\ E(\in_{\bar{x}_1}\in_{\bar{x}_2})=\theta_1C_x^2,\quad E(\in_{\bar{x}_2}\in_{\bar{z}_1})=\theta_1\rho_{xz}C_xC_z\\
&E(\in_{\bar{x}_2}\in_{\bar{z}_2})=\theta_2\rho_{xz}C_xC_z,\ E(\in_{\bar{z}_1}\in_{\bar{z}_2})=\theta_1C_z^2,\ \theta_1=\frac{1}{n_1}-\frac{1}{N},\ \theta_2=\frac{1}{n_2}-\frac{1}{N},\ \theta_3=\theta_2-\theta_1
\end{aligned}\right\} \tag{3.54}$$

Under case II

$$\left.\begin{aligned}
&\bar{y}_1 = \bar{Y}\left(1+\in_{\bar{y}_1}\right), \qquad \bar{x}_1 = \bar{X}\left(1+\in_{\bar{x}_1}\right), \qquad \bar{z}_1 = \bar{Z}\left(1+\in_{\bar{z}_1}\right)\\
&\bar{y}_2 = \bar{Y}\left(1+\in_{\bar{y}_2}\right), \qquad \bar{x}_2 = \bar{X}\left(1+\in_{\bar{x}_2}\right), \qquad \bar{z}_2 = \bar{Z}\left(1+\in_{\bar{z}_2}\right)\\
&E\left(\in_{\bar{x}_1}\right) = E\left(\in_{\bar{z}_1}\right) = E\left(\in_{\bar{y}_2}\right) = E\left(\in_{\bar{x}_2}\right) = E\left(\in_{\bar{z}_2}\right) = 0\\
&E\left(\in^2_{\bar{y}_2}\right) = \theta_2 C_y^2, \quad E\left(\in^2_{\bar{x}_1}\right) = \theta_1 C_x^2, \quad E\left(\in^2_{\bar{z}_1}\right) = \theta_1 C_z^2, \; E\left(\in^2_{\bar{x}_2}\right) = \theta_2 C_x^2\\
&E\left(\in^2_{\bar{z}_2}\right) = \theta_2 C_z^2, E\left(\in_{\bar{y}_2}\in_{\bar{x}_2}\right) = \theta_2 \rho_{xy} C_y C_x, \; E\left(\in_{\bar{y}_2}\in_{\bar{z}_2}\right) = \theta_2 \rho_{yz} C_y C_z,\\
&E\left(\in_{\bar{x}_1}\in_{\bar{z}_1}\right) = \theta_1 \rho_{xz} C_x C_z, \; E\left(\in_{\bar{x}_2}\in_{\bar{z}_2}\right) = \theta_2 \rho_{xz} C_x C_z\\
&E\left(\in_{\bar{x}_1}\in_{\bar{z}_2}\right) = E\left(\in_{\bar{x}_1}\in_{\bar{x}_2}\right) = E\left(\in_{\bar{x}_2}\in_{\bar{z}_1}\right) = E\left(\in_{\bar{y}_2}\in_{\bar{x}_1}\right) = E\left(\in_{\bar{y}_2}\in_{\bar{z}_1}\right) = E\left(\in_{\bar{z}_1}\in_{\bar{z}_2}\right) = 0
\end{aligned}\right\} \quad (3.55)$$

Theorem 3.4.9: To the first degree of approximation, the modified factor-type estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ can be expressed in terms of error terms $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}$ and $\in_{\bar{y}_2}$ as

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left[1+\in_{\bar{y}_2}\right]\left[1+\psi_1 \in_{\bar{x}_1} + \psi_2 \in_{\bar{x}_2}\right]\left[1+\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right]^{-1}\left[1+\delta_z \in_{\bar{z}_1}\right]^{-\beta_1} \quad (3.56)$$

where $\delta_z = \dfrac{a\bar{Z}}{a\bar{Z}+b}$

Proof:

Expressing equation (3.7) in terms of $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}$ and $\in_{\bar{y}_2}$, we have

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left(1+\in_{\bar{y}_2}\right)\left[\frac{(A+C)\bar{X}\left(1+\in_{\bar{x}_1}\right)+ fB\bar{X}\left(1+\in_{\bar{x}_2}\right)}{(A+fB)\bar{X}\left(1+\in_{\bar{x}_1}\right)+C\bar{X}\left(1+\in_{\bar{x}_2}\right)}\right]\left[\frac{a_z\bar{Z}+b_z}{a_z\left(1+\in_{\bar{z}_1}\right)\bar{Z}+b_z}\right]^{\beta_1} \quad (3.57)$$

Factorize $\bar{X}(A+fB+C)$ from the numerator and denominator of the right hand side of equation (3.57), we have

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[\frac{1+\psi_1\Delta_{\bar{x}_1}+\psi_2\Delta_{\bar{x}_2}}{1+\psi_3\Delta_{\bar{x}_1}+\psi_4\Delta_{\bar{x}_2}}\right]\left[\frac{a_z\bar{Z}+b_z}{a_z\bar{Z}+b_z+a_z\bar{Z}\in_{\bar{z}_1}}\right]^{\beta_1} \quad (3.58)$$

Factorize $\left(a_z\bar{Z}+b_z\right)$ from the numerator and denominator of the right hand side of equation (3.58), we have

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left(1+\in_{\bar{y}_2}\right)\left[\frac{1+\psi_1 \in_{\bar{x}_1} +\psi_2 \in_{\bar{x}_2}}{1+\psi_3 \in_{\bar{x}_1} +\psi_4 \in_{\bar{x}_2}}\right]\left[\frac{1}{1+\delta_z \in_{\bar{z}_1}}\right]^{\beta_1} \tag{3.59}$$

which can be written equivalently as

$$\bar{y}_{FTAA}^{\beta_1(d)} = \bar{Y}\left(1+\in_{\bar{y}_2}\right)\left[1+\psi_1 \in_{\bar{x}_1} +\psi_2 \in_{\bar{x}_2}\right]\left[1+\psi_3 \in_{\bar{x}_1} +\psi_4 \in_{\bar{x}_2}\right]^{-1}\left[1+\delta_z \in_{\bar{z}_1}\right]^{-\beta_1} \tag{3.60}$$

Theorem 3.4.10: To the first degree of approximation, the biases denoted by $Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I$ and $Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{(d)}$ under Cases I and II respectively are;

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}\left[\theta_3 PC_x^2\left(\rho_{xy}C_xC_y - \psi_4\right) + \theta_1 C_z^2\left(\frac{\beta_1\left(\beta_1+1\right)}{2}\delta_z^2 - \beta_1\delta_z\rho_{yz}C_yC_z\right)\right] \tag{3.61}$$

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}\left[\theta_3 P\psi_3 C_x^2 + P\theta_2 C_x^2\left(C_{yx} - \psi_4\right) + \beta_1\delta_z\theta_3 C_z^2\left(\delta_z\left(\beta_1+1\right)/2 + PC_{xz}\right)\right] \tag{3.62}$$

Proof:

Here the assumption is that in (3.60), $\left|\psi_3 \in_{\bar{x}_1} +\psi_4 \in_{\bar{x}_2}\right| < 1$ and $\left|\delta_z \in_{\bar{z}_1}\right| < 1$ so that $\left(1+\psi_3 \in_{\bar{x}_1} +\psi_4 \in_{\bar{x}_2}\right)^{-1}$ and $\left(1+\delta_z \in_{\bar{z}_1}\right)^{\beta_1}$ are expandable.

Subtract $\bar{Y}$ from both sides of equation (3.60) and using power series expansion, the simplification of equation (3.60) up to first order approximation $O(n^{-1})$ is given by

$$\bar{y}_{FTAA}^{\beta_1(d)} - \bar{Y} = \bar{Y}\left[\in_{\bar{y}_2} -\beta_1\delta_z \in_{\bar{z}_1} -P \in_{\bar{x}_1} +P \in_{\bar{x}_2} +\frac{\beta_1\left(\beta_1+1\right)}{2}\delta_z^2 \in_{\bar{z}_1}^2 +\beta_1\delta_z P \in_{\bar{x}_1}\in_{\bar{z}_1} -\beta_1\delta_z \in_{\bar{x}_2}\in_{\bar{z}_1} +\right.$$
$$\left.\psi_3 P \in_{\bar{x}_1}^2 -\psi_4 P \in_{\bar{x}_2}^2 +\left(2\psi_3\psi_4 - \psi_1\psi_4 - \psi_2\psi_3\right)\in_{\bar{x}_1}\in_{\bar{x}_2} -\beta_1\delta_z \in_{\bar{y}_2}\in_{\bar{z}_1} -P \in_{\bar{y}_2}\in_{\bar{x}_1} +P \in_{\bar{y}_2}\in_{\bar{x}_2}\right] \tag{3.63}$$

Taking expectation of (3.63) and apply the results in equation (3.54), the bias of the modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I is obtained as

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}\left[\theta_3 P C_x^2\left(\rho_{xy}C_xC_y - \psi_4\right) + \theta_1 C_z^2\left(\frac{\beta_1(\beta_1+1)}{2}\delta_z^2 - \beta_1\delta_z\rho_{yz}C_yC_z\right)\right] \quad (3.64)$$

Also, taking expectation of (3.63) and using results of (3.55), the bias of the modified estimator to terms of order n^{-1} under case II is obtained as

$$Bias\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}\left[\theta_3 P\psi_3 C_x^2 + P\theta_2 C_x^2\left(C_{yx} - \psi_4\right) + \beta_1\delta_z\theta_3 C_z^2\left(\delta_z\left(\beta_1+1\right)/2 + PC_{xz}\right)\right] \quad (3.65)$$

Theorem 3.4.11: To the first degree of approximation, the biases denoted by $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under Cases I and II respectively are;

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2 P\left(P + 2C_{yx}\right) + \theta_1\beta_1\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz}\right)\right] \quad (3.66)$$

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + C_x^2 P\left\{\theta_2\left(P + 2C_{yx}\right) + \theta_3 P\right\} + \theta_3\beta_1\delta_z C_z^2\left(\beta_1\delta_z + 2PC_{xz}\right)\right] \quad (3.67)$$

Proof:

Square both sides of (3.63), taking expectation and using the results in equation (3.54), we obtain the MSE of the modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2 P\left(P + 2C_{yx}\right) + \theta_1\beta_1\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz}\right)\right] \quad (3.68)$$

Also, square both sides of (3.63), taking expectation and using the results in equation (3.55), we obtain the MSE of the modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\left[\theta_2 C_y^2 + C_x^2 P\left\{\theta_2\left(P + 2C_{yx}\right) + \theta_3 P\right\} + \theta_3\beta_1\delta_z C_z^2\left(\beta_1\delta_z + 2PC_{xz}\right)\right] \quad (3.69)$$

Theorem 3.4.12: The $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II}$ are at minimum under the optimal conditions that $P_{31}^{opt} = -C_{yx}, \beta_{11}^{opt} = C_{yz}/\delta_z$ and $P_{32}^{opt} = -\theta_2 C_{yx}/\left(\theta_4 - \theta_1\rho_{xz}^2\right)$, $\beta_{12}^{opt} = \dfrac{-\theta_2 C_{xy} C_{xz}}{\delta_z\left(\theta_4 - \theta_1\rho_{xz}^2\right)}$ respectively.

Proof of theorem 3.4.12

Differentiate equation (3.68) with respect to P and equate to zero,

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[2P\theta_3 C_x^2 + 2\theta_3\rho_{xy}C_x C_y\right] \tag{3.70}$$

$$\bar{Y}^2\left[2P\theta_3 C_x^2 + 2\theta_3\rho_{xy}C_x C_y\right] = 0 \tag{3.71}$$

$$P = -C_{yx} = P_{31}^{opt} \tag{3.72}$$

Differentiate equation (3.68) with respect to β_1 and equate to zero,

$$\frac{\partial}{\partial \beta_1} MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2\left[2\theta_1 C_z^2\left(\beta_1\delta_z^2 - \delta_z C_{xz}\right)\right] \tag{3.73}$$

$$\bar{Y}^2\left[2\theta_1 C_z^2\left(\beta_1\delta_z^2 - \delta_z C_{xz}\right)\right] = 0 \tag{3.74}$$

$$\beta_1 = C_{yz}/\delta_z = \beta_{11}^{opt} \tag{3.75}$$

Substituting the values of P_{31}^{opt} and β_{11}^{opt} in equation (3.68) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I = \bar{Y}^2 C_y^2\left[\theta_2 - \theta_3\rho_{xy}^2 - \theta_1\rho_{yz}^2\right] \tag{3.76}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case I, $P = -C_{yx}$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = -C_{yx} \tag{3.77}$$

$$\frac{fB-C}{A+C+fB}=-C_{yx} \tag{3.78}$$

$$\begin{aligned}(u-1)d^3+(fu+f-8u+9)d^2-(5fu+5f-23u+26)d \\ +(4fu+4f-22u+24)=0\end{aligned} \tag{3.79}$$

where $u=-C_{yx}$

By solving (3.79), at most 3 zeros d_1 , d_2 and d_3 of the polynomials for which (3.7) is optimal under case I will be obtained.

Lemma 3.4.2: The values of unknown constant $d\in\Re^+$ in estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ when $S_2\subset S_1$ is independent of auxiliary variable Z .

Also, differentiate equation (3.69) with respect to P and β_1 and equate to zero, we obtain:

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II}=\bar{Y}^2\left[2\theta_2C_x^2\left(P+C_{yx}\right)+2\theta_1\left(PC_x^2+\beta_1\delta_zC_z^2C_{xz}\right)\right] \tag{3.80}$$

$$\bar{Y}^2\left[2\theta_2C_x^2\left(P+C_{yx}\right)+2\theta_1\left(PC_x^2+\beta_1\delta_zC_z^2C_{xz}\right)\right]=0 \tag{3.81}$$

$$P=\frac{-\left[\theta_2C_x^2C_{yx}+\theta_1\beta_1\delta_zC_z^2C_{xz}\right]}{\theta_4C_x^2} \tag{3.82}$$

$$\frac{\partial}{\partial\beta_1}MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II}=\bar{Y}^2\theta_1\left[2C_z^2\left(\beta_1\delta_z^2+\delta_zPC_{xz}\right)\right] \tag{3.83}$$

$$\bar{Y}^2\theta_1\left[2C_z^2\left(\beta_1\delta_z^2+\delta_zPC_{xz}\right)\right]=0 \tag{3.84}$$

$$\beta_1=-PC_{xz}/\delta_z \tag{3.85}$$

Solving P and β_1 simultaneously, the optimum values of P and β_1 are

$$P=-\theta_2C_{yx}/\left(\theta_4-\theta_1\rho_{xz}^2\right)=P_{32}^{opt} \tag{3.86}$$

$$\beta_{12}^{opt}=\frac{-\theta_2C_{xy}C_{xz}}{\delta_z\left(\theta_4-\theta_1\rho_{xz}^2\right)} \tag{3.87}$$

Substituting the values of P_{32}^{opt} and β_{12}^{opt} in equation (3.69) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{II} = \bar{Y}^2\theta_2 C_y^2\left[1-\rho_{xy}^2/\left(1+\left(\theta_1/\theta_2\right)-\left(\theta_1/\theta_2\right)\rho_{xz}^2\right)\right] \tag{3.88}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ under case II, $P = -\theta_2 C_{yx}/\left(\theta_4 - \theta_1\rho_{xz}^2\right)$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = -\theta_2 C_{yx}/\left(\theta_4 - \theta_1\rho_{xz}^2\right) \tag{3.89}$$

$$\frac{fB-C}{A+C+fB} = -\theta_2 C_{yx}/\left(\theta_4 - \theta_1\rho_{xz}^2\right) \tag{3.90}$$

$$\begin{aligned}(t-1)d^3+(ft+f-8t+9)d^2-(5ft+5f-23t+26)d \\ +(4ft+4f-22t+24)=0\end{aligned} \tag{3.91}$$

where $t = -\theta_2 C_{yx}/\left(\theta_4 - \theta_1\rho_{xz}^2\right)$

By solving (3.91), at most 3 zeros d_1, d_2 and d_3 of the polynomials for which (3.7) is optimal under case II will be obtained.

3.4.4 Bias and MSE of the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$

In order to obtain the bias and MSE of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$, the definitions of error term $\in_{\bar{y}_2}$, $\in_{\bar{x}_1}$, $\in_{\bar{x}_2}$, $\in_{\bar{z}_1}$ and $\in_{\bar{z}_2}$ defined in section 3.4.3 are used.

Theorem 3.4.13: To the first degree of approximation, the modified factor-type estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ can be expressed in terms of error terms $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}, \in_{\bar{z}_2}$ and $\in_{\bar{y}_2}$ is

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{Y}\left[1+\in_{\bar{y}_2}\right]\left[1+\psi_1\in_{\bar{x}_1}+\psi_2\in_{\bar{x}_2}\right]\left[1+\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}\right]^{-1}\left[1+\lambda_z\in_{\bar{z}_1}\right]^{-\beta_2}\left[1+\lambda_z\in_{\bar{z}_2}\right]^{-\beta_2} \tag{3.92}$$

Proof:

Expressing equation (3.8) in terms $\in_{\bar{x}_1}, \in_{\bar{x}_2}, \in_{\bar{z}_1}, \in_{\bar{z}_2}$ and $\in_{\bar{y}_2}$, we have

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{Y}\left(1+\in_{\bar{y}_2}\right)\left[\frac{(A+C)\bar{X}\left(1+\in_{\bar{x}_1}\right)+fB\bar{X}\left(1+\in_{\bar{x}_2}\right)}{(A+fB)\bar{X}\left(1+\in_{\bar{x}_1}\right)+C\bar{X}\left(1+\in_{\bar{x}_2}\right)}\right]\left[\frac{a_z\left(1+\in_{\bar{z}_1}\right)\bar{Z}+b_z}{a_z\left(1+\in_{\bar{z}_2}\right)\bar{Z}+b_z}\right]^{\beta_2} \quad (3.93)$$

Factorize $\bar{X}(A+fB+C)$ from the numerator and denominator of the right hand side of equation (3.93), we have

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{Y}\left(1+\Delta_{\bar{y}_2}\right)\left[\frac{1+\psi_1\Delta_{\bar{x}_1}+\psi_2\Delta_{\bar{x}_2}}{1+\psi_3\Delta_{\bar{x}_1}+\psi_4\Delta_{\bar{x}_2}}\right]\left[\frac{a_z\bar{Z}+b_z+a_z\bar{Z}\in_{\bar{z}_1}}{a_z\bar{Z}+b_z+a_z\bar{Z}\in_{\bar{z}_2}}\right]^{\beta_2} \quad (3.94)$$

Factorize $\left(a_z\bar{Z}+b_z\right)$ from the numerator and denominator of the right hand side of equation (3.94), we have

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{Y}\left(1+\in_{\bar{y}_2}\right)\left[\frac{1+\psi_1\in_{\bar{x}_1}+\psi_2\in_{\bar{x}_2}}{1+\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}}\right]\left[\frac{1+\delta_z\in_{\bar{z}_1}}{1+\delta_z\in_{\bar{z}_2}}\right]^{\beta_2} \quad (3.95)$$

$$\bar{y}_{FTAA}^{\beta_2(d)} = \bar{Y}\left[1+\in_{\bar{y}_2}\right]\left[1+\psi_1\in_{\bar{x}_1}+\psi_2\in_{\bar{x}_2}\right]\left[1+\psi_3\in_{\bar{x}_1}+\psi_4\in_{\bar{x}_2}\right]^{-1}\left[1+\delta_z\in_{\bar{z}_1}\right]^{\beta_2}\left[1+\delta_z\in_{\bar{z}_2}\right]^{-\beta_2} \quad (3.96)$$

Theorem 3.4.14: To the first degree of approximation, the biases denoted by $Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I$ and $Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under Case I and Case II respectively are;

$$Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}\Big[\theta_2 PC_{yz}C_z^2-\theta_1 PC_{yx}C_x^2-\theta_2\psi_4 PC_x^2-\theta_3\beta_2\delta_z C_{yz}C_z^2 +\theta_2\frac{\beta_2(\beta_2+1)}{2}\delta_z^2C_z^2+\theta_1\delta_z^2C_z^2\left(\frac{\beta_2(\beta_2-1)}{2}-\beta_2^2\right)\Big] \quad (3.97)$$

$$Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}\Big[\frac{\beta_2(\beta_2+1)}{2}\delta_z^2\theta_2C_z^2+\frac{\beta_2(\beta_2-1)}{2}\delta_z^2\theta_1C_z^2-\beta_2\delta_z P\theta_2C_{xz}C_z^2 +\psi_3 P\theta_1C_x^2-\psi_4 P\theta_2C_x^2+P\theta_2C_x^2C_{yx}-\beta_2\delta_z\theta_2C_{yz}C_z^2\Big] \quad (3.98)$$

Proof:

Here, we now assume that in (3.96), $\left|\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right| < 1$ and $\left|\delta_z \in_{\bar{z}_2}\right| < 1$ so that $\left(1+\psi_3 \in_{\bar{x}_1} + \psi_4 \in_{\bar{x}_2}\right)^{-1}$ and $\left(1+\delta_z \in_{\bar{z}_2}\right)^{-\beta_2}$ are expandable. Expanding the right hand side of (3.96) up to second degree approximation and then subtract $\bar{Y}$, we have,

$$\bar{y}_{FTAA}^{\beta_2(d)} - \bar{Y} = \bar{Y}\Bigg[\in_{\bar{y}_2} - \beta_2\delta_z \in_{\bar{z}_2} - P \in_{\bar{x}_1} + P \in_{\bar{x}_2} + \frac{\beta_2(\beta_2+1)}{2}\delta_z^2 \in_{\bar{z}_2}^2 + \beta_2\delta_z P \in_{\bar{x}_1}\in_{\bar{z}_2} -$$
$$\beta_2\delta_z P \in_{\bar{x}_2}\in_{\bar{z}_1} + \psi_3 P \in_{\bar{x}_1}^2 - \psi_4 P \in_{\bar{x}_2}^2 + (\psi_4 - \psi_3) P \in_{\bar{x}_1}\in_{\bar{x}_2} + \beta_2\delta_z \in_{\bar{y}_2}\in_{\bar{z}_1} - P \in_{\bar{y}_2}\in_{\bar{x}_1} + P \in_{\bar{y}_2}\in_{\bar{x}_2}$$
$$-\beta_2^2\delta_z^2 \in_{\bar{z}_1}\in_{\bar{z}_2} + \frac{\beta_2(\beta_2-1)}{2}\delta_z^2 \in_{\bar{z}_1}^2 - \beta_2\delta_z P \in_{\bar{z}_2}\in_{\bar{z}_1} + \beta_2\delta_z P \in_{x_2}\in_{\bar{z}_1} - \beta_2\delta_z \in_{\bar{y}_2}\in_{\bar{z}_2} \Bigg] \tag{3.99}$$

Taking expectation of equation (3.99) and apply the results in equation (3.54), the bias of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ when $S_2 \subset S_1$ is obtained as

$$Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}\Big[\theta_2 P C_{yz} C_z^2 - \theta_1 P C_{yx} C_x^2 - \theta_3\psi_4 P C_x^2 - \theta_3\beta_2\delta_z C_{yz} C_z^2$$
$$+\theta_2\frac{\beta_2(\beta_2+1)}{2}\delta_z^2 C_z^2 + \theta_1\delta_z^2 C_z^2\left(\frac{\beta_2(\beta_2-1)}{2} - \beta_2^2\right)\Big] \tag{3.100}$$

Also, taking expectation of (3.99) and using results of (3.55), the bias of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ to terms of order n^{-1} when $S_2 \subset \Omega_N$ is obtained as:

$$Bias\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}\Bigg[\frac{\beta_2(\beta_2+1)}{2}\delta_z^2\theta_2 C_z^2 + \frac{\beta_2(\beta_2-1)}{2}\delta_z^2\theta_1 C_z^2 - \beta_2\delta_z P\theta_2 C_{xz} C_z^2$$
$$+\psi_3 P\theta_1 C_x^2 - \psi_4 P\theta_2 C_x^2 + P\theta_2 C_x^2 C_{yx} - \beta_2\delta_z\theta_2 C_{yz} C_z^2\Bigg] \tag{3.101}$$

Theorem 3.4.15: To the first degree of approximation, the biases denoted by $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}$, of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under Cases I and II respectively are;

$$MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2 P\left(P - 2C_{yx}\right) + \theta_3\beta_2\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz} - 2PC_{xz}\right)\right] \quad (3.107)$$

$$\begin{aligned} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\Big[&\theta_2 C_y^2 + \theta_2 C_x^2 P\left(P + 2C_{yx}\right) + \theta_3\beta_2\delta_z C_z^2\left(\beta_2\delta_z - 2PC_{xz}\right) \\ &\theta_2\delta_z\beta_2 C_z^2\left(\delta_z\beta_2 - 2C_{yz} - 2PC_{xz}\right) + \theta_3 P^2 C_x^2\Big] \end{aligned} \quad (3.108)$$

Proof:

Square both sides of (3.99), taking expectation and using the results in equation (3.54), we obtain the MSE of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I as:

$$MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2\left[\delta_2 C_y^2 + \delta_3 C_x^2 P\left(P - 2C_{yx}\right) + \delta_3\beta_2\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz} - 2PC_{xz}\right)\right] \quad (3.109)$$

Also, square both sides of (3.99), taking expectation and using the results in equation (3.55), we obtain the MSE of the modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II as:

$$\begin{aligned} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\Big[&\theta_2 C_y^2 + \theta_2 C_x^2 P\left(P + 2C_{yx}\right) + \theta_1\beta_2\delta_z C_z^2\left(\beta_2\lambda_z - 2PC_{xz}\right) \\ &\theta_2\delta_z\beta_2 C_z^2\left(\delta_z\beta_2 - 2C_{yz} - 2PC_{xz}\right) + \theta_1 P^2 C_x^2\Big] \end{aligned} \quad (3.110)$$

Theorem 3.4.16: The $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I$ and $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}$ are at minimum under the optimal conditions that

$P = \left(C_z^2 C_{xz} C_{yz} + C_{yx} C_x^2\right) / \left(C_x^2 - C_z^2 C_{xz}^2\right)$, $\beta_2 = \left(C_{yz} C_x^2 + C_x^2 C_{yx} C_{xz}\right) / \left(C_x^2 - C_z^2 C_{xz}^2\right)\lambda_z$ and

$P = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right] / \theta_4 C_x\left[1 - \rho_{xz}^2\right]$, $\beta_2 = \theta_2\left[C_{yz} - C_{yx}C_{xz}\right] / \theta_4\lambda_z\left(1 - \rho_{xz}^2\right)$ respectively.

Proof:

Differentiate equation (3.109) with respect to P and equate to zero,

$$\frac{\partial}{\partial P} MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I = \bar{Y}^2\left[2P\theta_3 C_x^2 - 2\theta_3\rho_{xy} C_x C_y - 2\theta_3\beta_2\delta_z\rho_{xz} C_x C_z\right] \quad (3.111)$$

$$\bar{Y}^2\left[2P\theta_3C_x^2-2\theta_3\rho_{xy}C_xC_y-2\theta_3\beta_2\delta_z\rho_{xz}C_xC_z\right]=0 \tag{3.112}$$

$$P=\left[\rho_{xy}C_y+\beta_2\delta_z\rho_{xz}C_z\right]/C_x \tag{3.113}$$

Differentiate equation (3.109) with respect to β_2 and equate to zero,

$$\frac{\partial}{\partial\beta_2}MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I=\bar{Y}^2\theta_3\delta_zC_z^2\left[2\beta_2\delta_z-2C_{yz}-2PC_{xz}\right] \tag{3.114}$$

$$\bar{Y}^2\theta_3\delta_zC_z^2\left[2\beta_2\delta_z-2C_{yz}-2PC_{xz}\right]=0 \tag{3.115}$$

$$\beta_2=\left[C_{yz}+PC_{xz}\right]/\delta_z \tag{3.116}$$

Solving P and β_1 simultaneously, the optimum values of P and β_1 are

$$P=\left[\rho_{xy}C_y+\rho_{xz}C_zC_{yz}\right]/\left[C_x-\rho_{xz}C_zC_{xz}\right]=P_{41}^{opt} \tag{3.117}$$

$$\beta_2=\left[C_xC_{yz}+\rho_{xy}C_yC_{xz}\right]/\delta_z\left[C_x-\rho_{xz}C_zC_{xz}\right]=\beta_{21}^{opt} \tag{3.118}$$

Substituting the values of P_{41}^{opt} and β_{21}^{opt} in equation (3.109) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_I=\bar{Y}^2C_y^2\left[\theta_2-\theta_3\left(\rho_{xy}^2+\rho_{yz}^2+2\rho_{xy}\rho_{xz}\rho_{yz}\right)/\left(1-\rho_{xz}^2\right)\right] \tag{3.119}$$

In order to estimate unknown constant $d\in\Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case I, $P=\left[\rho_{xy}C_y+\rho_{xz}C_zC_{yz}\right]/\left[C_x-\rho_{xz}C_zC_{xz}\right]$ and $P=\psi_3-\psi_1$ are equated as

$$\psi_3-\psi_1=\left[\rho_{xy}C_y+\rho_{xz}C_zC_{yz}\right]/\left[C_x-\rho_{xz}C_zC_{xz}\right] \tag{3.120}$$

$$\frac{fB-C}{A+C+fB}=\left[\rho_{xy}C_y+\rho_{xz}C_zC_{yz}\right]/\left[C_x-\rho_{xz}C_zC_{xz}\right] \tag{3.121}$$

$$(m-1)d^3+(fm+f-8m+9)d^2-(5fm+5f-23m+26)d+(4fm+4f-22m+24)=0 \quad (3.122)$$

where $m=\left[\rho_{xy}C_y+\rho_{xz}C_zC_{yz}\right]/\left[C_x-\rho_{xz}C_zC_{xz}\right]$

By solving (3.122), at most 3 zeros d_1 , d_2 and d_3 of the polynomials for which (3.8) is optimal under case I will be obtained.

Also, differentiate equation (3.110) with respect to P and β_2 and equate to zero, we obtain:

$$\frac{\partial}{\partial P}MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}=\bar{Y}^2\left[2C_x^2\left(\theta_4P+\theta_2C_{yx}\right)-2\theta_4\beta_2\delta_zC_z^2C_{xz}\right] \quad (3.123)$$

$$\bar{Y}^2\left[2C_x^2\left(\theta_4P+\theta_2C_{yx}\right)-2\theta_4\beta_2\delta_zC_z^2C_{xz}\right]=0 \quad (3.124)$$

$$P=\frac{\theta_4\beta_2\delta_zC_z^2C_{xz}+\theta_2C_x^2C_{yx}}{\theta_4C_x^2} \quad (3.125)$$

$$\frac{\partial}{\partial \beta_2}MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II}=\bar{Y}^2\left[2\theta_3\beta_2\delta_z^2C_z^2-2\theta_3P\delta_zC_z^2C_{xz}-2\theta_2\delta_zC_z^2C_{xz}\right] \quad (3.126)$$

$$\bar{Y}^2\left[2\theta_4\beta_2\delta_z^2C_z^2-2\theta_4P\delta_zC_z^2C_{xz}-2\theta_2\delta_zC_z^2C_{xz}\right]=0 \quad (3.127)$$

$$\beta_2=\left[\theta_2C_{xz}+\theta_4PC_{xz}\right]/\theta_4\delta_z \quad (3.128)$$

Solving P and β_2 simultaneously, the optimum values of P and β_2 are

$$P_{42}^{opt}=\theta_2\left[\rho_{yz}\rho_{xz}-C_y\rho_{xy}\right]/\theta_4C_x\left[1-\rho_{xz}^2\right] \quad (3.129)$$

$$\beta_{22}^{opt}=\frac{\theta_2\left[C_{yz}-C_{yx}C_{xz}\right]}{\theta_4\lambda_z\left(1-\rho_{xz}^2\right)} \quad (3.130)$$

Substituting the values of P_{42}^{opt} and β_{22}^{opt} in equation (3.110) and simplify, we obtain minimum mean square error of $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II as

$$MSE_{\min}\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{II} = \bar{Y}^2\theta_2\left[C_y^2 + \frac{\left(\rho_{yz}\rho_{xz} - \rho_{xy}C_y\right)}{\theta_4^2\left(1-\rho_{xz}^2\right)^2}\left\{\rho_{yz}\rho_{xz}\theta_2\theta_4\left(1-2C_y\right) + \theta_4\rho_{xy}C_y\left(2-\theta_2\right)\right.\right.$$
$$\left. 2\theta_4\rho_{xz}^2\rho_{xy}C_y\left(\theta_2-1\right)\right\} + \frac{\left(C_{yz} - C_{yx}C_{xz}\right)}{\theta_4^2\left(1-\rho_{xz}^2\right)^2}\left\{\theta_2\rho_{yz}C_yC_z\left(\theta_4-2\right)\right.$$
$$\left.\left. + \theta_2\rho_{xz}C_yC_z\left(2\rho_{yz}\rho_{xz} - \theta_4\rho_{xy}\right)\right]\right. \tag{3.131}$$

In order to estimate unknown constant $d \in \Re^+$ in the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ under case II, $P = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4C_x\left[1-\rho_{xz}^2\right]$ and $P = \psi_3 - \psi_1$ are equated as

$$\psi_3 - \psi_1 = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4C_x\left[1-\rho_{xz}^2\right] \tag{3.132}$$

$$\frac{fB-C}{A+C+fB} = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4C_x\left[1-\rho_{xz}^2\right] \tag{3.133}$$

$$\left(q-1\right)d^3 + \left(fq + f - 8q + 9\right)d^2 - \left(5fq + 5f - 23q + 26\right)d + \left(4fq + 4f - 22q + 24\right) = 0 \tag{3.134}$$

where $q = \theta_2\left[\rho_{yz}\rho_{xz} - C_y\rho_{xy}\right]/\theta_4C_x\left[1-\rho_{xz}^2\right]$

By solving (3.134), at most 3 zeros d_1 , d_2 and d_3 of the polynomials for which (3.8) is optimal under case II will be obtained.

3.5 EFFICIENCY COMPARISONS

In this section, the efficiency of modified estimators are compared theoretically with that of some existing estimators in the literature and the conditions for which the modified estimators performed better have been established.

3.5.1 Efficiency of $\bar{y}_{FTAA}$ over some related Estimators

1. The efficiency of modified estimator $\bar{y}_{FTAA}$ and sample mean $\bar{y}$ estimator are compared as,

$$\text{var}(\bar{y}) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.135}$$

$$\bar{Y}^2 \frac{1-f}{n} C_y^2 - \bar{Y}^2 \frac{1-f}{n} \left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P \rho_{xy} C_x C_y \right] > 0$$

$$-\delta_x^2 P^2 C_x^2 - 2\delta_x P \rho_{xy} C_x C_y > 0$$

$$P > -\frac{2\rho_{xy} C_y}{\delta_x C_x} \tag{3.136}$$

Theorem 3.5.1: The modified estimator $\bar{y}_{FTAA}$ is more efficient than sample mean $\bar{y}$ if $P > -\frac{2\rho_{xy} C_y}{\delta_x C_x}$.

2. The efficiency of modified estimator $\bar{y}_{FTAA}$ and traditional ratio estimator $\bar{y}_r$ suggested by Cochran (1942) are compared as,

$$MSE(\bar{y}_r) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.137}$$

$$\bar{Y}^2 \frac{1-f}{n} \left[C_y^2 + C_x^2 - 2\rho_{xy} C_x C_y \right] - \bar{Y}^2 \frac{1-f}{n} \left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P \rho_{xy} C_x C_y \right] > 0$$

$$C_x^2 - 2\rho_{xy} C_x C_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P \rho_{xy} C_x C_y > 0$$

$$\left[C_x - \rho_{xy} C_y \right]^2 > \left[\delta_x P C_x + \rho_{xy} C_y \right]^2$$

$$P < \frac{C_x - 2\rho_{xy} C_y}{\delta_x C_x} \tag{3.138}$$

Theorem 3.5.2: The modified estimator $\bar{y}_{FTAA}$ is more efficient than ratio estimator $\bar{y}_r$ suggested by Cochran (1942) if $P < \frac{C_x - 2\rho_{xy} C_y}{\delta_x C_x}$.

3. The efficiency of modified estimator $\bar{y}_{FTAA}$ and traditional product estimator $\bar{y}_r$ suggested by Murthy (1964) are compared as if,

$$MSE\left(\bar{y}_p\right) - MSE\left(\bar{y}_{FTAA}\right) > 0 \tag{3.139}$$

$$\bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + C_x^2 + 2\rho_{xy}C_xC_y\right] - \bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P\rho_{xy}C_xC_y\right] > 0$$

$$C_x^2 + 2\rho_{xy}C_xC_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P\rho_{xy}C_xC_y > 0$$

$$\left[C_x + \rho_{xy}C_y\right]^2 > \left[\delta_x PC_x + \rho_{xy}C_y\right]^2$$

$$P < \frac{1}{\delta_x} \tag{3.140}$$

Theorem 3.5.3: The modified estimator $\bar{y}_{FTAA}$ is more efficient than product estimator $\bar{y}_p$ suggested by Murthy (1964) if $P < \frac{1}{\delta_x}$.

4. The efficiency of modified estimator $\bar{y}_{FTAA}$ and traditional dual to ratio estimator $\bar{y}_r$ suggested by Srivenkataramana (1980) are compared as,

$$MSE\left(\bar{y}_{st}\right) - MSE\left(\bar{y}_{FTAA}\right) > 0 \tag{3.141}$$

$$\bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \left(\frac{n}{N-n}\right)^2 C_x^2 - 2\left(\frac{n}{N-n}\right)\rho_{xy}C_xC_y\right] - \bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P\rho_{xy}C_xC_y\right] > 0$$

$$\left(\frac{n}{N-n}\right)^2 C_x^2 - 2\left(\frac{n}{N-n}\right)\rho_{xy}C_xC_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P\rho_{xy}C_xC_y > 0$$

$$\left[\left(\frac{n}{N-n}\right)C_x - \rho_{xy}C_y\right]^2 > \left[\delta_x PC_x + \rho_{xy}C_y\right]^2$$

$$P < \frac{nC_x/(N-n) - 2\rho_{xy}C_y}{\delta_x C_x} \tag{3.142}$$

Theorem 3.5.4: The modified estimator $\bar{y}_{FTAA}$ is more efficient than dual to ratio estimator $\bar{y}_r$ suggested by Srivenkataramana (1980) if $P < \dfrac{nC_x/(N-n) - 2\rho_{xy}C_y}{\delta_x C_x}$.

5. The efficiency of modified estimator $\bar{y}_{FTAA}$ is compare to that of the estimators t_2, t_3, t_4, t_5 and t_6 suggested by Sisodia and Diwivedi (1981), Singh and Tailor (2003), Singh et. al. (2004) and Upadhyaya and Singh (1999) respectively as,

$$MSE(t_i) - MSE(\bar{y}_{FTAA}) > 0, \qquad i = 2,3,4,5,6 \tag{3.143}$$

$$\bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \eta_i^2 C_x^2 - 2\eta_i \rho_{xy} C_x C_y\right] - \bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P \rho_{xy} C_x C_y\right] > 0$$

where

$$\eta_2 = \frac{\bar{X}}{\bar{X} + C_x}, \eta_3 = \frac{\bar{X}}{\bar{X} + \rho_{xy}}, \eta_4 = \frac{\bar{X}}{\bar{X} + \beta_{2(x)}}, \eta_5 = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_{2(x)}}, \eta_6 = \frac{\beta_{2(x)} \bar{X}}{\beta_{2(x)} \bar{X} + C_x}$$

$$\eta_i^2 C_x^2 - 2\eta_i \rho_{xy} C_x C_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P \rho_{xy} C_x C_y > 0$$

$$\left[\eta_i C_x - \rho_{xy} C_y\right]^2 > \left[\delta_x P C_x + \rho_{xy} C_y\right]^2$$

$$P < \frac{\eta_i C_x - 2\rho_{xy} C_y}{\delta_x C_x} \tag{3.144}$$

Theorem 3.5.5: The modified estimator $\bar{y}_{FTAA}$ is more efficient than the estimators t_2, t_3, t_4, t_5 and t_6 suggested by Sisodia and Diwivedi (1981), Singh and Tailor (2003), Singh et. al. (2004), Upadhyaya and Singh (1999) and Upadhyaya and Singh (1999) if $P < \dfrac{\bar{X}C_x/(\bar{X} + C_x) - 2\rho_{xy}C_y}{\delta_x C_x}$, $P < \dfrac{\bar{X}C_x/(\bar{X} + \rho_{xy}) - 2\rho_{xy}C_y}{\delta_x C_x}$, $P < \dfrac{\bar{X}C_x/(\bar{X} + \beta_{2(x)}) - 2\rho_{xy}C_y}{\delta_x C_x}$, $P < \dfrac{\bar{X}C_x^2/(C_x\bar{X} + \beta_{2(x)}) - 2\rho_{xy}C_y}{\delta_x C_x}$ and $P < \dfrac{\bar{X}\beta_{2(x)}C_x/(\beta_{2(x)}\bar{X} + C_x) - 2\rho_{xy}C_y}{\delta_x C_x}$ respectively.

6. The efficiency of modified estimator $\bar{y}_{FTAA}$ is compare to the estimators $t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}$ and t_{16} modified by Pandey and Dubey (1988), Singh and Tailor (2003), Singh (2003), Singh et. al. (2004), Upadhyaya and Singh (1999) respectively as,

$$MSE(t_i) - MSE(\bar{y}_{FTAA}) > 0, \qquad i = 8,9,10,11,12,13,14,15,16 \tag{3.145}$$

$$\bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \Gamma_i^2 C_x^2 + 2\Gamma_i \rho_{xy} C_x C_y\right] - \bar{Y}^2 \frac{1-f}{n}\left[C_y^2 + \delta_x^2 P^2 C_x^2 + 2\delta_x P \rho_{xy} C_x C_y\right] > 0$$

Where $\Gamma_8 = \frac{\bar{X}}{\bar{X} + C_x}, \Gamma_9 = \frac{\bar{X}}{\bar{X} + \rho_{xy}}, \Gamma_{10} = \frac{\bar{X}}{\bar{X} + \sigma_x}, \Gamma_{11} = \frac{\bar{X}}{\bar{X} + \beta_{2(x)}}, \Gamma_{12} = \frac{C_x \bar{X}}{C_x \bar{X} + \beta_{2(x)}}$

$$\Gamma_{13} = \frac{\beta_{1(x)} \bar{X}}{\beta_{1(x)} \bar{X} + C_x}, \Gamma_{14} = \frac{\beta_{1(x)} \bar{X}}{\beta_{1(x)} \bar{X} + \sigma_x}, \Gamma_{15} = \frac{\beta_{2(x)} \bar{X}}{\beta_{2(x)} \bar{X} + C_x}, \Gamma_{16} = \frac{\beta_{2(x)} \bar{X}}{\beta_{2(x)} \bar{X} + \sigma_x}$$

$$\Gamma_i^2 C_x^2 + 2\Gamma_i \rho_{xy} C_x C_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P \rho_{xy} C_x C_y > 0$$

$$\left[\Gamma_i C_x + \rho_{xy} C_y\right]^2 > \left[\delta_x P C_x + \rho_{xy} C_y\right]^2$$

$$P < \frac{\eta_i}{\delta_x} \tag{3.146}$$

Theorem 3.5.6: The modified estimator $\bar{y}_{FTAA}$ is more efficient than the estimators $t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}$ and t_{16} suggested by Pandey and Dubey (1988), Singh and Tailor (2003), Singh et. al. (2004), Upadhyaya and Singh (1999), and Singh (2003) if

$$P < \frac{\bar{X}}{\delta_x(\bar{X} + C_x)}, \qquad P < \frac{\bar{X}}{\delta_x(\bar{X} + \rho_{xy})}, \qquad P < \frac{\bar{X}}{\delta_x(\bar{X} + \sigma_x)}, \qquad P < \frac{\bar{X}}{\delta_x(\bar{X} + \beta_{2(x)})},$$

$P < \frac{\bar{X}C_x}{\delta_x(C_x\bar{X}+\beta_{2(x)})}$, $P < \frac{\bar{X}\beta_{1(x)}}{\delta_x(\beta_{1(x)}\bar{X}+C_x)}$, $P < \frac{\bar{X}\beta_{1(x)}}{\delta_x(\beta_{1(x)}\bar{X}+\sigma_x)}$, $P < \frac{\bar{X}\beta_{2(x)}}{\delta_x(\beta_{2(x)}\bar{X}+C_x)}$ and

$P < \frac{\bar{X}\beta_{2(x)}}{\delta_x(\beta_{2(x)}\bar{X}+\sigma_x)}$ respectively.

7. The efficiency of modified estimator $\bar{y}_{FTAA}$ is compare to traditional factor-type estimator $\bar{y}_{FT}$ suggested by Singh and Shukla (1987, 1993) as,

$$MSE(\bar{y}_{FT}) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.147}$$

$$\bar{Y}^2\frac{1-f}{n}\left[C_y^2 + P^2C_x^2 + 2P\rho_{xy}C_xC_y\right] - \bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \delta_x^2P^2C_x^2 + 2\delta_xP\rho_{xy}C_xC_y\right] > 0$$

$$PC_x^2 + 2\rho_{xy}C_xC_y - \delta_x^2PC_x^2 - 2\delta_x\rho_{xy}C_xC_y < 0$$

$$-PC_x\left[\delta_x^2 - 1\right] > 2\rho_{xy}C_y\left[\delta_x - 1\right]$$

$$P < -\frac{2\rho_{xy}C_y}{\left[\delta_x + 1\right]C_x} \tag{3.148}$$

Theorem 3.5.7: The modified estimator $\bar{y}_{FTAA}$ is more efficient than traditional factor-type estimator $\bar{y}_{FT}$ suggested by Singh and Shukla (1987, 1993) if $P < -\frac{2\rho_{xy}C_y}{\left[\delta_x + 1\right]C_x}$.

8. The efficiency of modified estimator $\bar{y}_{FTAA}$ and factor-type estimator $\bar{y}_{FTRP}$ suggested by Thakur and Gupta (2013) as,

$$MSE(\bar{y}_{FTRP}) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.149}$$

$$\bar{Y}^2\frac{1-f}{n}\left[C_y^2 + (2f-1)^2P^2C_x^2 + 2(2f-1)P\rho_{xy}C_xC_y\right]$$
$$-\bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \delta_x^2P^2C_x^2 + 2\delta_xP\rho_{xy}C_xC_y\right] > 0$$

$$(2f-1)^2 PC_x + 2(2f-1)\rho_{xy}C_y - \delta_x^2 PC_x - 2\delta_x\rho_{xy}C_y > 0$$

$$-P\delta C_x\left[\delta_x^2 - (2f-1)^2\right] > 2\rho_{xy}C_y\left[\delta_x - (2f-1)\right]$$

$$P < -\frac{2\rho_{xy}C_y}{\left[\delta_x + 2f - 1\right]C_x} \tag{3.150}$$

Theorem 3.5.8: The modified estimator $\bar{y}_{FTAA}$ is more efficient than traditional factor-type estimator $\bar{y}_{FTRP}$ suggested by Thakur and Gupta (2013) if, $P < -\frac{2\rho_{xy}C_y}{\left[\delta_x + 2f - 1\right]C_x}$.

9. The efficiency of modified estimator $\bar{y}_{FTAA}$ and factor-type estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ suggested by Jain and Shukla (2013) as,

$$MSE(\bar{y}_{FT1}) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.151}$$

$$\bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \left(\frac{2f-3}{2f+9}\right)^2 C_x^2 + 2\left(\frac{2f-3}{2f+9}\right)\rho_{xy}C_xC_y\right]$$

$$-\bar{Y}^2\frac{1-f}{n}\left[C_y^2 + \delta_x^2P^2C_x^2 + 2\delta_xP\rho_{xy}C_xC_y\right] > 0$$

$$\left(\frac{2f-3}{2f+9}\right)^2 C_x^2 + 2\left(\frac{2f-3}{2f+9}\right)\rho_{xy}C_xC_y - \delta_x^2P^2C_x^2 - 2\delta_xP\rho_{xy}C_xC_y > 0$$

$$\left[\left(\frac{2f-3}{2f+9}\right)C_x + \rho_{xy}C_y\right]^2 > \left[\delta_xPC_x + \rho_{xy}C_y\right]^2$$

$$P < \frac{1}{\delta_x}\left(\frac{2f-3}{2f+9}\right) \tag{3.152}$$

$$MSE(\bar{y}_{FT2}) - MSE(\bar{y}_{FTAA}) > 0 \tag{3.153}$$

$$\bar{Y}^2\frac{1-f}{n}\left[C_y^2+\left(\frac{5f-12}{5f+22}\right)^2C_x^2+2\left(\frac{5f-12}{5f+22}\right)\rho_{xy}C_xC_y\right]$$
$$-\bar{Y}^2\frac{1-f}{n}\left[C_y^2+\delta_x^2P^2C_x^2+2\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$\left(\frac{5f-12}{5f+22}\right)^2C_x^2+2\left(\frac{5f-12}{5f+22}\right)\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\delta_xP\rho_{xy}C_xC_y>0$$

$$\left[\left(\frac{5f-12}{5f+22}\right)C_x+\rho_{xy}C_y\right]^2>\left[\delta_xPC_x+\rho_{xy}C_y\right]^2$$

$$P<\frac{1}{\delta_x}\left(\frac{5f-12}{5f+22}\right) \tag{3.154}$$

Theorem 3.5.9: The modified estimator $\bar{y}_{FTAA}$ is more efficient than factor-type estimators $\bar{y}_{FT1}$ and $\bar{y}_{FT2}$ suggested by Jain and Shukla (2013) if $P<\frac{1}{\delta_x}\left(\frac{2f-3}{2f+9}\right)$ and $P<\frac{1}{\delta_x}\left(\frac{5f-12}{5f+22}\right)$ respectively.

10. The efficiency of modified estimator $\bar{y}_{FTAA}$ and factor-type estimator $\bar{y}_{TFT}$ suggested by Shukla et. al. (2013) as,

$$MSE\left(\bar{y}_{TFT}\right)-MSE\left(\bar{y}_{FTAA}\right)>0 \tag{3.155}$$

$$\bar{Y}^2\frac{1-f}{n}\left[C_y^2+\left(\frac{P}{d-1}\right)^2C_x^2+2\left(\frac{P}{d-1}\right)\rho_{xy}C_xC_y\right]$$
$$-\bar{Y}^2\frac{1-f}{n}\left[C_y^2+\delta_x^2P^2C_x^2+2\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$\left(\frac{P}{d-1}\right)^2PC_x+2\left(\frac{P}{d-1}\right)\rho_{xy}C_y-\delta_x^2PC_x-2\delta_x\rho_{xy}C_y>0$$

$$-PC_x\left[\delta_x^2-\left(\frac{P}{d-1}\right)^2\right]>2\rho_{xy}C_y\left[\delta_x-\left(\frac{P}{d-1}\right)\right]$$

$$P<-\frac{2\rho_{xy}C_y}{\left[\delta_x+1/(d-1)\right]C_x},\qquad \forall\, d>1 \tag{3.156}$$

Theorem 3.5.10: The modified estimator $\bar{y}_{FTAA}$ is more efficient than traditional factor-type estimator $\bar{y}_{TFT}$ suggested by Shukla et. al. (2013) if $P<-\frac{2\rho_{xy}C_y}{\left[\delta_x+1/(d-1)\right]C_x},\ \forall\ d>1.$

3.5.2 Efficiency of $\bar{y}_{FTAA}^{(d)}$ over some related Estimators

1. The efficiency of modified estimator $\bar{y}_{FTAA}$ and estimator $t_1^{(d)}$ suggested by Sukhatme (1962) are compared as,

$$MSE\left(t_1^{(d)}\right)_I-MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I>0 \tag{3.157}$$

$$\bar{Y}^2\left[\theta_2C_y^2+\theta_3C_x^2-2\theta_3\rho_{xy}C_xC_y\right]-\bar{Y}^2\left[\theta_2C_y^2+\theta_3\delta_x^2P^2C_x^2+2\theta_3\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$C_x^2-2\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\delta_xP\rho_{xy}C_xC_y>0$$

$$\left[\delta_xPC_x+\rho_{xy}C_y\right]^2<\left[C_x-\rho_{xy}C_y\right]^2$$

$$P<\frac{C_x-2\rho_{xy}C_y}{\delta_xC_x} \tag{3.158}$$

$$MSE\left(t_1^{(d)}\right)_{II}-MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II}>0 \tag{3.159}$$

$$\bar{Y}^2\left[\theta_2C_y^2+\theta_4C_x^2-2\theta_2\rho_{xy}C_xC_y\right]-\bar{Y}^2\left[\theta_2C_y^2+\theta_4\delta_x^2P^2C_x^2+2\theta_2\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$C_x^2-2\frac{\theta_2}{\theta_4}\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\frac{\theta_2}{\theta_4}\delta_xP\rho_{xy}C_xC_y>0$$

$$\left[\delta_xPC_x+\frac{\theta_2}{\theta_4}\rho_{xy}C_y\right]^2<\left[C_x-\frac{\theta_2}{\theta_4}\rho_{xy}C_y\right]^2$$

$$P < \frac{C_x - 2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\delta_x C_x} \tag{3.160}$$

Theorem 3.5.11: The modified estimator $\bar{y}_{FTAA}$ is more efficient than estimator $t_1^{(d)}$ suggested by Sukhatme (1962) if $P < \frac{C_x - 2\rho_{xy}C_y}{\delta_x C_x}$ and $P < \frac{C_x - 2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\delta_x C_x}$ when $S_2 \subset S_1$ and $S_2 \subset \Omega_N$ respectively.

2. The efficiency of modified estimator $\bar{y}_{FTAA}$ and estimator $t_3^{(d)}$ suggested by Malik and Tailor (2013) are compared as,

$$MSE\left(t_3^{(d)}\right)_I - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I > 0 \tag{3.161}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)^2 C_x^2 - 2\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)\theta_3\rho_{xy}C_xC_y\right]$$
$$-\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3\delta_x^2 P^2 C_x^2 + 2\theta_3\delta_x P\rho_{xy}C_xC_y\right] > 0$$

$$\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)^2 C_x^2 - 2\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)\rho_{xy}C_xC_y - \delta_x^2 P^2 C_x^2 - 2\delta_x P\rho_{xy}C_xC_y > 0$$

$$\left[\delta_x P C_x + \rho_{xy}C_y\right]^2 < \left[\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)C_x - \rho_{xy}C_y\right]^2$$

$$P < \frac{C_x\bar{X}/(\bar{X}+\rho_{xy}) - 2\rho_{xy}C_y}{\delta_x C_x} \tag{3.162}$$

$$MSE\left(t_3^{(d)}\right)_{II} - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} > 0 \tag{3.163}$$

$$\bar{Y}^2\left[\theta_2 C_y^2+\theta_4\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)^2 C_x^2-2\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)\theta_2\rho_{xy}C_xC_y\right]$$

$$-\bar{Y}^2\left[\theta_2 C_y^2+\theta_4\delta_x^2P^2C_x^2+2\theta_2\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)^2 C_x^2-2\frac{\theta_2}{\theta_4}\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\frac{\theta_2}{\theta_4}\delta_xP\rho_{xy}C_xC_y>0$$

$$\left[\delta_xPC_x+\frac{\theta_2}{\theta_4}\rho_{xy}C_y\right]^2<\left[\left(\frac{\bar{X}}{\bar{X}+\rho_{xy}}\right)C_x-\frac{\theta_2}{\theta_4}\rho_{xy}C_y\right]^2$$

$$P<\frac{C_x\bar{X}/(\bar{X}+\rho_{xy})-2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\delta_xC_x} \tag{3.164}$$

Theorem 3.5.12: The modified estimator $\bar{y}_{FTAA}$ is more efficient than estimator $t_3^{(d)}$ suggested by Malik and Tailor (2013) if $P<\frac{C_x\bar{X}/(\bar{X}+\rho_{xy})-2\rho_{xy}C_y}{\delta_xC_x}$ and

$P<\frac{C_x\bar{X}/(\bar{X}+\rho_{xy})-2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\delta_xC_x}$ when $S_2\subset S_1$ and $S_2\subset\Omega_N$ respectively.

3. The efficiency of modified estimator $\bar{y}_{FTAA}$ and traditional product estimator $t_7^{(d)}$ under two-phase sampling are compared as,

$$MSE\left(t_7^{(d)}\right)_I-MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I>0 \tag{3.165}$$

$$\bar{Y}^2\left[\theta_2C_y^2+\theta_3C_x^2+2\theta_3\rho_{xy}C_xC_y\right]-\bar{Y}^2\left[\theta_2C_y^2+\theta_3\delta_x^2P^2C_x^2+2\theta_3\delta_xP\rho_{xy}C_xC_y\right]>0$$

$$C_x^2+2\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\delta_xP\rho_{xy}C_xC_y>0$$

$$\left[\delta_xPC_x+\rho_{xy}C_y\right]^2<\left[C_x+\rho_{xy}C_y\right]^2$$

$$P < \frac{1}{\delta_x} \tag{3.166}$$

$$MSE\left(t_7^{(d)}\right)_{II} - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} > 0 \tag{3.167}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_4 C_x^2 + 2\theta_2 \rho_{xy} C_x C_y\right] - \bar{Y}^2\left[\theta_2 C_y^2 + \theta_4 \delta_x^2 P^2 C_x^2 + 2\theta_2 \delta_x P \rho_{xy} C_x C_y\right] > 0$$

$$C_x^2 + 2\frac{\theta_2}{\theta_4}\rho_{xy} C_x C_y - \delta_x^2 P^2 C_x^2 - 2\frac{\theta_2}{\theta_4}\delta_x P \rho_{xy} C_x C_y > 0$$

$$\left[\delta_x P C_x + \frac{\theta_2}{\theta_4}\rho_{xy} C_y\right]^2 < \left[C_x + \frac{\theta_2}{\theta_4}\rho_{xy} C_y\right]^2$$

$$P < \frac{1}{\delta_x} \tag{3.168}$$

Theorem 3.5.13: The modified estimator $\bar{y}_{FTAA}$ is more efficient than traditional product estimator $t_7^{(d)}$ under two-phase sampling if $P < \frac{1}{\delta_x}$ when $S_2 \subset S_1$ and $S_2 \subset \Omega_N$.

4. The efficiency of modified estimator $\bar{y}_{FTAA}$ and estimator $t_1^{(d)}$ suggested by Shukla (2002) are compared as,

$$MSE\left(\bar{y}_{FTd}\right)_I - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_I > 0 \tag{3.169}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P^2 C_x^2 + 2\theta_3 P \rho_{xy} C_x C_y\right] - \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 \delta_x^2 P^2 C_x^2 + 2\theta_3 \delta_x P \rho_{xy} C_x C_y\right] > 0$$

$$PC_x^2 + 2\rho_{xy} C_x C_y - \delta_x^2 P C_x^2 - 2\delta_x \rho_{xy} C_x C_y > 0$$

$$-P\delta C_x\left[\delta_x^2 - 1\right] > 2\rho_{xy} C_y\left[\delta_x - 1\right]$$

$$P < -\frac{2\rho_{xy} C_y}{\left[\delta_x + 1\right] C_x} \tag{3.170}$$

$$MSE\left(\bar{y}_{FTd}\right)_{II} - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II} > 0 \tag{3.171}$$

$$\bar{Y}^2\left[\theta_2 C_y^2+\theta_4 P^2 C_x^2+2\theta_2 P\rho_{xy}C_xC_y\right]-\bar{Y}^2\left[\theta_2 C_y^2+\theta_4\delta_x^2 P^2 C_x^2+2\theta_2\delta_x P\rho_{xy}C_xC_y\right]>0$$

$$P^2C_x^2+2\frac{\theta_2}{\theta_4}P\rho_{xy}C_xC_y-\delta_x^2P^2C_x^2-2\frac{\theta_2}{\theta_4}\delta_x P\rho_{xy}C_xC_y>0$$

$$-P\delta C_x\left[\delta_x^2-1\right]>2\frac{\theta_2}{\theta_4}\rho_{xy}C_y\left[\delta_x-1\right]$$

$$P<-\frac{2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\left[\delta_x+1\right]C_x} \tag{3.172}$$

Theorem 3.5.14: The modified estimator $\bar{y}_{FTAA}$ is more efficient than estimator $\bar{y}_{FTd}$ suggested by Shukla (2002) if $P<-\frac{2\rho_{xy}C_y}{\left[\delta_x+1\right]C_x}$ and $P<-\frac{2\frac{\theta_2}{\theta_4}\rho_{xy}C_y}{\left[\delta_x+1\right]C_x}$ when $S_2\subset S_1$ and $S_2\subset\Omega_N$ respectively.

3.5.3 Efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}$ over some related Estimators

1. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator $\bar{y}_{RP}^{(dc)}$ suggested by Choudhury and Singh (2012) are compared as,

$$MSE\left(\bar{y}_{RP}^{(dc)}\right)_{I\min}-MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{I\min}>0 \tag{3.173}$$

$$\bar{Y}^2\left[\theta_2C_y^2-\frac{\left(\theta_3C_x^2C_{yx}+\theta_1C_z^2C_{yz}\right)^2}{\left(\theta_3C_x^2+\theta_1C_z^2\right)}\right]-\bar{Y}^2\left[\theta_2C_y^2-\theta_3C_x^2C_{yx}^2-\theta_1C_z^2C_{yz}^2\right]>0$$

$$\theta_1\theta_3C_x^2C_z^2\left[C_{yx}^2+C_{yz}^2-2C_{yx}C_{yz}\right]>0$$

$$\left[C_{yx}-C_{yz}\right]^2>0$$

$$\rho_{yz}<\frac{\rho_{xy}C_z}{C_x} \tag{3.174}$$

$$MSE\left(\bar{y}_{RP}^{(dc)}\right)_{II\min} - MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{II\min} > 0 \tag{3.175}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 - \frac{\theta_2^2 C_{yx}^2 C_x^4}{\left(\theta_4 C_x^2 + \theta_1 C_z^2\left\{1-2C_{xz}\right\}\right)}\right] - \bar{Y}^2\left[\theta_2 C_y^2 - \frac{\theta_2^2 C_{yx}^2 C_x^4}{\left(\theta_4 C_x^2 - \theta_1 C_z^2 C_{xz}^2\right)}\right] > 0$$

$$\left(\theta_4 C_x^2 + \theta_1 C_z^2\left\{1-2C_{xz}\right\}\right) > \left(\theta_4 C_x^2 - \theta_1 C_z^2 C_{xz}^2\right)$$

$$\left[C_{xz} - 1\right]^2 > 0$$

$$\rho_{xz} > \frac{C_z}{C_x} \tag{3.176}$$

Theorem 3.5.15: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator $\bar{y}_{RP}^{(dc)}$ suggested by Choudhury and Singh (2012) if $\rho_{yz} < \frac{\rho_{xy} C_z}{C_x}$ and $\rho_{xz} > \frac{C_z}{C_x}$ when $S_2 \subset S_1$ and $S_2 \subset \Omega_N$ respectively.

2. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{31} suggested by Chand (1975) are compared as,

$$MSE\left(t_{31}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{3.177}$$

$$\bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 C_x^2\left(1-2C_{yx}\right) + \theta_1 C_z^2\left(1-2C_{yz}\right)\right] - \bar{Y}^2\left[\theta_2 C_y^2 + \theta_3 P C_x^2\left(P+2C_{yx}\right) + \theta_2\beta_1 C_z^2\delta_z\left(\beta_1\delta_z - 2C_{yz}\right)\right] > 0$$

$$\left[P + C_{yx}\right]^2 < \frac{\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[C_z^2 - 2\rho_{yz}C_y C_z - \beta_1 C_z^2\delta_z\left(\delta_z\beta_1 - 2C_{yz}\right)\right]}{\theta_3 C_x^2}$$

Set $\beta_1 = C_{yz}/\delta_z$ as obtained in section 3.3.3

$$\left[P + C_{yx}\right]^2 < \frac{\theta_3\left[\rho_{xy}C_y - C_x\right]^2 + \theta_1\left[\rho_{yz}C_y - C_z\right]^2}{\theta_3 C_x^2}$$

$$P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \tag{3.178}$$

Theorem 3.5.16: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator t_{31} modified suggested by Chand (1975) if $P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx}$ when $S_2 \subset S_1$.

3. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{32} suggested by Singh and Upadhayaya (2001) are compared as,

$$MSE\left(t_{32}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \tag{3.179}$$

$$\bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 C_x^2 \left(1 - 2C_{yx}\right) + \theta_1 \frac{\bar{Z}}{\bar{Z} + C_z} C_z^2 \left(\frac{\bar{Z}}{\bar{Z} + C_z} - 2C_{yz} \right) \right]$$
$$- \bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 P C_x^2 \left(P + 2C_{yx} \right) + \theta_2 \beta_1 C_z^2 \delta_z \left(\beta_1 \delta_z - 2C_{yz} \right) \right] > 0$$

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\left\{ \frac{\bar{Z}}{\bar{Z} + C_z} \right\}^2 C_z^2 - 2 \frac{\bar{Z}}{\bar{Z} + C_z} \rho_{yz} C_y C_z - \beta_1 C_z^2 \delta_z \left(\delta_z \beta_1 - 2C_{yz} \right) \right]}{\theta_3 C_x^2}$$

Set $\beta_1 = C_{yz} / \delta_z$ as obtained in section 3.3.3

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \frac{\bar{Z}}{\bar{Z} + C_z} C_z \right]^2}{\theta_3 C_x^2}$$

$$P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \frac{\bar{Z}}{\bar{Z} + C_z} C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \qquad (3.180)$$

Theorem 3.5.17: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator t_{32} suggested by Singh and Upadhayaya (2001) if

$$P < \left\{ \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \bar{Z} C_z / \left(\bar{Z} + C_z \right) \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \right\} \text{ when } S_2 \subset S_1 .$$

4. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{33} suggested by Singh et. al (2007) are compared as,

$$MSE\left(t_{33}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \qquad (3.181)$$

$$\bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 C_x^2 \left(1 - 2C_{yx}\right) + \theta_1 \frac{\bar{Z}}{\bar{Z} + \rho_{xz}} C_z^2 \left(\frac{\bar{Z}}{\bar{Z} + \rho_{xz}} - 2C_{yz} \right) \right]$$
$$- \bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 P C_x^2 \left(P + 2C_{yx} \right) + \theta_2 \beta_1 C_z^2 \delta_z \left(\beta_1 \delta_z - 2C_{yz} \right) \right] > 0$$

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\left\{ \frac{\bar{Z}}{\bar{Z} + \rho_{xz}} \right\}^2 C_z^2 - 2 \frac{\bar{Z}}{\bar{Z} + \rho_{xz}} \rho_{yz} C_y C_z - \beta_1 C_z^2 \delta_z \left(\delta_z \beta_1 - 2C_{yz} \right) \right]}{\theta_3 C_x^2}$$

Set $\beta_1 = C_{yz} / \delta_z$ as obtained in section 3.3.3

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \frac{\bar{Z}}{\bar{Z} + \rho_{xz}} C_z \right]^2}{\theta_3 C_x^2}$$

$$P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \frac{\bar{Z}}{\bar{Z} + \rho_{xz}} C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \qquad (3.182)$$

Theorem 3.5.18: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator t_{33} modified by Singh et. al (2007) if

$$P < \left\{ \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \bar{Z} C_z / \left(\bar{Z} + \rho_{xz} \right) \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \right\} \text{ when } S_2 \subset S_1 .$$

5. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{34} suggested by Upadhayaya and Singh (2001) are compared as,

$$MSE\left(t_{34}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \qquad (3.183)$$

$$\bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 C_x^2 \left(1 - 2C_{yx}\right) + \theta_1 \frac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} C_z^2 \left(\frac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} - 2C_{yz} \right) \right]$$
$$- \bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 P C_x^2 \left(P + 2C_{yx} \right) + \theta_2 \beta_1 C_z^2 \delta_z \left(\beta_1 \delta_z - 2C_{yz} \right) \right] > 0$$

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\left\{ \frac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} \right\}^2 C_z^2 - 2 \frac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} \rho_{yz} C_y C_z - \beta_1 C_z^2 \delta_z \left(\delta_z \beta_1 - 2C_{yz} \right) \right]}{\theta_3 C_x^2}$$

Set $\beta_1 = C_{yz} / \delta_z$ as obtained in section 3.3.3

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \frac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} C_z \right]^2}{\theta_3 C_x^2}$$

$$P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \dfrac{C_z \bar{Z}}{C_z \bar{Z} + \beta_{2(z)}} C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \qquad (3.184)$$

Theorem 3.5.19: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator t_{34} suggested by Upadhayaya and Singh (2001) if

$$P < \left\{ \left(\left[\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \bar{Z} C_z^2 / \left(C_z \bar{Z} + \beta_{2(z)} \right) \right]^2 \right] / \theta_3 C_x^2 \right)^{1/2} - C_{yx} \right\} \text{ when } S_2 \subset S_1$$

6. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ and estimator t_{35} suggested by Singh (2001) are compared as,

$$MSE\left(t_{35}\right)_I - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_I > 0 \qquad (3.185)$$

$$\bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 C_x^2 \left(1 - 2C_{yx}\right) + \theta_1 \frac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} C_z^2 \left(\frac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} - 2C_{yz} \right) \right]$$
$$- \bar{Y}^2 \left[\theta_2 C_y^2 + \theta_3 P C_x^2 \left(P + 2C_{yx} \right) + \theta_2 \beta_1 C_z^2 \delta_z \left(\beta_1 \delta_z - 2C_{yz} \right) \right] > 0$$

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\left\{ \dfrac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} \right\}^2 C_z^2 - 2 \dfrac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} \rho_{yz} C_y C_z - \beta_1 C_z^2 \delta_z \left(\delta_z \beta_1 - 2C_{yz} \right) \right]}{\theta_3 C_x^2}$$

Set $\beta_1 = C_{yz} / \delta_z$ as obtained in section 3.3.3

$$\left[P + C_{yx} \right]^2 < \frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \dfrac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} C_z \right]^2}{\theta_3 C_x^2}$$

$$P < \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \dfrac{\beta_{1(z)} \bar{Z}}{\beta_{1(z)} \bar{Z} + \sigma_z} C_z \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \tag{3.186}$$

Theorem 3.5.20: The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is more efficient than estimator t_{35} suggested by Singh (2001) if

$$P < \left\{ \left[\frac{\theta_3 \left[\rho_{xy} C_y - C_x \right]^2 + \theta_1 \left[\rho_{yz} C_y - \beta_{1(z)} \bar{Z} C_z / \left(\beta_{1(z)} \bar{Z} + \sigma_z \right) \right]^2}{\theta_3 C_x^2} \right]^{1/2} - C_{yx} \right\} \text{ when } S_2 \subset S_1 .$$

3.5.4 Efficiency of $\bar{y}_{FTAA}^{\beta_2(d)}$ over some related Estimators

1. The efficiency of modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ and modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ are compared as,

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)} \right)_{I\min} - MSE\left(\bar{y}_{FTAA}^{\beta_2(d)} \right)_{I\min} > 0 \tag{3.187}$$

$$\bar{Y}^2 C_y^2 \left\{ \left[\theta_2 - \theta_3 \rho_{xy}^2 - \theta_1 \rho_{yz}^2 \right] - \left[\theta_2 - \frac{\theta_3 \left[\rho_{xy}^2 + \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz} \right]}{1 - \rho_{xz}^2} \right] \right\} > 0$$

$$\theta_3 \left[\rho_{yz} + \rho_{xy}\rho_{xz} \right]^2 - \theta \left[1 - \rho_{xz}^2 \right] > 0$$

$$\frac{\rho_{xy}\rho_{xz}}{\rho_{yz}} > \left[\frac{\theta_1}{\theta_3} \left(1 - \rho_{xz}^2 \right) \right]^{1/2} - 1$$

$$\rho_{yz} > \rho_{xy}\rho_{xz} \Big/ \left\{ \left[\frac{\theta_1}{\theta_3} \left(1 - \rho_{xz}^2 \right) \right]^{1/2} - 1 \right\} \tag{3.188}$$

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)} \right)_{II\min} - MSE\left(\bar{y}_{FTAA}^{\beta_2(d)} \right)_{II\min} > 0 \tag{3.189}$$

$$\bar{Y}^2 C_y^2 \left[\left(1 - 2C_y \right) \theta_2 \rho_{xz}\rho_{yz} + \left(2 - \theta_2 \right) \rho_{xy} C_y + 2\left(\theta_2 - 1 \right) \rho_{xz}^2 \rho_{xy} C_y \right] > 0$$

$$\rho_{yz} > \frac{-\left[\left(2-\theta_2\right)\rho_{xy}C_y + 2\left(\theta_2-1\right)\rho_{xz}^2\rho_{xy}C_y\right]}{\left(1-2C_y\right)\theta_2\rho_{xz}}$$

$$\rho_{yz} > \frac{\rho_{xy}C_y\left[2\left(1-\rho_{xy}^2\right)-\theta_2\left(1-2\rho_{xy}^2\right)\right]}{\left(2C_y-1\right)\theta_2\rho_{xz}} \tag{3.190}$$

Theorem 3.5.21: The modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ is more efficient than modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ if $\rho_{yz} > \rho_{xy}\rho_{xz} \Big/ \left\{\left[\frac{\theta_1}{\theta_3}\left(1-\rho_{xz}^2\right)\right]^{1/2} - 1\right\}$ and $\rho_{yz} > \frac{\rho_{xy}C_y\left[2\left(1-\rho_{xy}^2\right)-\theta_2\left(1-2\rho_{xy}^2\right)\right]}{\left(2C_y-1\right)\theta_2\rho_{xz}}$

when $S_2 \subset S_1$ and $S_2 \subset \Omega_N$ respectively.

3.6 PROPERTIES OF THE MODIFIED FACTOR-TYPE ESTIMATORS UNDER NON-RESPONSE MODEL

Therefore, if $\bar{y}$ is replaced by $\bar{y}^*$ in the modified estimators then the modified estimators can be written under non-response model as;

$$\bar{y}_{FTAA}^{*} = \left(\frac{n_{S1}\bar{y}_{s1} + n_{s2}\bar{y}_{sh2}}{n}\right)\left(\frac{\left(\left(A+C\right)\bar{X} + fB\bar{x}\right)a_x + \left(A+C+fB\right)b_x}{\left(\left(A+fB\right)\bar{X} + C\bar{x}\right)a_x + \left(A+C+fB\right)b_x}\right) \tag{3.191}$$

$$\bar{y}_{FTAA}^{(d)*} = \left(\frac{n_{S1}\bar{y}_{s1} + n_{s2}\bar{y}_{sh2}}{n_2}\right)\left(\frac{\left(\left(A+C\right)\bar{x}_1 + fB\bar{x}_2\right)a_x + \left(A+C+fB\right)b_x}{\left(\left(A+fB\right)\bar{x}_1 + C\bar{x}_2\right)a_x + \left(A+C+fB\right)b_x}\right) \tag{3.192}$$

$$\bar{y}_{FTAA}^{\beta_1(d)*} = \left(\frac{n_{S1}\bar{y}_{s1} + n_{s2}\bar{y}_{sh2}}{n_2}\right)\left(\frac{\left(A+C\right)\bar{x}_1 + fB\bar{x}_2}{\left(A+fB\right)\bar{x}_1 + C\bar{x}_2}\right)\left(\frac{a_z\bar{Z} + b_z}{a_z\bar{z}_1 + b_z}\right)^{\beta_1} \tag{3.193}$$

$$\bar{y}_{FTAA}^{\beta_2(d)*} = \left(\frac{n_{S1}\bar{y}_{s1} + n_{s2}\bar{y}_{sh2}}{n_2}\right)\left(\frac{\left(A+C\right)\bar{x}_1 + fB\bar{x}_2}{\left(A+fB\right)\bar{x}_1 + C\bar{x}_2}\right)\left(\frac{a_z\bar{z}_1 + b_z}{a_z\bar{z}_2 + b_z}\right)^{\beta_2} \tag{3.194}$$

In order to get the MSEs of $\bar{y}^{*}_{FTAA}$, $\bar{y}^{(d)*}_{FTAA}$, $\bar{y}^{\beta_1(d)*}_{FTAA}$ and $\bar{y}^{\beta_2(d)*}_{FTAA}$, we defined an error term $\Delta^{*}_{\bar{y}} = \left(\bar{y}^{*} - \bar{Y}\right)/\bar{Y}$ such that $\left|\Delta^{*}_{\bar{y}}\right| < 1$ and $E\left(\Delta^{*}_{\bar{y}}\right) = 0$, $E\left(\Delta^{*2}_{\bar{y}}\right) = \frac{1}{\bar{Y}^2}\left(\left(\frac{1}{n} - \frac{1}{N}\right)S_y^2 + \frac{W_2(k-1)}{n}S^2_{y(s2)}\right)$, $E\left(\Delta_{\bar{x}}\Delta^{*}_{\bar{y}}\right) = E\left(\Delta^{*}_{\bar{y}}\Delta_{\bar{z}}\right) = 0$, then

$$MSE\left(\bar{y}^{*}_{FTAA}\right) = MSE\left(\bar{y}_{FTAA}\right) + \frac{W_2(k-1)}{n\bar{Y}^2}S^2_{y(s2)} \tag{3.195}$$

$$MSE\left(\bar{y}^{(d)*}_{FTAA}\right) = MSE\left(\bar{y}^{(d)}_{FTAA}\right) + \frac{W_2(k-1)}{n_2\bar{Y}^2}S^2_{y(s2)} \tag{3.196}$$

$$MSE\left(\bar{y}^{\beta_1(d)*}_{FTAA}\right) = MSE\left(\bar{y}^{\beta_1(d)}_{FTAA}\right) + \frac{W_2(k-1)}{n_2\bar{Y}^2}S^2_{y(s2)} \tag{3.197}$$

$$MSE\left(\bar{y}^{\beta_2(d)*}_{FTAA}\right) = MSE\left(\bar{y}^{\beta_2(d)}_{FTAA}\right) + \frac{W_2(k-1)}{n_2\bar{Y}^2}S^2_{y(s2)} \tag{3.198}$$

Where $W_2 = \frac{N}{N_{s2}}$ and $k = \frac{n_{s2}}{n_{h2}}$.

3.7 OPTIMUM SAMPLE SIZE

In this section, the functions of optimum sample sizes for which the modified estimators attain maximum precision for a given cost and minimum cost for a specified precision have been obtained using Lagrange multiplier.

3.7.4 Determination of n_1 and n_2 in $\bar{y}^{(d)}_{FTAA}$ for the Fixed Cost

Let c_0 be the overhead cost, c_1 be the cost of selecting and processing a single unit in the first phase sampling n_1 and c_2 be the cost of selecting and processing a single unit in the second phase sampling n_2. The expected total cost of the survey is given by

$$C = c_0 + c_1n_1 + c_2n_2 \tag{3.199}$$

Let us defined a function L_1 for minimizing the $MSE\left(\bar{y}_{FTAA}^{(d)}\right)$ and obtaining the optimum values of n_1 and n_2 for a fixed cost C which is given as

$$L_1 = MSE\left(\bar{y}_{FTAA}^{(d)}\right) - \lambda_1(C - c_0 - c_1 n_1 - c_2 n_2) \tag{3.200}$$

where λ_1 is the Lagrange multiplier.

Now, differentiating L_1 with respect to n_1 and n_2 and equate to zero, we have

$$n_1 = \bar{Y}\sqrt{\frac{-2E_2 - E_1}{\lambda_1 c_1}} \tag{3.201}$$

$$n_2 = \bar{Y}\sqrt{\frac{C_y^2 + E_1 + 2E_2}{\lambda_1 c_2}} \tag{3.202}$$

where $E_1 = \delta_x^2 P^2 C_x^2$, $E_2 = \delta_x P \rho_{xy} C_x C_y$

By substituting the values of n_1 and n_2 from (3.201) and (3.202) in (3.199), we have

$$\sqrt{\lambda_1} = \bar{Y}E_3 / \left(C - c_0\right) \tag{3.203}$$

where $E_3 = \sqrt{c_1\left[-2E_2 - E_1\right]} + \sqrt{c_2\left[C_y^2 + E_1 + 2E_2\right]}$

Substitute (3.203) back in (3.201) and (3.202), we have

$$n_1 = \frac{\left(C - c_0\right)}{E_3}\sqrt{\frac{-2E_2 - E_1}{c_1}} \tag{3.204}$$

$$n_2 = \frac{\left(C - c_0\right)}{E_3}\sqrt{\frac{C_y^2 + E_1 + 2E_2}{c_2}} \tag{3.205}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = -C_{yx} / \delta_x$ in equations (2.15) and (2.16) and we have

$$n_1^{opt} = \frac{(C - c_0)\rho_{xy}}{\left(c_1\rho_{xy} + \sqrt{c_1 c_2\left[1-\rho_{xy}^2\right]}\right)} \tag{3.206}$$

$$n_2^{opt} = \frac{(C - c_0)\sqrt{1-\rho_{xy}^2}}{\left(\sqrt{c_1 c_2}\,\rho_{xy} + c_2\sqrt{1-\rho_{xy}^2}\right)} \tag{3.207}$$

Hence, for the optimum values of n_1 and n_2, the minimum MSE of $\bar{y}_{FTAA}^{(d)}$ having neglecting the term of order $1/N$ is

$$MSE\left(\bar{y}_{FTAA}^{(d)}\right)_{\min} = \bar{Y}^2 C_y^2\left[\frac{1}{(C-c_0)}\left\{\sqrt{\left(1-\rho_{xy}^2\right)c_2} + \sqrt{c_1}\rho_{xy}\right\}^2\right] \tag{3.208}$$

3.7.5 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{\beta_1(d)}$ for the Fixed Cost

Let us defined a function L_2 for minimizing the $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)$ and obtaining the optimum values of n_1 and n_2 for a fixed cost C which is given as

$$L_2 = MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right) - \lambda_2(C - c_0 - c_1 n_1 - c_2 n_2) \tag{3.209}$$

where λ_2 is the Lagrange multiplier.

Now, differentiating L_2 with respect to n_1 and n_2 and equate to zero,

$$\frac{\partial}{\partial n_1}\left[MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right) - \lambda_2(C - c_0 - c_1 n_1 - c_2 n_2)\right] = 0 \tag{3.210}$$

$$\frac{\partial}{\partial n_2}\left[MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right) - \lambda_2(C - c_0 - c_1 n_1 - c_2 n_2)\right] = 0 \tag{3.211}$$

Solve for n_1 and n_2, we have

$$n_1 = \bar{Y}\sqrt{\frac{E_4 - E_5}{\lambda_2 c_1}} \tag{3.212}$$

$$n_2 = \bar{Y}\sqrt{\frac{C_y^2 + E_5}{\lambda_2 c_2}} \tag{3.213}$$

where $E_4 = \beta_1\delta_z C_z^2\left(\beta_1\delta_z - 2C_{yz}\right)$, $E_5 = C_x^2 P\left(P + 2C_{yx}\right)$

By substituting the values of n_1 and n_2 from (3.212) and (3.213) in (3.199), we have

$$\sqrt{\lambda_2} = \bar{Y}E_6 / \left(C - c_0\right) \tag{3.214}$$

where $E_6 = \sqrt{c_1\left[E_4 - E_5\right]} + \sqrt{c_2\left[C_y^2 + E_5\right]}$

Substituting (3.214) in (3.212) and (3.213), we have

$$n_1 = \frac{\left(C - c_0\right)}{E_6}\sqrt{\frac{E_4 - E_5}{c_1}} \tag{3.215}$$

$$n_2 = \frac{\left(C - c_0\right)}{E_6}\sqrt{\frac{C_y^2 + E_5}{c_2}} \tag{3.216}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = -C_{yx}$ and $\beta_1 = C_{yz} / \delta_z$ we have

$$n_1^{opt} = \frac{\left(C - c_0\right)\sqrt{\rho_{xy}^2 - \rho_{yz}^2}}{\left(c_1\sqrt{\rho_{xy}^2 - \rho_{yz}^2} + \sqrt{c_1 c_2\left(1 - \rho_{xy}^2\right)}\right)} \tag{3.217}$$

$$n_2^{opt} = \frac{\left(C - c_0\right)\sqrt{1 - \rho_{xy}^2}}{\left(\sqrt{c_1 c_2\left(\rho_{xy}^2 - \rho_{yz}^2\right)} + c_2\sqrt{1 - \rho_{xy}^2}\right)} \tag{3.218}$$

Hence, for the optimum values of n_1 and n_2, the minimum MSE of $\bar{y}_{FTAA}^{\beta_1(d)}$ having neglecting the term of order $1/N$ is

$$MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{\min} = \frac{\bar{Y}^2 C_y^2}{\left(C-c_0\right)}\left(\left(c_1+\sqrt{\frac{c_1c_2\left(1-\rho_{xy}^2\right)}{\rho_{xy}^2-\rho_{yz}^2}}\right)\left(1-\rho_{xy}^2\right)+\left(c_2+\sqrt{\frac{c_1c_2\left(\rho_{xy}^2-\rho_{yz}^2\right)}{1-\rho_{xy}^2}}\right)\left(1-\rho_{yz}^2\right)\right) \tag{3.219}$$

3.7.6 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{\beta_2(d)}$ for the Fixed Cost

Let us defined a function L_2 for minimizing the $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)$ and obtaining the optimum values of n_1 and n_2 for a fixed cost C which is given as

$$L_3 = MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)-\lambda_3(C-c_0-c_1n_1-c_2n_2) \tag{3.220}$$

where λ_3 is the Lagrange multiplier.

Now, differentiating L_3 with respect to n_1 and n_2 and equate to zero,

$$\frac{\partial}{\partial n_1}\left[MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)-\lambda_3(C-c_0-c_1n_1-c_2n_2)\right]=0 \tag{3.221}$$

$$\bar{Y}^2\left(n_1^{-2}C_x^2P\left(P-2C_{yx}\right)-n_1^{-2}\beta_2\delta_zC_z^2\left(\beta_2\delta_z-2C_{yz}-2PC_{xz}\right)\right)+\lambda_3c_1=0 \tag{3.222}$$

$$\frac{\partial}{\partial n_2}\left[MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)-\lambda_3(C-c_0-c_1n_1-c_2n_2)\right]=0 \tag{3.223}$$

$$\bar{Y}^2\left(-n_2^{-2}C_y^2-n_2^{-2}C_x^2P\left(P-2C_{yx}\right)-n_2^{-2}\beta_2\delta_zC_z^2\left(\beta_2\delta_z-2C_{yz}-2PC_{xz}\right)\right)+\lambda_3c_2=0 \tag{3.224}$$

Solve for n_1 and n_2, we have

$$n_1=\bar{Y}\sqrt{\frac{E_7-E_8}{\lambda_3c_1}} \tag{3.225}$$

$$n_2=\bar{Y}\sqrt{\frac{C_y^2+E_7+E_8}{\lambda_3c_2}} \tag{3.226}$$

where $E_7=\beta_2\delta_zC_z^2\left(\beta_2\delta_z-2C_{yz}-2PC_{xz}\right)$, $E_8=C_x^2P\left(P-2C_{yx}\right)$

By substituting the values of n_1 and n_2 from (3.225) and (3.226) in (3.199), we have

$$\sqrt{\lambda_3} = \bar{Y}E_9 / (C - c_0) \tag{3.227}$$

where $E_9 = \sqrt{c_1\left[E_7 - E_8\right]} + \sqrt{c_2\left[C_y^2 + E_7 + E_8\right]}$

Substituting (3.277) in (3.225) and (3.226), we have

$$n_1 = \frac{(C - c_0)}{E_9}\sqrt{\frac{E_7 - E_8}{c_1}} \tag{3.228}$$

$$n_2 = \frac{(C - c_0)}{E_9}\sqrt{\frac{C_y^2 + E_7 + E_8}{c_2}} \tag{3.229}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = \frac{C_y\left(\rho_{xz}\rho_{yz} + \rho_{xy}\right)}{C_x\left(1 - \rho_{xz}^2\right)}$

and $\beta_2 = \frac{C_y\left(\rho_{xy}\rho_{xz} + \rho_{yz}\right)}{C_z\left(1 - \rho_{xz}^2\right)\delta_z}$ we have

$$n_1^{opt} = \frac{(C - c_0)\sqrt{K_{xyz}Q_{xyz}}}{\left(c_1\sqrt{K_{xyz}Q_{xyz}} + \sqrt{c_1c_2\left(1 - K_{xyz}Q_{xyz}\right)}\right)} \tag{3.230}$$

$$n_2^{opt} = \frac{(C - c_0)\sqrt{1 - K_{xyz}Q_{xyz}}}{\left(\sqrt{c_1c_2K_{xyz}Q_{xyz}} + c_2\sqrt{\left(1 - K_{xyz}Q_{xyz}\right)}\right)} \tag{3.231}$$

where $K_{xyz} = \frac{\rho_{xz}\rho_{yz} + \rho_{xy}}{\left(1 - \rho_{xz}^2\right)^2}$, $Q_{xyz} = \rho_{xy} + \rho_{yz}\left(1 - \rho_{xz}\right) + \rho_{xy}\rho_{xz}\left(1 - 2\rho_{xz}\right)$

Hence, for the optimum values of n_1 and n_2, the minimum MSE of $\bar{y}_{FTAA}^{(d)}$ having neglecting the term of order $1/N$ is

$$MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{\min} = \frac{\bar{Y}^2 C_y^2}{\left(C-c_0\right)}\left(\left(c_1+\sqrt{\frac{c_1c_2\left(1-K_{xyz}Q_{xyz}\right)}{K_{xyz}Q_{xyz}}}\right)\right. \\ \left.+\left(c_2+\sqrt{\frac{c_1c_2K_{xyz}Q_{xyz}}{1-K_{xyz}Q_{xyz}}}\right)\left(1-\frac{\rho_{xy}^2+\rho_{yz}^2+2\rho_{xy}\rho_{xz}\rho_{yz}}{1-\rho_{xz}^2}\right)\right) \tag{3.232}$$

3.7.4 Determination of n_1 and n_2 in $\bar{y}_{FTAA}^{(d)}$ for Specified Precision

Let V be the specified variance fixed in advance. Then, for the estimator $\bar{y}_{FTAA}^{(d)}$ we have

$$V = MSE\left(\bar{y}_{FTAA}^{(d)}\right) \tag{3.233}$$

To minimize the cost C for the specified variance $MSE\left(\bar{y}_{FTAA}^{(d)}\right) = V$ and to obtain the optimum values of n_1 and n_2 we defined a function ψ_1 given as

$$\psi_1 = \left(c_0 + c_1n_1 + c_2n_2\right) - \mu_1\left[V - MSE\left(\bar{y}_{FTAA}^{(d)}\right)\right] \tag{3.234}$$

where μ_1 is the Lagrange multiplier.

Now, differentiating ψ_1 with respect to n_1 and n_2 and equate to zero, we have

$$n_1 = \bar{Y}\sqrt{\frac{-\mu_1\left(E_1+2E_2\right)}{c_1}} \tag{3.235}$$

$$n_2 = \bar{Y}\sqrt{\frac{\mu_1\left(C_y^2+E_1+2E_2\right)}{c_2}} \tag{3.236}$$

By substituting the values of n_1 and n_2 from (3.235) and (3.236) in (3.233), we have

$$\sqrt{\mu_1} = \bar{Y}E_{12} / \left(V + \bar{Y}^2C_y^2 / N\right) \tag{3.237}$$

where $E_{12} = \sqrt{\frac{c_2}{C_y^2+E_1+2E_2}}C_y^2 + \left(\sqrt{\frac{c_2}{C_y^2+E_1+2E_2}} - \sqrt{\frac{c_1}{-E_1-2E_2}}\right)\left(E_1+2E_2\right)$

Substituting (3.237) in (3.235) and (3.236), we have

$$n_1 = \bar{Y}^2 \frac{E_{12}}{V + \bar{Y}^2 C_y^2 / N} \sqrt{\frac{-E_1 - 2E_2}{c_1}} \tag{3.238}$$

$$n_2 = \bar{Y}^2 \frac{E_{12}}{V + \bar{Y}^2 C_y^2 / N} \sqrt{\frac{C_y^2 + E_1 + 2E_2}{c_2}} \tag{3.239}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = -C_{yx} / \delta_x$ in (3.238) and (3.239) and we have

$$n_1^{opt} = \bar{Y}^2 C_y^2 \frac{\sqrt{c_2\left[1-\rho_{xy}^2\right]} + \sqrt{c_1\rho_{xy}^2}}{V + \bar{Y}^2 C_y^2 / N} \frac{\rho_{xy}}{\sqrt{c_1}} \tag{3.240}$$

$$n_2^{opt} = \bar{Y}^2 C_y^2 \frac{\sqrt{c_2\left[1-\rho_{xy}^2\right]} + \sqrt{c_1\rho_{xy}^2}}{V + \bar{Y}^2 C_y^2 / N} \sqrt{\frac{\left[1-\rho_{xy}^2\right]}{c_2}} \tag{3.241}$$

Hence, for the optimum values of n_1 and n_2, the minimum total cost to be incurred on the use of $\bar{y}_{FTAA}^{(d)}$ for the specified precision having neglecting the term of order $1/N$ is

$$C\left(\bar{y}_{FTAA}^{(d)}\right)_{\min} = c_0 + \bar{Y}^2 C_y^2 \left\{\sqrt{c_2\left[1-\rho_{xy}^2\right]} + \sqrt{c_1\rho_{xy}^2}\right\}^2 / V \tag{`3.242}$$

3.7.5 Determination of n_1 and n_2 for $\bar{y}_{FTAA}^{\beta_1(d)}$ Specified Precision V

For the estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ we have

$$V = MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right) \tag{3.243}$$

To minimize the cost C for the specified variance $MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right) = V$ and to obtain the optimum values of n_1 and n_2 we defined a function ψ_1 given as

$$\psi_2 = (c_0 + c_1 n_1 + c_2 n_2) - \mu_2\left[V - MSE\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)\right] \tag{3.244}$$

where μ_1 is the Lagrange multiplier.

Now, differentiating ψ_1 with respect to n_1 and n_2 and equate to zero, we have

$$n_1 = \bar{Y}\sqrt{\frac{\mu_2\left(E_4 - E_5\right)}{c_1}} \tag{3.245}$$

$$n = \bar{Y}\sqrt{\frac{\mu_2\left(C_y^2 + E_5\right)}{c_2}} \tag{3.246}$$

By substituting the values of n_1 and n_2 from (3.245) and (3.246) in (3.243), we have

$$\sqrt{\mu_1} = \bar{Y}E_6 / \left(V + \bar{Y}^2 C_y^2 / N\right) \tag{3.247}$$

Substituting (3.247) in (3.245) and (3.246), we have

$$n_1 = \bar{Y}^2 \frac{E_6}{V + \bar{Y}^2 C_y^2 / N}\sqrt{\frac{-E_1 - 2E_2}{c_1}} \tag{3.248}$$

$$n_2 = \bar{Y}^2 \frac{E_6}{V + \bar{Y}^2 C_y^2 / N}\sqrt{\frac{C_y^2 + E_1 + 2E_2}{c_2}} \tag{3.249}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = -C_{yx}$ and $\beta_1 = C_{yz} / \delta_z$ in (3.248) and (3.249), we have

$$n_1^{opt} = \bar{Y}^2 C_y^2 \frac{\sqrt{c_1}\left(\rho_{xy}^2 - \rho_{yz}^2\right) + \sqrt{c_2\left(1 - \rho_{xy}^2\right)\left(\rho_{xy}^2 - \rho_{yz}^2\right)}}{\sqrt{c_1}\left(V + \bar{Y}^2 C_y^2 / N\right)} \tag{3.250}$$

$$n_2^{opt} = \bar{Y}^2 C_y^2 \frac{\sqrt{c_2}\left(1 - \rho_{xy}^2\right) + \sqrt{c_1\left(1 - \rho_{xy}^2\right)\left(\rho_{xy}^2 - \rho_{yz}^2\right)}}{\sqrt{c_2}\left(V + \bar{Y}^2 C_y^2 / N\right)} \tag{3.251}$$

Hence, for the optimum values of n_1 and n_2, the minimum total cost to be incurred on the use of $\bar{y}_{FTAA}^{\beta_1(d)}$ for the specified precision having neglecting the term of order $1/N$ under case I is

$$C\left(\bar{y}_{FTAA}^{\beta_1(d)}\right)_{\min} = c_0 + \bar{Y}^2 C_y^2 \left\{\sqrt{c_2\left[1-\rho_{xy}^2\right]} + \sqrt{c_1\left(\rho_{xy}^2 - \rho_{yz}^2\right)}\right\}^2 / V \tag{3.252}$$

3.7.6 Determination of n_1 and n_2 for $\bar{y}_{FTAA}^{\beta_2(d)}$ Specified Precision V

For the estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ we have

$$V = MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right) \tag{3.253}$$

To minimize the cost C for the specified variance $MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right) = V$ and to obtain the optimum values of n_1 and n_2 we defined a function ψ_1 given as

$$\psi_3 = (c_0 + c_1 n_1 + c_2 n_2) - \mu_3\left[V - MSE\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)\right] \tag{3.254}$$

where μ_3 is the Lagrange multiplier.

Now, differentiating ψ_3 with respect to n_1 and n_2 and equate to zero, we have

$$n_1 = \bar{Y}\sqrt{\frac{\mu_3\left(-E_7 - E_8\right)}{c_1}} \tag{3.255}$$

$$n_2 = \bar{Y}\sqrt{\frac{\mu_3\left(C_y^2 + E_7 + E_8\right)}{c_2}} \tag{3.256}$$

By substituting the values of n_1 and n_2 from (3.255) and (3.256) in (3.253), we have

$$\sqrt{\mu_3} = \bar{Y}E_{78} / \left(V + \bar{Y}^2 C_y^2 / N\right) \tag{3.257}$$

where $E_{78} = \sqrt{\frac{c_2}{C_y^2 + E_7 + E_8}}C_y^2 + \left(\sqrt{\frac{c_2}{C_y^2 + E_7 + E_8}} - \sqrt{\frac{c_1}{-E_7 - E_8}}\right)\left(E_7 + E_8\right)$

Substituting (3.257) in (3.255) and (3.256), we have

$$n_1 = \bar{Y}^2 \frac{E_{78}}{V + \bar{Y}^2 C_y^2 / N}\sqrt{\frac{-E_7 - E_8}{c_1}} \tag{3.258}$$

$$n_2 = \bar{Y}^2 \frac{E_{78}}{V + \bar{Y}^2 C_y^2 / N} \sqrt{\frac{C_y^2 + E_7 + E_8}{c_2}} \tag{3.259}$$

The optimum n_1 and n_2 are obtained by putting the optimum value of $P = \frac{C_y\left(\rho_{xz}\rho_{yz} + \rho_{xy}\right)}{C_x\left(1-\rho_{xz}^2\right)}$

and $\beta_2 = \frac{C_y\left(\rho_{xy}\rho_{xz} + \rho_{yz}\right)}{C_z\left(1-\rho_{xz}^2\right)\delta_z}$ in (3.258) and (3.259) and we have

$$n_1^{opt} = \bar{Y}^2 C_y^2 \frac{\sqrt{\frac{c_2 K_{xyz} Q_{xyz}}{c_1\left(1-K_{xyz}Q_{xyz}\right)}} + K_{xyz}Q_{xyz}\left(\sqrt{\frac{c_2 K_{xyz} Q_{xyz}}{c_1\left(1-K_{xyz}Q_{xyz}\right)}} - 1\right)}{V + \bar{Y}^2 C_y^2 / N} \tag{3.260}$$

$$n_2^{opt} = \bar{Y}^2 C_y^2 \frac{1 + K_{xyz}Q_{xyz}\left(1 - \sqrt{\frac{c_1\left(1-K_{xyz}Q_{xyz}\right)}{c_2 K_{xyz} Q_{xyz}}}\right)}{V + \bar{Y}^2 C_y^2 / N} \tag{3.261}$$

Hence, for the optimum values of n_1 and n_2, the minimum total cost to be incurred on the use of $\bar{y}_{FTAA}^{\beta_2(d)}$ for the specified precision having neglecting the term of order $1/N$ is

$$C\left(\bar{y}_{FTAA}^{\beta_2(d)}\right)_{\min} = c_0 + \left(\bar{Y}^2 C_y^2 c_2 + \left(1 + K_{xyz}Q_{xyz}\right)\sqrt{\frac{c_1 c_2 K_{xyz} Q_{xyz}}{1-K_{xyz}Q_{xyz}}} + \left(c_2 - c_1\right)K_{xyz}Q_{xyz} - K_{xyz}Q_{xyz}\sqrt{\frac{c_1 c_2\left(1-K_{xyz}Q_{xyz}\right)}{K_{xyz}Q_{xyz}}}\right)/V \tag{3.262}$$

CHAPTER FOUR
EMPIRICAL STUDY AND DISCUSSION

6.0 INTRODUCTION

This chapter presents empirical analyses on the efficiency and robustness of the modified estimators suggested in this research work and that of some existing related ratio, product and factor-type estimators using both real and simulated data.

4.1 EMPIRICAL STUDY USING REAL LIFE DATA

In this section, efficiency of these modified modified factor-type estimators over some existing related estimators were investigated using real life data to support the theoretical comparisons in Chapter 3.

Thirteen (13) real life data were used for the computation of MSEs and PREs of these modified factor-type estimators and some existing related estimators.

Data sets A: Data used for Bias, MSE and PRE

Data A1: Akkus (2016)
y: The amount of produced table olive (tons) , x: The number of fruit trees in that age

z: Collective areas of fruit (decar)
$N = 287;\ n_1 = 165;\ n_2 = 106;\ \bar{Y} = 1306.62;\ \bar{X} = 124081.969;\ \bar{Z}$=7672.369;
$C_y = 2.26099;\ C_x = 3.51001;\ C_z = 3.51944;\ \rho_{yx} = 0.8045;\ \rho_{yz} = 0.8188,\ \rho_{xz} = 0.965$
$\beta_1(x) = 9.3154,\ \beta_2(x) = 110.140;\ \beta_1(z) = 11.382;\ \beta_2(z) = 9.3154;$

Data A2: Anderson (1958)
Y: Head length of second son , **Z:** Head breadth of first son , **X:** Head length of first son
$N = 25;\ n_1 = 10;\ n_2$=7; $\bar{Y} = 183.34;\ \bar{X} = 185.72;\ \bar{Z}$=151.12;
$\rho_{xy} = 0.7108;\ \rho_{yz}$=0.6932; ρ_{xz}=0.7346; $C_y = 0.0546;\ C_x = 0.2422;$
C_z=0.0488; β_1(z)=0.002; β_2(z)=2.6519

Data A3: Handiquer et. al. (2011)
Y: forest timber volume in cubic meter (Cum) in 0.1 ha sample plot, X: average tree height in the sample plot in meter (m), Z: average crown diameter in the sample plot in meter (m)

$N = 2500;\ n_2 = 25;\ n_1$=200, $\bar{Y} = 4.63;\ \bar{X}$=21.09; $\bar{Z}$=13.55, $\rho_{xy} = 0.79;$
$C_y = 0.95;\ C_x = 0.98;\ C_z = 0.64;\ \rho_{xz} = 0.66;\ \rho_{yz} = 0.72$

Data A4: Johnson (1972)
y: Percentage of hives affected by disease, x: Date of flowering of a particular summer species, z: average January temperature.

$N = 10;\ n_2 = 4;\ n_1$=7, $\bar{Y} = 52;\ \bar{X}$=200; $\bar{Z}$=42, $\rho_{xy} = -0.94;$
$C_y^2 = 0.0244;\ C_x^2 = 0.0021;\ C_z^2 = 0.017;\ \rho_{xz} = -0.73;\ \rho_{yz} = 0.80$

Data A5: Singh (1967)
y: Number of female employed, x: Number of educated female, z: Number of female in service

$N = 61;\ n_2 = 20;\ n_1$=25, $\bar{Y} = 7.46;\ \bar{X}$=179.0; $\bar{Z}$=5.31, $\rho_{xy} = -0.207;$
$C_y^2 = 0.5046;\ C_x^2 = 0.0633;\ C_z^2 = 0.5737;\ \rho_{xz} = -0.0033;\ \rho_{yz} = 0.7737$

Data A6: Steel and Torrie (1960)
y: Log of leaf burn in sec, x: Chlorine percentage, z: Potassium percentage

$N = 30;\ \mathrm{n}_2 = 6;\ n_1$=14, $\bar{\mathrm{Y}} = 0.686;\ \bar{\mathrm{X}}$=0.8077; $\bar{\mathrm{Z}}$=4.654, $\rho_{xy} = -0.4996;$
$\mathrm{C}_y = 0.4803;\ \mathrm{C}_x = 0.7493;\ \mathrm{C}_z = 0.2295;\ \rho_{xz} = 0.4074;\ \rho_{yz} = 0.1794$

Data sets B: Data used for cost analysis

Data B1: Singh (2003)
$\bar{Y} = 555.43, \bar{X} = 878.16, C_x^2 = 1.5256, C_y^2 = 1.1086, \rho_{xy} = 0.8038$
$N = 50, C = \$5000, c_0 = \$2,000, c_1 = \$50, c_2 = \$500, V = 27765.22$

Data sets C: Data used for Non-response model

Data C1: Khare and Rehman (2015)

Y: Number of Agricultural labor, X: Area of the Village (hectares), Z: Number of cultivators in the village.

$\bar{Y} = 137.9271,\ \bar{X} = 144.8720,\ \bar{Z} = 185.188,\ S_y = 182.5012, C_x = 0.8115$
$S_{y(2)} = 287.4204,\ \rho_{xy} = 0.773,\ \rho_{yz} = 0.786,\ \rho_{xz} = 0.819,\ W_2 = 0.25$
$N = 96,\ n_2 = 24,\ n_1 = 60$

Data C2: Singh and Kumar (2011)

Y: Weight (kg) of the children, X: Skull circumference (cm) of the children, Z: Chest circumference (cm) of the children

$\bar{Y} = 19.4968,\ \bar{X} = 51.1726,\ \bar{Z} = 55.1726,\ C_y = 0.15613, C_x = 0.03006$
$C_{y(2)} = 287.4204,\ \rho_{xy} = 0.328,\ \rho_{yz} = 0.846,\ \rho_{xz} = 0.297,\ W_2 = 0.25$
$N = 95,\ n_2 = 24,\ n_1 = 35$

Dataset C3: Sanaullah et al. (2015)

Y: Food expenditure, X: Household earning, Z: Total expenditure in month of May

$\bar{Y} = 47.9805,\ \bar{X} = 18746.55,\ \bar{Z} = 19124.75,\ S_y = 21.4256, S_x = 16625.33$
$S_{y(2)} = 20.4752,\ \rho_{xy} = -0.4777,\ \rho_{yz} = -0.4422,\ \rho_{xz} = 0.9138,\ W_2 = 0.10$
$N = 6940,\ n_2 = 750,\ n_1 = 1874$

Dataset C4: Sanaullah et al. (2015)

Y: Food expenditure, X: House earn , Z: Total expenditure in month of May

$\bar{Y} = 48.0556,\ \bar{X} = 14303.98,\ \bar{Z} = 14742.47,\ S_y = 22.1319, S_x = 12861.4$

$S_{y(2)} = 21.7407,\ \rho_{xy} = -0.4406,\ \rho_{yz} = -0.3547,\ \rho_{xz} = 0.8035,\ W_2 = 0.10$

$N = 1678,\ n_2 = 181,\ n_1 = 453$

Data sets D: Data used for Super-population model

Data D1: Singh (2003)

Y: Apples' Season average price (in $) in 1996**, X:** Apples' Season average price (in $) in 1995**, Z:** Apples' Season average price (in $) in 1994

$N = 36, n_1 = 20, n_2 = 16, \bar{X}^{(1)} = 0.1856, \bar{X}^{(2)} = 0.03996, \bar{X}^{(3)} = 0.00987, \bar{Z} = 0.1709,\ \bar{x}_s^{(1)} = 0.1967,$

$\bar{x}_s^{(2)} = 0.04466, \bar{x}_s^{(3)} = 0.01153, \sigma_x = 0.0752, C_x = 40.5, \beta_{1(x)} = 0.81, \beta_{2(x)} = 0.59, \sigma_z = 0.0634,$

$C_z = 37.06, \beta_{1(z)} = 0.62, \beta_{2(z)} = 0.1, \bar{z}_1 = 0.1852, \bar{z}_2 = 0.153, \bar{y} = 0.2263, B_0 = 0.1309, B_1 = -1.196;$

$B_2 = 16.29, B_3 = -53.8, \sigma_e = 0.03978, \bar{x}_{\bar{s}} = 0.1718, \rho_{xy} = 0.878,\ \rho_{yz} = 0.858,\ \rho_{xz} = 0.879$

Data D2: Singh (2003)

Y: Expectation of life at birth, X: Infant mortality rate, Z: Crude death rate

$N = 96, n_1 = 40, n_2 = 36, \bar{X}^{(1)} = 50.14, \bar{X}^{(2)} = 4067, \bar{X}^{(3)} = 390935, \bar{Z} = 10.875,\ \bar{x}_s^{(1)} = 47.81,$

$\bar{x}_s^{(2)} = 38.41, \bar{x}_s^{(3)} = 364360, \sigma_x = 39.61, C_x = 79.01, \beta_{1(x)} = 0.52, \beta_{2(x)} = -1.01, \sigma_z = 5.183,$

$C_z = 47.67, \beta_{1(z)} = 1.06, \beta_{2(z)} = 0.75, \bar{z}_1 = 10.89, \bar{z}_2 = 10.845, \bar{y} = 64.4, B_0 = 79.212, B_1 = -0.3145;$

$B_2 = 0.000629, B_3 = -0.00001512, \sigma_e = 5.36149, \bar{x}_{\bar{s}} = 51.36, \rho_{xy} = -0.91,\ \rho_{yz} = -0.828,\ \rho_{xz} = 0.661$

4.1.1 Efficiency Comparison of $\bar{y}_{FTAA}$ with some existing related Estimators

In this subsection, relative efficiency of $\bar{y}_{FTAA}$ with respect to some existing related ratio, product and factor-type estimators were investigated.

4.1.1.1 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related ratio estimators were estimated under the condition that the study and auxiliary variables are positively correlated.

Table 4.1A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators when $\rho_{xy} > 0$

Estimators	Data A1			Data A2			Data A3		
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE

Sample mean $\bar{y}$	0	5145375	100	0	51873.87	100	0	0.766	100
Cochran (1942) $\bar{y}_r$	459.3	4693439	109.6	65.9	745485.3	6.96	0.041	0.333	230.3
Srivenkataramana (1980) $\bar{y}_{st}$	1075.7	10445262	49.3	591.6	8182732	0.634	0.002	0.746	102.7
Sisodia and Diwivedi (1981) t_2	459.3	4693101	109.6	65.9	745482	6.958	0.032	0.317	241.4
Singh and Tailor (2003) t_3	459.3	4693362	109.63	65.9	745475.5	6.958	0.034	0.320	239.4
Singh et. al. (2004) t_4	458.1	4682852	109.9	65.89	745443.9	6.958	0.017	0.298	257.3
Upadhyaya and Singh (1999) t_5	459.0	4690419	109.7	65.89	745314.2	6.960	0.017	0.297	257.6
Upadhyaya and Singh (1999) t_6	459.3	4693436	109.6	65.90	745484.2	6.96	0.038	0.327	234.2
Modified $\bar{y}_{FTAA}$	163	1815184	283.5	13.0	25665.29	202.1	0.02	0.288	266.0

Table 4.1B: Efficiency conditions of $\bar{y}_{FTAA}$ over Ratio estimators in Table 4.1A

Estimators	Data A1	Data A2	Data A3
Sample mean $\bar{y}$	P>-1.536544	P>-3.222019	P>-1.531633
Cochran (1942) $\bar{y}_r$	P<-0.03644864	P<0.6831826	P< -0.5316327
Srivenkataramana (1980) $\bar{y}_{st}$	P< 0.316044	P< -1.507119	P< -1.460969
Sisodia and Diwivedi (1981) t_2	P< -0.03647693	P<0.6818731	P< -0.5760368
Singh and Tailor (2003) t_3	P< -0.03645513	P<0.6793494	P<-0.5677387
Singh et. al. (2004) t_4	P< -0.03733558	P<0.6672004	P<-0.6561657
Upadhyaya and Singh (1999) t_5	P< -0.03670149	P<0.6203217	P< -0.658385
Upadhyaya and Singh (1999) t_6	P<-1.091108	P<2.690023	P<1.335155
Modified $\bar{y}_{FTAA}$	P= -1.518272	P= -1.611009	P= -1.4658163

COMMENT: Table 4.1A show the biases, MSEs and percentage relative efficiency (PRE) of $\bar{y}_{FTAA}$ and other existing related ratio estimators under simple random sampling scheme when the study and auxiliary variables are positively correlated using Data A1, A2 and A3

respectively. The results showed that all the estimators with the exception of sample mean estimator are biased. The above results revealed that $\bar{y}_{FTAA}$ has minimum MSE and high PRE.

Table 4.1B shows the efficiency conditions of $\bar{y}_{FTAA}$ over other existing related ratio estimators using Data A1, A2 and A3 and the results revealed that the conditions are satisfied.

4.1.1.2 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related product estimators were estimated under the condition that the study and auxiliary variables are negatively correlated.

Table 4.2A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some related Product estimators when $\rho_{xy} < 0$

Estimators	Data A4			Data A5			Data A6		
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Sample mean $\bar{y}$	0	9.8966	100	0	0.9437	100	0	0.0145	100
Product	-0.036	5.2901	187.08	0.0066	0.9237	102.17	0.0349	0.0271	53.333
Pandey and Dubey (1988) t_8	-0.036	5.2910	187.05	0.0066	0.9236	102.18	0.0053	0.0123	118.16
Singh and Tailor (2003) t_9	-0.0362	5.2724	187.71	0.0066	0.9239	102.15	0.3098	0.1974	73.31
Singh (2003) t_{10}	-0.035	5.4401	181.92	0.0028	0.9089	103.83	0.0077	0.0132	109.34
Singh et. al. (2004) t_{11}	-0.0358	5.3457	185.13	0.0062	0.9221	102.34	-0.001	0.0113	128.39
Upadhyaya and Singh (1999) t_{12}	-0.030	6.2678	157.90	0.0053	0.9181	102.80	-0.001	0.0117	123.93
Singh (2003) t_{13}	0.0000	9.8966	100.0	0.000	0.9437	100.0	0.000	0.0145	100.0
Singh (2003) t_{14}	0.0000	9.8966	100.0	0.000	0.9437	100.0	0.00 0	0.0145	100.0
Upadhyaya and Singh (1999) t_{15}	-0.036	5.2904	187.07	0.0066	0.9237	102.17	0.0174	0.0178	81.35
Singh (2003) t_{16}	0.0411	5.3411	185.29	0.9169	0.9169	102.92	0.0192	0.0192	75.56
Modified $\bar{y}_{FTAA}$	0.003	1.1520	859.11	0.006	0.9033	104.48	0.014	0.0109	133.26

Table 4.2B: Efficiency conditions of $\bar{y}_{FTAA}$ over Product estimators in Table 4.2A

Estimators	Data A4	Data A5	Data A6
Sample mean $\bar{y}$	P<1.822449	P<4.013265	P<6.404855
Product	P<1.000234	P<1.0000021	P<1.00001
Pandey and Dubey (1988) t_8	P<0.9951239	P< 0.994555	P<0.518754
Singh and Tailor (2003) t_9	P<1.004722	P<1.001158	P<2.621551
Singh (2003) t_{10}	P<0.9746821	P<0.9782017	P<0.1890007
Singh et. al. (2004) t_{11}	P<0.9852217	P<0.9835165	P<0.2121228
Upadhyaya and Singh (1999) t_{12}	P<0.9849246	P<0.9831857	P<1.1678709
Singh (2003) t_{13}	P<1.0004190	P<1.000310	P<1.0000420
Singh (2003) t_{14}	P<1.000231	P<1.00032	P<1.000110
Upadhyaya and Singh (1999) t_{15}	P<0.9983693	P< 0.9981784	P<0.7638066
Singh (2003) t_{16}	P<0.9914158	P<0.9926267	P<0.411467
Modified $\bar{y}_{FTAA}$	P=0. 112245	P=0.2006633	P=0.202427

COMMENT: Table 4.2A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and other existing related product estimators under simple random sampling scheme when the study and auxiliary variables are negatively correlated using Data A4, A5 and A6 respectively. The results showed that all the estimators with the exception of sample mean estimator are biased. The above results revealed that $\bar{y}_{FTAA}$ has minimum MSE and high PRE.

Table 4.2B shows the efficiency conditions of $\bar{y}_{FTAA}$ over other existing related product estimators using Data A4, A5 and A6 and the results revealed that the conditions are satisfied.

4.1.1.3 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related factor-type estimators were

estimated under the condition that the study and auxiliary variables are both positively and negatively correlated.

Table 4.3A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} > 0$

Estimators	Values of d	Data A1			Data A2			Data A3		
		Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	-384.6	1978608	260.1	-0.180	5.1012	202.1	**0**	0.3600	240.8
	5	-108.0	3475775	148.0	-0.214	5.6240	183.3	-0.03	0.4881	157
	10	-173.4	2238983	229.8	-0.617	8.8811	116.1	0.033	0.4091	187.3
	15	-86.07	2896460	177.6	-0.680	9.1840	112.2	0.059	0.4218	181.6
	20	-54.51	3342618	153.9	-0.704	9.2637	111.3	0.070	0.4300	178.2
	30	-23.35	3800992	135.4	-0.723	9.8079	105.1	0.079	0.4382	174.8
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
	5	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
	10	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
	15	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
	20	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
	30	-38.69	3214611	160.1	0.044	6.9301	148.7	0.443	0.4426	173.1
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
	5	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
	10	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
	15	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
	20	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
	30	-12.52	2099293	245.1	0.177	22.483	45.9	-0.02	0.3288	233.0
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	-29.86	4413841	116.6	0.013	16.00	64.40	0.001	0.7723	99.2
	5	24.575	4858597	105.9	0.077	18.094	56.97	0.049	1.1543	66.37
	10	114.6	4534395	113.5	0.697	23.198	44.43	0.220	2.2547	33.98
	15	605.06	3763082	136.7	0.898	33.528	30.74	0.260	2.4911	30.75
	20	682.60	3681954	139.7	0.983	37.910	27.19	0.275	2.5820	29.67
	30	754.29	3611680	142.5	1.059	41.786	24.66	0.288	2.6578	28.83
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	-60.87	3057449	168.3	-0.008	6.3607	162.1	-1.00	0.2630	291.3
	5	-17.43	4674058	110.1	-0.006	7.4425	138.5	-1.01	0.2854	268.4
	10	-28.70	4343616	118.5	-0.007	7.3254	140.7	-1.01	0.2912	263.1
	15	-25.49	4440419	115.9	-0.007	8.0927	127.4	-1.01	0.3196	239.7
	20	-20.61	4583373	112.3	-0.006	8.6825	118.7	-1.01	0.3170	241.7
	30	-14.69	4750791	108.3	-0.005	9.4157	109.5	-1.00	0.3258	235.2
Modified $\bar{y}_{FTAA}$	3	-25.0	1815184	283.5	-0.027	4.1169	250.4	-0.00	0.2614	293.1
	5	-13.0	1815184	283.5	0.018	4.3577	236.5	-0.02	0.2755	278.1
	10	-39.46	1815184	283.5	0.043	4.8423	212.9	-0.02	0.2689	284.9
	15	17.75	1815184	283.5	0.058	6.5908	156.4	-0.01	0.2547	300.9
	20	18.29	1815184	283.5	0.064	7.6620	134.5	-0.01	0.2514	304.8
	30	20.93	1815184	283.5	0.069	7.6964	134.0	-0.01	0.2495	307.0

Table 4.3B: Efficiency conditions of $\bar{y}_{FTAA}$ over Factor-type estimators in Table 4.3A

Estimators	Data A1	Data A2	Data A3
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	P>-5.182228	P>-4.1602382	P>-2.7051143
Jain and Shukla (2013) $\bar{y}_{FT1}$	P<-0.3130701	P<-0.3130699	P<-0.3130699
Jain and Shukla (2013) $\bar{y}_{FT2}$	P<-0.5214422	P<-0.5214419	P<-0.5214419
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	P<-4.462442	P<-2.307429	P<-1.02776
Shukla et. al. (2013) $\bar{y}_{TFT}$	P<-0.6909638	P<-0.2136509	P<-1.021088
Modified $\bar{y}_{FTAA}$	P=-4.51822	P=-3.1602382	P=-1.7658163

COMMENT: Table 4.3A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and other existing related factor-type estimators under simple random sampling scheme when the study and auxiliary variables are positively correlated for some arbitrary values of $d=3, d=5, d=10, d=15, d=20$ and $d=30$ using Data A1, A2 and A3 respectively. The results of the analysis revealed that $\bar{y}_{FTAA}$ has minimum MSE and high PRE.

Table 4.3B shows the efficiency conditions of $\bar{y}_{FTAA}$ over other existing related factor-type estimators using Data A1, A2 and A3 and the results revealed that the conditions are satisfied.

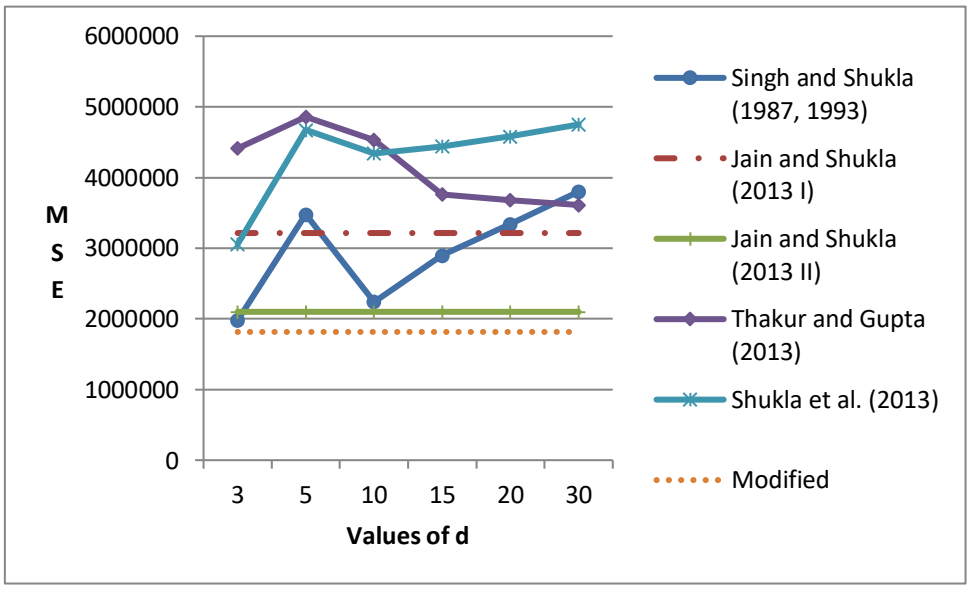


Figure 4.1.1A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A1

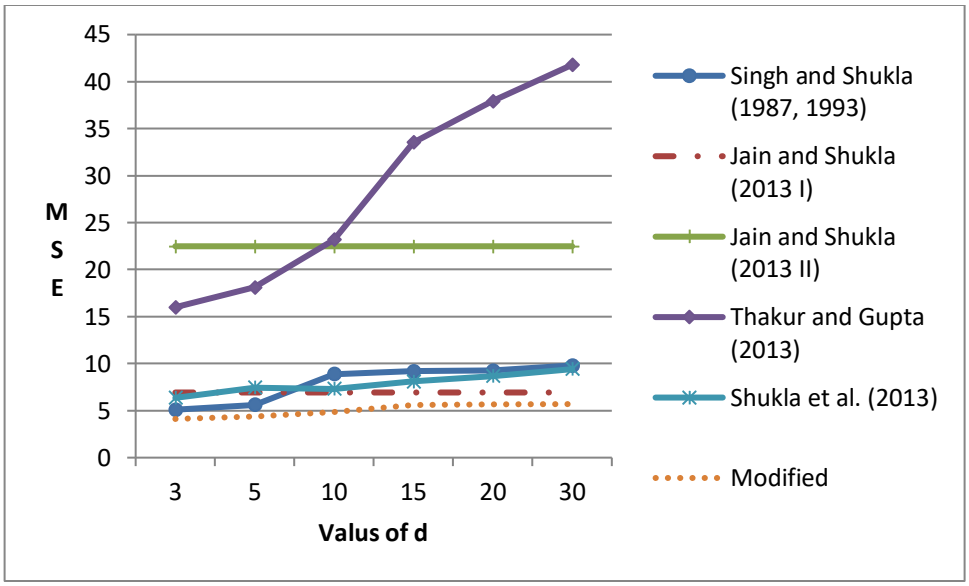


Figure 4.1.2A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A2

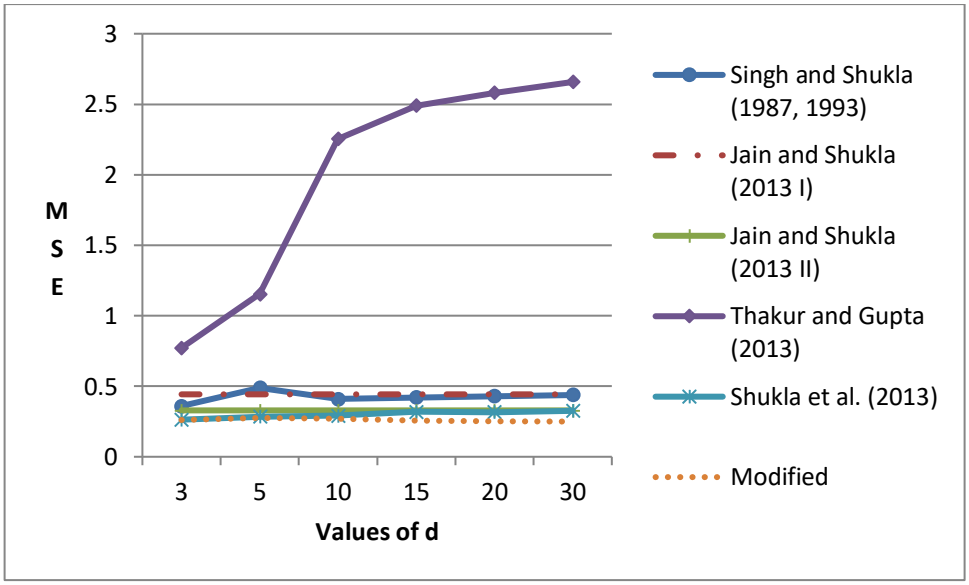


Figure 4.1.3A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A3

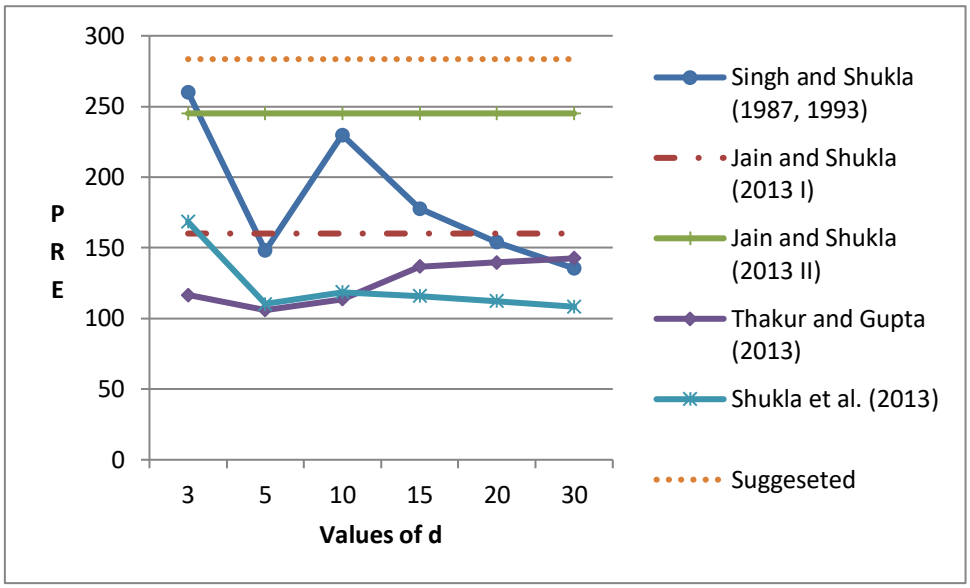


Figure 4.1.1B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A1

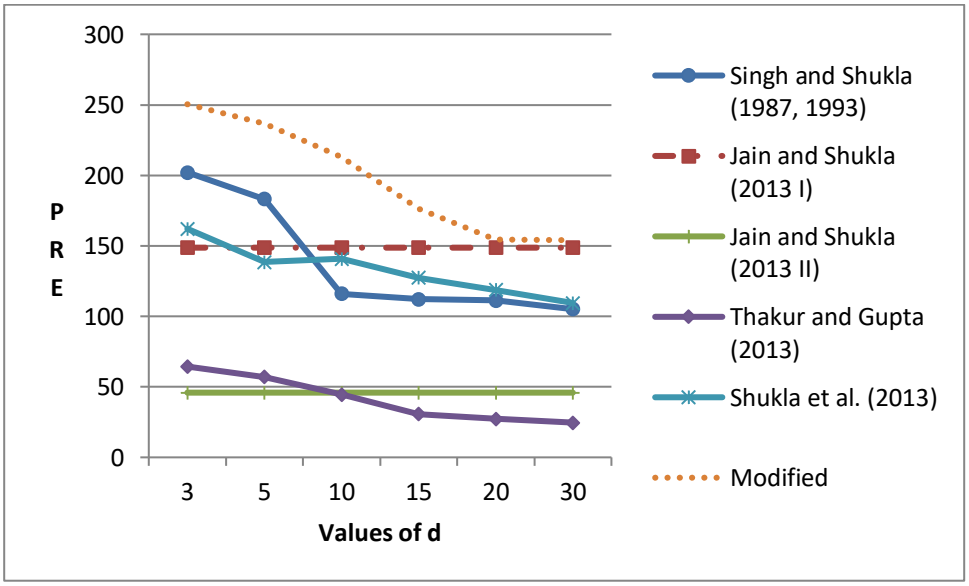


Figure 4.1.2B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A2

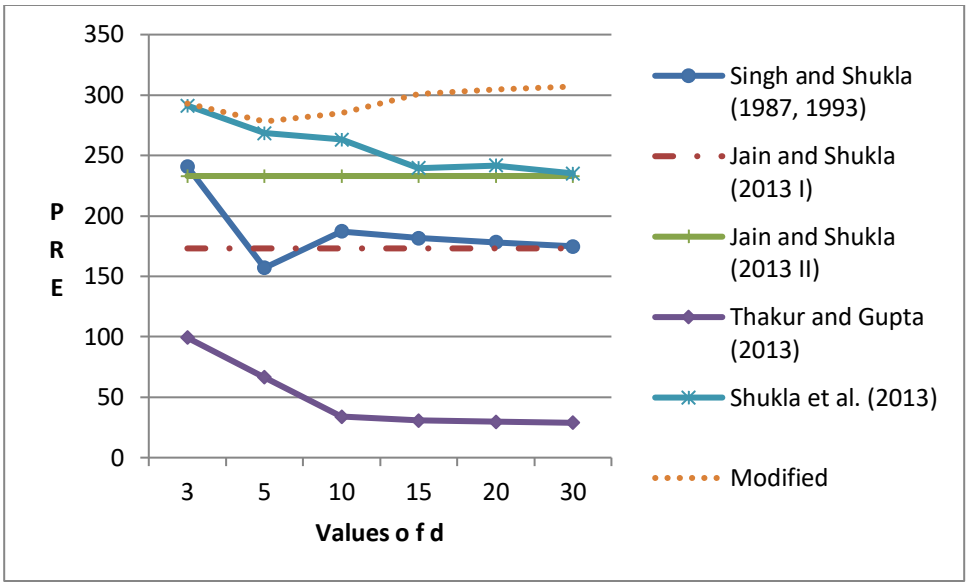


Figure 4.1.3B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A3

Comment: Figures (4.1.1A – 4.1.3A), and (4.1.1B – 4.1.3B) show the behavioural pattern of MSEs and PRE of estimators considered in Table 4.3A respectively. The plots generally exhibit upward and downward movement respectively as the values of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

Table 4.4A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} < 0$

Estimators	Values of d	Data A4			Data A5			Data A6		
		Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	-0.0041	7.31	135.3	-0.003	0.675	139.7	-0.005	0.0124	116.6
	5	-0.0080	6.94	142.5	-0.004	0.676	139.5	-0.014	0.0114	127.5
	10	-0.0694	9.25	107.0	-0.022	0.809	116.7	-0.059	0.0125	115.9
	15	-0.0902	9.07	109.1	-0.027	0.846	111.5	-0.070	0.0137	106.0
	20	-0.0994	9.41	105.1	-0.029	0.863	109.4	-0.074	0.0131	110.4
	30	-0.1076	9.71	101.9	-0.031	0.877	107.6	-0.078	0.0122	118.4
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
	5	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
	10	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
	15	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
	20	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
	30	0.0129	7.16	138.1	0.0034	0.684	137.9	0.0091	0.0124	116.6
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
	5	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
	10	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
	15	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
	20	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
	30	0.0253	8.32	119.0	0.0076	0.727	129.8	0.0207	0.0133	108.6
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	-0.0026	6.63	149.4	-0.001	0.635	148.6	-0.001	0.0121	119.3
	5	-0.0012	6.69	147.8	0.0002	0.635	148.7	0.0008	0.0110	131.0
	10	0.0015	7.13	138.8	0.0066	0.616	153.1	0.0232	0.0105	137.7
	15	0.0031	7.51	131.9	0.0091	0.613	153.9	0.0308	0.0113	128.4
	20	0.0039	7.45	132.8	0.0101	0.662	142.5	0.0340	0.0117	124.2
	30	0.0046	7.41	133.6	0.0111	0.661	142.7	0.0369	0.0120	120.6
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	0.0068	6.59	150.1	0.0011	0.658	143.3	0.0011	0.0108	133.6
	5	0.0025	6.15	160.8	0.0005	0.651	145.0	0.0011	0.0109	133.1
	10	0.0043	7.34	134.9	0.0009	0.656	143.9	0.0018	0.0116	124.4
	15	0.0032	7.23	136.9	0.0006	0.653	144.6	0.0012	0.0110	131.5
	20	0.0025	7.15	138.3	0.0005	0.651	145.1	0.0009	0.0106	136.1
	30	0.0017	7.57	130.7	0.0003	0.648	145.6	0.0006	0.0102	141.3
Modified Estimator $\bar{y}_{FTAA}$	3	0.0129	5.79	170.8	0.0017	0.473	199.7	0.0015	0.0098	148.1
	5	0.0104	5.43	182.3	0.0024	0.474	199.3	0.0050	0.0097	149.1
	10	0.0459	6.68	148.1	0.0139	0.592	159.5	0.0300	0.0094	154.0
	15	0.0550	6.49	152.4	0.0173	0.595	158.6	0.0370	0.0099	145.6
	20	0.0588	6.83	144.9	0.0188	0.609	154.9	0.0399	0.0104	139.2
	30	0.0622	6.92	142.9	0.0200	0.622	151.8	0.0425	0.0098	147.1

Table 4.4B: Efficiency conditions of $\bar{y}_{FTAA}$ over Factor-type estimators in Table 4A

Estimators	Data A4	Data A5	Data A6
$\bar{y}_{FT}$	P<2.890903	P<0.5443429	P<0.3202427
$\bar{y}_{FT1}$	P<-0.3130699	P<-0.3130699	P<-0.3130699
$\bar{y}_{FT2}$	P<-0.5214419	P<-0.5214419	P<-0.5214419
$\bar{y}_{FTRP}$	P<41.773	P< 8.415978	P<4.611495
$\bar{y}_{TFT}$	P<3.867871	P< 0.7792572	P<0.4269903
$\bar{y}_{FTAA}$	P=-2.900903	P=-0.5844429	P<-0.602327

COMMENT: Table 4.4A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and some existing related factor-type estimators under simple random sampling scheme when the study and auxiliary variables are negatively correlated for some arbitrary values of $d=3, d=5, d=10, d=15, d=20$ and $d=30$ using Data A4, A5 and A6 respectively. The results of the analysis revealed that $\bar{y}_{FTAA}$ has minimum MSE and high PRE.

Table 4.4B shows the efficiency conditions of $\bar{y}_{FTAA}$ over other existing related factor-type estimators using Data A4, A5 and A6 and the results revealed that the conditions are satisfied.

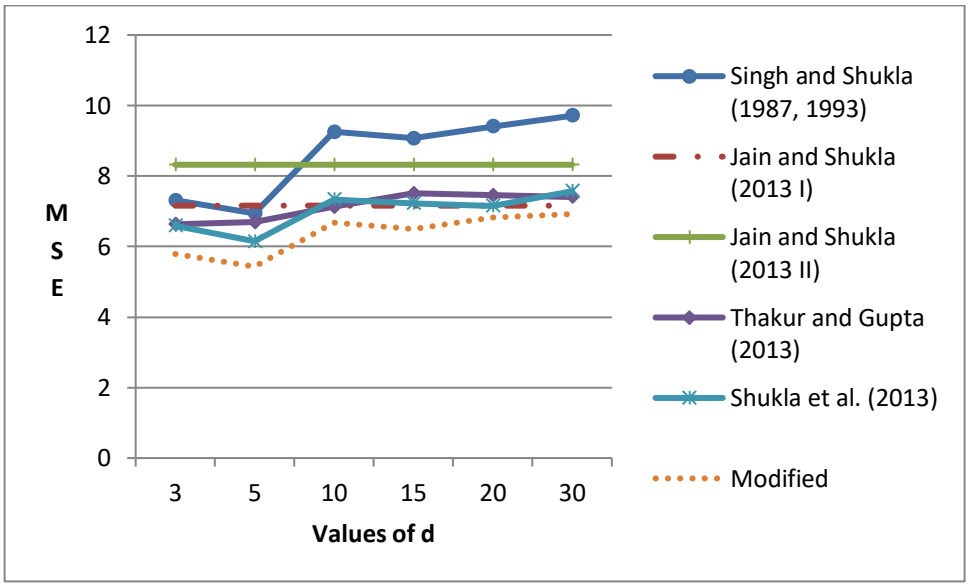


Figure 4.1.4A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A4

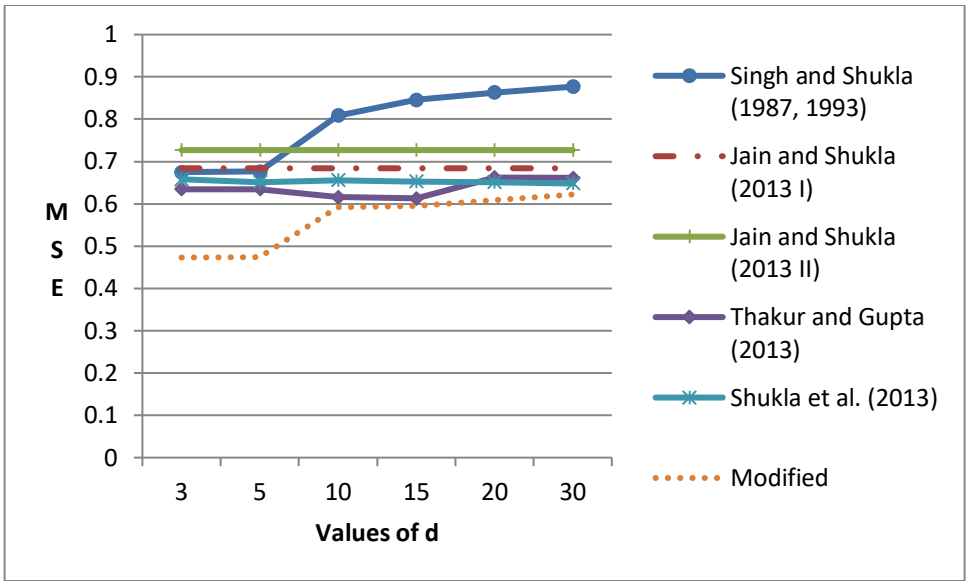


Figure 4.1.5A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A5

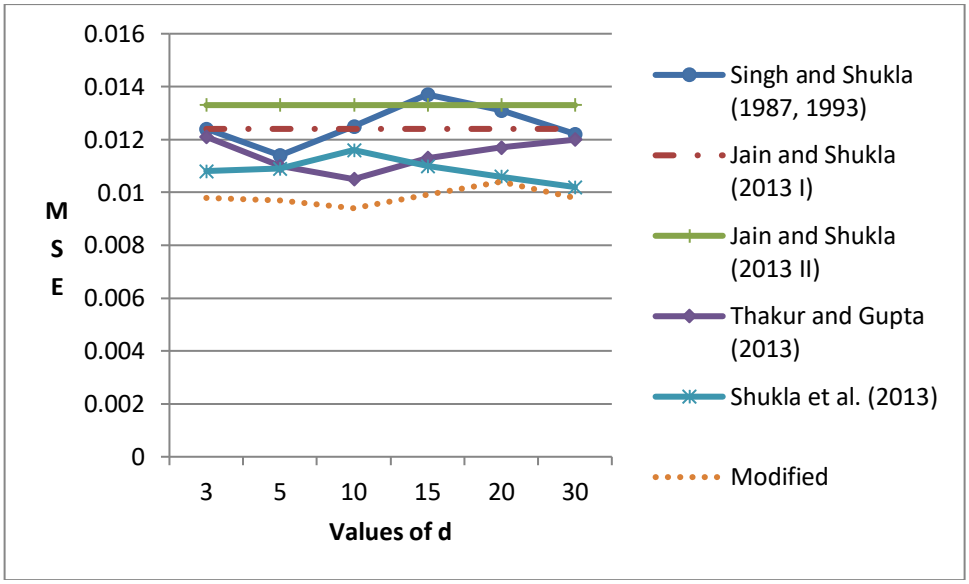


Figure 4.1.6A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A6

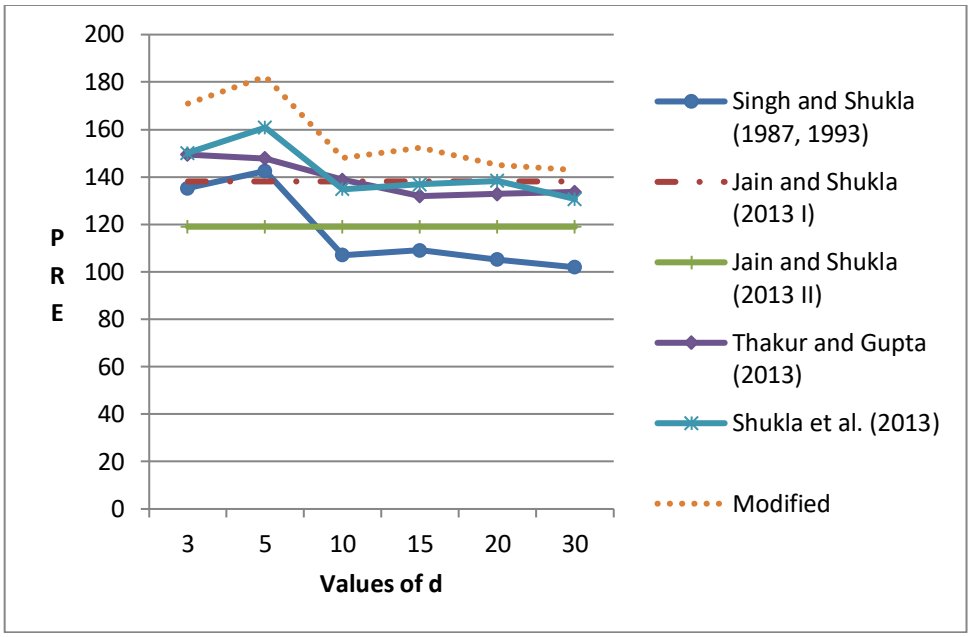


Figure 4.1.4B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A4

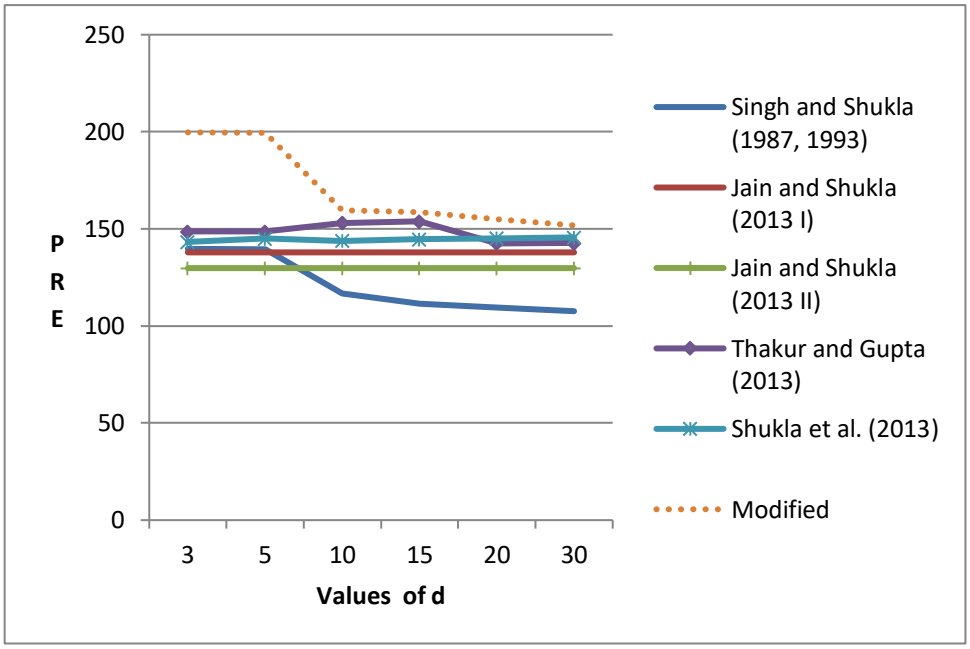


Figure 4.1.5B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A5

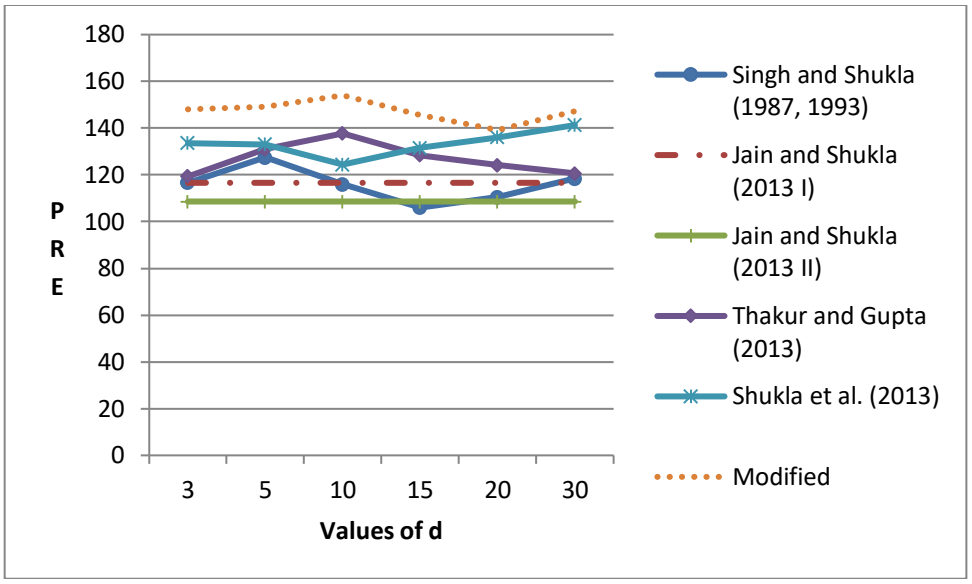


Figure 4.1.6B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators using Data A6

Comment: Figures (4.1.4A – 4.1.6A), and (4.1.4B – 4.1.6B) show the behavioural pattern of MSEs and PRE of estimators considered in Table 4.4B respectively. The plots generally exhibit upward and downward movement respectively as the values of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

4.1.2 Efficiency Comparison of $\bar{y}_{FTAA}^{(d)}$ with some existing related Estimators

In this subsection, relative efficiency of estimator $\bar{y}_{FTAA}^{(d)}$ with respect to some existing related ratio, product and factor-type estimators were investigated.

4.1.2.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some existing related Ratio Estimators

Bias, MSE and PRE of estimator $\bar{y}_{FTAA}^{(d)}$ and some related ratio estimators were estimated under the condition that the study and auxiliary variables are positively correlated.

Table 4.5A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related Ratio estimators when $\rho_{xy} > 0$

Estimators	Data A1			Data A2			Data A3		
	Case I								
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Sample mean $\bar{y}$	0	5145375	100	0	51873.87	100	0	0.7661	100
Sukhatme (1962) t_1	260.4	4889134	105.2	27.46	340878.6	15.21	0.0364	0.3830	200.1
Srivenkataramana (1988) two-phase	349.1	19463315	26.44	43.18	613478.9	8.456	0.3940	2.5904	29.57
Malik and Tailor (2013)	260.4	4889090	105.2	27.459	340874.5	15.22	0.0297	0.3718	206.1
Modified $\bar{y}_{FTAA}^{(d)}$	127.7	4287813	120.2	18.34	40953.63	126.7	0.004	0.3435	223.0
	Case II								
Sample mean $\bar{y}$	0	5145375	100	0	51873.87	100	0	0.7661	100
Sukhatme (1962) t_1	872.2	10063010	51.13	111.68	1340912	38.68	0.0617	0.4274	179.2
Srivenkataramana (1988) two-phase	1860.3	35767723	14.39	136.83	1995153	2.600	0.3313	2.9248	26.19
Malik and Tailor (2013)	872.1	10062863	51.13	111.68	1340896	38.69	0.0526	0.4080	187.8
Modified $\bar{y}_{FTAA}^{(d)}$	439.2	2821470	182.3	54.98	35321.08	146.9	0.0137	0.3378	226.8

Table 4.5B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Ratio estimators in Table 4.5A

Estimators	Data A1	Data A2	Data A3
CASE I			
Sample mean $\bar{y}$	P>-2.036446	P>-2.3204763	P>-2.531633
Sukhatme (1962) t_1	P<-0.03644519	P<0.6795237	P<-0.5316327
Srivenkataramana (1988) two-phase	P<0.3160141	P<-1.499048	P<-1.460969
Malik and Tailor (2013)	P< -0.03645168	P<0.675711	P<-0.5677387
Modified $\bar{y}_{FTAA}^{(d)}$	P<-1.5182229	P= -1.602382	P=-1.7658163
CASE II			
Sample mean $\bar{y}$	P<-1.036446	P<-0.3204763	P<-1.531633
Sukhatme (1962) t_1	P<0.1739838	P< 0.7975939	P<-0.372232
Srivenkataramana (1988) two-phase	P<1.472835	P< 0.4642606	P<-1.285275
Malik and Tailor (2013)	P<0.1739773	P<- 0.7937812	P<-0.408338
Modified $\bar{y}_{FTAA}^{(d)}$	P= -1.4130085	P=-1.101203	P=-1.686116

Table 4.5A show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and other related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables

are positively correlated using Data A1, A2 and A3 respectively. These properties (Bias, MSE and PRE) were computed under cases I and II respectively. The results of the analysis revealed that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE.

Table 4.5B shows the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over other existing related ratio estimators using Data A1, A2 and A3 and the results revealed that the conditions are satisfied.

4.1.2.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some Related Product Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related product estimators were estimated under the condition that the study and auxiliary variables are negatively correlated.

Table 4.6A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related Ratio estimators when $\rho_{xy} < 0$

Estimators	Data A4			Data A5			Data A6		
	Case I								
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Sample mean $\bar{y}$	0	9.8966	100	0	0.9437	100	0	0.0145	100
Product $\bar{y}_p$	0.049	14.403	68.71	0.0075	1.0201	92.51	0.0484	0.0558	126.0
Srivenkataramana (1988) two-phase	-0.063	6.606	149.8	-0.001	0.9378	100.6	0.0132	0.0235	61.54
Malik and Tailor (2013)	0.050	14.428	68.59	0.0075	1.0203	92.50	0.2829	0.2297	6.302
Modified $\bar{y}_{FTAA}^{(d)}$	0.034	3.6504	271.1	0.0028	0.9317	101.3	0.029	0.0119	121.7
	Case II								
Sample mean $\bar{y}$	0	9.8966	100	0	0.9437	100	0	0.0145	100
Product $\bar{y}_p$	0.074	16.450	60.16	0.036	1.2836	73.52	0.0824	0.0823	17.58
Srivenkataramana (1988) two-phase	-0.03	5.5334	178.9	0.0177	1.0069	93.73	0.0496	0.0372	38.91
Malik and Tailor (2013)	0.074	16.486	60.03	0.0364	1.2843	73.48	0.4969	0.3849	37.61
Modified $\bar{y}_{FTAA}^{(d)}$	0.009	3.0952	319.7	0.005	0.9200	102.6	0.0256	0.0117	124.1

Table 4.6B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Product estimators in Table 4.6A

Estimators	Data A4	Data A5	Data A6
CASE I			
Sample mean $\bar{y}$	P<1.822449	P<0.4013265	P<0.6404855
Sukhatme (1962) t_1	P<2.822449	P<1.401327	P<1.640485
Srivenkataramana (1988) two-phase	P<1.7579329	P<1.185212	P<-0.5817368
Malik and Tailor (2013)	P<2.827171	P<1.402484	P<3.262037
Modified $\bar{y}_{FTAA}^{(d)}$	P=0.9112245	P=0.2006633	P=0.3202427
CASE II			
Sample mean $\bar{y}$	P<1.822449	P<0.4013265	P<0.6404855
Sukhatme (1962) t_1	P<2.41746	P<1.235736	P<1.498155
Srivenkataramana (1988) two-phase	P< 3.750794	P<0.9301807	P<1.373155
Malik and Tailor (2013)	P< 2.422183	P<1.236894	P< 3.119707
Modified $\bar{y}_{FTAA}^{(d)}$	P= 0.7087302	P=0.1178681	P=0.2490777

Comment: Table 4.6A show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and other related product estimators under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated using Data A4, A5 and A6 respectively. The results of the analysis revealed that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE.

Table 4.6B shows the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over other existing related ratio estimators using Data A4, A5 and A6 and the results revealed that the conditions are satisfied.

4.1.2.4 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$

Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ were estimated under the condition that the study and auxiliary variables are both positively and negatively correlated.

Table 4.7A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} > 0$

Estimators	Values of d	Data A1			Data A2			Data A3		
		Case I								
		Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Shukla (2002) $\bar{y}_{FTd}$	3	-17.95	4638975	110.9	0.0002	8.1384	126.7	-0.001	0.5206	147.2
	5	-36.12	3796482	135.5	0.0049	8.3562	123.4	-0.021	0.4204	182.2
	10	106.87	3716254	138.5	0.0093	8.7966	117.2	0.005	0.4446	172.3
	15	169.70	4159331	123.7	0.0843	9.0062	114.5	0.018	0.4557	168.1
	20	196.88	4369052	117.8	0.1173	9.4561	109.0	0.024	0.4630	165.5
	30	220.97	4561779	112.8	0.1471	9.3912	109.8	0.029	0.4702	162.9
Modified Estimator $\bar{y}_{FTAA}^{(d)}$	3	-17.95	4639093	110.9	-0.001	7.1449	144.3	-0.00	0.4720	162.3
	5	-36.12	3796571	135.5	0.0032	7.2452	142.3	-0.018	0.3711	206.5
	10	106.83	3715989	138.5	-0.022	7.7805	132.5	-0.015	0.3820	200.6
	15	169.65	4158910	123.7	0.0420	8.0923	127.4	-0.01	0.3894	196.7
	20	196.82	4368564	117.8	0.0702	8.7052	118.4	-0.007	0.3865	198.2
	30	220.90	4561231	112.8	0.0957	8.8863	116.0	-0.005	0.3849	199.1
	Case II									
Shukla (2002) $\bar{y}_{FTd}$	3	-29.34	4282358	120.2	0.0176	8.2367	125.1	-0.001	0.5599	136.8
	5	-39.67	3078888	167.1	0.0406	8.8952	115.9	-0.022	0.4950	154.8
	10	35.64	5839452	88.11	0.0042	9.1220	113.0	0.019	0.5503	139.2
	15	66.41	7530757	68.32	0.3007	8.8057	117.1	0.037	0.5778	132.6
	20	93.55	8280435	62.14	0.4309	9.2181	111.8	0.045	0.5920	129.4
	30	71.51	8951687	57.48	0.5479	9.4291	109.3	0.051	0.6051	126.6
Modified $\bar{y}_{FTAA}^{(d)}$	3	-29.34	4282431	120.2	0.0132	7.8175	131.9	-0.001	0.5114	149.8
	5	-39.68	3078986	167.1	0.0325	7.8978	130.5	-0.019	0.4494	170.5
	10	35.53	5838373	88.13	0.0097	8.0353	128.3	-0.01	0.4536	168.9
	15	66.25	7529229	68.34	0.2638	8.1993	125.7	-0.001	0.4679	163.8
	20	93.37	8278716	62.15	0.3754	8.3086	124.1	0.002	0.4680	163.7
	30	61.31	8949800	57.49	0.4757	8.5498	120.6	0.005	0.4689	163.4

Table 4.7B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimator in Table 4.7A

Estimators	Data A1	Data A2	Data A3
Case I			
Shukla (2002) $\bar{y}_{FTd}$	P<-0.5182473	P<-0.1602382	P<-0.7658163
Modified $\bar{y}_{FTAA}^{(d)}$	P=-0.533272	P=-0.1912312	P=-0.7848123
Case II			
Shukla (2002) $\bar{y}_{FTd}$	P<-0.04172721	P<-0.101203	P<-0.686116
Modified $\bar{y}_{FTAA}^{(d)}$	P=-0.04172919	P=-0.301001	P=-0.886315

Table 4.7A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated

for some arbitrary values of $d=3, d=5, d=10, d=15, d=20$ and $d=30$ using Data A1, A2 and A3 respectively. The results of the analysis revealed that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE.

Table 4.7B shows the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimators using Data A1, A2 and A3 and the results revealed that the conditions are satisfied.

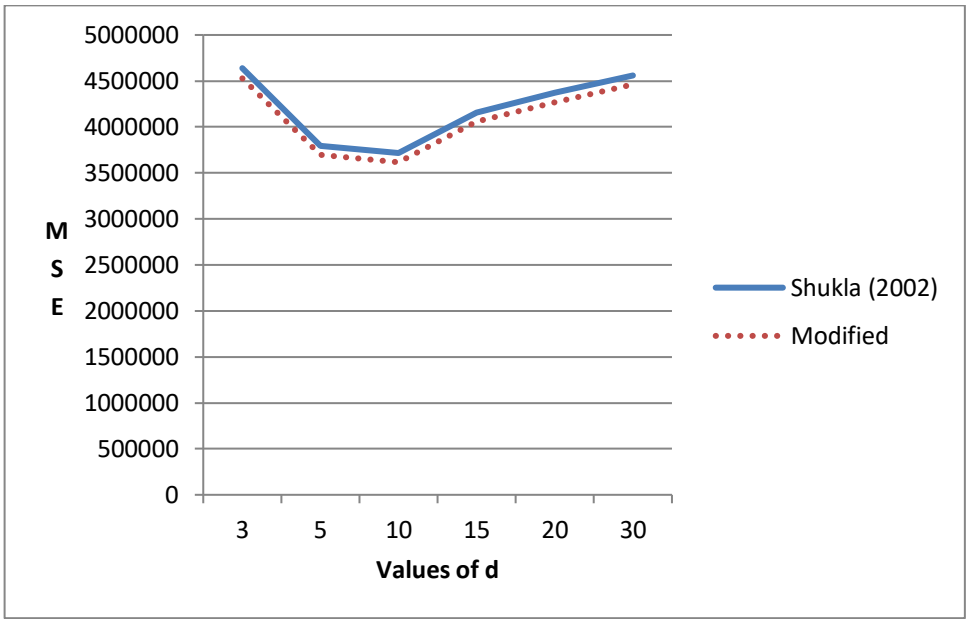


Figure 4.1.7A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A1

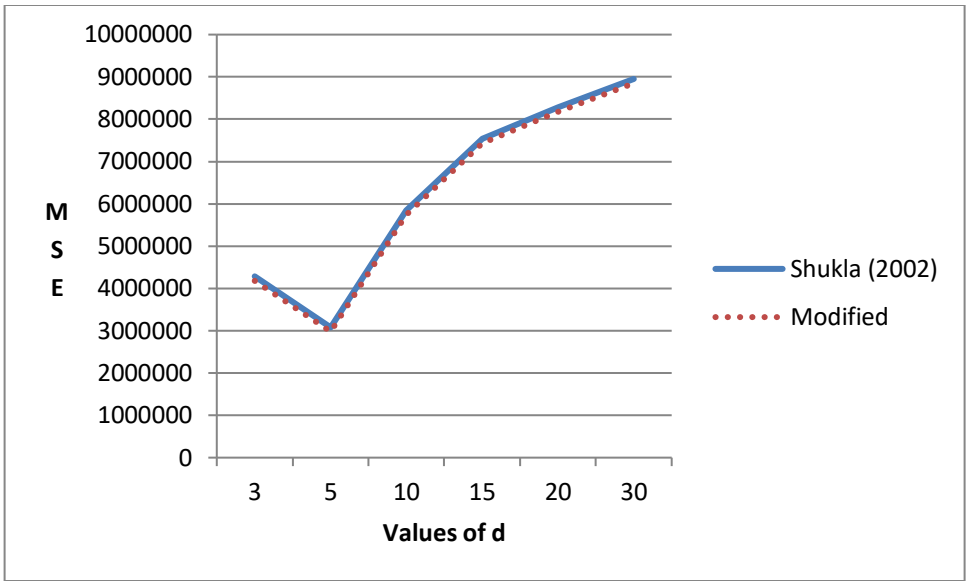


Figure 4.1.8A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A1

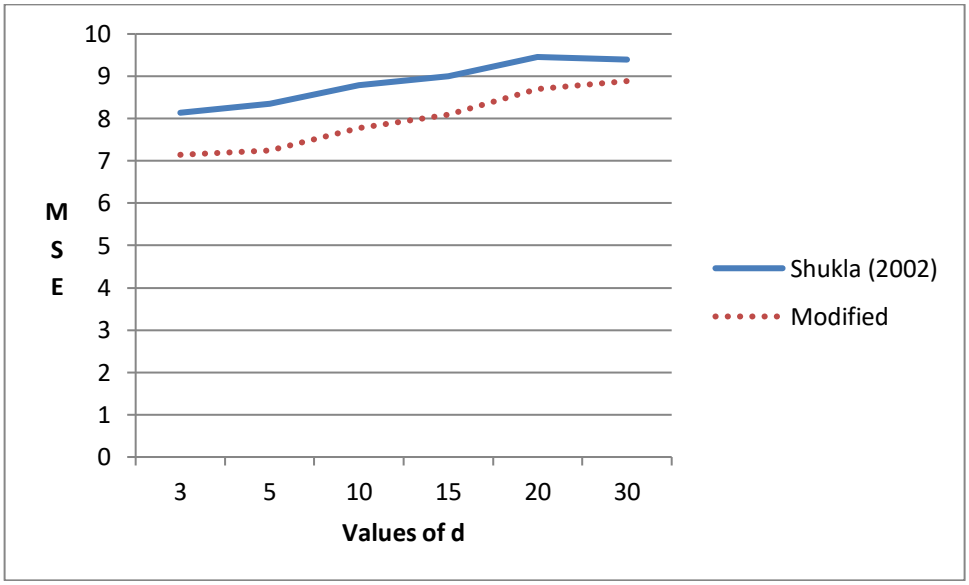


Figure 4.1.9A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A2

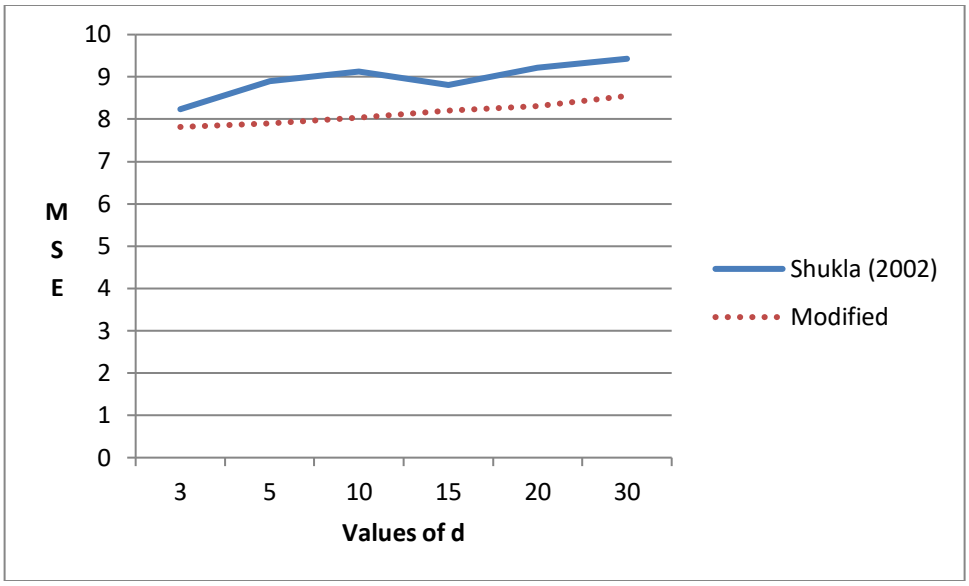


Figure 4.1.10A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A2

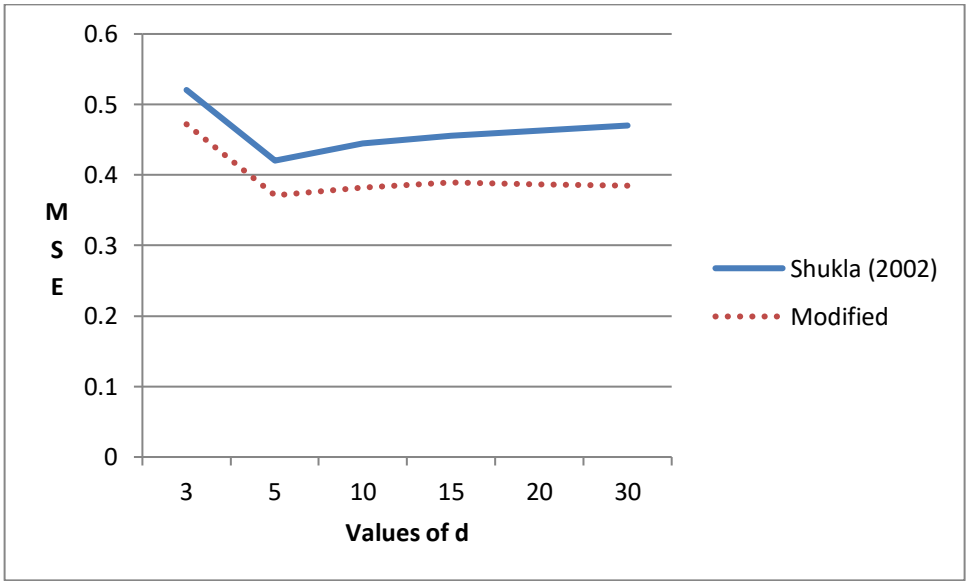


Figure 4.1.11A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A3

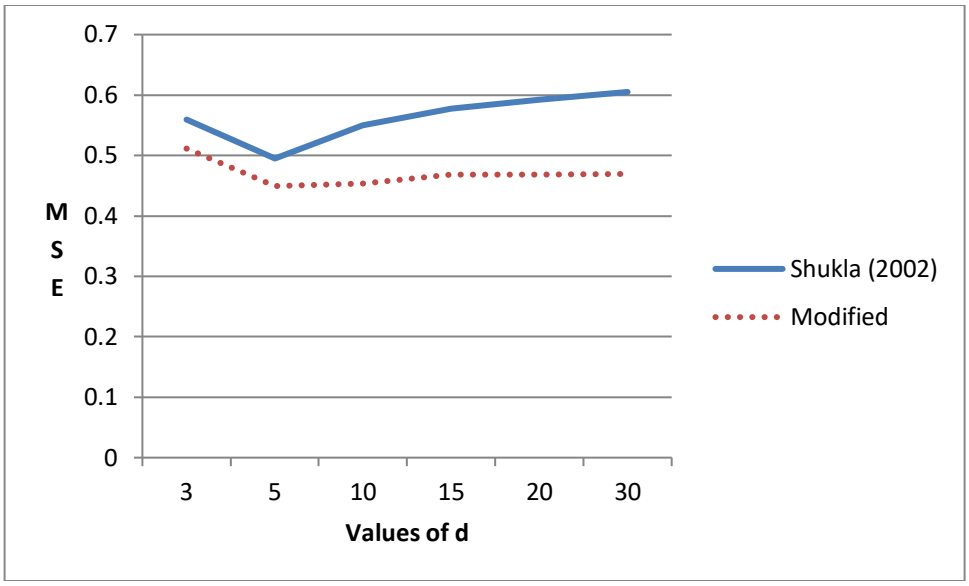


Figure 4.1.12A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A3

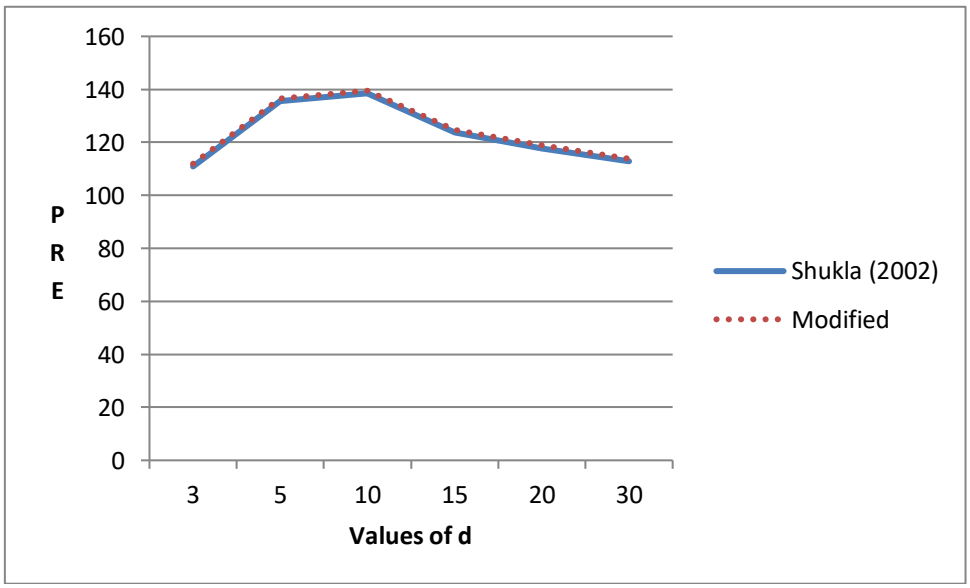


Figure 4.1.7B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 1

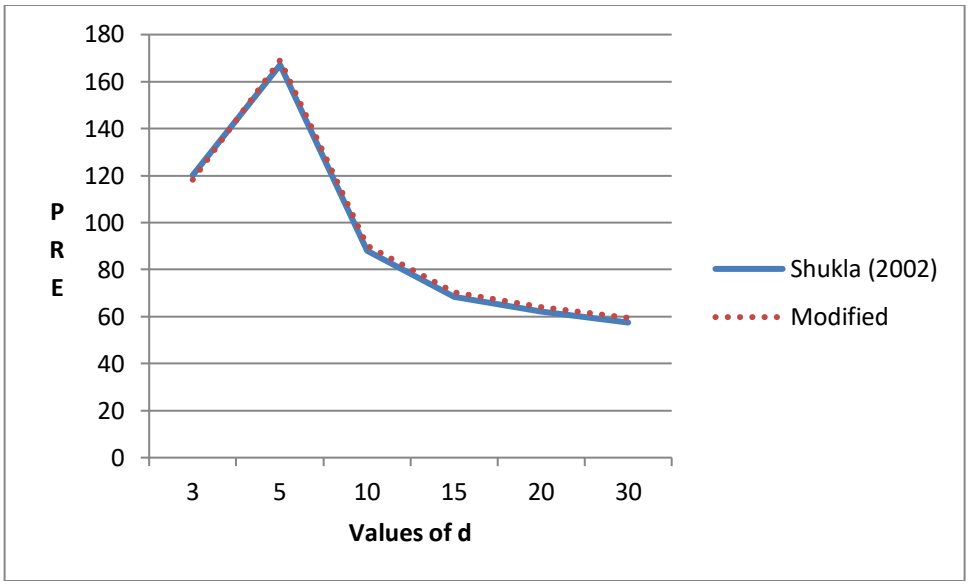


Figure 4.1.8B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 1

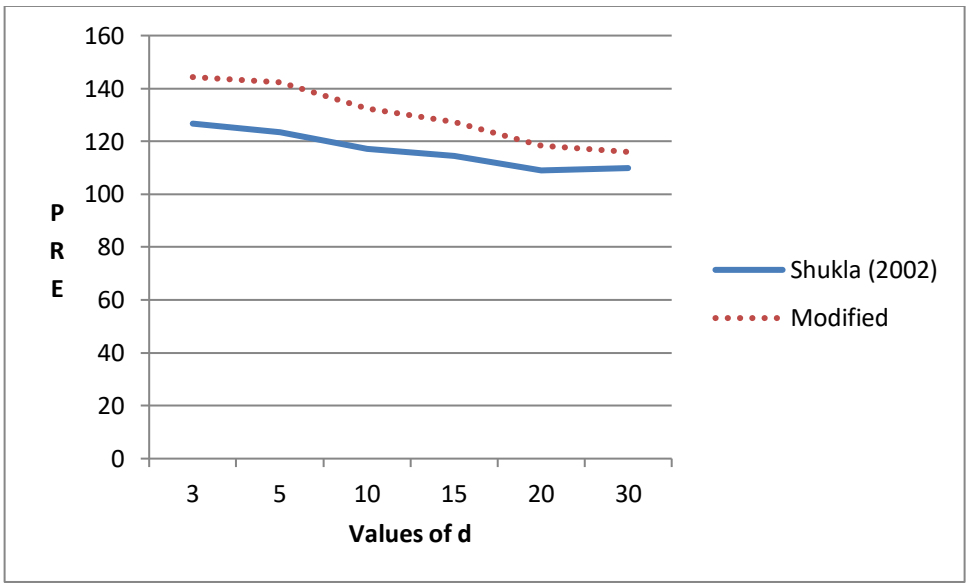


Figure 4.1.9B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 2

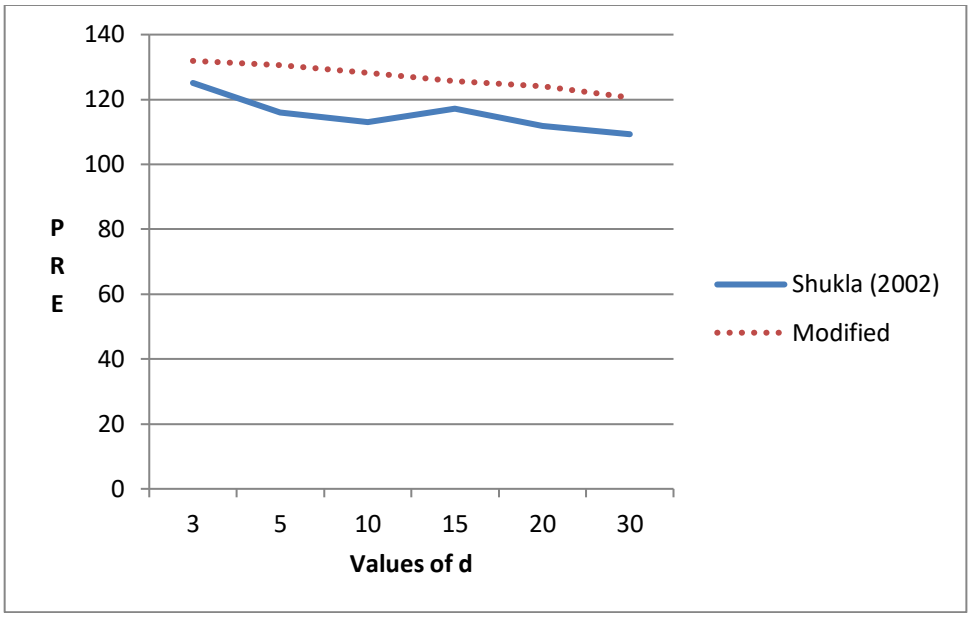


Figure 4.1.10B: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data 2

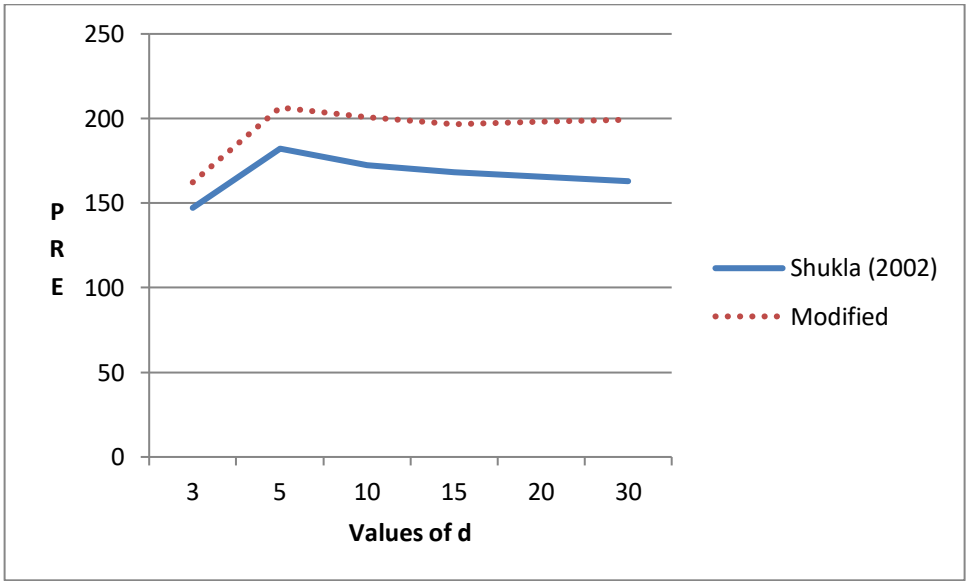


Figure 4.1.11: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data 3

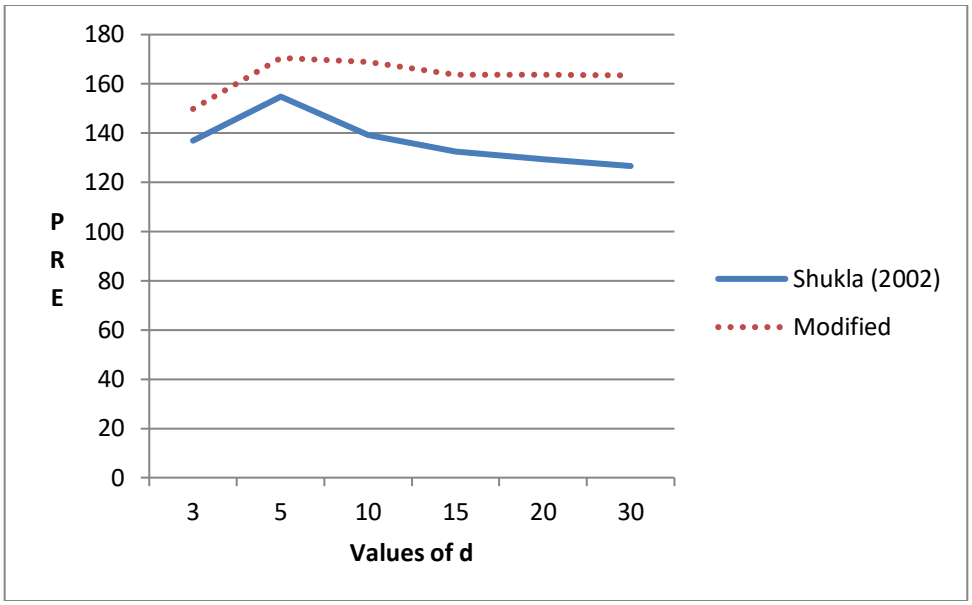


Figure 4.1.12B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data3

COMMENT: Figures (4.1.7A – 4.1.12A), and (4.1.7B – 4.1.12B) show the behavioural pattern of MSEs and PRE of estimators considered in Table 4.7A respectively and the plots generally exhibit upward and downward movement respectively as the values of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

Table 4.8: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} < 0$

Estimators	Values of d	Data A4			Data A5			Data A6		
		Case I								
		Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Shukla (2002) $\bar{y}_{FTd}$	3	0.010	6.9094	143.2	0.0007	0.7532	125.3	0.002	0.0112	125.0
	5	0.007	6.645	148.9	0.0007	0.7534	125.3	0.005	0.0111	130.9
	10	0.033	8.004	123.6	0.0046	0.7928	119.0	0.030	0.0114	127.4
	15	0.040	8.592	115.2	0.0057	0.8041	117.4	0.038	0.0126	115.0
	20	0.043	8.837	112.0	0.0063	0.808	116.7	0.041	0.0130	111.1
	30	0.045	9.052	109.3	0.0067	0.8131	116.1	0.044	0.0136	106.8
Modified $\bar{y}_{FTAA}^{(d)}$	3	0.001	6.8951	145.5	0.0007	0.7523	128.4	0.001	0.0106	136.3
	5	0.007	6.6342	151.2	0.0007	0.7526	128.1	0.003	0.0098	147.9
	10	0.032	7.9577	126.4	0.0040	0.7877	121.8	0.021	0.0103	140.8
	15	0.039	8.5366	117.9	0.0051	0.7977	118.9	0.026	0.0108	133.8
	20	0.042	8.7770	114.8	0.0055	0.8019	118.5	0.028	0.0117	124.0
	30	0.044	8.9882	113.5	0.0059	0.8056	117.9	0.030	0.0122	118.6
Case II										
Shukla (2002) $\bar{y}_{FTd}$	3	0.014	7.3297	135.0	0.0029	0.7786	121.2	0.003	0.0125	115.4
	5	0.011	6.9524	142.3	0.059	0.7797	121.0	0.007	0.0131	110.7
	10	0.048	8.3720	118.2	0.063	0.8531	110.6	0.051	0.0129	112.2
	15	0.059	9.2417	107.1	0.067	0.8563	110.2	0.064	0.0132	109.4
	20	0.063	9.6047	103.0	0.014	0.8792	107.3	0.070	0.0133	108.5
	30	0.067	9.6243	102.8	0.010	0.8996	104.9	0.074	0.0137	105.6
Modified $\bar{y}_{FTAA}^{(d)}$	3	0.014	7.3093	137.4	0.048	0.7754	123.7	0.002	0.0114	127.5
	5	0.010	6.9375	144.7	0.058	0.7764	123.6	0.005	0.0116	124.3
	10	0.048	8.3038	122.2	0.062	0.8300	113.8	0.035	0.0123	117.9
	15	0.058	9.1589	109.1	0.0239	0.8261	114.2	0.043	0.0123	118.1
	20	0.062	9.5158	106.0	0.0262	0.8461	111.5	0.047	0.0119	121.7
	30	0.066	9.5298	104.8	0.0281	0.8639	109.2	0.051	0.0124	117.0

Table 4.8B: Efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimator in Table 4.8A

Estimators	Data A4	Data A5	Data A6
CASE I			
Shukla (2002) $\bar{y}_{FTd}$	P>0.7724345	P>0.1700999	P>0.3202427
Modified $\bar{y}_{FTAA}^{(d)}$	P=0.7924105	P=0.1771949	P=0.4122431
CASE II			
Shukla (2002) $\bar{y}_{FTd}$	P>0.6007824	P>0.09991543	P>0.2490777
Modified $\bar{y}_{FTAA}^{(d)}$	P=0.6417801	P=0.10921503	P=0.26190175

Table 4.8A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated for some arbitrary values of $d = 3, d = 5, d = 10, d = 15, d = 20$ and $d = 30$ using Data A4,

A5 and A6 respectively. The results of the analysis revealed that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE.

Table 4.8B shows the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over Shukla (2002) estimators using Data A4, A5 and A6 and the results revealed that the conditions are satisfied.

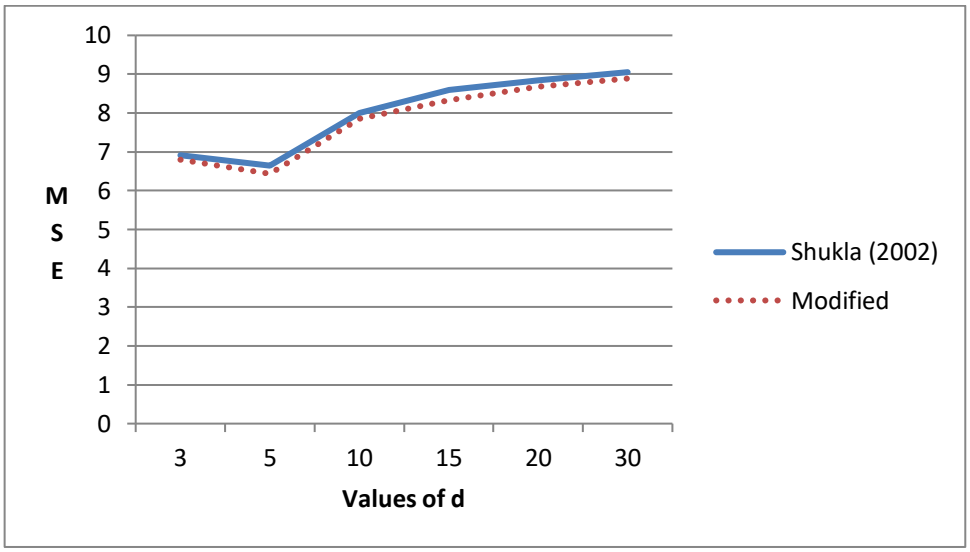


Figure 4.1.13A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A4

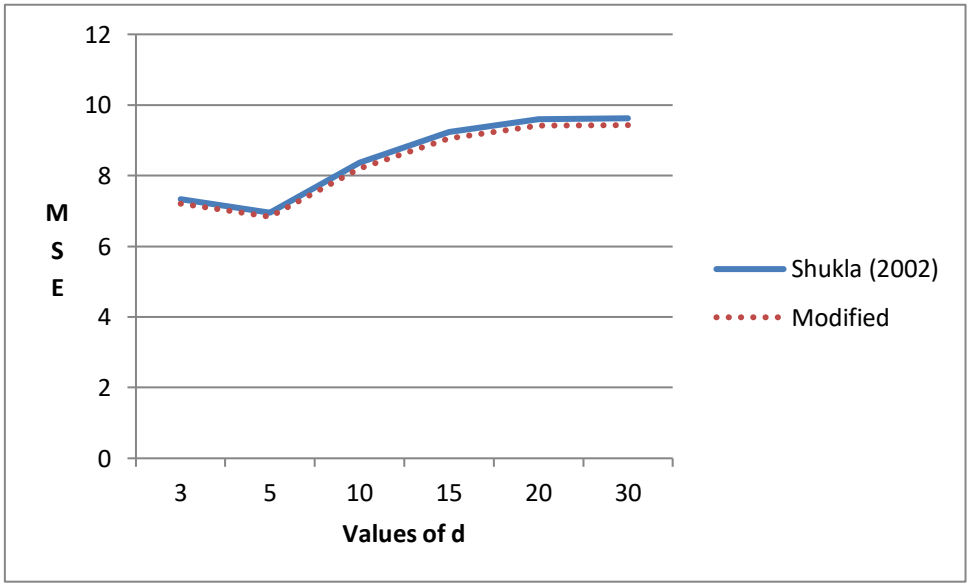


Figure 4.1.14A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A4

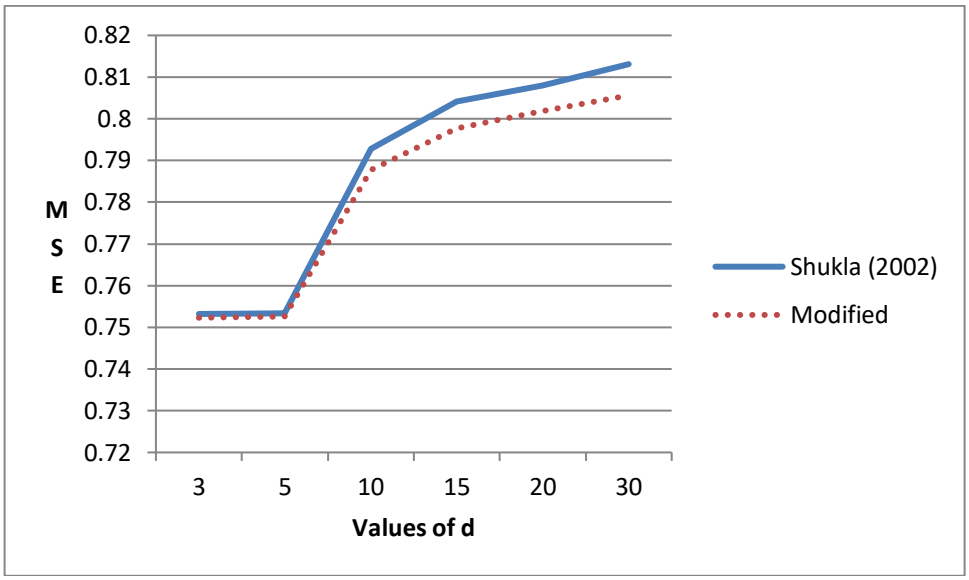


Figure 4.1.15A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A5

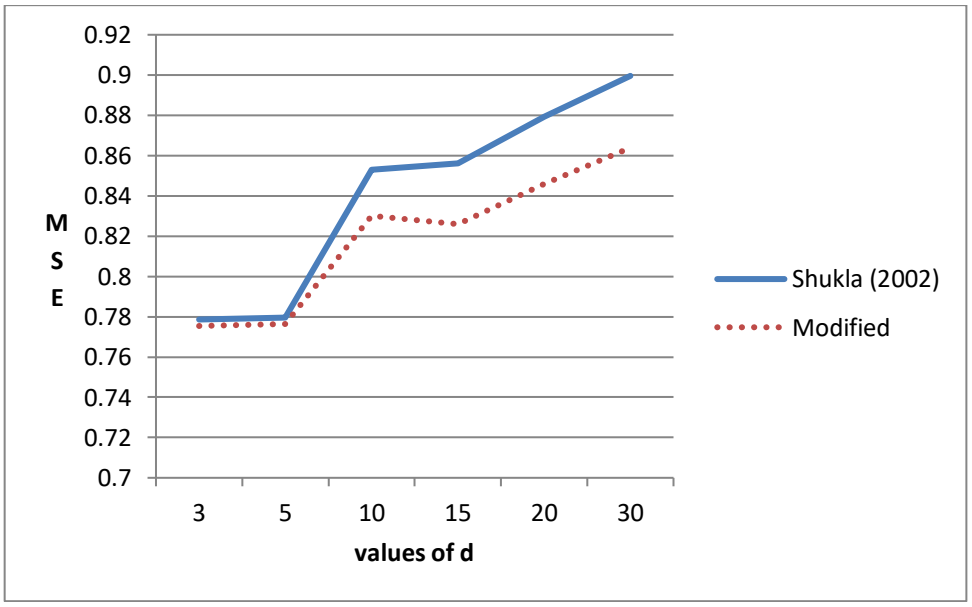


Figure 4.1.16A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A5

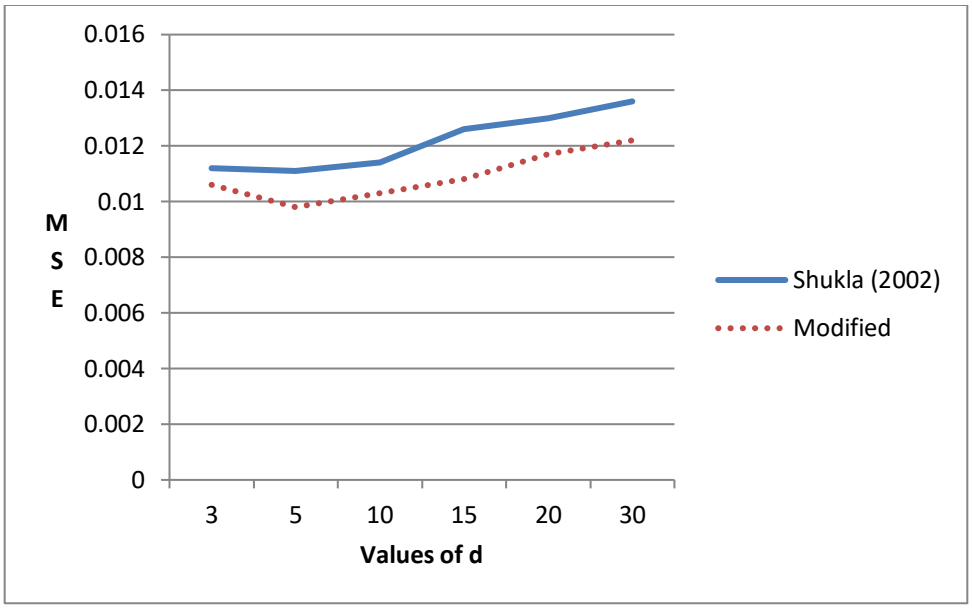


Figure 4.1.17A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A6

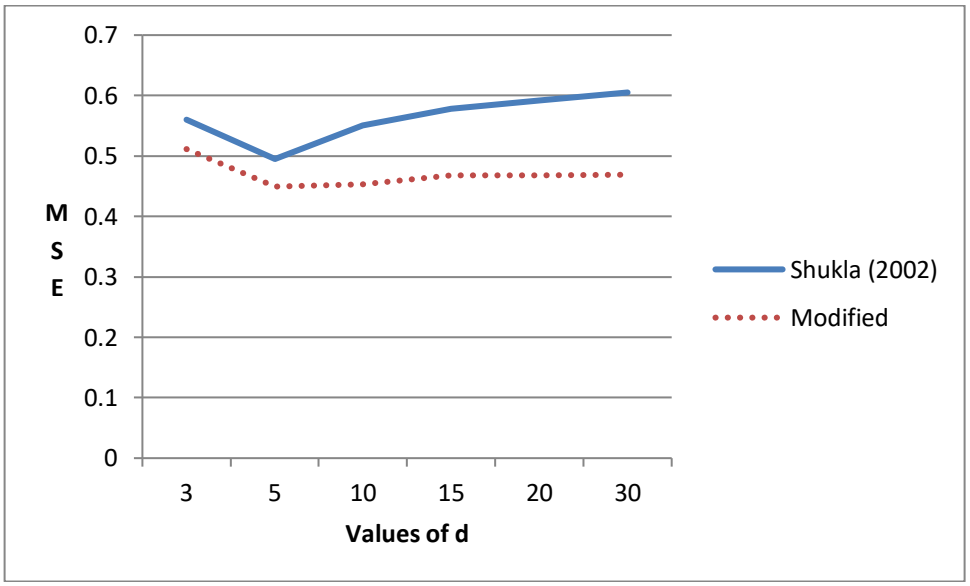


Figure 4.1.18A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A6

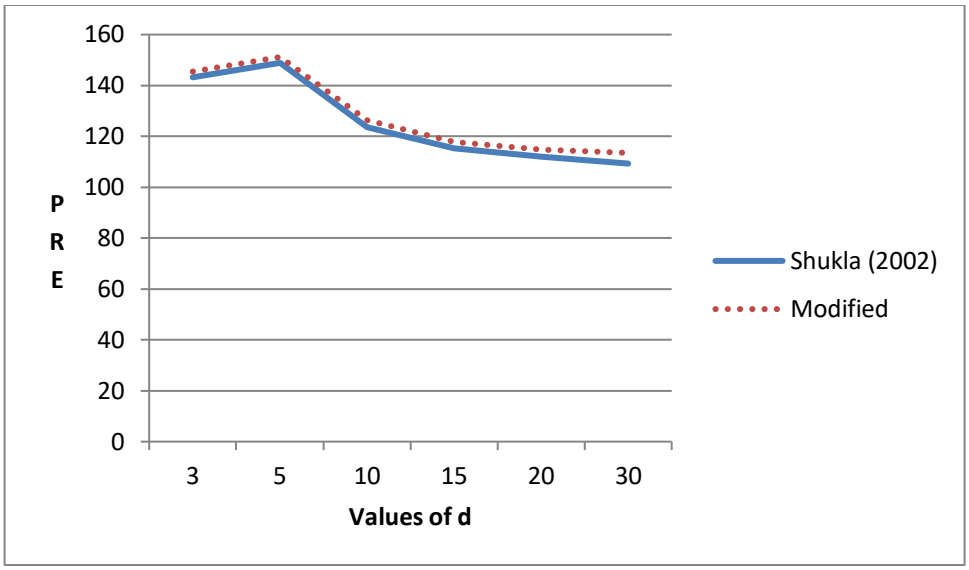


Figure 4.1.13B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A4

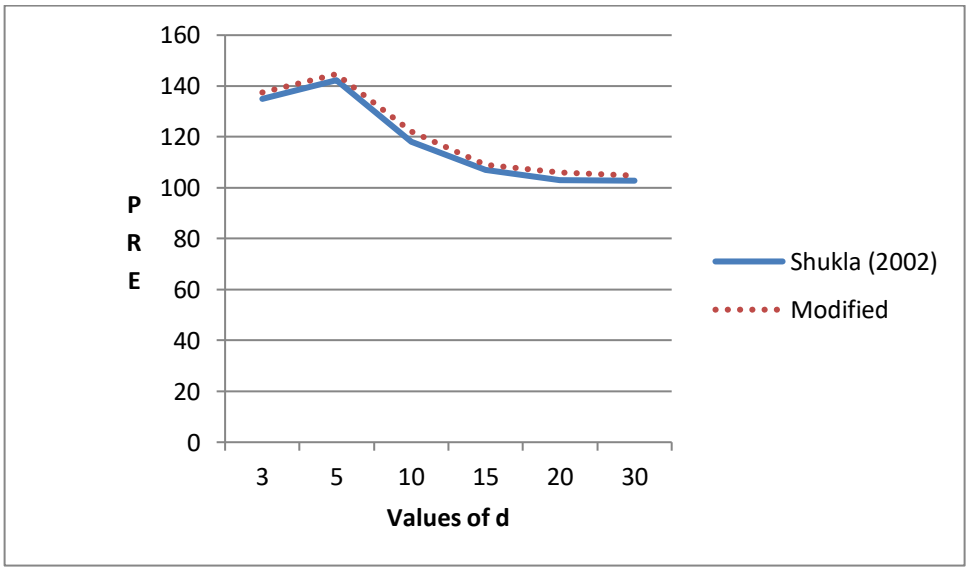


Figure 4.1.14B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A4

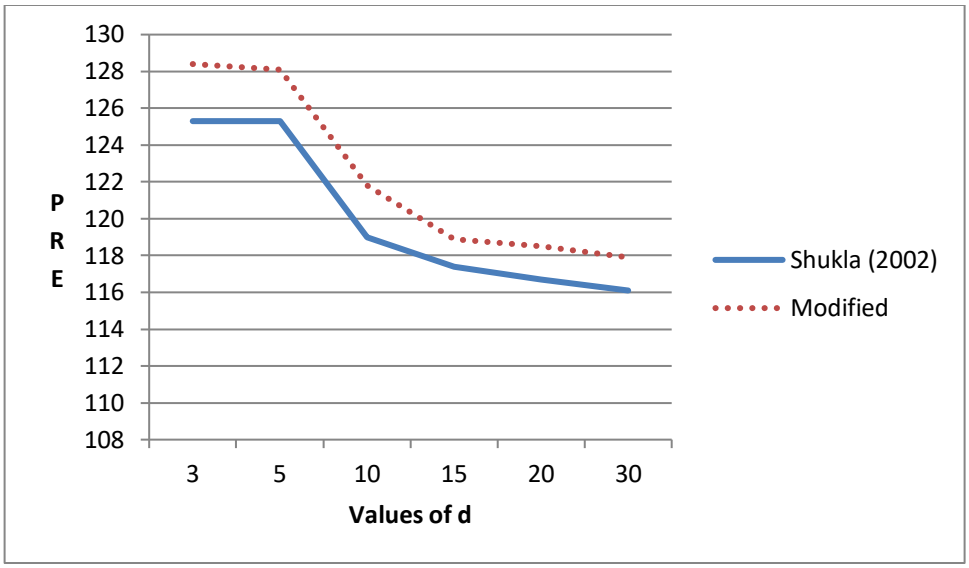


Figure 4.1.15B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A5

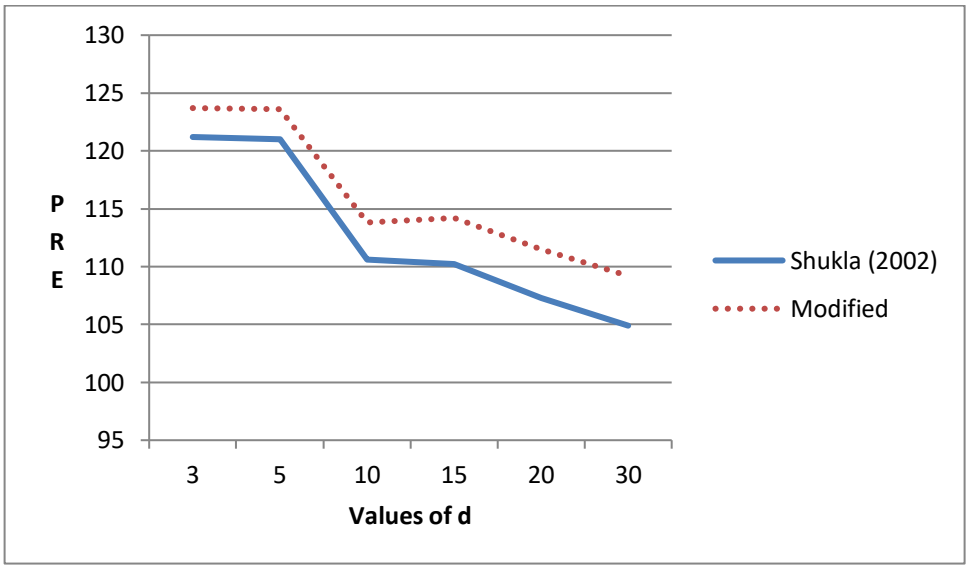


Figure 4.1.16B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A5

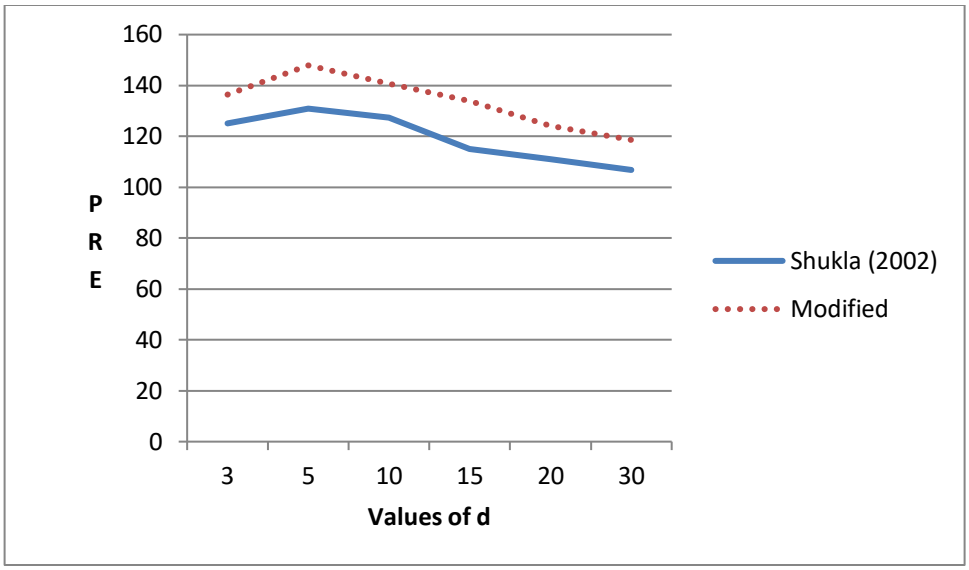


Figure 4.1.17: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I using Data A6

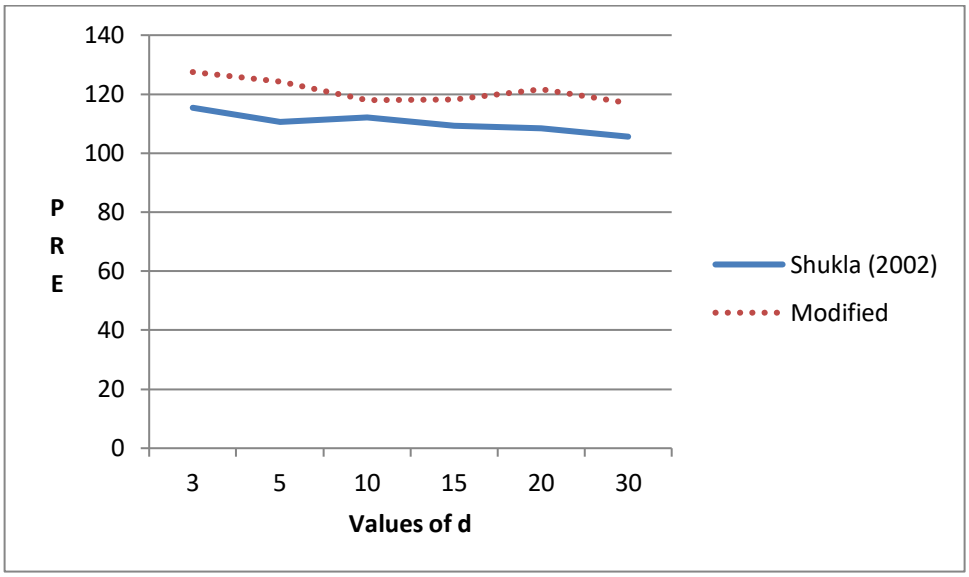


Fig. 4.1.18B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II using Data A6

Figures (4.1.13A – 4.1.18A), and (4.1.13B – 4.1.18B) show the behaviourial pattern of MSEs and PRE of estimators considered in Table 4.8A respectively.

4.1.3 Efficiency Comparison of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ with some existing related Estimators

In this subsection, relative efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ with respect to some existing related ratio and product estimators were investigated.

4.1.3.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Ratio Estimators

Table 4.9A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related ratio estimators when $\rho_{xy} > 0$

Estimators	Data A1			Data A2			Data A3		
	Case I								
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Sample mean	0	5145375	100	0	51873.9	100	0	0.7661	100
Choudhury and Singh (2012)	49.70	1814947.1	283.5	0.084	25667.4	202.1	0.018	0.2879	266.1
Chand (1975)	96.88	3961027.7	129.9	0.116	17728.6	292.6	0.024	0.5821	131.6
Singh and Upadhayaya (2001)	20.97	4220980.31	121.9	0.147	16610.3	312.3	0.029	0.3047	251.4
Singh et al. (2007)	-17.95	176322.48	291.8	-0.01	26412.4	196.4	-0.00	0.2974	257.6
Singh and Upadhayaya (2001)	-36.12	4220980.3	121.9	0.003	16546.7	313.5	-0.08	0.3079	248.8
Singh (2001)	10.83	3396287.1	151.5	-0.02	33273.8	115.9	-0.02	0.3369	227.4
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	19.65	1619060.7	317.8	0.042	14555.0	356.4	-0.01	0.2664	287.6
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	16.82	1720285.9	299.1	0.009	15591.8	332.7	0.005	0.2828	270.9
	Case II								
Sample mean	0	5145375	100	0	51873.8	100	0	0.7661	100
Choudhury and Singh (2012)	49.64	29570112.1	174.0	0.004	15829.7	327.7	0.019	0.2879	266.1
Chand (1975)	68.41	4529379.4	113.6	0.302	32199.8	161.1	0.037	0.5823	131.9
Singh and Upadhayaya (2001)	63.55	4664891.2	110.3	0.431	39568.2	131.1	0.045	0.3976	192.7
Singh et al. (2007)	71.51	3127887.5	164.5	0.547	13803.6	375.8	0.051	0.2975	257.8
Singh and Upadhayaya (2001)	-29.34	4751038.78	108.3	0.013	28424.1	182.5	-0.06	0.3479	220.2
Singh (2001)	-39.68	2695324.8	190.9	0.033	35505.7	146.1	-0.02	0.4579	167.3
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	45.53	2231298.8	230.6	0.009	13089.6	396.3	-0.01	0.2823	271.4
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	62.25	2588216.9	198.8	0.263	12881.5	402.7	0.001	0.2501	306.4

Table 4.9B: Efficiency Conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over other Estimators in Table 4.9A

Estimators	Data A1	Data A2	Data A3
CASE I			
Sample mean	P<-1.036445	P<-0.3204763	P<-1.531633
Choudhury and Singh (2012)	P< 0.8066614	P< 0.1432165	P< 0.5159184
Chand (1975)	P< 0.117864	P<0.6812262	P<-0.5310677
Singh and Upadhayaya (2001)	P< 0.1176018	P<0.6812213	P<-0.5300865
Singh et al. (2007)	P< 0.1177921	P< 0.6811536	P< -0.5300499
Singh and Upadhayaya (2001)	P< 0.1176668	P< 0.679578	P< -0.520896
Singh (2001)	P<0.1178395	P<0.6959813	P<-0.4222940
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-0.518272	P=-1.016231	P<-0.610252
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P= -1.107168	P=-2.648007	P=-1.7165601
CASE II			
Sample mean	P<-1.036445	P<-0.3204763	P<-1.531633
Choudhury and Singh (2012)	P<1.002687	P< 0.2014864	P<0.6530612
Chand (1975)	P<0.117864	P<0.6812262	P< -0.5310677
Singh and Upadhayaya (2001)	P<0.1176018	P<0.6812213	P<-0.5300865
Singh et al. (2007)	P< 0.1177921	P< 0.6811536	P<-0.5300499
Singh and Upadhayaya (2001)	P< 0.1176668	P< 0.679578	P< -0.520896
Singh (2001)	P<0.1178395	P<0.6959813	P<-0.4222940
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-0.5032359	P=-1.263165	P=-0.7186974
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=-3.62536	P=-1.571182	P=-4.30034

COMMENT: Table 4.9A shows the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and other existing related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated. The biases and MSEs are obtained using Data A1, A2 and A3 respectively. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ have minimum MSE and high PRE.

Table 4.9B shows the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ over some related estimators using Data A1, A2 and A3 and the results revealed that the conditions are satisfied.

4.1.3.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Product Estimators

Table 4.10A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} < 0$

Estimators	Data A4			Data A5			Data A6		
	Case I								
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
Sample mean	0	9.8966	100	0	0.9437	100	0	0.0144	100
Choudhury and Singh (2012)	0.040	5.2335	189.1	0.044	0.9031	104.5	0.138	0.0108	133.3
Chand (1975)	0.043	7.3526	134.6	0.105	1.4408	65.5	0.141	0.0094	152.9
Singh and Upadhayaya (2001)	0.045	7.4579	132.7	0.106	0.7729	122.1	0.144	0.0128	112.4
Singh et al. (2007)	0.001	6.4389	153.7	0.067	0.5350	176.4	0.101	0.0117	123.1
Singh and Upadhayaya (2001)	0.007	7.5145	131.7	0.107	0.4921	191.8	0.003	0.0128	112.8
Singh (2001)	0.032	9.1129	108.6	0.117	0.6233	151.4	0.021	0.0114	126.0
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.039	5.0596	195.6	0.204	0.4815	196.4	0.026	0.0093	155.1
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.042	4.6419	213.2	0.215	0.5338	176.8	0.028	0.0089	161.7
	Case II								
Sample mean	0	9.8966	100	0	0.9437	100	0	0.0144	100
Choudhury and Singh (2012)	0.059	6.2204	159.1	0.059	0.8315	113.5	0.051	0.0108	133.2
Chand (1975)	0.063	7.3745	134.2	0.063	0.9710	96.3	0.064	0.0094	152.9
Singh and Upadhayaya (2001)	0.067	8.9159	111.0	0.067	0.6576	143.5	0.070	0.0171	84.28
Singh et al. (2007)	0.014	7.1223	137.6	0.014	0.5350	176.4	0.074	0.0117	123.1
Singh and Upadhayaya (2001)	0.010	5.7074	173.4	0.010	0.4563	206.8	0.002	0.0189	76.3
Singh (2001)	0.048	9.4433	104.8	0.048	0.8156	115.7	0.005	0.0126	113.9
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.058	4.4260	223.6	0.058	0.4481	210.6	0.035	0.0087	165.2
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.062	4.8018	206.1	0.062	0.4505	209.5	0.043	0.0091	157.5

Table 4.10B: Efficiency Condition of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over other Estimators in Table 4.10A

Estimators	Data A4	Data A5	Data A6
CASE I			
Sample mean	P<1.822449	P<0.4013265	P<0.6404855
Choudhury and Singh (2012)	P< 1.6138776	P< 1.1351837	P< 1.1530204
Chand (1975)	P< 2.824017	P< 1.410532	P< 1.646017
Singh and Upadhayaya (2001)	P< 2.824278	P< 1.428501	P< 1.645217
Singh et al. (2007)	P< 2.823735	P< 1.410455	P< 1.644685
Singh and Upadhayaya (2001)	P<2.825878	P<1.53791	P<1.640596
Singh (2001)	P<2.882203	P<1.862882	P<1.642481
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=1.13762	P=0.0123419	P=0.621011
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=1.20863	P=0.0012734	P=0.481847
CASE II			
Sample mean	P<1.822449	P<0.4013265	P<0.6404855
Choudhury and Singh (2012)	P< 0.6530612	P<0.6530612	P<0.3062859
Chand (1975)	P<2.824017	P< 1.410532	P<1.646017
Singh and Upadhayaya (2001)	P<2.824278	P<1.428501	P<1.645217
Singh et al. (2007)	P<2.823735	P<1.410455	P<1.644685
Singh and Upadhayaya (2001)	P<2.825878	P<1.53791	P<1.640596
Singh (2001)	P<2.882144	P<1.857761	P<1.642461
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=0.8039338	P=0.2704137	P=0.2586163
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=3.152276	P=0.3711047	P=0.267815

Table 4.10A show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and other related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated using Data A4, A5 and A6 respectively. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ have minimum MSEs and high PRE.

Table 4.10B shows the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ over some related estimators using Data A4, A5 and A6 and the results revealed that the conditions are satisfied.

4.1.4 Cost Analysis

In this subsection, the expected cost analysis of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{\beta_1(d)}$ and some related estimators is estimated using Data B1.

Table 4.11: Expected cost of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{\beta_1(d)}$ and some existing related Estimators

Estimators	Data B1			Estimators	Data B1		
	n_1^{opt}	n_2^{opt}	Expected Cost $		n_1^{opt}	n_2^{opt}	Expected Cost $
Sample mean $\bar{y}$	NA	10	5,000	Singh and Shukla $\bar{y}_{FT}$ (1987, 1993)	NA	8	4,000
Cochran (1942) $\bar{y}_r$	NA	16	8,000	Jain and Shukla $\bar{y}_{FT1}$ (2013)	NA	8	4,000
Srivenkataramana (1980) $\bar{y}_{st}$	NA	9	4,500	Jain and Shukla $\bar{y}_{FT2}$ (2013)	NA	7	4,200
Sisodia and Diwivedi (1981) t_2	NA	16	8,000	Shukla et al. $\bar{y}_{TFT}$ (2013)	NA	8	4,800
Singh and Tailor (2003) t_3	NA	16	8,000	Thakur and Gupta $\bar{y}_{FTRP}$ (2013)	NA	10	5,000
Singh et. al. (2004) t_4	NA	12	6,000	Sukhatme (1962)	21	6	4,050
Upadhyaya and Singh (1999) t_5	NA	15	7,500	Malik and Tailor estimator (2013)	21	6	4,050
Upadhyaya and Singh (1999) t_6	NA	16	8,000	Shukla estimator (2002)	21	5	3,550
Modified $\bar{y}_{FTAA}$	NA	6	3,000	Modified $\bar{y}_{FTAA}^{(d)}$	19	5	3,450
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	21	5	3,550	Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	20	5	3,500

NA means "Not Applicable"

Table 4.11 shows the optimum preliminary (n_1^{opt}) and secondary (n_2^{opt}) sample sizes and the expected cost using $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{\beta_1(d)}$ and some existing related estimators. The results revealed that the expected expenses using $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{\beta_1(d)}$ are likely to be slightly cheaper than the existing related estimators considered.

4.1.5 Robustness under Non-Response Model

In this section, robustness of the modified estimators were investigated with respect to that of some existing related ratio, product, dual-to-ratio and factor-type estimators under non-response model defined in chapter three.

Table 4.12: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C1

Estimators	$(1/k)$				Average	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Choudhury and Singh (2012)	256.1	255.1	254.1	253.1	254.6	1st
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	392.5	392.47	392.43	392.39	392.45	2nd
Singh et al. (2007)	397.3	396.3	395.3	394.3	395.8	3rd
Modified $\bar{y}_{FTAA}$	396.7	396.6	396.5	396.4	396.55	4th
Singh (2001)	417.4	416.4	415.4	414.4	415.9	5th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	423.39	423.44	423.48	423.52	423.46	6th
Singh and Tailor (2003)	426.9	425.9	424.9	423.93	425.4	7th
Sisodia and Diwivedi (1981)	426.9	425.94	424.9	423.9	425.41	8th
Upadhyaya and Singh (1999)	426.9	425.94	424.9	423.9	425.41	9th
Upadhyaya and Singh (1999)	426.9	425.94	424.9	423.9	425.41	9th
Cochran (1942)	426.94	425.94	424.94	423.94	425.44	11th
Jain and Shukla (2013) $\bar{y}_{FT2}$	482.3	481.3	480.3	479.4	480.83	12th
Malik and Tailor (2013)	545.7	544.8	543.9	542.8	544.3	13th
Sukhatme (1962)	546	545	544	543	544.5	14th
Shukla et. al. (2013)	590.3	589.9	588.3	587.9	589.1	15th
Modified $\bar{y}_{FTAA}^{(d)}$	614.28	614.24	614.2	614.1	614.21	16th
Jain and Shukla (2013) $\bar{y}_{FT1}$	629.3	628.3	627.3	626.3	627.8	17th
Singh and Upadhayaya (2001)	727.2	726.2	725.2	724.2	725.7	18th
Singh and Upadhayaya (2001)	727.2	726.2	725.2	724.2	725.7	18th
Sample mean	985.32	985.32	985.32	985.32	985.32	20th
Singh and Shukla (1987, 1993)	990.3	989.3	988.3	987.3	988.8	21st
Thakur and Gupta (2013)	990.3	989.3	988.3	987.3	988.8	22nd
Shukla (2002)	990.3	989.3	988.4	987.3	988.83	23rd
Srivenkataramana (1988) –two-phase	2789.1	2788.1	2787.1	2786.1	2787.6	24th
Chand (1975)	5494.6	5493.6	5492.6	5491.6	5493.1	25th
Srivenkataramana (1980)	9039.9	9038.9	9037.9	9036.9	9038.4	26th
Singh et. al. (2004)	9039.9	9038.9	9037.9	9036.9	9038.4	27th

COMMENT: Table 4.12 shows the biases and MSEs of the modified estimators and those of some existing related estimators under Non-response model. Estimation of biases and

MSEs under the model was done using Data C1. The robustness of the considered estimators were obtained by averaging their expected MSEs and ranked accordingly to their level of efficiency. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and $\bar{y}_{FTAA}^{(d)}$ are ranked second, fourth, sixth and sixteenth respectively.

Table 4.13: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C2

Estimators	$(1/k)$				Average	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.1564	0.1563	0.1561	0.156	0.1562	1st
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.1577	0.1575	0.1574	0.1572	0.15745	2nd
Modified $\bar{y}_{FTAA}^{(d)}$	0.2312	0.231	0.2309	0.2307	0.23095	3rd
Modified $\bar{y}_{FTAA}$	0.2581	0.258	0.2578	0.2577	0.2579	4th
Sample mean	0.2886	0.2886	0.2886	0.2886	0.2886	5th
Singh et al. (2007)	9.211	6.9472	4.6834	2.4196	5.8153	6th
Singh and Shukla (1987, 1993)	9.3437	7.0799	4.8161	2.5523	5.948	7th
Shukla (2002)	9.3437	7.0799	4.8161	2.5523	5.948	7th
Thakur and Gupta (2013)	9.3437	7.0799	4.8161	2.5523	5.948	7th
Shukla et. al. (2013)	9.3437	7.0799	4.8161	2.5523	5.948	10th
Singh and Upadhayaya (2001)	9.3768	7.113	4.8492	2.5854	5.9811	11th
Singh and Upadhayaya (2001)	9.3768	7.113	4.8492	2.5854	5.9811	11th
Chand (1975)	9.78	7.5162	5.2524	2.9886	6.3843	13th
Choudhury and Singh (2012)	10.331	8.0669	5.8032	3.5393	6.9351	14th
Jain and Shukla (2013) $\bar{y}_{FT1}$	10.645	8.3811	6.1173	3.8535	7.24923	15th
Jain and Shukla (2013) $\bar{y}_{FT2}$	13.473	11.209	8.9455	6.6817	10.0773	16th
Singh (2001)	16.189	13.925	11.661	9.3971	12.793	17th
Malik and Tailor (2013)	16.303	14.039	11.775	9.5114	12.9071	18th
Sukhatme (1962)	16.308	14.044	11.781	9.5168	12.9125	19th
Srivenkataramana (1988) –two-phase	17.568	15.304	13.04	10.776	14.172	20th
Sisodia and Diwivedi (1981)	25.905	23.641	21.378	19.113	22.5093	21st
Cochran (1942)	25.905	23.641	21.378	19.114	22.5095	22nd
Singh and Tailor (2003)	25.905	23.641	21.378	19.114	22.5095	23rd
Upadhyaya and Singh (1999)	25.905	23.641	21.378	19.114	22.5095	24th
Upadhyaya and Singh (1999)	25.905	23.641	21.378	19.114	22.5095	24th
Srivenkataramana (1980)	292.3	290.03	287.77	285.5	288.9	26th
Singh et. al. (2004)	292.3	290.03	287.77	285.5	288.9	27th

COMMENT: Table 4.13 shows the biases and MSEs of the modified estimators and those of some existing related estimators under Non-response model. Computations of biases and

MSEs under the model were done using Data C2. The robustness of the considered estimators were obtained by averaging their expected MSEs and ranked accordingly to their level of efficiency. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTAA}$ are ranked first, second, third and fourth respectively.

Table 4.14: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C3

Estimators	$(1/k)$				Average	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Choudhury and Singh (2012)	0.2552	0.2552	0.2552	0.2552	0.2552	1st
Chand (1975)	0.2825	0.2825	0.2824	0.2824	0.28245	2nd
Modified $\bar{y}_{FTAA}$	0.4215	0.4214	0.4214	0.4214	0.421425	3rd
Singh et al. (2007)	0.4273	0.4273	0.4272	0.4272	0.42725	4th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.4273	0.4273	0.4272	0.4272	0.42725	5th
Modified $\bar{y}_{FTAA}^{(d)}$	0.4623	0.4622	0.4622	0.4621	0.4622	6th
Singh and Shukla (1987, 1993)	0.5462	0.546	0.5459	0.5458	0.545975	7th
Shukla (2002)	0.5462	0.546	0.5459	0.5458	0.545975	7th
Sample mean	0.546	0.546	0.546	0.546	0.546	9th
Thakur and Gupta (2013)	0.546	0.546	0.546	0.546	0.546	9th
Shukla et. al. (2013)	0.546	0.546	0.546	0.546	0.546	9th
Singh and Upadhayaya (2001)	0.8178	0.8178	0.8177	0.8177	0.81775	12th
Singh and Upadhayaya (2001)	0.8178	0.8178	0.8177	0.8177	0.81775	12th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0814	1.0813	1.08132	1.0813	1.08133	14th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.256	1.256	1.2561	1.2561	1.25605	15th
Product	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Pandey and Dubey (1988)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh and Tailor (2003)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh (2003)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh et. al. (2004)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Upadhyaya and Singh (1999)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh (2003)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh (2003)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Upadhyaya and Singh (1999)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Singh (2003)	1.6634	1.6634	1.6634	1.6634	1.6634	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.6716	1.6716	1.6716	1.6716	1.6716	26th
Singh (2001)	2.6556	2.6555	2.6554	2.6555	2.6555	27th
Malik and Tailor (2013)	2.6907	2.6906	2.6906	2.6906	2.690625	28th

COMMENT: Table 4.14 shows the biases and MSEs of the modified estimators and those of some existing related estimators under Non-response model. Computations of biases and

MSEs under the model were done using Data C3. The robustness of the considered estimators were obtained by averaging their expected MSEs and ranked accordingly to their level of efficiency. The results revealed that out of the twenty-eight competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked third, fifth, sixth and fifteenth respectively.

Table 4.15: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model using Data C4

Estimators	$(1/k)$				Average	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.2039	0.2036	0.2034	0.2031	0.2035	1st
Choudhury and Singh (2012)	1.1089	1.1085	1.10819	1.1079	1.10837	2nd
Chand (1975)	1.4976	1.4973	1.497	1.4967	1.49715	3rd
Modified $\bar{y}_{FTAA}$	1.9467	1.9465	1.9462	1.9459	1.946325	4th
Singh et al. (2007)	2.0007	2.0004	2.0001	1.9998	2.00025	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	2.0007	2.0004	2.0001	1.9998	2.00025	5th
Modified $\bar{y}_{FTAA}^{(d)}$	2.1	2.0997	2.0994	2.0991	2.09955	7th
Sample mean	2.4154	2.4151	2.41485	2.4146	2.41499	8th
Singh and Shukla (1987, 1993)	2.4154	2.4151	2.4149	2.4146	2.415	8th
Shukla (2002)	2.4154	2.4151	2.4149	2.4146	2.415	8th
Thakur and Gupta (2013)	2.4154	2.4151	2.4149	2.4146	2.415	8th
Shukla et. al. (2013)	2.4154	2.4151	2.4149	2.4146	2.415	8th
Singh and Upadhayaya (2001)	3.6034	3.6031	3.6029	3.6026	3.603	13th
Singh and Upadhayaya (2001)	3.6034	3.6032	3.6029	3.6026	3.60303	13th
Jain and Shukla (2013) $\bar{y}_{FT1}$	4.6177	4.6174	4.6172	4.6169	4.6173	15th
Jain and Shukla (2013) $\bar{y}_{FT2}$	7.0834	7.0831	7.0828	7.0826	7.08298	16th
Product	7.4642	7.464	7.4637	7.4634	7.46383	17th
Pandey and Dubey (1988)	7.4642	7.464	7.4637	7.4634	7.46383	17th
Singh (2003)	7.4643	7.464	7.4637	7.4633	7.46383	17th
Upadhyaya and Singh (1999)	7.4643	7.464	7.4637	7.4633	7.46383	17th
Singh (2003)	7.4643	7.464	7.4637	7.4633	7.46383	17th
Singh and Tailor (2003)	7.4643	7.464	7.46372	7.4633	7.46383	17th
Singh (2003)	7.4643	7.464	7.46372	7.4633	7.46383	17th
Singh et. al. (2004)	7.4643	7.464	7.46372	7.4633	7.46383	17th
Upadhyaya and Singh (1999)	7.4643	7.464	7.46372	7.4633	7.46383	17th
Singh (2003)	7.4643	7.464	7.46372	7.4633	7.46383	17th
Singh (2001)	11.305	11.305	11.304	11.303	11.3043	27th
Malik and Tailor (2013)	11.405	11.405	11.404	11.403	11.4043	28th

COMMENT: Table 4.15 shows the biases and MSEs of the modified estimators and those of some existing related estimators under Non-response model. Computations of biases and MSEs under the model were done using Data C4. The robustness of the considered estimators were obtained by averaging their expected MSEs and ranked accordingly to their level of efficiency. The results revealed that out of the twenty-eight competing estimators, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{(d)}$ are ranked first, second, fourth, fifth and seventh respectively.

4.1.6 Robustness under Super-Population Model

In this section, robustness of the modified estimators were investigated with respect to that of some existing related ratio, product, dual-to-ratio and factor-type estimators under super-population model defined chapter three.

Using Data D1, the following models are obtained;

$$\left.\begin{aligned} &Model\ I: y_i = 0.1309 - 1.196x_i \\ &Model\ II: y_i = 0.1309 - 1.196x_i + 16.29x_i^2 \\ &Model\ III: y_i = 0.1309 - 1.196x_i + 16.29x_i^2 - 53.8x_i^3 \end{aligned}\right\} \quad (4.1)$$

Using Data D2, the following models are obtained;

$$\left.\begin{aligned} &Model\ I: y_i = 79.212 - 0.3154x_i \\ &Model\ II: y_i = 79.212 - 0.3154x_i + 0.000629x_i^2 \\ &Model\ III: y_i = 79.212 - 0.3154x_i + 0.000629x_i^2 - 0.00001512x_i^3 \end{aligned}\right\} \quad (4.2)$$

Biases and MSEs of these modified estimators and some existing related estimators considered in the study are obtained as shown in Tables 4.16 and 4.17 for Models I, II and III using (4.1) and (4.2) respectively.

Table 4.16A: Bias and MSE of modified Estimators and some related Estimators under some Super-population models I using Data D1 $\left(\rho_{xy} > 0\right)$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.33	0.0019	1st
Product	0.104	0.0107	2nd
Thakur and Gupta (2013)	0.104	0.0108	3rd
Upadhyaya and Singh (1999)	0.105	0.0109	4th
Singh (2003)	0.108	0.0116	5th
Singh (2003)	0.108	0.0117	6th
Singh (2003)	0.109	0.0119	7th
Singh et. al. (2004)	0.114	0.0131	8th
Singh and Tailor (2003)	0.115	0.0133	9th
Sample mean	0.118	0.0138	10th
Pandey and Dubey (1988)	0.118	0.0138	10th
Singh (2003)	0.118	0.0138	10th
Upadhyaya and Singh (1999)	0.118	0.0138	10th
Sisodia and Diwivedi (1981)	0.118	0.0139	14th
Upadhyaya and Singh (1999)	0.118	0.0139	14th
Singh and Tailor (2003)	0.12	0.0144	16th
Singh et. al. (2004)	0.121	0.0146	17th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.122	0.0149	18th
Malik and Tailor (2013)	0.123	0.0152	19th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.125	0.0156	20th
Upadhyaya and Singh (1999)	0.13	0.0169	21st
Cochran (1942)	0.131	0.0171	22nd
Singh and Shukla (1987, 1993)	0.132	0.0175	23rd
Shukla et al. (2013)	0.133	0.0176	24th
Modified $\bar{y}_{FTAA}^{(d)}$	-0.14	0.0182	25th
Modified $\bar{y}_{FTAA}$	-0.1	0.0196	26th
Srivenkataramana (1980)	0.148	0.0218	27th
Shukla (2002)	0.15	0.0226	28th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.151	0.0228	29th
Singh and Upadhayaya (2001)	0.162	0.0261	30th
Singh et al. (2007)	0.296	0.0876	31st
Singh (2001)	0.33	0.109	32nd
Chand (1975) 20	-3.2	9.8994	33rd

Table 4.16B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II using Data D1 $\left(\rho_{xy} > 0\right)$

Estimators	Bias	MSE	Rank
Singh and Upadhayaya (2001)	0.102	0.0104	1st
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.125	0.0156	2nd
Shukla (2002)	0.126	0.0158	3rd
Srivenkataramana (1980)	0.132	0.0173	4th
Modified $\bar{y}_{FTAA}^{(d)}$	0.722	0.0217	5th
Shukla et al. (2013)	0.163	0.0265	6th
Singh and Shukla (1987, 1993)	0.163	0.0267	7th
Cochran (1942)	0.166	0.0277	8th
Upadhyaya and Singh (1999)	0.168	0.0283	9th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.909	0.0315	10th
Singh et al. (2007)	-0.18	0.032	11th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.179	0.0322	12th
Malik and Tailor (2013)	0.183	0.0334	13th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.185	0.0343	14th
Singh et. al. (2004)	0.187	0.0351	15th
Singh and Tailor (2003)	0.189	0.0358	16th
Sample mean	0.194	0.0371	17th
Modified $\bar{y}_{FTAA}$	0.733	0.0375	18th
Sisodia and Diwivedi (1981)	0.194	0.0377	19th
Upadhyaya and Singh (1999)	0.194	0.0377	19th
Upadhyaya and Singh (1999)	0.194	0.0377	19th
Pandey and Dubey (1988)	0.194	0.0378	22nd
Singh (2003)	0.194	0.0378	22nd
Singh and Tailor (2003)	0.199	0.0397	24th
Singh et. al. (2004)	0.201	0.0405	25th
Singh (2003)	0.212	0.0448	26th
Singh (2003)	0.214	0.0457	27th
Singh (2003)	0.215	0.0463	28th
Upadhyaya and Singh (1999)	0.222	0.0491	29th
Thakur and Gupta (2013)	0.223	0.0496	30th
Product	0.224	0.05	31st
Singh (2001)	-0.3	0.0627	32nd
Chand (1975)	7.023	49.32	33rd

Table 4.16C: Bias and MSE of modified Estimators and some related Estimators under some Super-population models III using Data D1 $\left(\rho_{xy} > 0\right)$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.013	0.0002	1st
Modified $\bar{y}_{FTAA}^{(d)}$	-0.04	0.0014	2nd
Modified $\bar{y}_{FTAA}$	-0.04	0.0017	3rd
Product	0.092	0.0085	4th
Upadhyaya and Singh (1999)	0.093	0.0086	5th
Thakur and Gupta (2013)	0.093	0.0086	5th
Singh (2003)	0.096	0.009	7th
Singh (2003)	0.095	0.009	7th
Singh (2003)	0.096	0.0091	9th
Singh and Tailor (2003)	0.099	0.0097	10th
Singh et. al. (2004)	0.098	0.0097	10th
Sample mean	0.1	0.01	12th
Sisodia and Diwivedi (1981)	0.1	0.01	12th
Upadhyaya and Singh (1999)	0.1	0.01	12th
Pandey and Dubey (1988)	0.1	0.01	12th
Singh (2003)	0.1	0.01	12th
Upadhyaya and Singh (1999)	0.1	0.01	12th
Singh and Tailor (2003)	0.101	0.0103	18th
Singh et. al. (2004)	0.102	0.0104	19th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.102	0.0105	20th
Malik and Tailor (2013)	0.103	0.0106	21st
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.104	0.0108	22nd
Upadhyaya and Singh (1999)	0.107	0.0114	23rd
Cochran (1942)	0.107	0.0115	24th
Singh and Shukla (1987, 1993)	0.101	0.0117	25th
Shukla et al. (2013)	0.108	0.0117	25th
Srivenkataramana (1980)	0.116	0.0135	27th
Shukla (2002)	0.118	0.0139	28th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.118	0.014	29th
Singh and Upadhayaya (2001)	0.124	0.0154	30th
Singh et al. (2007)	0.197	0.0389	31st
Singh (2001)	0.216	0.0465	32nd
Chand (1975) 20	-1.68	2.8134	33rd

COMMENT: Tables 4.16A, 4.16B and 4.16C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively using data D1 with $\rho_{xy} > 0$. The robustness of the estimators were obtained by overall average of the MSEs and ranked accordingly with respect to their efficiency. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}$ emerged second, fifth, tenth, eighteenth positions respectively in Table 4.16A, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, twenty-nineth respectively in Table 4.16B and $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, twenty-nineth respectively in Table 4.16C.

From these results, it is deduced that:

i. The modified estimators $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{(d)}$ were highly efficient and robust competitively under linear model in comparisons with other estimators considered in the study.

ii. The modified estimators $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTAA}$ were highly efficient and robust competitively but $\bar{y}_{FTAA}^{\beta_1(d)}$ was outperformed by other other estimators considered in the study under quadratic and cubic models.

iii. The modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ outperformed all other three modified estimators.

iv. It is generally observed that the MSEs of the estimators increases as the order of the models increases. This implies that the more the relationship $\left(\rho_{xy}\right)$ between the study and auxiliary variables approaches linearity, the better their performance.

Table 4.17A: Bias and MSE of modified Estimators and some related Estimators under some Super-population models I using Data D2 $\left(\rho_{xy} < 0\right)$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	38.16	1467.05	1st
Modified $\bar{y}_{FTAA}^{(d)}$	39.34	1809.95	2nd
Singh et al. (2007)	42.71	2153.28	3rd
Singh (2001)	62.89	4065.17	4th
Singh and Upadhayaya (2001)	76.28	5850.67	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	78.68	6215.51	6th
Shukla (2002)	78.69	6215.59	7th
Srivenkataramana (1980)	78.69	6215.91	8th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	62.89	6227.19	9th
Malik and Tailor (2013)	78.81	6234.2	10th
Singh and Shukla (1987, 1993)	79.12	6283.08	11th
Shukla et al. (2013)	79.13	6284.64	12th
Singh and Tailor (2003)	79.2	6295.02	13th
Cochran (1942)	79.21	6297.24	14th
Upadhyaya and Singh (1999)	79.21	6297.26	15th
Singh et. al. (2004)	79.22	6298.63	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	79.57	6353.41	17th
Sisodia and Diwivedi (1981)	79.67	6368.69	18th
Jain and Shukla (2013) $\bar{y}_{FT1}$	79.72	6377.2	19th
Upadhyaya and Singh (1999)	79.75	6381.81	20th
Sample mean	79.95	6412.17	21st
Upadhyaya and Singh (1999)	80.14	6442.22	22nd
Singh (2003)	80.18	6449.64	23rd
Pandey and Dubey (1988)	80.22	6455.03	24th
Singh (2003)	80.25	6459.38	25th
Singh (2003)	80.3	6468.09	26th
Singh (2003)	80.34	6473.9	27th
Thakur and Gupta (2013)	80.62	6518.25	28th
Singh et. al. (2004)	80.64	6521.62	29th
Upadhyaya and Singh (1999)	80.65	6522.9	30th
Product	80.65	6522.91	31st
Singh and Tailor (2003)	80.66	6524.97	32nd
Chand (1975)	90.69	8234.28	33rd

Table 4.17B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II using Data D2 $\left(\rho_{xy} < 0\right)$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{(d)}$	36.71	1084.2	1st
Modified $\bar{y}_{FTAA}$	35.54	1747.97	2nd
Singh et al. (2007)	40.24	1948.05	3rd
Singh (2001)	60.38	3756.17	4th
Singh and Upadhayaya (2001)	73.75	5471.39	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	76.15	5823.47	6th
Shukla (2002)	76.15	5823.54	7th
Srivenkataramana (1980)	76.16	5823.85	8th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	60.36	5834.74	9th
Malik and Tailor (2013)	76.27	5841.52	10th
Singh and Shukla (1987, 1993)	76.59	5888.72	11th
Shukla et al. (2013)	76.6	5890.22	12th
Singh and Tailor (2003)	76.66	5900.25	13th
Cochran (1942)	76.68	5902.39	14th
Upadhyaya and Singh (1999)	76.68	5902.41	15th
Singh et. al. (2004)	76.69	5903.74	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	77.04	5956.65	17th
Sisodia and Diwivedi (1981)	77.14	5971.42	18th
Jain and Shukla (2013) $\bar{y}_{FT1}$	77.19	5979.63	19th
Upadhyaya and Singh (1999)	77.22	5984.09	20th
Sample mean	77.41	6013.42	21st
Upadhyaya and Singh (1999)	77.6	6042.46	22nd
Singh (2003)	77.65	6049.63	23rd
Pandey and Dubey (1988)	77.68	6054.83	24th
Singh (2003)	77.71	6059.04	25th
Singh (2003)	77.77	6067.46	26th
Singh (2003)	77.8	6073.09	27th
Thakur and Gupta (2013)	78.08	6115.93	28th
Singh et. al. (2004)	78.1	6119.19	29th
Upadhyaya and Singh (1999)	78.11	6120.42	30th
Product	78.11	6120.44	31st
Singh and Tailor (2003)	78.13	6122.43	32nd
Chand (1975)	88.14	7778.05	33rd

Table 4.17C: Bias and MSE of modified Estimators and some related Estimators under some Super-population models III using Data D2 $\left(\rho_{xy} < 0\right)$

Estimators	Bias	MSE	Rank
Singh et al. (2007)	27.03	1859.93	1st
Modified $\bar{y}_{FTAA}$	157.2	2091.92	2nd
Modified $\bar{y}_{FTAA}^{(d)}$	158.8	2614.66	3rd
Singh (2001)	54.55	3086.12	4th
Singh and Upadhayaya (2001)	72.81	5333.79	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	76.09	5814.43	6th
Shukla (2002)	76.09	5814.53	7th
Srivenkataramana (1980)	76.1	5814.96	8th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	76.2	5829.9	9th
Malik and Tailor (2013)	76.26	5839.2	10th
Singh and Shukla (1987, 1993)	76.69	5904.05	11th
Shukla et al. (2013)	76.7	5906.12	12th
Singh and Tailor (2003)	76.79	5919.92	13th
Cochran (1942)	76.81	5922.86	14th
Upadhyaya and Singh (1999)	76.81	5922.89	15th
Singh et. al. (2004)	76.82	5924.71	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	77.3	5997.54	17th
Sisodia and Diwivedi (1981)	77.44	6017.88	18th
Jain and Shukla (2013) $\bar{y}_{FT1}$	77.51	6029.2	19th
Upadhyaya and Singh (1999)	77.55	6035.35	20th
Sample mean	77.81	6075.79	21st
Upadhyaya and Singh (1999)	78.08	6115.87	22nd
Singh (2003)	78.14	6125.77	23rd
Pandey and Dubey (1988)	78.19	6132.96	24th
Singh (2003)	78.22	6138.76	25th
Singh (2003)	78.3	6150.39	26th
Singh (2003)	78.35	6158.15	27th
Thakur and Gupta (2013)	78.73	6217.4	28th
Singh et. al. (2004)	78.76	6221.91	29th
Upadhyaya and Singh (1999)	78.77	6223.62	30th
Product	78.77	6223.64	31st
Singh and Tailor (2003)	78.79	6226.39	32nd
Chand (1975)	92.47	8559.51	33rd

COMMENT: Table 4.17A, 4.17B and 4.17C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively using data D2 with $\rho_{xy} < 0$. The robustness of the estimators were obtained by overall average of the MSEs and ranked accordingly with respect to their efficiency. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged first, second, sixth, ninth positions respectively in Table 4.17A, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged first, second, sixth, ninth positions respectively in Table 4.17B and $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged second, third, sixth, ninth positions respectively in Table 4.17C.

From these results, it is deduced that:

i. The four modified estimators performed efficiently under linear, quadratic and cubic models as they emerged between first ten positions under all the models.

ii. The modified estimator $\bar{y}_{FTAA}$ outperformed all the remaining three modified estimators suggested.

iii. Generally, it is observed that the MSEs of the estimators decreases as the order of the models increases. This implies that the more the relationship (ρ_{xy}) between the study and auxiliary variables diverges from linearity, the better their performance.

4.2 EMPIRICAL STUDY USING SIMULATED DATA

In order to justify the results obtained in our last seminar using real life data, a numerical simulation study is conducted to investigate the efficiency and robustness of the modified factor-type estimators in comparison with existing related estimators considered in the study. The correlation coefficients between the study and auxiliary variables are assumed to be $\rho_{xy} = \pm 0.2$, $\rho_{xy} = \pm 0.5$, $\rho_{xy} = \pm 0.8$ and $\rho_{xy} = \pm 0.99$ using multivariate normal with the following parameters:

$N = 200,\ \bar{Y} = 1.51,\ \bar{X} = 2.02,\ \bar{Z} = 3.05,\ S_y = 0.241, S_x = 0.387,\ S_z = 0.239$

4.2.1 Efficiency Comparisons of $\bar{y}_{FTAA}$ with some related Estimators

In this subsection, predictive ability and relative efficiency of the modified estimator $\bar{y}_{FTAA}$ with respect to some related ratio, product and factor-type estimators are investigated.

4.2.1.1 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related ratio estimators were estimated under the condition that the study and auxiliary variables are positively correlated. The results are presented in Tables 4.18A and 4.18B.

Table 4.18A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators

Estimators	$\rho_{xy} = 0.2$				$\rho_{xy} = 0.5$			
	Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Sample mean $\bar{y}$	P>-5.579	0	$11.38X10^{-4}$	100	P>-13.984	0	$8.54X10^{-4}$	100
Cochran (1942) $\bar{y}_r$	P<11.636	$9.62X10^{-4}$	$23.10X10^{-4}$	49.273	P<3.275	$5.1\ X10^{-4}$	$11.1X10^{-4}$	77.58
Srivenkataramana (1980) $\bar{y}_{st}$	P< -1.276	$0.25X10^{-4}$	$11.06X10^{-4}$	102.90	P<-5.231	$-2.1X10^{-5}$	$6.47X10^{-4}$	131.95
Sisodia and Diwivedi (1981) t_2	P<10.431	$8.21X10^{-4}$	$21.15X10^{-4}$	53.813	P<2.066	$4.0X10^{-4}$	$9.74X10^{-4}$	87.62
Singh and Tailor (2003) t_3	P<10.402	$8.17X10^{-4}$	$21.11X10^{-4}$	53.927	P<0.482	$2.7X10^{-4}$	$8.45X10^{-4}$	101.07
Singh et. al. (2004) t_4	P<1.4202	$10.2X10^{-4}$	$23.80X10^{-4}$	47.826	P<-6.978	$-1.7x10^{-5}$	$6.45X10^{-4}$	132.45
Upadhyaya and Singh (1999) t_5	P<-3.551	$12.7X10^{-4}$	$27.34X10^{-4}$	41.643	P<-1.96	$-2.5X10^{-5}$	$7.66X10^{-4}$	111.44
Upadhyaya and Singh (1999) t_6	P<54.941	$18.4X10^{-4}$	$30.05X10^{-4}$	37.887	P<46.85	$4.8X10^{-4}$	$10.6X10^{-4}$	80.32
Modified $\bar{y}_{FTAA}$	P=-2.789	$-11.3X10^{-4}$	$10.72X10^{-4}$	106.15	P=-6.992	$-14X10^{-4}$	$6.40X10^{-4}$	133.33

Table 4.18B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Ratio estimators

Estimators	$\rho_{xy} = 0.8$				$\rho_{xy} = 0.99$			
	Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Sample mean $\bar{y}$	P>-8.387	0	$6.64X10^{-4}$	100	P>-20.497	0	$6.64X10^{-4}$	100
Cochran (1942) $\bar{y}_r$	P<-1.918	$2.4X10^{-4}$	$3.64X10^{-4}$	182.31	P< -7.721	$1.33X10^{-4}$	$5.28X10^{-4}$	125.75
Srivenkataramana (1980) $\bar{y}_{st}$	P<-1.918	$-6.3X10^{-5}$	$2.88X10^{-4}$	230.69	P< -17.30	$-9.3X10^{-4}$	$3.22X10^{-4}$	206.49
Sisodia and Diwivedi (1981) t_2	P<-2.371	$1.6X10^{-4}$	$3.09\ X10^{-4}$	214.61	P<-8.6152	$7.9\ X10^{-4}$	$2.97X10^{-4}$	223.27
Singh and Tailor (2003) t_3	P<-3.444	$3.3X10^{-5}$	$2.44X10^{-4}$	272.43	P<-11.254	$-3.8\ X10^{-4}$	$1.95X10^{-4}$	340.92
Singh et. al. (2004) t_4	P<-4.127	$-6.1X10^{-5}$	$2.82X10^{-4}$	235.43	P<-15.202	$-1.1X10^{-4}$	$1.65X10^{-4}$	401.74
Upadhyaya and Singh (1999) t_5	P<-3.318	$-4.6X10^{-5}$	$5.07X10^{-4}$	131.05	P<-18.948	$-5.5\ X10^{-4}$	$4.82X10^{-4}$	137.72
Upadhyaya and Singh (1999) t_6	P< 6.955	$2.03X10^{-4}$	$3.40X10^{-4}$	195.52	P<22.9896	$1.17X10^{-4}$	$4.52X10^{-4}$	147.07
Modified $\bar{y}_{FTAA}$	P=-4.194	$-12X10^{-4}$	$2.39X10^{-4}$	277.78	P= -20.25	$-5.5X10^{-4}$	$1.32X10^{-4}$	502.51

COMMENT: Tables 4.18A and 4.18B show the biases, MSEs and percentage relative efficiency (PRE) of $\bar{y}_{FTAA}$ and other existing related ratio estimators under simple random sampling scheme when the correlation between the study and auxiliary variables is at 0.2, 0.5, and 0.8, 0.99 respectively as well as the efficiency conditions of $\bar{y}_{FTAA}$ over others. The results showed that all the estimators with exception of sample mean estimator are biased. The results also revealed that when $P = -2.789$, $P = -6.992$, $P = -4.194$ and $P = -20.25$, $\bar{y}_{FTAA}$ has minimum MSE and high PRE and its efficiency conditions over others are satisfied.

4.2.1.2 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related product estimators were estimated under the condition that the study and auxiliary variables are negatively correlated. The results are presented in Tables 4.19A and 4.19B.

Table 4.19A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product estimators

Estimators	$\rho_{xy} = -0.2$				$\rho_{xy} = -0.5$			
	Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Sample mean $\bar{y}$	P<2.77677	0	$11.38X10^{-4}$	100	P<13.58	0	$8.54X10^{-4}$	100
Product	P<8.56757	$9.63\ X10^{-4}$	$23.10X10^{-4}$	49.272	P<16.76	$5.1X10^{-4}$	$11.01X10^{-4}$	77.58
Pandey and Dubey (1988) t_8	P<7.96768	$8.21\ X10^{-4}$	$21.15X10^{-4}$	53.814	P<15.59	$4.2X10^{-4}$	$9.99X10^{-4}$	85.50
Singh and Tailor (2003) t_9	P<9.28452	$11.5X10^{-4}$	$25.66X10^{-4}$	44.369	P<20.77	$8.9X10^{-4}$	$15.45X10^{-4}$	55.26
Singh (2003) t_{10}	P<7.44731	$7.06X10^{-4}$	$19.61X10^{-4}$	58.079	P<14.60	$3.5X10^{-4}$	$9.23X10^{-4}$	92.54
Singh et. al. (2004) t_{11}	P<4.01266	$48.6\ X10^{-4}$	$78.80X10^{-4}$	14.446	P<7.472	$10.1X10^{-4}$	$16.92X10^{-4}$	50.46
Upadhyaya and Singh (1999) t_{12}	P< 1.52602	$4.90\ X10^{-4}$	$20.41X10^{-4}$	55.788	P<22.72	$24.3X10^{-4}$	$39.82X10^{-4}$	24.42
Singh (2003) t_{13}	P<5.70437	$8.97\ X10^{-4}$	$22.21X10^{-4}$	51.253	P<21.8	$4.2\ X10^{-4}$	$9.98X10^{-4}$	85.56
Singh (2003) t_{14}	P<17.6578	$8.38\ X10^{-4}$	$21.41X10^{-4}$	53.183	P<14.46	$3.5\ X10^{-4}$	$9.22\ X10^{-4}$	92.65
Upadhyaya and Singh (1999) t_{15}	P<8.35364	$10.9X10^{-4}$	$24.84X10^{-4}$	45.832	P<16.38	$7.3\ X10^{-4}$	$13.47X10^{-4}$	63.38
Singh (2003) t_{16}	P<8.15055	$26.9X10^{-4}$	$26.92X10^{-4}$	42.291	P<16.02	$17.4X10^{-4}$	$17.39X10^{-4}$	49.07
Modified $\bar{y}_{FTAA}$	P=1.38839	$-3.46X10^{-4}$	$10.93X10^{-4}$	104.17	P=6.79	$-2.9X10^{-4}$	$6.40X10^{-4}$	133.3

Table 4.19B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Product estimators

Estimators	$\rho_{xy} = -0.8$				$\rho_{xy} = -0.99$			
	Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Sample mean $\bar{y}$	P<84.30	0	$9.80X10^{-4}$	100	P<9.3905	0	$6.64\ X10^{-4}$	100
Product	P<65.02	$3.48X10^{-4}$	$5.38X10^{-4}$	182.31	P<5.8533	$1.33\ X10^{-4}$	$5.28\ X10^{-4}$	125.75
Pandey and Dubey (1988) t_8	P<60.47	$2.6X10^{-4}$	$4.7X10^{-4}$	207.93	P<5.4434	$7.96\ X10^{-4}$	$2.97\ X10^{-4}$	223.27
Singh and Tailor (2003) t_9	P<94.08	$11.4X10^{-4}$	$13.05X10^{-4}$	75.09	P<9.475	$8.85\ X10^{-4}$	$6.88\ X10^{-4}$	96.561
Singh (2003) t_{10}	P<56.16	$1.8X10^{-4}$	$4.22X10^{4}$	232.18	P<5.0857	$3.88\ X10^{-4}$	$1.77\ X10^{-4}$	374.83
Singh et. al. (2004) t_{11}	P<28.02	$2.3X10^{-4}$	$4.50X10^{-4}$	217.66	P<7.919	$-1.04X10^{-4}$	$1.88X10^{-4}$	352.60
Upadhyaya and Singh (1999) t_{12}	P<8.37	$-10.1X10^{-4}$	$3.53\ X10^{-4}$	277.77	P<8.838	$-6.56X10^{-4}$	$4.42\ X10^{-4}$	150.20
Singh (2003) t_{13}	P<23.52	$4.7X10^{-4}$	$6.37X10^{-4}$	153.94	P<11.056	$13.6X10^{-4}$	$4.28\ X10^{-4}$	154.99
Singh (2003) t_{14}	P<13.95	$6.5\ X10^{-4}$	$7.99\ X10^{-4}$	122.73	P<17.271	$1.98\ X10^{-4}$	$2.87X10^{-4}$	231.72
Upadhyaya and Singh (1999) t_{15}	P<63.62	$1.07X10^{-4}$	$3.81\ X10^{-4}$	257.21	P<5.7021	$1.12\ X10^{-4}$	$4.31\ X10^{-4}$	153.91
Singh (2003) t_{16}	P<62.15	$3.5X10^{-4}$	$3.531X10^{-4}$	277.61	P<5.5579	$3.58\ X10^{-4}$	$3.52\ X10^{-4}$	188.72
Modified $\bar{y}_{FTAA}$	P=4.215	$3.65\ X10^{-4}$	$3.529X10^{-4}$	277.78	P=4.6952	$0.01X10^{-4}$	$1.32\ X10^{-4}$	502.51

COMMENT: Table 4.19A and 4.19B shows the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and other existing related product estimators under simple random sampling scheme when the study and auxiliary variables are negatively correlated with -0.2, -0.5 and -0.8, -0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}$ over others. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}$ over other product estimators considered in the study are satisfied when $P = 1.38839$, $P = 6.79$, $P = 4.215$ and $P = 4.6952$ and $\bar{y}_{FTAA}$ has minimum MSE and high PRE among others.

4.2.1.3 Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related factor-type estimators were estimated under the condition that the study and auxiliary variables are both positively and negatively correlated. The results are presented in Tables 4.20A-4.21B.

Table 4.20A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$

Estimators	Values of d	$\rho_{xy} = 0.2$				$\rho_{xy} = 0.5$			
		Efficiency Condition	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	P<-0.3063	-1.28X10^{-4}	10.97X10^{-4}	103.74	P<-0.766	-1.2X10^{-4}	7.30X10^{-4}	134.32
	5		-2.40X10^{-4}	11.04X10^{-4}	103.45		-1.48X10^{-4}	6.86 X10^{-4}	142.86
	10		-6.51X10^{-4}	17.13X10^{-4}	66.449		-2.4X10^{-4}	7.94 X10^{-4}	123.42
	15		-7.11X10^{-4}	19.56X10^{-4}	58.213		-2.1X10^{-4}	9.09 X10^{-4}	107.79
	20		-7.33X10^{-4}	20.61X10^{-4}	55.236		-1.99X10^{-4}	9.64 X10^{-4}	101.72
	30		-7.50X10^{-4}	21.55X10^{-4}	52.828		-1.86X10^{-4}	10.1X10^{-4}	96.68
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	P<-4.7619	0.49X10^{-4}	11.16X10^{-4}	102.04	P<-5.4033	-20.2X10^{-4}	6.67X10^{-4}	147.07
	5		0.49X10^{-4}	11.16X10^{-4}	102.04		-20.2X10^{-4}	6.67X10^{-4}	147.07
	10		0.49 X10^{-4}	11.16X10^{-4}	102.04		-20.2X10^{-4}	6.67X10^{-4}	147.07
	15		0.49X10^{-4}	11.16X10^{-4}	102.04		-20.2X10^{-4}	6.67X10^{-4}	147.07
	20		0.49X10^{-4}	11.16X10^{-4}	102.04		-20.2X10^{-4}	6.67 X10^{-4}	147.07
	30		0.49X10^{-4}	11.16X10^{-4}	102.04		-20.2X10^{-4}	6.67X10^{-4}	147.07
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	P<-8.2338	1.98X10^{-4}	12.66X10^{-4}	89.904	P<-4.9996	44.1X10^{-4}	6.45X10^{-4}	152.07
	5		1.98 X10^{-4}	12.66X10^{-4}	89.904		44.1X10^{-4}	6.45X10^{-4}	152.07
	10		1.98 X10^{-4}	12.66X10^{-4}	89.904		44.1 X10^{-4}	6.45 X10^{-4}	152.07
	15		1.98 X10^{-4}	12.66X10^{-4}	89.904		44.1X10^{-4}	6.45X10^{-4}	152.07
	20		1.98 X10^{-4}	12.66X10^{-4}	89.904		44.1X10^{-4}	6.45X10^{-4}	152.07
	30		1.98 X10^{-4}	12.66X10^{-4}	89.904		44.1X10^{-4}	6.45X10^{-4}	152.07
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	P<0.59807	0.12 X10^{-4}	11.84X10^{-4}	96.183	P<1.00882	24.9X10^{-4}	9.36X10^{-4}	104.77
	5		0.94 X10^{-4}	12.48X10^{-4}	91.234		86.8 X10^{-4}	9.84 X10^{-4}	99.65
	10		7.70 X10^{-4}	17.55X10^{-4}	64.850		6.34 X10^{-4}	14.3X10^{-4}	68.51
	15		9.77X10^{-4}	19.01X10^{-4}	59.898		8.02X10^{-4}	15.5X10^{-4}	63.35
	20		10.63X10^{-4}	19.61X10^{-4}	58.050		8.72X10^{-4}	15.9 X10^{-4}	61.44
	30		11.38X10^{-4}	20.14X10^{-4}	56.524		9.34 X10^{-4}	16.4X10^{-4}	59.86
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	P<-0.5807	-0.07 X10^{-4}	11.13X10^{-4}	102.33	P<-1.4522	-20.5X10^{-4}	7.85X10^{-4}	124.85
	5		-0.07X10^{-4}	11.12X10^{-4}	102.37		-16.4X10^{-4}	8.01X10^{-4}	122.48
	10		-0.08X10^{-4}	11.03X10^{-4}	103.18		-23.1X10^{-4}	7.75 X10^{-4}	126.47
	15		-0.07X10^{-4}	11.10X10^{-4}	102.54		-18.2X10^{-4}	7.94 X10^{-4}	123.47
	20		-0.06X10^{-4}	11.15X10^{-4}	102.06		-14.6X10^{-4}	8.07X10^{-4}	121.53
	30		-0.05X10^{-4}	11.22X10^{-4}	101.47		-10.4X10^{-4}	8.21X10^{-4}	119.41
Modified $\bar{y}_{FTAA}$	3	P=-9.7899	-0.01X10^{-4}	10.32X10^{-4}	110.31	P=-6.992	-28.9X10^{-4}	5.37X10^{-4}	158.99
	5		-0.02X10^{-4}	10.38X10^{-4}	109.63		-42.3X10^{-4}	5.22X10^{-4}	163.60
	10		-0.06X10^{-4}	10.35X10^{-4}	109.92		-13.5X10^{-4}	5.08X10^{-4}	167.95
	15		-0.06X10^{-4}	10.52X10^{-4}	108.15		-15.2X10^{-4}	5.18X10^{-4}	164.85
	20		-0.07X10^{-4}	10.61X10^{-4}	107.23		-15.9X10^{-4}	5.33X10^{-4}	160.20
	30		-0.07X10^{-4}	10.81X10^{-4}	105.30		-16.4X10^{-4}	5.42X10^{-4}	157.50

Table 4.20B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$

Estimators	Values of d	$\rho_{xy} = 0.8$				$\rho_{xy} = 0.99$			
		Efficiency Condition	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	P<-1.123	-1.12X10^{-4}	4.64 X10^{-4}	211.20	P<-1.4879	-7.15X10^{-4}	59.41X10^{-4}	67.63
	5		-92.1 X10^{-4}	4.36 X10^{-4}	224.81		-11.9X10^{-4}	63.11X10^{-4}	63.66
	10		27.3X10^{-4}	2.47 10^{-4}	366.46		-67.2X10^{-4}	146.1X10^{-4}	27.52
	15		77.7 X10^{-4}	2.81X10^{-4}	348.40		-83.2X10^{-4}	168.4X10^{-4}	23.86
	20		1.12 X10^{-4}	3.02 X10^{-4}	324.19		-89.9X10^{-4}	177.8X10^{-4}	22.60
	30		1.44 X10^{-4}	3.24 X10$^{-4.}$	302.77		-95.8X10^{-4}	186.1X10^{-4}	21.60
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	P<-2.025	-56.2X10^{-4}	3.99X10^{-4}	245.41	P<-3.53380	11.29X10^{-4}	68.55X10^{-4}	58.61
	5		-56.2X10^{-4}	3.99X10^{-4}	245.41		11.29X10^{-4}	68.55X10^{-4}	58.61
	10		-56.2X10^{-4}	3.99X10^{-4}	245.41		11.29 X10^{-4}	68.55X10^{-4}	58.61
	15		-56.2 X10^{-4}	3.99 X10^{-4}	245.41		11.29 X10^{-4}	68.55X10^{-4}	58.61
	20		-56.2 X10^{-4}	3.99 X10^{-4}	245.41		11.29 X10^{-4}	68.55X10^{-4}	58.61
	30		-56.2 X10^{-4}	3.99 X10^{-4}	245.41		11.29 X10^{-4}	68.55X10^{-4}	58.61
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	P<-3.374	-41.2 X10^{-4}	2.81 X10^{-4}	349.99	P<-6.11029	23.77 X10^{-4}	96.27X10^{-4}	41.74
	5		-41.2 X10^{-4}	2.81 X10^{-4}	349.99		23.77 X10^{-4}	96.27X10^{-4}	41.74
	10		-41.2 X10^{-4}	2.81 X10^{-4}	349.99		23.77 X10^{-4}	96.27X10^{-4}	41.74
	15		-41.2 X10^{-4}	2.81 X10^{-4}	349.99		23.77 X10^{-4}	96.27X10^{-4}	41.74
	20		-41.2 X10^{-4}	2.81 X10^{-4}	349.99		23.77 X10^{-4}	96.27X10^{-4}	41.74
	30		-41.2 X10^{-4}	2.81 X10^{-4}	349.99		23.77 X10^{-4}	96.27X10^{-4}	41.74
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	P<1.83486	30.7 X10^{-4}	7.62 X10^{-4}	128.70	P<3.07499	-2.30X10^{-4}	37.64X10^{-4}	106.8
	5		71.8 X10^{-4}	7.80 X10^{-4}	125.76		-0.52X10^{-4}	37.30X10^{-4}	107.7
	10		5.12 X10^{-4}	11.39X10^{-4}	86.03		13.6X10^{-4}	32.62X10^{-4}	123.2
	15		6.49 X10^{-4}	12.29X10^{-4}	79.76		19.2X10^{-4}	30.68X10^{-4}	131.1
	20		7.07X10^{-4}	12.66X10^{-4}	77.44		21.7X10^{-4}	30.46X10^{-4}	131.9
	30		7.59X10^{-4}	12.98X10^{-4}	75.52		23.9X10^{-4}	29.58X10^{-4}	135.8
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	P<-1.98057	-33.1 X10^{-4}	5.56 X10^{-4}	176.24	P<-2.77432	3.18 X10^{-4}	49.32X10^{-4}	81.47
	5		-20.7 X10^{-4}	5.99 X10^{-4}	163.67		1.79X10^{-4}	45.42X10^{-4}	88.46
	10		-31.1 X10^{-4}	5.63 X10^{-4}	174.02		2.94 X10^{-4}	48.64X10^{-4}	82.60
	15		-23.9 X10^{-4}	5.88 X10^{-4}	166.73		2.13 X10^{-4}	46.39X10^{-4}	86.62
	20		-19.0 X10^{-4}	6.04 X10^{-4}	162.21		1.63 X10^{-4}	44.96X10^{-4}	89.37
	30		-13.3 X10^{-4}	6.23 X10^{-4}	157.42		1.09 X10^{-4}	43.42X10^{-4}	92.54
Modified $\bar{y}_{FTAA}$	3	P=-4.1937	-11.8 X10^{-4}	1.88 X10^{-4}	352.31	P=-10.2484	-0.98X10^{-4}	37.27X10^{-4}	107.8
	5		-13.0 X10^{-4}	1.78 X10^{-4}	374.41		-1.09X10^{-4}	36.79X10^{-4}	109.2
	10		-40.4 X10^{-4}	1.72 X10^{-4}	386.76		-3.42X10^{-4}	28.77X10^{-4}	139.7
	15		-45.3 X10^{-4}	1.73 X10^{-4}	384.39		-3.85X10^{-4}	27.17X10^{-4}	147.9
	20		-47.1 X10^{-4}	1.79 X10^{-4}	373.35		-4.01X10^{-4}	26.55X10^{-4}	151.3
	30		-48.7 X10^{-4}	1.82 X10$^{-4.}$	364.49		-4.16X10^{-4}	26.02X10^{-4}	154.4

Tables 4.20A and 4.20B show the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and other existing related factor-type estimators under simple random sampling scheme when the study and auxiliary variables are positively correlated with 0.2, 0.5 and 0.8, 0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}$ over others. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}$ over other product estimators considered in the study when $P = -9.7899$, $P = -6.992$, $P = -4.1937$ and $P = -10.2484$ are satisfied and that $\bar{y}_{FTAA}$ has minimum MSE and high PRE among others.

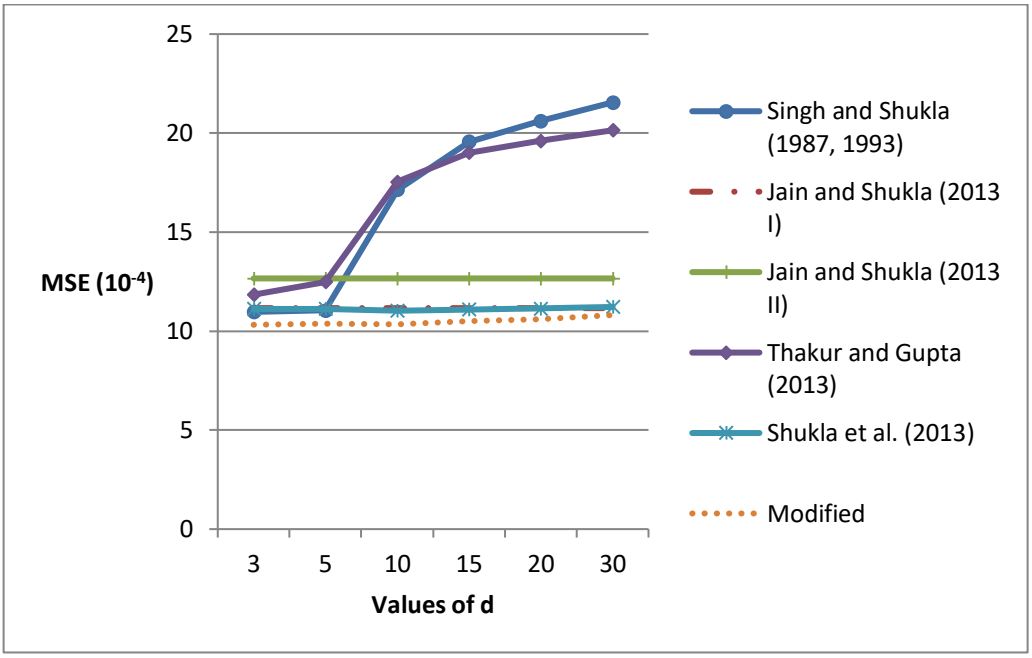


Figure 4.2.1A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.2$

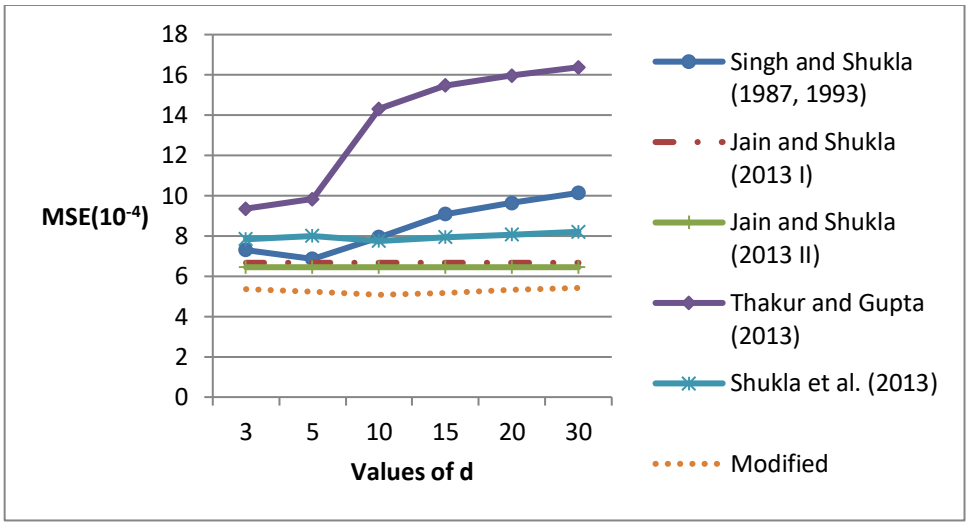


Figure 4.2.2A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.5$

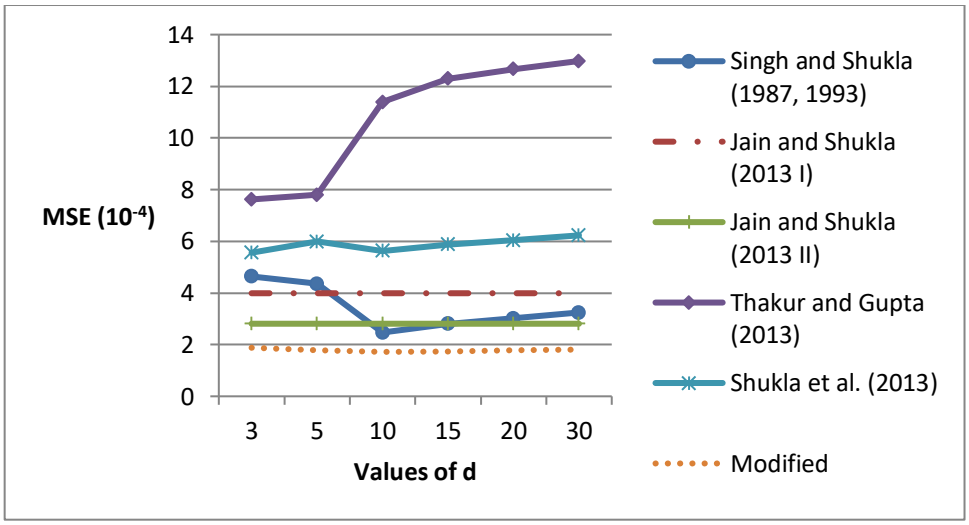


Figure 4.2.3A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.8$

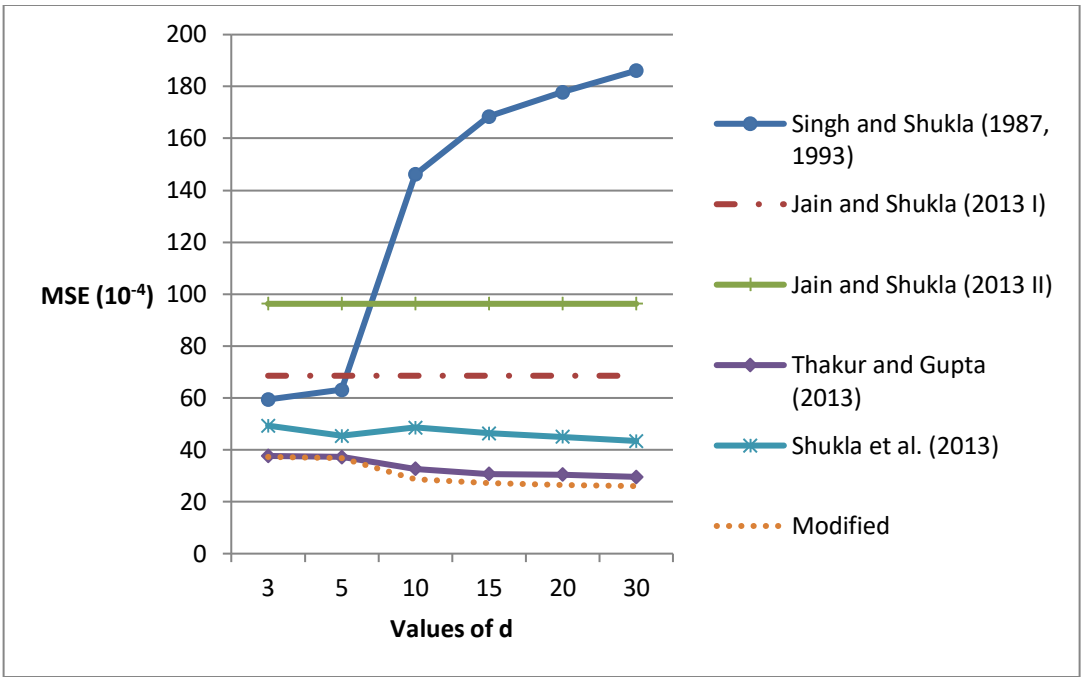


Figure 4.2.4A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.99$

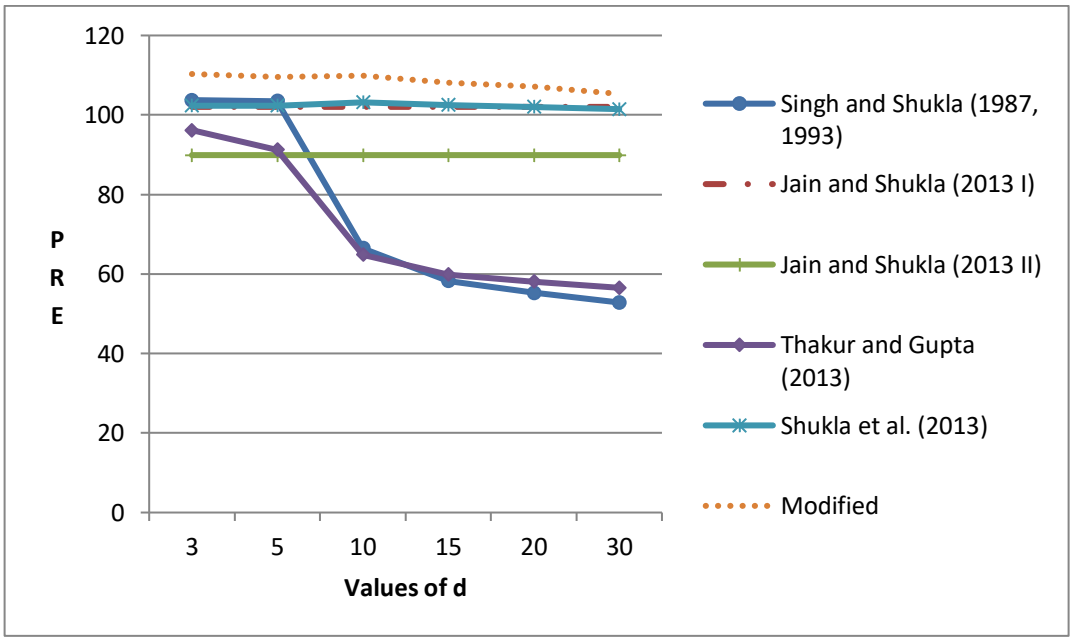


Figure 4.2.1B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.2$

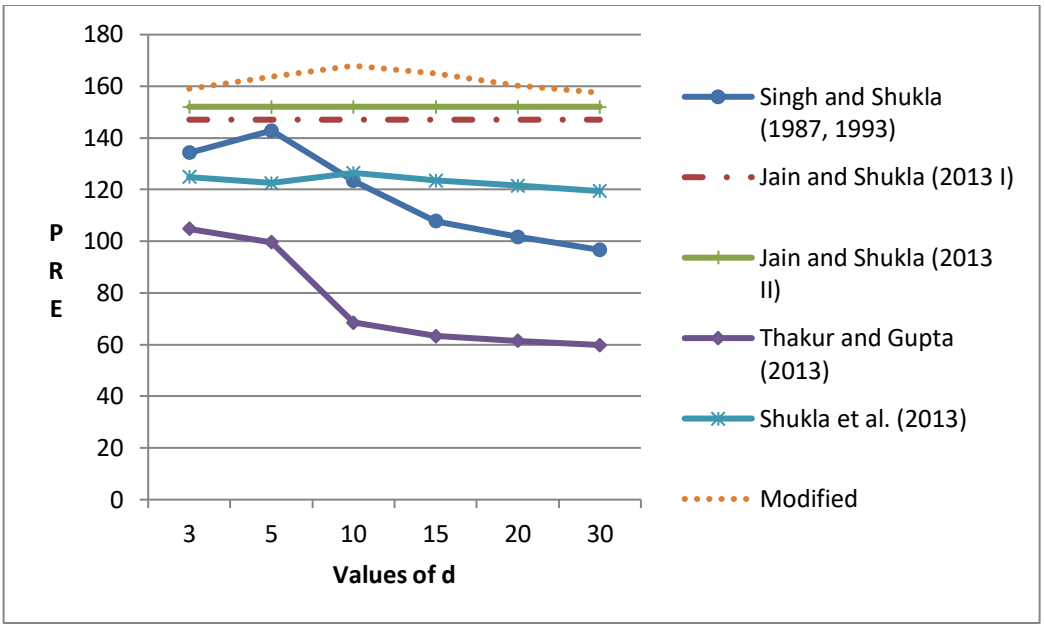


Figure 4.2.2B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.5$

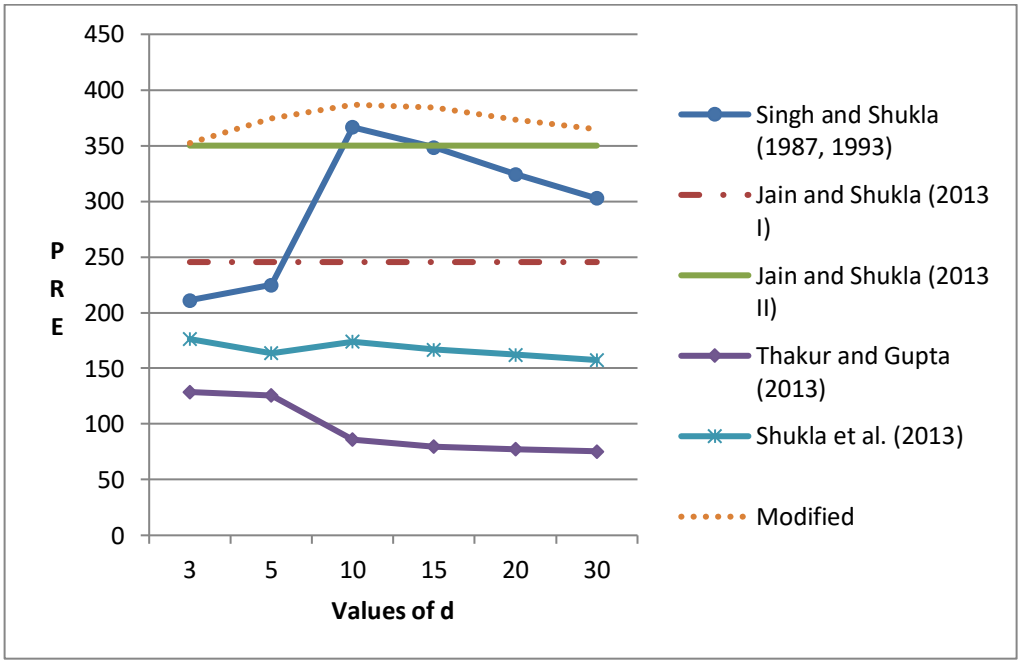


Figure 4.23B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.8$

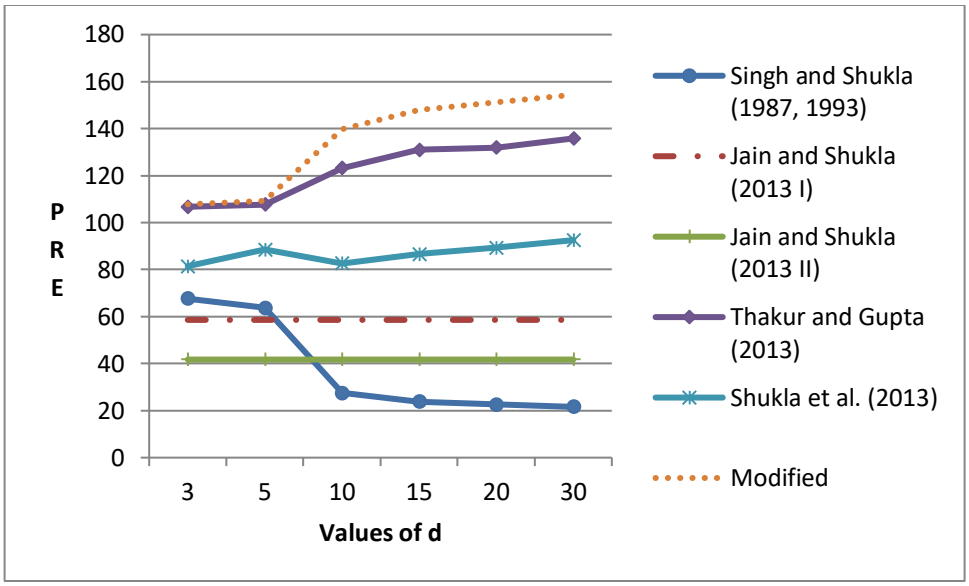


Figure 4.2.4B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = 0.99$

Figures (4.2.1A – 4.2.4A) and (4.2.1B – 4.2.4B) show the behavioural pattern of MSEs and PRE of estimators considered in Tables 4.20A and 4.20B respectively. The plots generally exhibit upward and downward movement respectively as the values of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

Table 4.21A: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$

Estimators	Values of d	$\rho_{xy} = -0.2$				$\rho_{xy} = -0.5$			
		Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	P>0.2902274	-1.28X10^{-4}	12.22X10^{-4}	93.139	P>0.862	-1.23X10^{-4}	10.31X10^{-4}	95.09
	5		-2.84X10^{-4}	13.57X10^{-4}	83.905		-2.27X10^{-4}	11.44X10^{-4}	85.66
	10		-10.9X10^{-4}	25.68X10^{-4}	44.337		-10.5X10^{-4}	23.73X10^{-4}	41.31
	15		-12.8X10^{-4}	29.31X10^{-4}	38.846		-12.7X10^{-4}	27.22X10^{-4}	36.01
	20		-13.6X10^{-4}	30.83X10^{-4}	36.926		-13.6X10^{-4}	28.69X10^{-4}	34.17
	30		-14.2X10^{-4}	32.17X10^{-4}	35.390		-14.4X10^{-4}	29.92X10^{-4}	32.71
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	P>-2.6825	1.53 X10^{-4}	14.27X10^{-4}	79.803	P>5.24725	1.63X10^{-4}	12.21X10^{-4}	80.28
	5		1.53X10^{-4}	14.27X10^{-4}	79.803		1.63X10^{-4}	12.21X10^{-4}	80.28
	10		1.53 X10^{-4}	14.27X10^{-4}	79.803		1.63X10^{-4}	12.21X10^{-4}	80.28
	15		1.53 X10^{-4}	14.27X10^{-4}	79.803		1.63X10^{-4}	12.21X10^{-4}	80.28
	20		1.53 X10^{-4}	14.27X10^{-4}	79.803		1.63 X10^{-4}	12.21X10^{-4}	80.28
	30		1.53 X10^{-4}	14.27X10^{-4}	79.803		1.63 X10^{-4}	12.21X10^{-4}	80.28
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	P>-4.4675	3.76 X10^{-4}	18.04X10^{-4}	63.111	P>5.73969	3.67X10^{-4}	16.2X10^{-4}	60.55
	5		3.76 X10^{-4}	18.04X10^{-4}	63.111		3.67X10^{-4}	16.2X10^{-4}	60.55
	10		3.76 X10^{-4}	18.04X10^{-4}	63.111		3.67X10^{-4}	16.19X10^{-4}	60.55
	15		3.76 X10^{-4}	18.04X10^{-4}	63.111		3.67X10^{-4}	16.19X10^{-4}	60.55
	20		3.76 X10^{-4}	18.04X10^{-4}	63.111		3.67X10^{-4}	16.19X10^{-4}	60.55
	30		3.76 X10^{-4}	18.04X10^{-4}	63.111		3.67X10^{-4}	16.19X10^{-4}	60.55
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	P>-0.4354	-0.12X10^{-4}	11.09X10^{-4}	102.68	P>0.878	-24.9X10^{-4}	7.85X10^{-4}	124.9
	5		0.44 X10^{-4}	10.94X10^{-4}	104.06		10.9 X10^{-4}	7.55X10^{-4}	129.9
	10		6.01 X10^{-4}	12.43X10^{-4}	91.60		3.73 X10^{-4}	6.42X10^{-4}	152.8
	15		7.83 X10^{-4}	13.16X10^{-4}	86.532		5.02 X10^{-4}	6.41X10^{-4}	152.9
	20		8.60X10^{-4}	13.48X10^{-4}	84.458		5.57X10^{-4}	6.43X10^{-4}	152.4
	30		9.27 X10^{-4}	13.77X10^{-4}	82.677		6.06X10^{-4}	6.46X10^{-4}	151.8
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	P>0.52553	0.14 X10^{-4}	11.75X10^{-4}	96.887	P>1.84009	29.3 X10^{-4}	9.36 X10^{-4}	104.8
	5		0.14 X10^{-4}	11.76X10^{-4}	96.796		21.5 X10^{-4}	9.15 X10^{-4}	107.2
	10		0.24 X10^{-4}	11.98X10^{-4}	95.006		35.1 X10^{-4}	9.51 X10^{-4}	103.1
	15		0.16 X10^{-4}	11.79X10^{-4}	96.485		24.7 X10^{-4}	9.23 X10^{-4}	106.2
	20		0.12 X10^{-4}	11.69X10^{-4}	97.360		18.5X10^{-4}	9.07 X10^{-4}	108.1
	30		0.73 X10^{-4}	11.59X10^{-4}	98.261		12.2X10^{-4}	8.89 X10^{-4}	110.2
Modified $\bar{y}_{FTAA}$	3	P=1.38839	0.24 X10^{-4}	10.34X10^{-4}	109.34	P=6.7902	-2.97X10^{-4}	6.58 X10^{-4}	129.7
	5		0.59 X10^{-4}	10.41X10^{-4}	108.57		-4.35X10^{-4}	6.22 X10^{-4}	137.2
	10		0.26 X10^{-4}	10.80X10^{-4}	104.72		-13.8X10^{-4}	5.34X10^{-4}	159.8
	15		0.31 X10^{-4}	10.89X10^{-4}	103.84		-15.6X10^{-4}	5.28X10^{-4}	161.9
	20		0.33 X10^{-4}	10.93X10^{-4}	103.50		-16.3X10^{-4}	5.34X10^{-4}	159.6
	30		0.35 X10^{-4}	10.96X10^{-4}	103.20		-16.9X10^{-4}	5.43 X10^{-4}	157.2

Table 4.21B: Bias, MSE and PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$

Estimators	Values of d	$\rho_{xy} = -0.8$				$\rho_{xy} = -0.99$			
		Efficiency conditions	Bias	MSE	PRE	Efficiency conditions	Bias	MSE	PRE
Singh and Shukla (1987, 1993) $\bar{y}_{FT}$	3	P>1.317	$-1.25X10^{-4}$	$12.50X10^{-4}$	78.44	P>1.93487	$-10.4X10^{-4}$	$87.79X10^{-4}$	67.63
	5		$-2.95X10^{-4}$	$14.85X10^{-4}$	65.99		$-17.5X10^{-4}$	$93.26X10^{-4}$	63.66
	10		$-15.1X10^{-4}$	$32.91X10^{-4}$	29.78		$-29.2X10^{-4}$	$215.8X10^{-4}$	27.52
	15		$18.3\ X10^{-4}$	$37.66X10^{-4}$	26.03		$-22.8X10^{-4}$	$248.9X10^{-4}$	23.86
	20		$-19.7X10^{-4}$	$39.62X10^{-4}$	24.74		$-32.\ X10^{-4}$	$262.7X10^{-4}$	22.60
	30		$-20.8X10^{-4}$	$41.33X10^{-4}$	23.72		$-46.3X10^{-4}$	$286.1X10^{-4}$	21.59
Jain and Shukla (2013) $\bar{y}_{FT1}$	3	P>20.37	$2.58\ X10^{-4}$	$16.11X10^{-4}$	60.87	P>1.46332	$16.7X10^{-4}$	$101.3X10^{-4}$	58.61
	5		$2.58X10^{-4}$	$16.11X10^{-4}$	60.87		$16.7\ X10^{-4}$	$101.3X10^{-4}$	58.61
	10		$2.58X10^{-4}$	$16.11X10^{-4}$	60.87		$16.7\ X10^{-4}$	$101.3X10^{-4}$	58.61
	15		$2.58X10^{-4}$	$16.11X10^{-4}$	60.87		$16.7\ X10^{-4}$	$101.3X10^{-4}$	58.61
	20		$2.58X10^{-4}$	$16.11X10^{-4}$	60.87		$16.7\ X10^{-4}$	$101.3X10^{-4}$	58.61
	30		$2.58X10^{-4}$	$16.11X10^{-4}$	60.87		$16.7\ X10^{-4}$	$101.3X10^{-4}$	58.61
Jain and Shukla (2013) $\bar{y}_{FT2}$	3	P>33.9058	$5.43X10^{-4}$	$22.21X10^{-4}$	44.14	P>2.61530	$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
	5		$5.43X10^{-4}$	$22.21X10^{-4}$	44.14		$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
	10		$5.43X10^{-4}$	$22.21X10^{-4}$	44.14		$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
	15		$5.43X10^{-4}$	$22.21X10^{-4}$	44.14		$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
	20		$5.43X10^{-4}$	$22.21X10^{-4}$	44.14		$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
	30		$5.43X10^{-4}$	$22.21X10^{-4}$	44.14		$35.1\ X10^{-4}$	$142.3X10^{-4}$	41.74
Thakur and Gupta (2013) $\bar{y}_{FTRP}$	3	P>1.4791	$-44.7X10^{-4}$	$8.53\ X10^{-4}$	114.98	P>2.81041	$-3.39X10^{-4}$	$49.58X10^{-4}$	119.7
	5		$-20.8X10^{-4}$	$7.66\ X10^{-4}$	128.04		$35.1X10^{-4}$	$47.59X10^{-4}$	124.7
	10		$3.18X10^{-4}$	$4.34\ X10^{-4}$	225.83		$20.1\ X10^{-4}$	$41.47X10^{-4}$	143.2
	15		$4.48X10^{-4}$	$3.98\ X10^{-4}$	246.50		$28.4\ X10^{-4}$	$42.12X10^{-4}$	140.9
	20		$5.05\ X10^{-4}$	$3.86\ X10^{-4}$	253.74		$31.9\ X10^{-4}$	$38.57X10^{-4}$	153.9
	30		$5.54X10^{-4}$	$3.78\ X10^{-4}$	259.34		$23.9X10^{-4}$	$38.07X10^{-4}$	155.8
Shukla et. al. (2013) $\bar{y}_{TFT}$	3	P>2.67510	$44.6X10^{-4}$	$11.90X10^{-4}$	88.40	P>4.87401	$4.71X10^{-4}$	$72.88X10^{-4}$	81.47
	5		$38.8X10^{-4}$	$10.93X10^{-4}$	89.71		$2.65X10^{-4}$	$67.12X10^{-4}$	88.46
	10		$60.7X10^{-4}$	$11.53X10^{-4}$	85.013		$4.34\ X10^{-4}$	$71.88X10^{-4}$	82.60
	15		$43.3X10^{-4}$	$11.05X10^{-4}$	88.682		$3.15\ X10^{-4}$	$68.54X10^{-4}$	86.62
	20		$32.8\ X10^{-4}$	$10.76X10^{-4}$	91.101		$2.41\ X10^{-4}$	$66.44X10^{-4}$	89.37
	30		$21.9\ X10^{-4}$	$10.45X10^{-4}$	93.825		$1.09\ X10^{-4}$	$43.42X10^{-4}$	92.54
Modified $\bar{y}_{FTAA}$	3	P=42.148	$-1.25X10^{-4}$	$8.18X10^{-4}$	119.77	P=5.69523	$-1.09X10^{-4}$	$47.20X10^{-4}$	125.8
	5		$-21.8X10^{-4}$	$7.32X10^{-4}$	133.74		$-1.22X10^{-4}$	$46.81X10^{-4}$	126.8
	10		$-7.31X10^{-4}$	$4.16X10^{-4}$	235.62		$3.98\ X10^{-4}$	$38.60X10^{-4}$	153.8
	15		$-8.34X10^{-4}$	$3.79X10^{-4}$	259.58		$4.65\ X10^{-4}$	$39.14X10^{-4}$	151.7
	20		$-8.74X10^{-4}$	$3.69X10^{-4}$	265.74		$4.92\ X10^{-4}$	$37.86X10^{-4}$	156.8
	30		$-9.09X10^{-4}$	$3.65X10^{-4}$	268.34		$4.05\ X10^{-4}$	$37.63X10^{-4}$	157.8

COMMENT: Tables 4.21A and 4.21B show the biases, MSEs and PRE of $\bar{y}_{FTAA}$ and other existing related factor-type estimators under simple random sampling scheme when the

study and auxiliary variables are negatively correlated with -0.2, -0.5 and -0.8, -0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}$ over others. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}$ over other product estimators considered in the study when $P=1.38839$, $P=6.7902$, $P=42.148$ and $P=5.69523$ are satisfied and that $\bar{y}_{FTAA}$ has minimum MSE and high PRE among others. These results are shown in figures 4.2.5A-4.2.8B below.

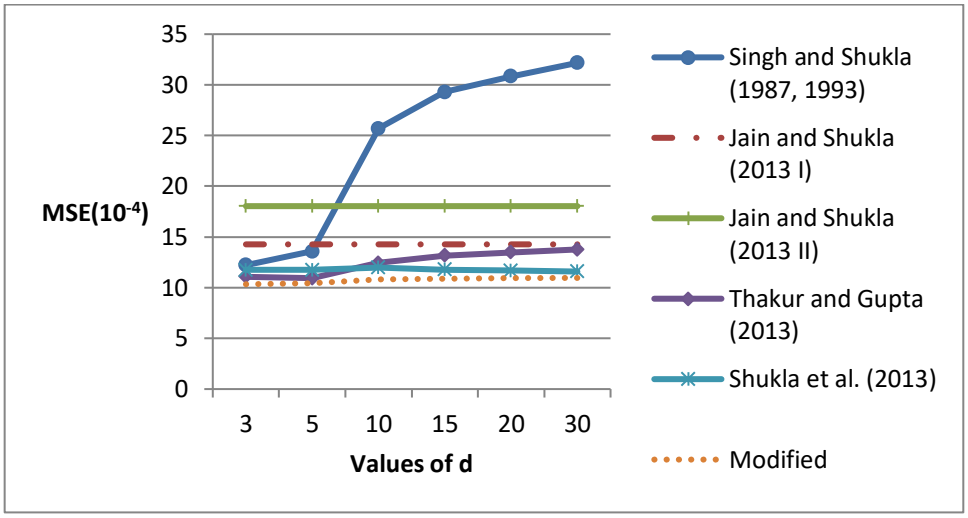


Figure 4.2.5A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy}=-0.2$

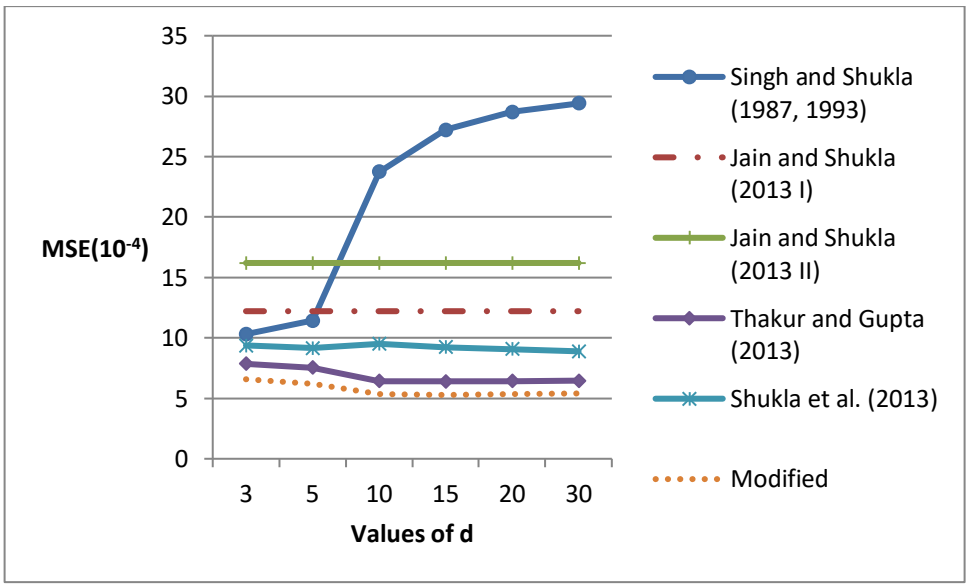


Figure 4.2.6A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.5$

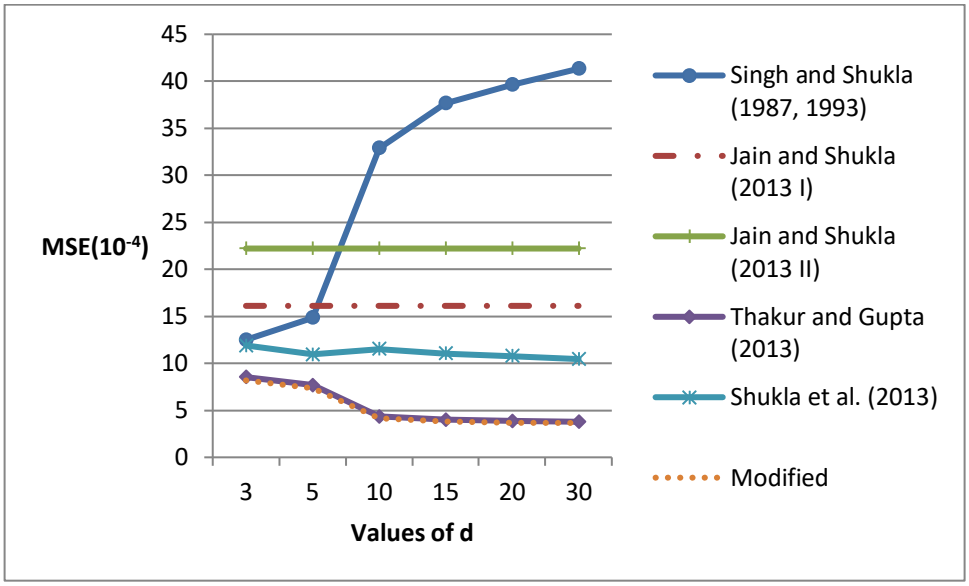


Figure 4.2.7A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.8$

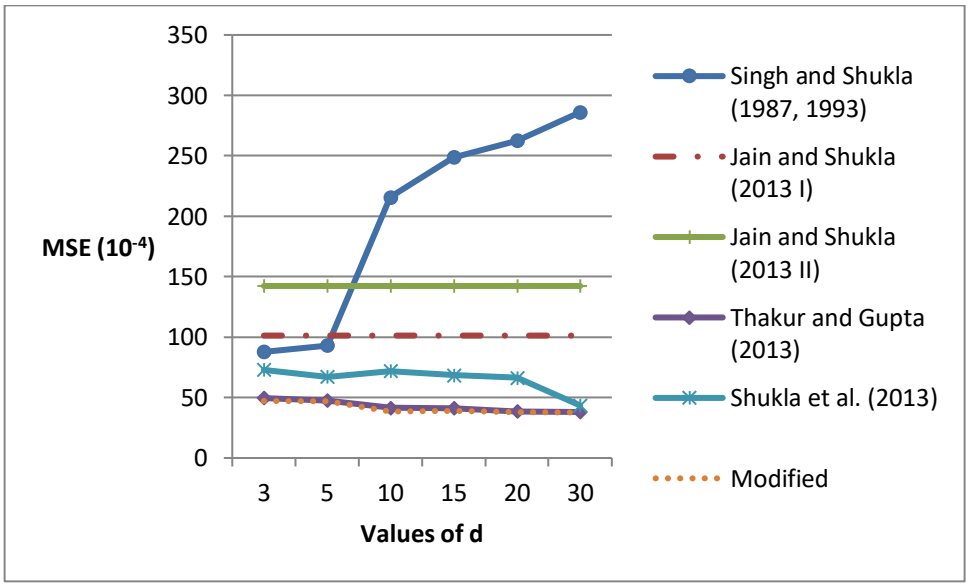


Figure 4.2.8A: Graph showing MSE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.99$

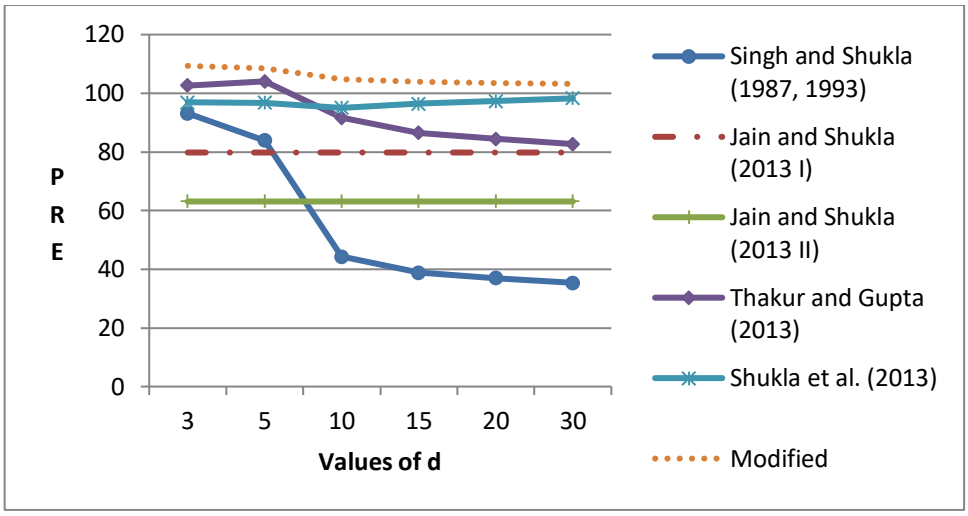


Figure 4.2.5B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.2$

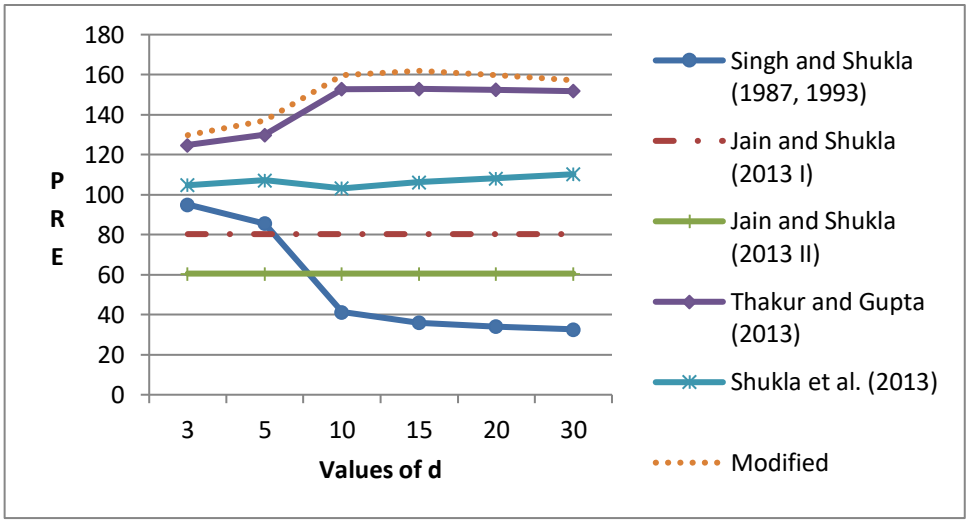


Figure 4.2.6B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.5$

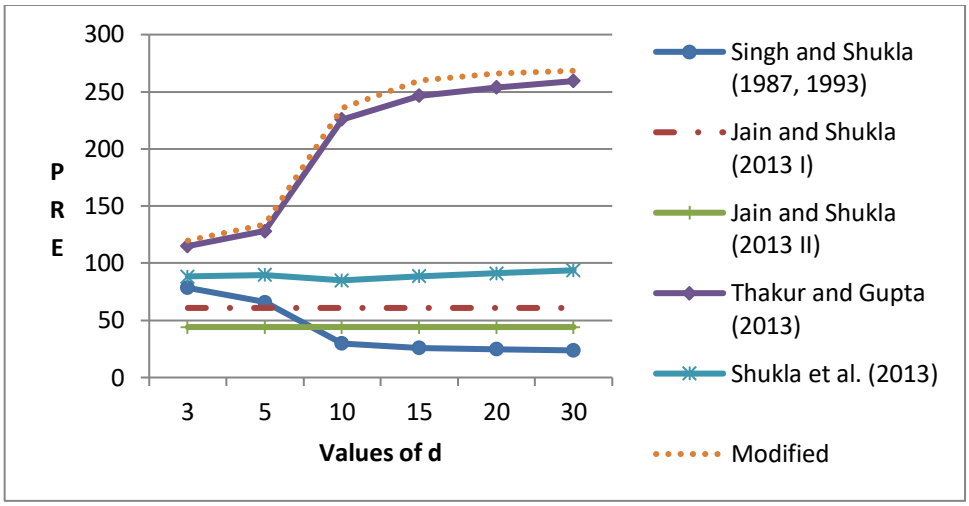


Figure 4.2.7B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.8$

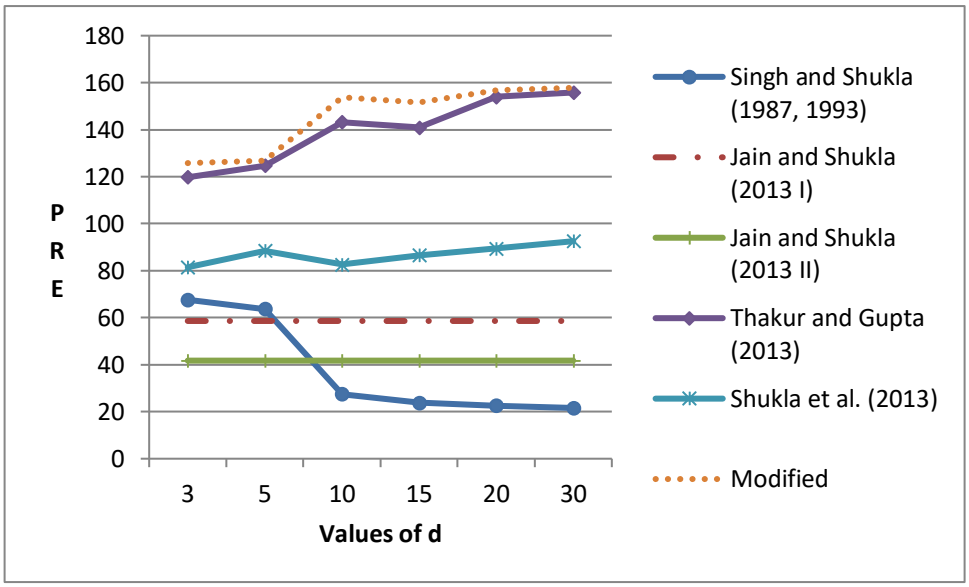


Figure 4.2.8B: Graph showing PRE of $\bar{y}_{FTAA}$ and some existing related Factor-type Estimators when $\rho_{xy} = -0.99$

Figures (4.2.5A – 4.2.8A), and (4.2.5B – 4.2.8B) show the behavioural pattern of MSEs and PRE of estimators considered in Tables 4A and 4B respectively. The plots generally exhibit upward and downward movement respectively as the values of d increases. This implies

that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

4.2.2 Efficiency Comparison of $\bar{y}_{FTAA}^{(d)}$ with some existing related Estimators

In this subsection, relative efficiency of estimator $\bar{y}_{FTAA}^{(d)}$ with respect to some existing related ratio, product and factor-type estimators for some correlation coefficients were investigated and the results are presented in Tables below.

4.2.2.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some existing related Ratio Estimators

Bias, MSE and PRE of estimator $\bar{y}_{FTAA}^{(d)}$ and some related ratio estimators were estimated under the condition that the study and auxiliary variables are positively correlated. The results are presented in Tables 4.22A and 4.22B.

Table 4.22A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$

Estimators	$\rho_{xy} = 0.2$				$\rho_{xy} = 0.5$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean $\bar{y}$	P<-1.57982	0	$11.38X10^{-4}$	100	P<-13.984	0	$8.54\ X10^{-4}$	100
Sukhatme (1962) t_1	P<11.6364	$7.22X10^{-4}$	$20.17X10^{-4}$	56.429	P<3.27481	$3.98X10^{-4}$	$10.46X10^{-4}$	81.65
Srivenkataramana (1980) two-phase	P<-1.27577	$11.4X10^{-4}$	$28.60X10^{-4}$	39.799	P< -8.231	$12.1X10^{-4}$	$26.85X10^{-4}$	31.80
Malik and Tailor (2013)	P<10.4023	$6.13X10^{-4}$	$18.68X10^{-4}$	60.947	P<0.4821	$2.13X10^{-4}$	$8.47X10^{-4}$	100.83
Modified $\bar{y}_{FTAA}^{(d)}$	P=-2.78991	$-29X10^{-4}$	$11.04X10^{-4}$	103.09	P=-16.99	$-23.9X10^{-4}$	$6.88X10^{-4}$	124.14
CASE II								
Sample mean $\bar{y}$	P<-1.57982	0	$11.38X10^{-4}$	100	P<-13.984	0	$8.54X10^{-4}$	100
Sukhatme (1962) t_1	P<12.7524	$12.5X10^{-4}$	$27.44X10^{-4}$	41.488	P<5.81741	$7.04\ X10^{-4}$	$13.90\ X10^{-4}$	61.444
Srivenkataramana (1988) two-phase	P<12.7524	$16.2X10^{-4}$	$38.68X10^{-4}$	29.432	P<14.447	$14.1X10^{-4}$	$34.97X10^{-4}$	24.415
Malik and Tailor (2013)	P<11.5182	$10.6X10^{-4}$	$24.85X10^{-4}$	45.819	P<3.02467	$3.97X10^{-4}$	$10.31X10^{-4}$	82.856
Modified $\bar{y}_{FTAA}^{(d)}$	P= -2.2319	$-12X10^{-4}$	$11.02X10^{-4}$	103.31	P=-15.721	$-13.2X10^{-4}$	$6.79X10^{-4}$	125.71

Table 4.22B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$

Estimators	$\rho_{xy} = 0.8$				$\rho_{xy} = 0.99$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean $\bar{y}$	P<-8.3875	0	6.64 X10^{-4}	100	P<-2.4967	0	6.64 X10^{-4}	100
Sukhatme (1962) t_1	P<-1.9177	1.35X10^{-4}	4.93 X10^{-4}	134.77	P<-7.72067	9.47X10^{-4}	2.27 X10^{-4}	291.96
Srivenkataramana (1980) two-phase	P<-6.2309	8.79X10^{-4}	19.91X10^{-4}	33.35	P<-1.3027	12.5X10^{-4}	25.46X10^{-4}	26.09
Malik and Tailor (2013)	P<-3.4445	18.7X10^{-4}	4.24X10^{-4}	156.66	P<-9.2537	-2.7X10^{-4}	2.034X10^{-4}	326.09
Modified $\bar{y}_{FTAA}^{(d)}$	P=-14.194	-24X10^{-4}	4.21X10^{-4}	157.66	P=-10.2484	-17X10^{-4}	1.99 X10^{-4}	333.41
CASE II								
Sample mean $\bar{y}$	P<-8.3875	0	6.64 X10^{-4}	100	P<-2.49673	0	6.64 X10^{-4}	100
Sukhatme (1962) t_1	P<0.59854	5.23X10^{-4}	7.98X10^{-4}	83.243	P<3.16584	3.24X10^{-4}	3.42 X10^{-4}	194.28
Srivenkataramana (1988) two-phase	P<0.59854	13.9X10^{-4}	34.20X10^{-4}	19.415	P<3.22219	13.9X10^{-4}	35.87X10^{-4}	18.511
Malik and Tailor (2013)	P<-0.9283	1.80X10^{-4}	4.66X10^{-4}	142.45	P<-6.69889	6.20X10^{-4}	1.71 X10^{-4}	388.93
Modified $\bar{y}_{FTAA}^{(d)}$	P=-12.936	-17X10^{-4}	3.67X10^{-4}	181.16	P=-7.97095	-11X10^{-4}	1.58 X10^{-4}	420.70

Tables 4.22A and 4.22B show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and other related ratio estimators under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated with 0.2, 0.5 and 0.8, 0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over others. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over other product estimators considered in the study when $P=-2.78991$, $P=-2.2319$, $P=-16.99$, $P=-15.721$, $P=-14.194$, $P=-12.936$, $P=-10.2484$ and $P=-7.97095$ are satisfied and that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE among others.

4.2.2.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some Related Product Estimators

Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related product estimators were estimated under the condition that the study and auxiliary variables are negatively correlated. The results are presented in Tables 4.23A and 4.23B.

Table 4.23A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$

Estimators	$\rho_{xy} = -0.2$				$\rho_{xy} = -0.5$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean $\bar{y}$	P<2.77677	0	11.38 X10^{-4}	100	P<13.5804	0	8.54 X10^{-4}	100
Product $\bar{y}_p$	P<11.3443	9.27X10^{-4}	27.33 X10^{-4}	41.656	P<30.3410	6.05X10^{-4}	20.31X10^{-4}	42.04
Srivenkataramana (1980) two-phase	P<11.3443	5.39X10^{-4}	19.52 X10^{-4}	58.311	P<19.1673	81.7X10^{-4}	9.77X10^{-4}	87.374
Malik and Tailor (2013)	P<12.0613	10.8X10^{-4}	29.75 X10^{-4}	38.260	P<34.3507	8.78X10^{-4}	25.05X10^{-4}	34.079
Modified $\bar{y}_{FTAA}^{(d)}$	P=1.38839	-15X10^{-4}	11.06786	102.86	P=6.79012	-12X10^{-4}	7.47X10^{-4}	114.29
CASE II								
Sample mean $\bar{y}$	P<2.77677	0	11.38 X10^{-4}	100	P<13.5804	0	8.54X10^{-4}	100
Product $\bar{y}_p$	P<10.6945	16.9X10^{-4}	39.64 X10^{-4}	28.717	P<25.8142	16.4X10^{-4}	38.58X10^{-4}	22.129
Srivenkataramana (1988) two-phase	P<9.13672	13.1X10^{-4}	28.40 X10^{-4}	40.081	P<20.2273	9.43X10^{-4}	17.51X10^{-4}	48.766
Malik and Tailor (2013)	P<11.4114	19.6X10^{-4}	44.06 X10^{-4}	25.838	P<29.8239	24.2X10^{-4}	51.55X10^{-4}	16.561
Modified $\bar{y}_{FTAA}^{(d)}$	P=1.06344	-22X10^{-4}	11.04 X10^{-4}	103.16	P=4.52680	-20X10^{-4}	7.12X10^{-4}	120.03

Table 4.23B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and some related estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$

Estimators	$\rho_{xy} = -0.8$				$\rho_{xy} = -0.99$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean $\bar{y}$	P<84.297	0	9.81 X10^{-4}	100	P<9.3905	0	3.84X10^{-4}	100
Product $\bar{y}_p$	P<149.32	7.51 X10^{-4}	25.61X10^{-4}	38.29	P<15.244	7.84 X10^{-4}	3.01 X10^{-4}	127.96
Srivenkataramana (1980) two-phase	P<103.17	-1.35X10^{-4}	7.76 X10^{-4}	126.27	P<10.854	-2.63 X10^{-4}	1.54 X10^{-4}	248.58
Malik and Tailor (2013)	P<178.38	13.8X10^{-4}	37.12X10^{-4}	26.411	P<18.865	17.01 X10^{-4}	2.37 X10^{-4}	162.33
Modified $\bar{y}_{FTAA}^{(d)}$	P=42.148	-23.9X10^{-4}	6.91 X10^{-4}	141.8	P=4.6952	-25.1X10^{-4}	1.39 X10^{-4}	275.05
CASE II								
Sample mean $\bar{y}$	P<84.2967	0	9.81X10^{-4}	100	P< 9.3904	0	6.64X10^{-4}	100
Product $\bar{y}_p$	P<119.791	21.6X10^{-4}	52.14X10^{-4}	18.800	P< 12.806	14.4 X10^{-4}	3.65X10^{-4}	181.78
Srivenkataramana (1988) two-phase	P<89.7801	8.81X10^{-4}	13.43X10^{-4}	73.005	P< 14.107	3.674227	4.07 X10^{-4}	162.96
Malik and Tailor (2013)	P<148.851	41.1X10^{-4}	85.93X10^{-4}	11.408	P<16.428	32.4 X10^{-4}	6.87 X10^{-4}	96.65
Modified $\bar{y}_{FTAA}^{(d)}$	P=27.3838	-17.0X10^{-4}	5.73 X10^{-4}	171.18	P=3.4763	-15.3X10^{-4}	1.82 X10^{-4}	364.50

COMMENT: Tables 4.23A and 4.23B show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and other related product estimators under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated with -0.2, -0.5 and -0.8, -0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over others. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}^{(d)}$ over other product estimators considered in the study when $P=1.38839$, $P=1.06344$, $P=6.79012$, $P=4.52680$, $P=42.1484$, $P=27.3838$, $P=4.6952$ and $P=3.4763$ are satisfied and that $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE among others.

4.2.2.3 Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and Shukla (2002) Estimator $\bar{y}_{FTd}$

Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ were estimated under the condition that the study and auxiliary variables are both positively and negatively correlated. The correlation coefficients considered are $\pm 0.2, \pm 0.5, \pm 0.8$ and ± 0.99. The results are presented in Tables 4.24A-4.25B.

Table 4.24A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = 0.2$ and $\rho_{xy} = 0.5$

Estimators	Values of d	$\rho_{xy} = 0.2$				$\rho_{xy} = 0.5$			
		Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I									
Shukla (2002) $\bar{y}_{FTd}$	3	P<-0.30631	-0.05X10^{-4}	11.36 X10^{-4}	100.23	P<-0.76588	-23.6X10^{-4}	7.66X10^{-4}	111.44
	5		0.13 X10^{-4}	11.33 X10^{-4}	100.47		-27.2X10^{-4}	7.21X10^{-4}	118.35
	10		-0.04X10^{-4}	15.70 X10^{-4}	72.532		1.77X10^{-4}	8.11X10^{-4}	105.23
	15		1.97 X10^{-4}	17.51 X10^{-4}	65.004		2.65X10^{-4}	9.01X10^{-4}	94.813
	20		5.85 X10^{-4}	18.30 X10^{-4}	62.197		3.04 X10^{-4}	9.42X10^{-4}	90.630
	30		6.37 X10^{-4}	19.01 X10^{-4}	59.891		3.40X10^{-4}	9.80X10^{-4}	87.096
Modified Estimator $\bar{y}_{FTAA}^{(d)}$	3	P=-2.78991	-0.01X10^{-4}	11.08 X10^{-4}	102.78	P=-6.99216	-19.6X10^{-4}	7.01X10^{-4}	121.75
	5		-0.02X10^{-4}	11.10 X10^{-4}	102.57		-33.9X10^{-4}	6.67X10^{-4}	127.95
	10		-0.04X10^{-4}	11.22 X10^{-4}	101.43		-10.6X10^{-4}	7.29X10^{-4}	117.13
	15		-0.05X10^{-4}	11.20 X10^{-4}	101.60		-11.9X10^{-4}	7.75X10^{-4}	110.23
	20		-0.05X10^{-4}	11.19 X10^{-4}	101.66		-12.4X10^{-4}	7.95X10^{-4}	107.42
	30		-0.06X10^{-4}	11.18 X10^{-4}	101.72		-12.8X10^{-4}	8.20 X10^{-4}	104.17
CASE II									
Shukla (2002) $\bar{y}_{FTd}$	3	P<-0.24505	-0.03X10^{-4}	11.35 X10^{-4}	100.31	P<-0.62663	-23.6X10^{-4}	7.66 X10^{-4}	111.44
	5		0.32 X10^{-4}	11.31 X10^{-4}	100.62		-25.5X10^{-4}	6.98 X10^{-4}	122.33
	10		6.88 X10^{-4}	19.64 X10^{-4}	57.972		3.36 X10^{-4}	9.64 X10^{-4}	88.588
	15		9.19 X10^{-4}	22.82 X10^{-4}	49.892		4.84 X10^{-4}	11.3X10^{-4}	75.595
	20		10.2 X10^{-4}	24.19 X10^{-4}	47.054		5.49 X10^{-4}	12.1X10^{-4}	70.867
	30		11.1 X10^{-4}	25.42 X10^{-4}	44.788		6.09 X10^{-4}	12.7X10^{-4}	67.05
Modified $\bar{y}_{FTAA}^{(d)}$	3	P=-2.23193	-0.01X10^{-4}	11.03 X10^{-4}	103.24	P=-5.72085	-19.6X10^{-4}	6.61 X10^{-4}	123.76
	5		-0.02X10^{-4}	11.23 X10^{-4}	101.37		43.3X10^{-4}	6.43 X10^{-4}	132.81
	10		-0.05X10^{-4}	11.18 X10^{-4}	101.84		-13.2X10^{-4}	7.09 X10^{-4}	120.52
	15		-0.06X10^{-4}	11.16 X10^{-4}	102.04		-14.8X10^{-4}	7.51X10^{-4}	113.65
	20		-0.06X10^{-4}	11.15 X10^{-4}	102.12		-15.4X10^{-4}	7.93X10^{-4}	107.71
	30		-0.06X10^{-4}	11.14 X10^{-4}	102.18		-15.9X10^{-4}	8.44X10^{-4}	101.18

Table 4.24B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = 0.8$ and $\rho_{xy} = 0.99$

Estimators	Values of d	$\rho_{xy} = 0.8$				$\rho_{xy} = 0.99$			
		Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I									
Shukla (2002) $\bar{y}_{FTd}$	3	P<-1.123	-27.2X10^{-4}	8.90 X10^{-4}	110.19	P<-1.48785	1.99 X10^{-4}	16.11X10^{-4}	41.230
	5		-43.7X10^{-4}	8.13X10^{-4}	120.63		-1.85X10^{-4}	9.83X10^{-4}	67.56
	10		38.8X10^{-4}	6.99X10^{-4}	140.08		1.06 X10^{-4}	11.85X10^{-4}	56.038
	15		86.6X10^{-4}	7.24X10^{-4}	135.36		0.12 X10^{-4}	11.95X10^{-4}	55.555
	20		1.08 X10^{-4}	7.38X10^{-4}	132.82		-0.69X10^{-4}	11.94X10^{-4}	55.607
	30		1.28 X10^{-4}	7.52X10^{-4}	130.42		2.01 X10^{-4}	11.56X10^{-4}	57.365
Modified $\bar{y}_{FTAA}^{(d)}$	3	P=-4.19373	39.3X10^{-4}	7.28X10^{-4}	117.32	P= -10.248	0.32 X10^{-4}	7.63 X10^{-4}	87.028
	5		-80.4X10^{-4}	6.82X10^{-4}	125.31		-0.17X10^{-4}	6.11 X10^{-4}	108.70
	10		-23.9X10^{-4}	5.94X10^{-4}	143.72		0.49 X10^{-4}	8.18 X10^{-4}	81.164
	15		-26.6X10^{-4}	6.19X10^{-4}	138.06		-0.89X10^{-4}	8.98 X10^{-4}	73.96
	20		-27.6X10^{-4}	6.28X10^{-4}	135.97		0.62 X10^{-4}	8.59 X10^{-4}	77.232
	30		-28.5X10^{-4}	6.40X10^{-4}	133.47		-1.01X10^{-4}	9.25 X10^{-4}	71.769
CASE II									
Shukla (2002) $\bar{y}_{FTd}$	3	P<-0.08595	-52.4X10^{-4}	7.94X10^{-4}	123.53	P<-1.15722	-0.72X10^{-4}	6.11 X10^{-4}	108.67
	5		-67.0X10^{-4}	6.58X10^{-4}	148.90		-0.61X10^{-4}	5.49 X10^{-4}	120.88
	10		3.92 X10^{-4}	8.37X10^{-4}	117.14		3.01 X10^{-4}	5.95 X10^{-4}	111.64
	15		5.89 X10^{-4}	10.3X10^{-4}	95.138		1.29 X10^{-4}	6.19 X10^{-4}	107.29
	20		6.75 X10^{-4}	11.2X10^{-4}	87.508		1.81 X10^{-4}	6.43 X10^{-4}	103.27
	30		7.53 X10^{-4}	12.3X10^{-4}	81.516		2.31 X10^{-4}	6.23 X10^{-4}	106.57
Modified $\bar{y}_{FTAA}^{(d)}$	3	P=-2.9305	85.4 X10^{-4}	4.94X10^{-4}	134.34	P=-7.97095	-0.17X10^{-4}	5.53 X10^{-4}	120.09
	5		-17.0X10^{-4}	4.25X10^{-4}	156.19		-0.06X10^{-4}	4.53 X10^{-4}	146.61
	10		-47.1X10^{-4}	5.24X10^{-4}	126.74		-0.26X10^{-4}	5.82 X10^{-4}	114.12
	15		-51.6X10^{-4}	5.81X10^{-4}	114.28		-0.31X10^{-4}	5.66 X10^{-4}	117.24
	20		-53.1X10^{-4}	6.14X10^{-4}	108.14		-0.32X10^{-4}	5.59 X10^{-4}	118.59
	30		-54.4X10^{-4}	6.31 X10^{-4}	105.16		-0.34 10^{-4}	5.54 X10^{-4}	119.80

COMMENT: Tables 4.24A and 4.24B show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and Shukla (2002) estimator $\bar{y}_{FTd}$ under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated with 0.2, 0.5 and 0.8, 0.99 coefficients respectively for some arbitrary values of $d = 3, d = 5, d = 10, d = 15, d = 20$ and $d = 30$ as well as the efficiency conditions of $\bar{y}_{FTAA}$ over $\bar{y}_{FTd}$. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}$ over $\bar{y}_{FTd}$ when $P = -2.78991$,

$P = -2.23193$, $P = -6.99216$, $P = -5.72085$, $P = -4.19373$, $P = -2.9305$, $P = -10.2484$ and $P = -7.97095$ are satisfied and $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE. These results are shown in the figures 4.2.9A-4.2.16B below.

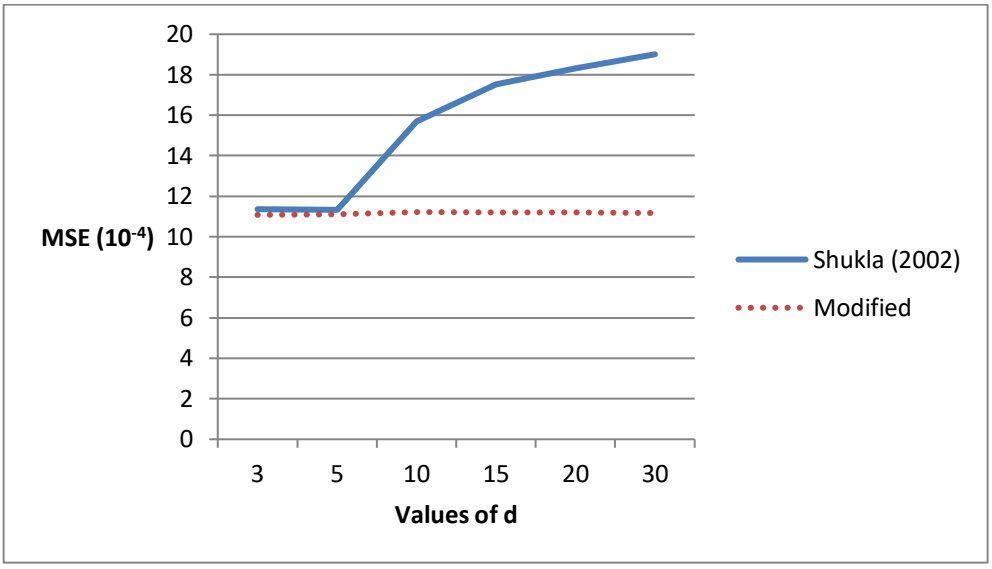


Figure 4.2.9A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.2$

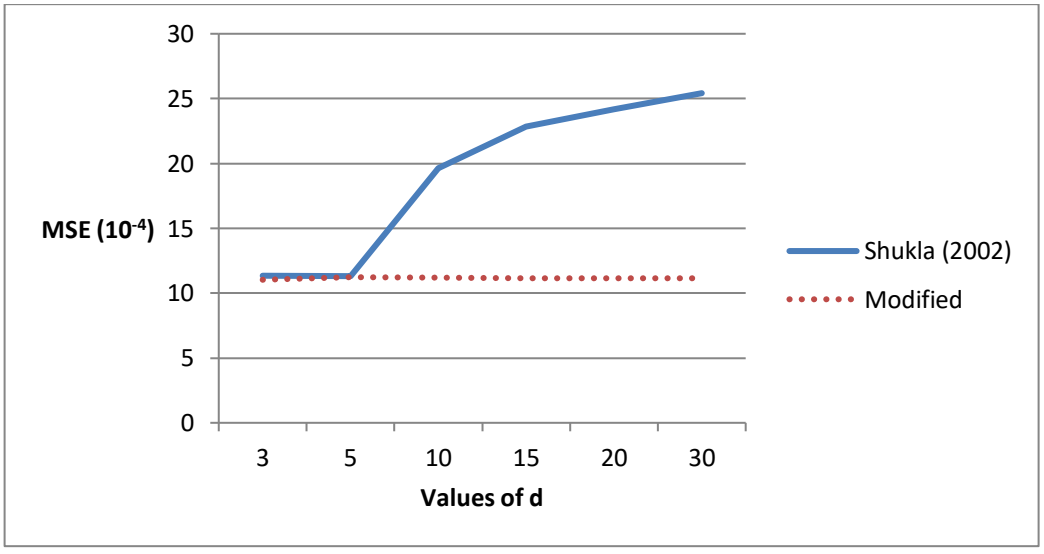


Figure 4.2.10A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.2$

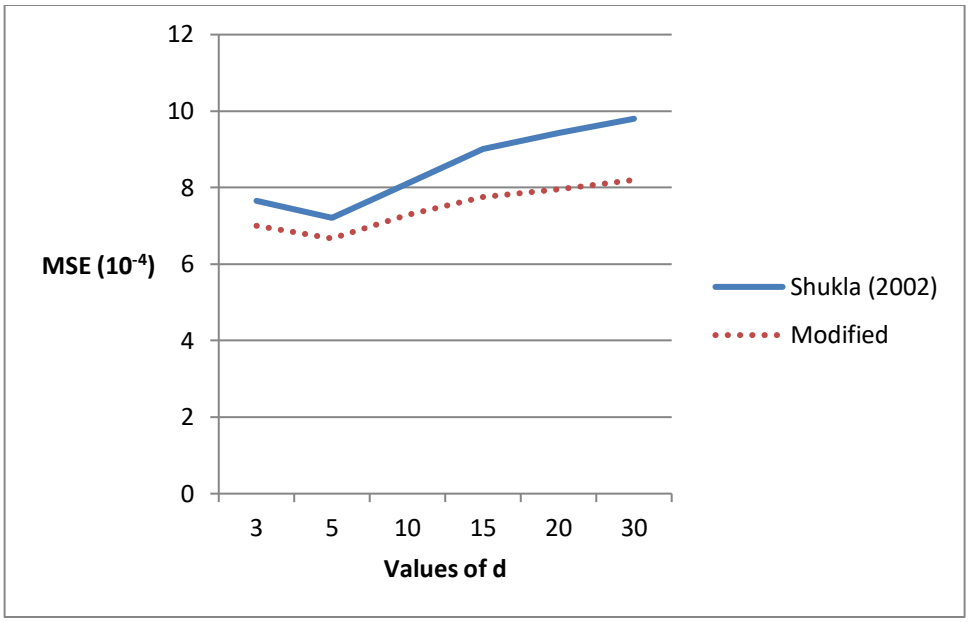


Figure 4.2.11A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.5$

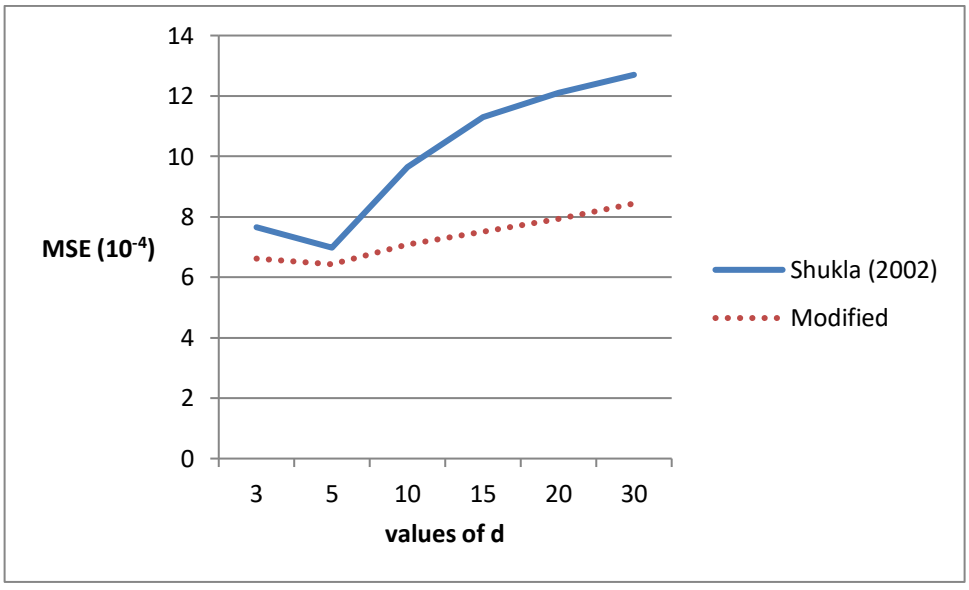


Figure 4.2.12A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.5$

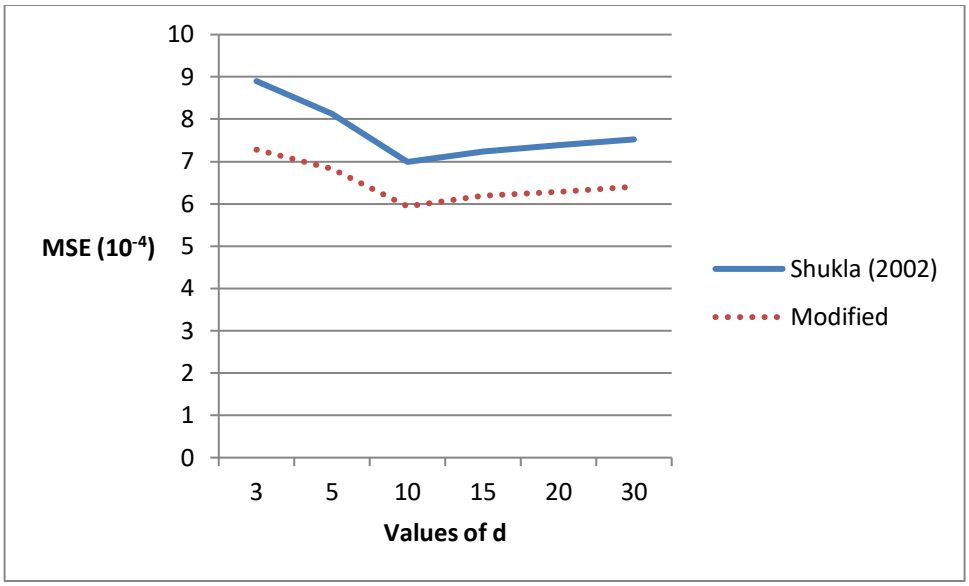


Figure 4.2.13A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.8$

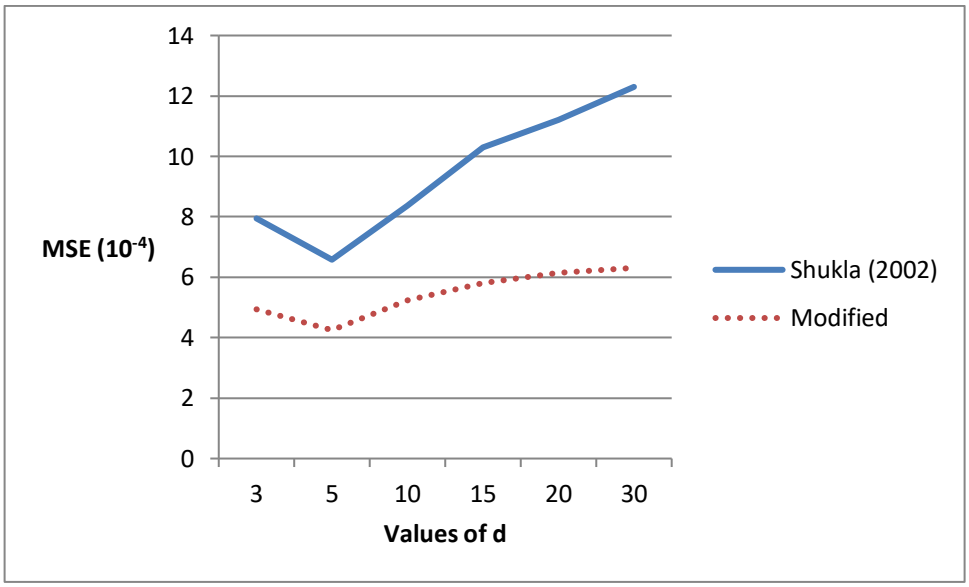


Figure 4.2.14A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.8$

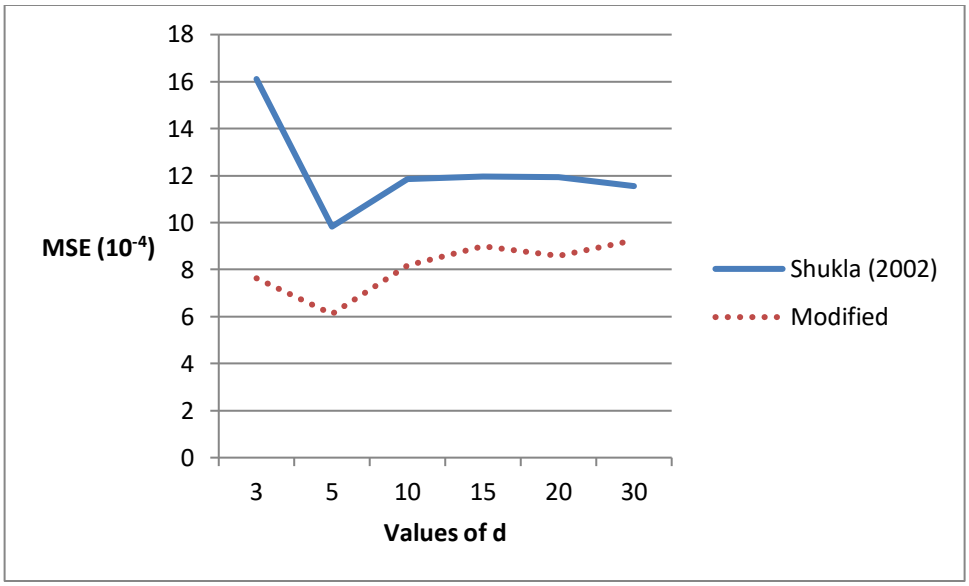


Figure 4.2.15A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when

$$\rho_{xy} = 0.99$$

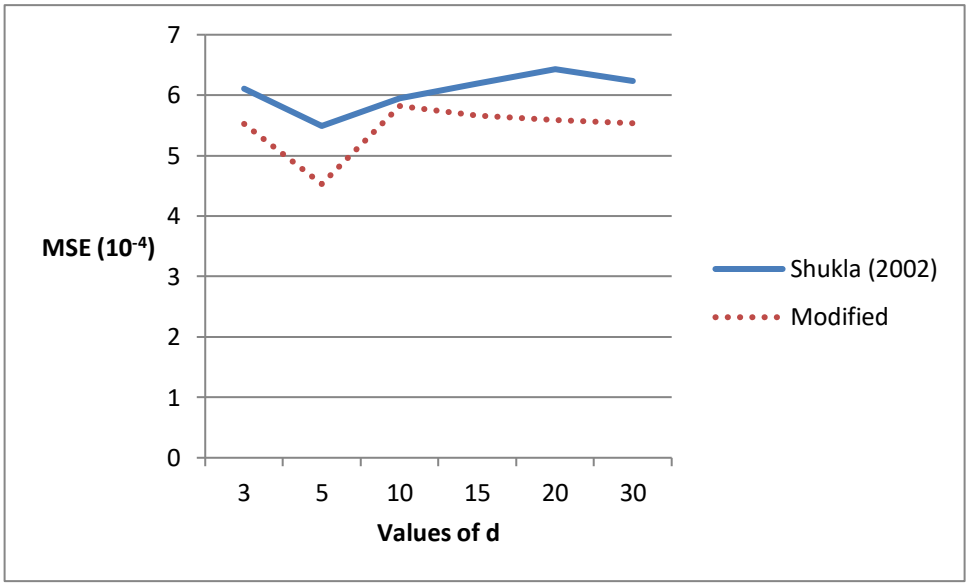


Figure 4.2.16A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when

$$\rho_{xy} = 0.99$$

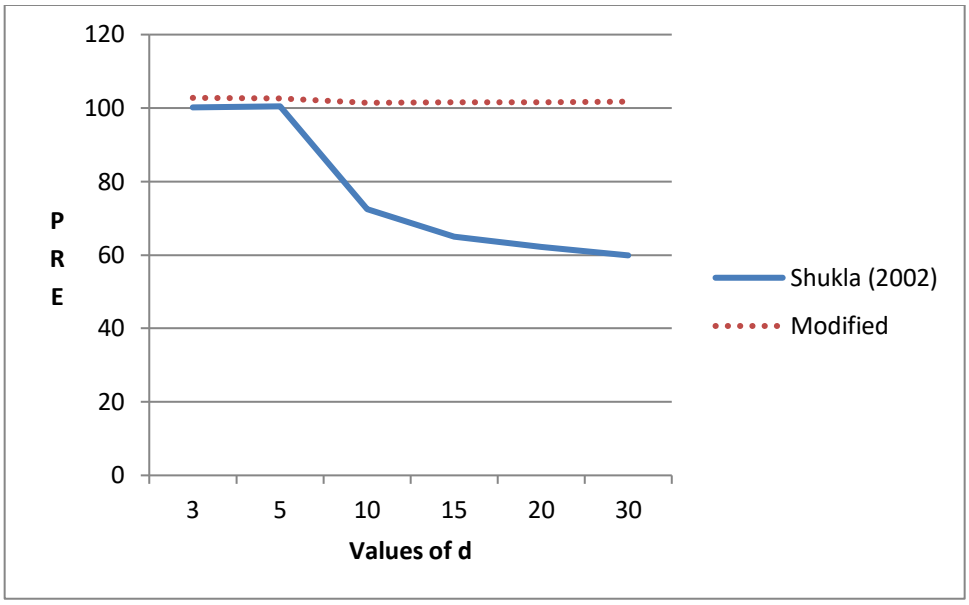


Figure 4.2.9B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.2$

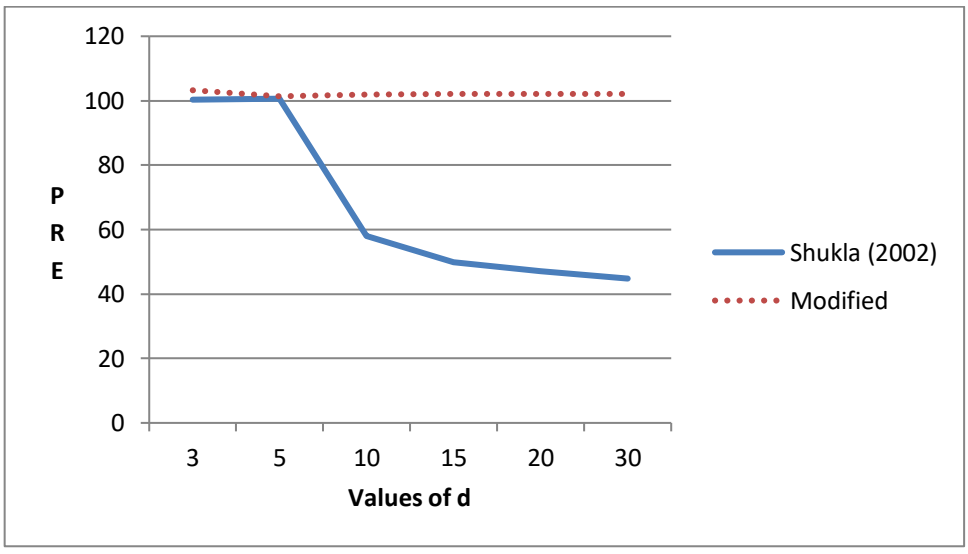


Figure 4.2.10B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.2$

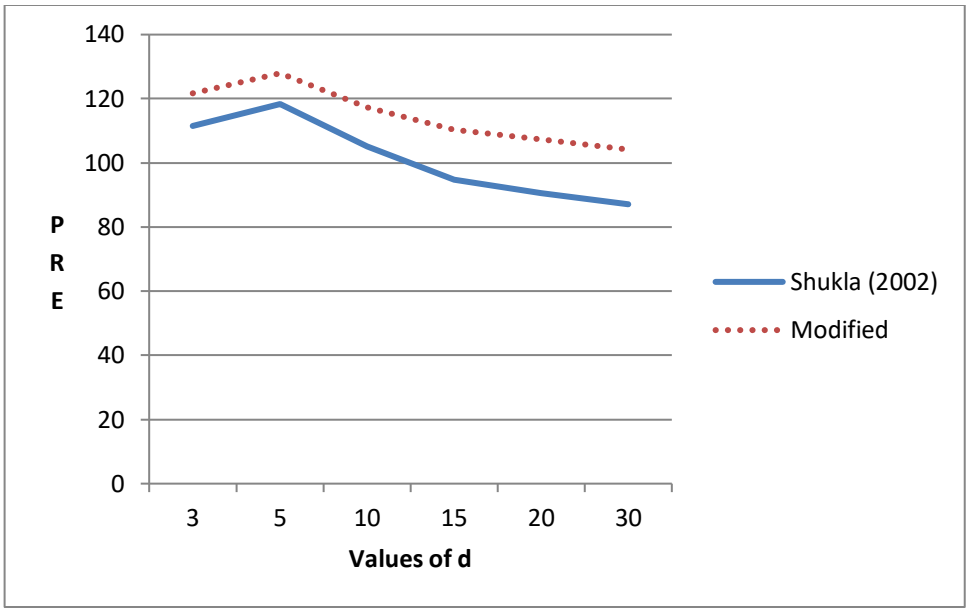


Figure 4.2.11B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.5$

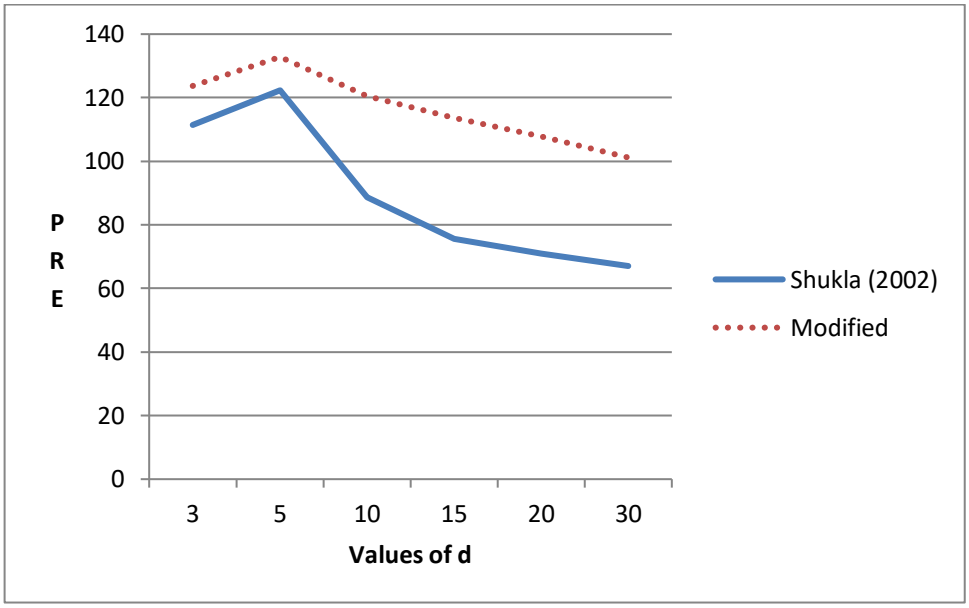


Figure 4.2.12B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.5$

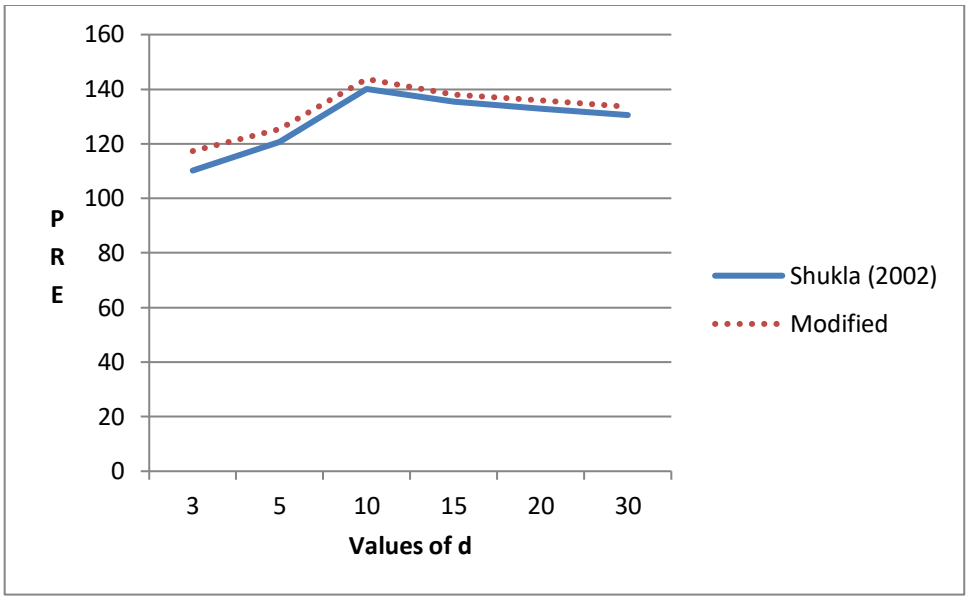


Figure 4.2.13B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.8$

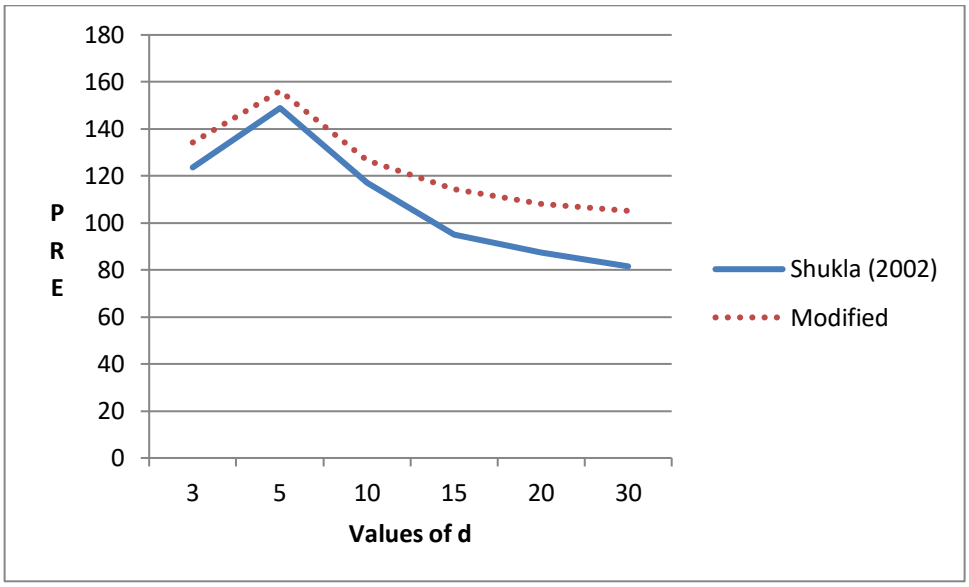


Figure 4.2.14B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.8$

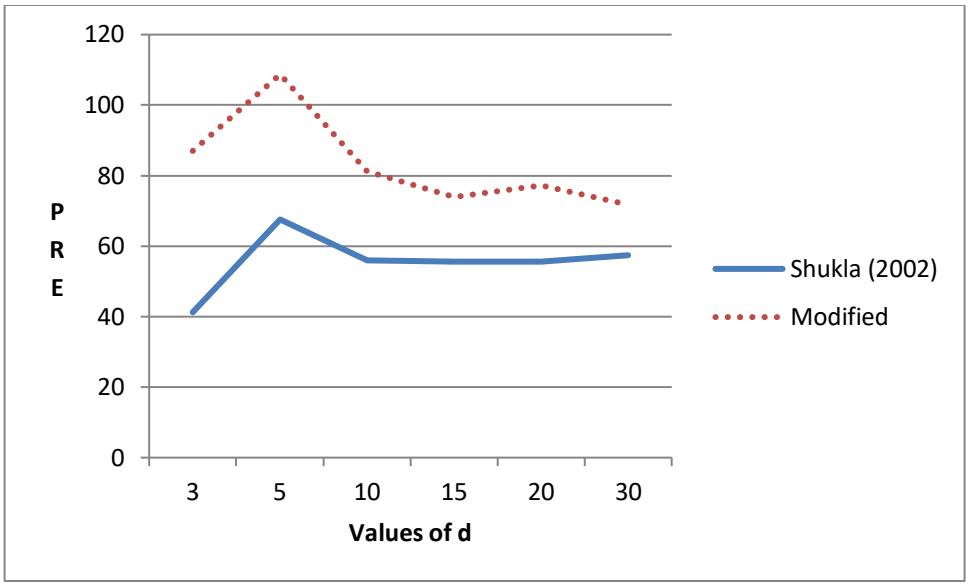


Figure 4.215B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = 0.99$

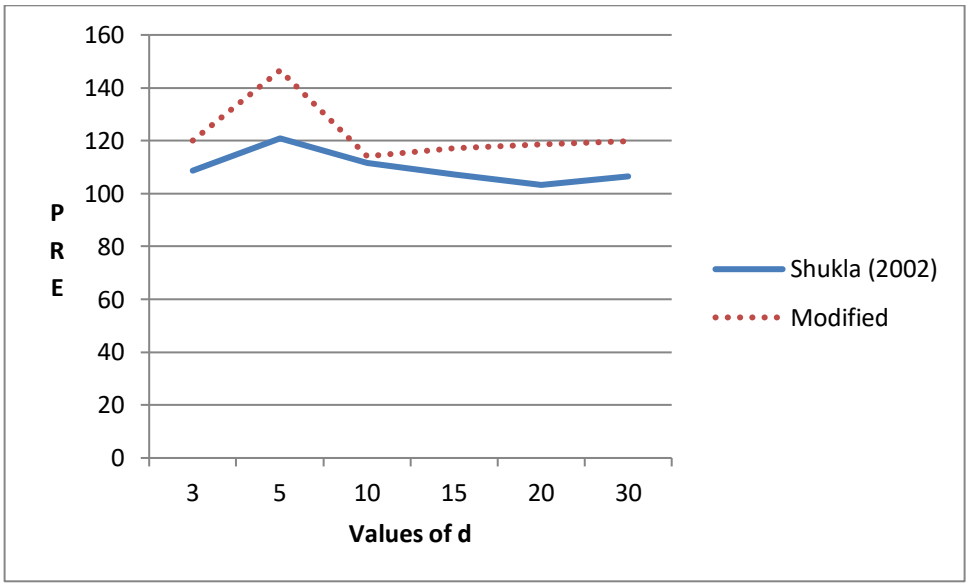


Figure 4.2.16B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = 0.99$

COMMENT: Figures (4.2.9A – 4.2.16A) and (4.2.9B – 4.2.16B) show the behavioural pattern of MSEs and PRE of estimators considered in Tables 4.24A and 4.24B respectively and the plots generally exhibit upward and downward movement respectively as the values of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators

Table 4.25A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$

Estimators	Values of d	$\rho_{xy} = -0.2$				$\rho_{xy} = -0.5$			
		Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I									
Shukla (2002) $\bar{y}_{FTd}$	3	P> 0.29023	$0.24 X10^{-4}$	$11.97X10^{-4}$	95.134	P>0.862	$29.1X10^{-4}$	$9.31 X10^{-4}$	91.704
	5		$0.71X10^{-4}$	$12.90X10^{-4}$	88.245		$60.2X10^{-4}$	$10.3X10^{-4}$	85.092
	10		$5.59 X10^{-4}$	$21.31X10^{-4}$	53.423		$3.77X10^{-4}$	$16.2X10^{-4}$	52.649
	15		$7.12 X10^{-4}$	$23.83X10^{-4}$	47.773		$4.72X10^{-4}$	$17.9X10^{-4}$	47.593
	20		$7.77 X10^{-4}$	$24.89X10^{-4}$	45.741		$5.12X10^{-4}$	$18.7X10^{-4}$	45.761
	30		$15.1X10^{-4}$	$36.9X10^{-4}$	30.855		$5.47X10^{-4}$	$19.3X10^{-4}$	44.269
Modified $\bar{y}_{FTAA}^{(d)}$	3	P= 1.38839	$0.02 X10^{-4}$	$11.44X10^{-4}$	99.540	P=6.7902	$13.1X10^{-4}$	$8.58 X10^{-4}$	99.545
	5		$0.04 X10^{-4}$	$11.50X10^{-4}$	99.022		$23.2X10^{-4}$	$8.61 X10^{-4}$	99.198
	10		$0.12 X10^{-4}$	$11.83X10^{-4}$	96.270		$84.5X10^{-4}$	$8.78 X10^{-4}$	97.235
	15		$0.21 X10^{-4}$	$11.90X10^{-4}$	95.643		$98.1X10^{-4}$	$8.82 X10^{-4}$	96.825
	20		$0.23 X10^{-4}$	$11.93X10^{-4}$	95.393		$10.4X10^{-4}$	$8.83 X10^{-4}$	96.664
	30		$0.39 X10^{-4}$	$12.28X10^{-4}$	92.712		$10.8X10^{-4}$	$8.85 X10^{-4}$	96.526
CASE II									
Shukla (2002) $\bar{y}_{FTd}$	3	P> 0.22230	$0.39 X10^{-4}$	$12.29X10^{-4}$	92.644	P>0.57469	$64.9X10^{-4}$	$10.2X10^{-4}$	83.811
	5		$1.20 X10^{-4}$	$13.84X10^{-4}$	82.235		$1.42X10^{-4}$	$11.9X10^{-4}$	72.037
	10		$10.1X10^{-4}$	$28.74X10^{-4}$	39.613		$9.99X10^{-4}$	$27.6X10^{-4}$	30.936
	15		$12.9 X10^{-4}$	$33.29X10^{-4}$	34.195		$12.7X10^{-4}$	$32.2X10^{-4}$	26.522
	20		$14.1 X10^{-4}$	$35.21X10^{-4}$	32.332		$13.8X10^{-4}$	$34.1X10^{-4}$	25.022
	30		$15.1 X10^{-4}$	$36.90X10^{-4}$	30.855		$14.8X10^{-4}$	$35.8X10^{-4}$	23.838
Modified $\bar{y}_{FTAA}^{(d)}$	3	P= 1.06344	$0.03 X10^{-4}$	$11.46X10^{-4}$	99.331	P=4.52703	$26.3X10^{-4}$	$8.62 X10^{-4}$	99.09
	5		$0.06 X10^{-4}$	$11.55X10^{-4}$	98.566		$47.1X10^{-4}$	8.677284	98.396
	10		$0.28 X10^{-4}$	$12.06X10^{-4}$	94.388		$17.7X10^{-4}$	$9.04 X10^{-4}$	94.488
	15		$0.34 X10^{-4}$	$12.19X10^{-4}$	93.426		$20.7X10^{-4}$	$9.11 X10^{-4}$	93.678
	20		$0.37 X10^{-4}$	$12.24X10^{-4}$	93.042		$21.9X10^{-4}$	$9.15 X10^{-4}$	93.359
	30		$0.39 X10^{-4}$	$12.28X10^{-4}$	92.712		$22.9X10^{-4}$	$9.17 X10^{-4}$	93.089

Table 4.25B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{(d)}$ and $\bar{y}_{FTd}$ when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$

Estimators	Values of d	$\rho_{xy} = -0.8$				$\rho_{xy} = -0.99$			
		Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I									
Shukla (2002) $\bar{y}_{FTd}$	3	P>1.317	$38.5X10^{-4}$	$10.88X10^{-4}$	90.108	P> 1.93487	$0.44\ X10^{-4}$	$7.89X10^{-4}$	84.138
	5		$91.1\ X10^{-4}$	$12.20X10^{-4}$	80.382		$1.02\ X10^{-4}$	$9.39X10^{-4}$	70.754
	10		$4.88\ X10^{-4}$	$20.56X10^{-4}$	47.679		$5.17\ X10^{-4}$	$18.45X10^{-4}$	36.011
	15		$5.99\ X10^{-4}$	$22.72X10^{-4}$	43.151		$6.30\ X10^{-4}$	$20.72X10^{-4}$	32.045
	20		$6.45\ X10^{-4}$	$23.60X10^{-4}$	41.533		$6.77\ X10^{-4}$	$21.65X10^{-4}$	30.670
	30		$6.86\ X10^{-4}$	$24.37X10^{-4}$	40.225		$7.18\ X10^{-4}$	$22.46X10^{-4}$	29.571
Modified $\bar{y}_{FTAA}^{(d)}$	3	P=42.148	$-39.2X10^{-4}$	$9.79\ X10^{-4}$	100.12	P= 4.6952	$-0.06X10^{-4}$	$6.44\ X10^{-4}$	103.07
	5		$-8.02X10^{-4}$	$9.78\ X10^{-4}$	100.25		$-0.13X10^{-4}$	$6.24\ X10^{-4}$	106.42
	10		$-2.65X10^{-4}$	$9.72\ X10^{-4}$	100.83		$-0.38X10^{-4}$	$5.38\ X10^{-4}$	123.36
	15		$-30.1X10^{-4}$	$9.71\ X10^{-4}$	100.94		$-0.42X10^{-4}$	$5.22\ X10^{-4}$	127.13
	20		$-31.6X10^{-4}$	$9.70\ X10^{-4}$	100.99		$-0.44X10^{-4}$	$5.16\ X10^{-4}$	128.64
	30		$-32.8X10^{-4}$	$9.69\ X10^{-4}$	101.03		$-0.45X10^{-4}$	$5.11\ X10^{-4}$	129.93
CASE II									
Shukla (2002) $\bar{y}_{FTd}$	3	P>0.85565	$90.1\ X10^{-4}$	$12.24X10^{-4}$	80.106	P> 1.43255	$0.71\ X10^{-4}$	$8.62\ X10^{-4}$	77.106
	5		$2.25\ X10^{-4}$	$15.41X10^{-4}$	63.601		$1.70\ X10^{-4}$	$11.05X10^{-4}$	60.084
	10		$13.7\ X10^{-4}$	$37.79X10^{-4}$	25.935		$9.31\ X10^{-4}$	$26.87X10^{-4}$	24.713
	15		$17.0\ X10^{-4}$	$43.89X10^{-4}$	22.337		$11.5X10^{-4}$	$30.99X10^{-4}$	21.424
	20		$18.4\ X10^{-4}$	$46.4\ X10^{-4}$	21.125		$12.4\ X10^{-4}$	$32.69X10^{-4}$	20.314
	30		$19.6\ X10^{-4}$	$48.59X10^{-4}$	20.171		$13.2\ X10^{-4}$	$34.16X10^{-4}$	19.439
Modified $\bar{y}_{FTAA}^{(d)}$	3	P=27.3836	$-84.9X10^{-4}$	$9.78\ X10^{-4}$	100.26	P=3.47628	$-0.10X10^{-4}$	$6.34\ X10^{-4}$	104.78
	5		$-17.4X10^{-4}$	$9.75\ X10^{-4}$	100.54		$-0.19X10^{-4}$	$6.30\ X10^{-4}$	110.14
	10		$-56.9X10^{-4}$	$9.63\ X10^{-4}$	101.81		$-0.55X10^{-4}$	$4.76X10^{-4}$	139.40
	15		$-64.8X10^{-4}$	$9.61\ X10^{-4}$	102.06		$-0.60X10^{-4}$	$4.54\ X10^{-4}$	146.40
	20		$-67.\ X10^{-4}$	$9.60\ X10^{-4}$	102.16		$-0.62X10^{-4}$	$4.45\ X10^{-4}$	149.24
	30		$-70.4X10^{-4}$	$9.59\ X10^{-4}$	102.25		$-0.63X10^{-4}$	$4.38\ X10^{-4}$	151.70

COMMENT: Tables 4.25A and 4.25B shows the biases, MSEs and PRE of $\bar{y}_{FTAA}^{(d)}$ and Shukla (2002) estimator $\bar{y}_{FTd}$ under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated with -0.2, -0.5 and -0.8, -0.99 coefficients respectively for some arbitrary values of $d = 3, d = 5, d = 10, d = 15, d = 20$ and $d = 30$ as well as the efficiency conditions of $\bar{y}_{FTAA}$ over $\bar{y}_{FTd}$. The results of the analysis revealed that the efficiency conditions of $\bar{y}_{FTAA}$ over $\bar{y}_{FTd}$ when $P = 1.38839,\ P = 1.06344,$

$P = 6.7902$, $P = 4.52703$, $P = 42.148$, $P = 27.3836$, $P = 4.69523$ and $P = 3.47628$ are satisfied and $\bar{y}_{FTAA}^{(d)}$ has minimum MSE and high PRE. These results are shown in the figures 4.2.17A-4.2.24B below.

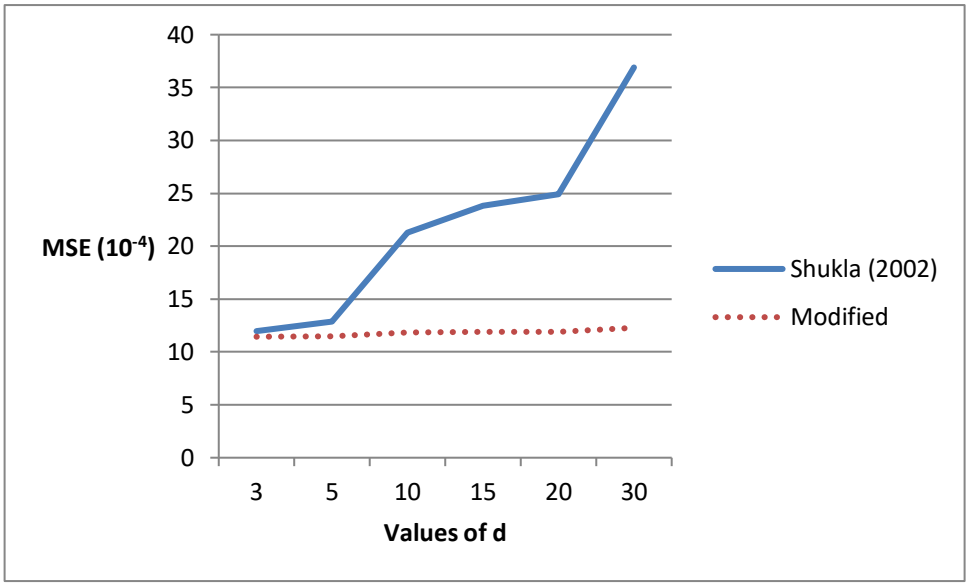


Figure 4.2.17A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.2$

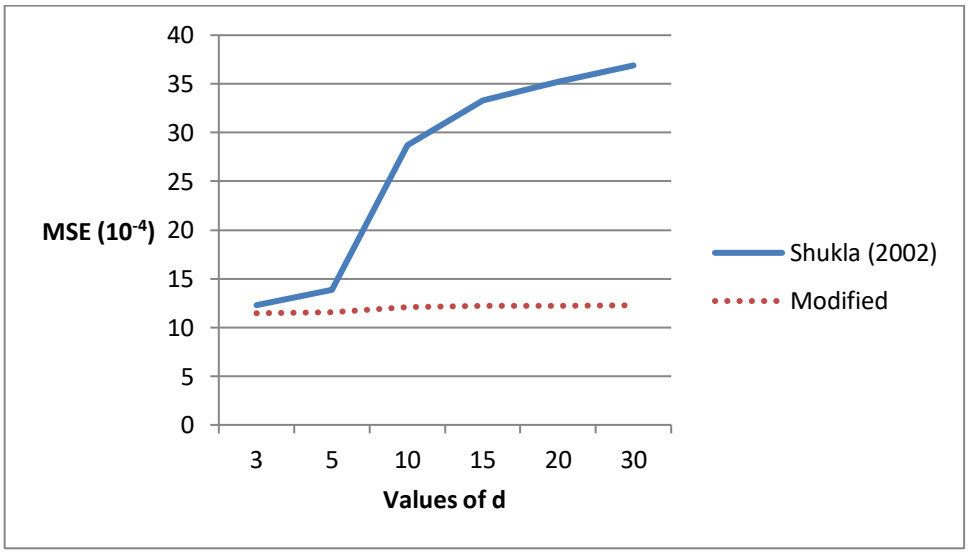


Figure 4.2.18A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.2$

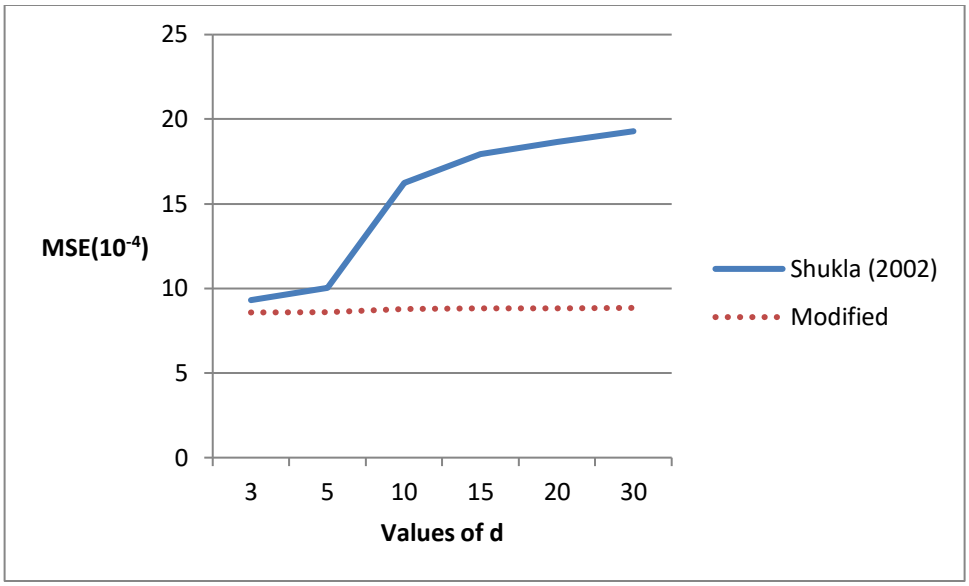


Figure 4.2.19A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.5$

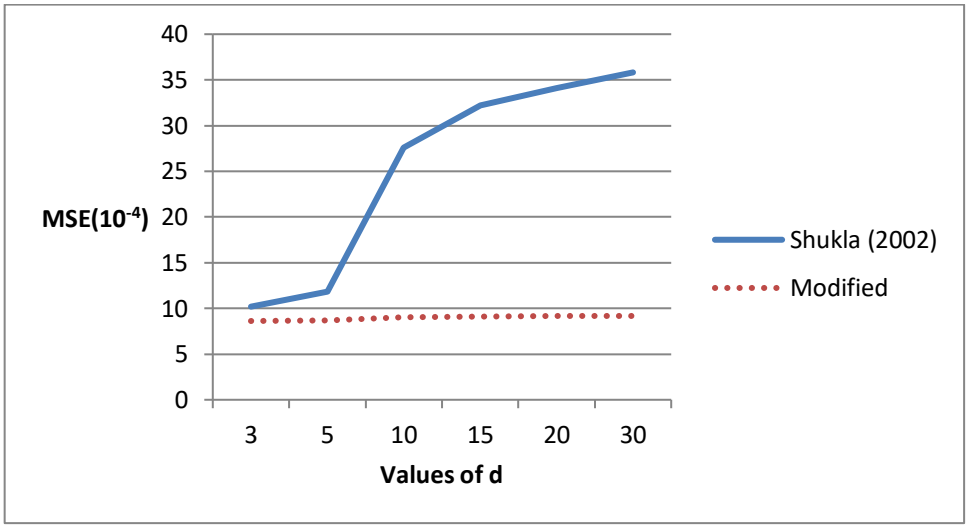


Figure 4.2.20A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.5$

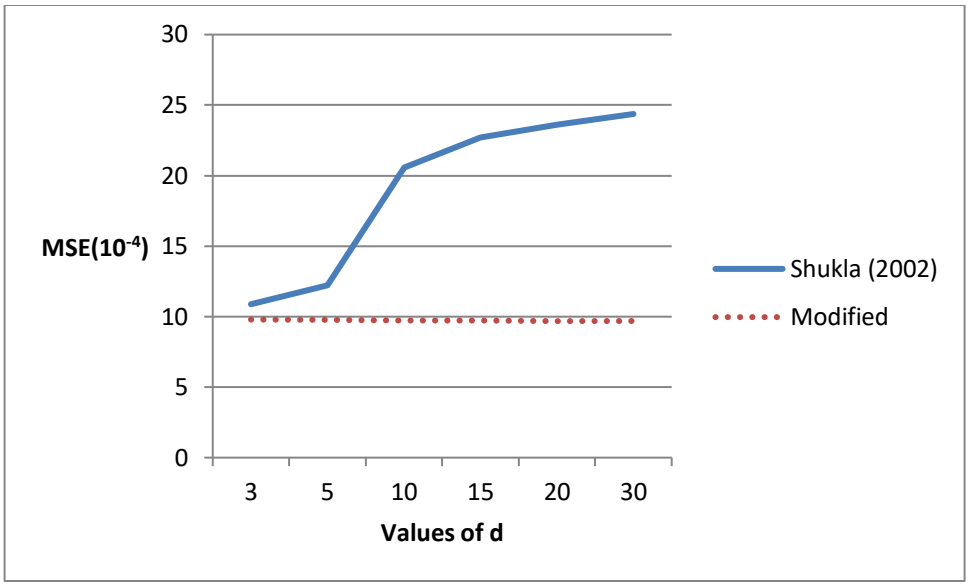


Figure 4.2.21A: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.8$

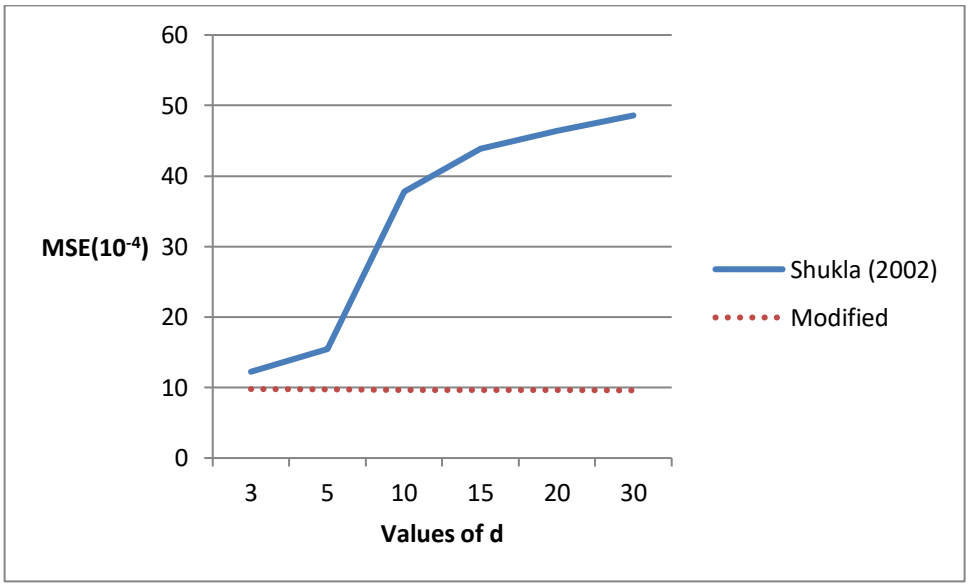


Figure 4.2.22A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.8$

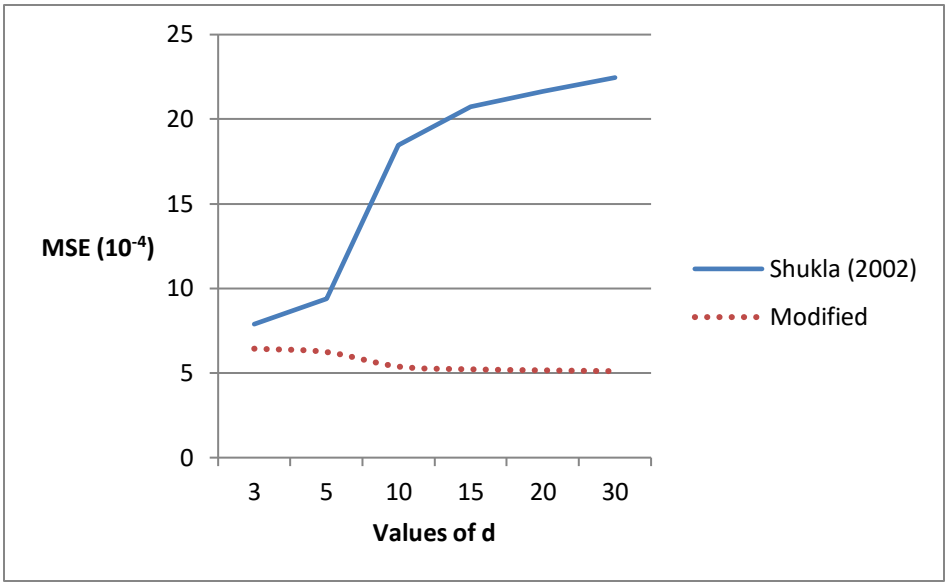


Figure 4.2.23A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.99$

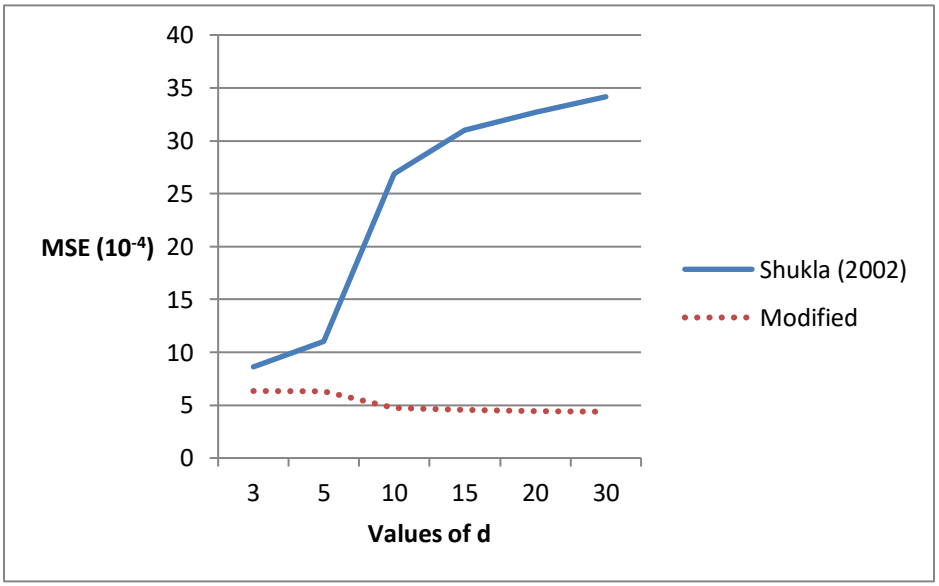


Figure 4.2.24A: Graph showing MSE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.99$

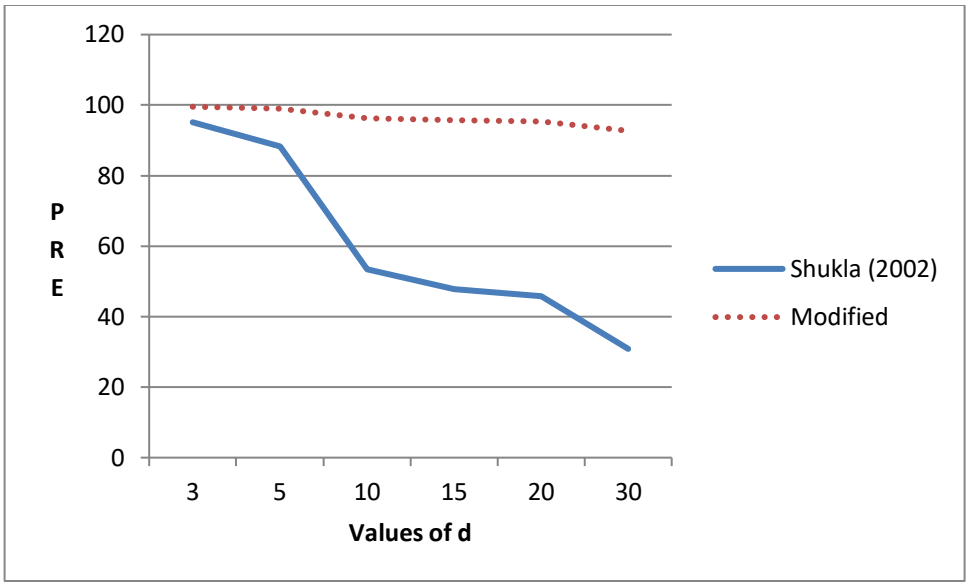


Figure 4.2.17B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.2$

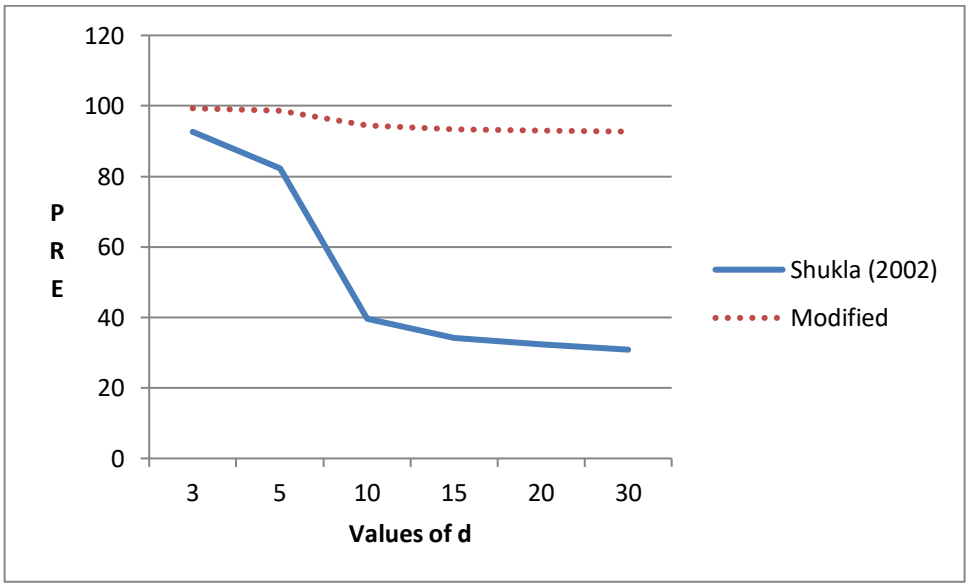


Figure 4.2.18B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.2$

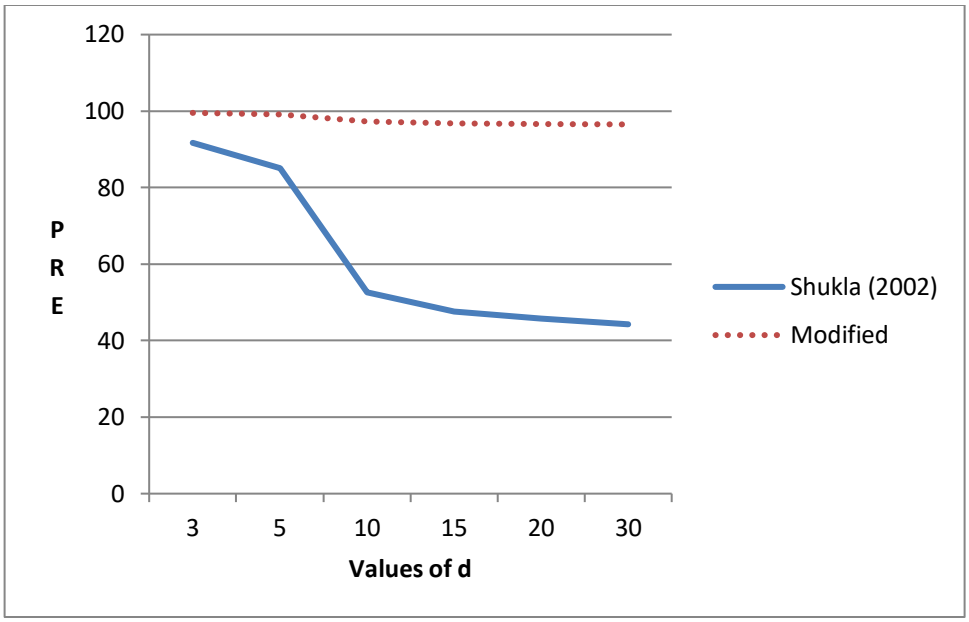


Figure 4.2.19B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.5$

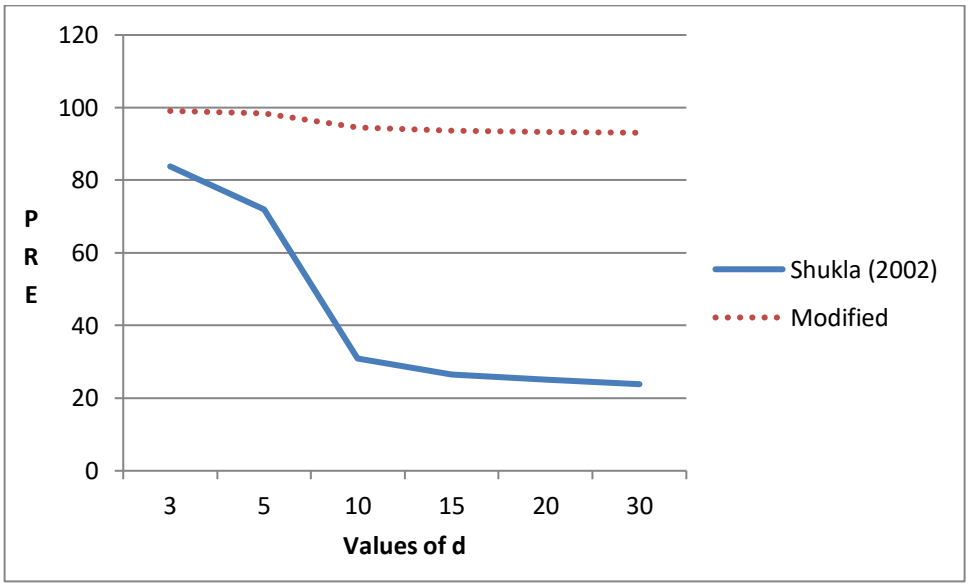


Figure 4.2.20B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.5$

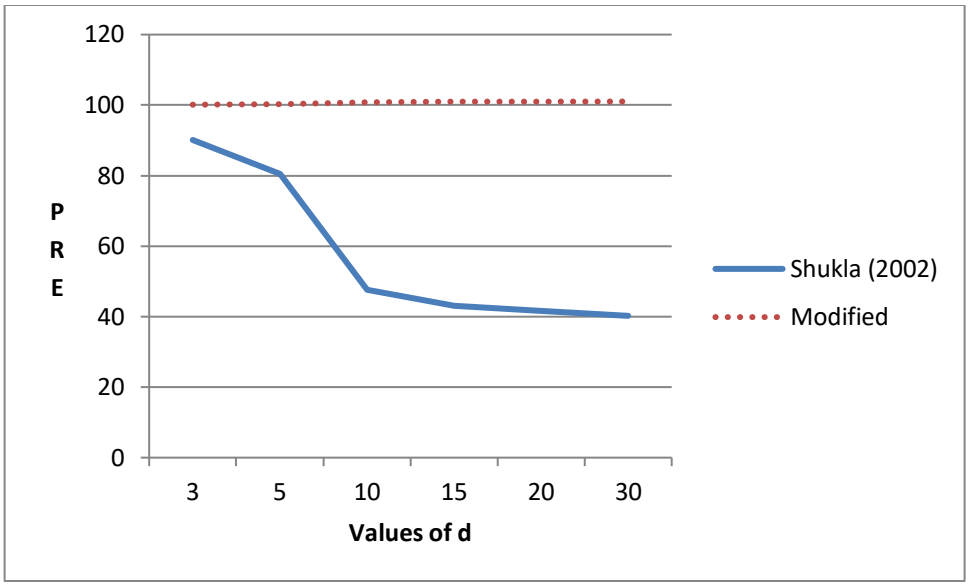


Figure 4.2.21B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.8$

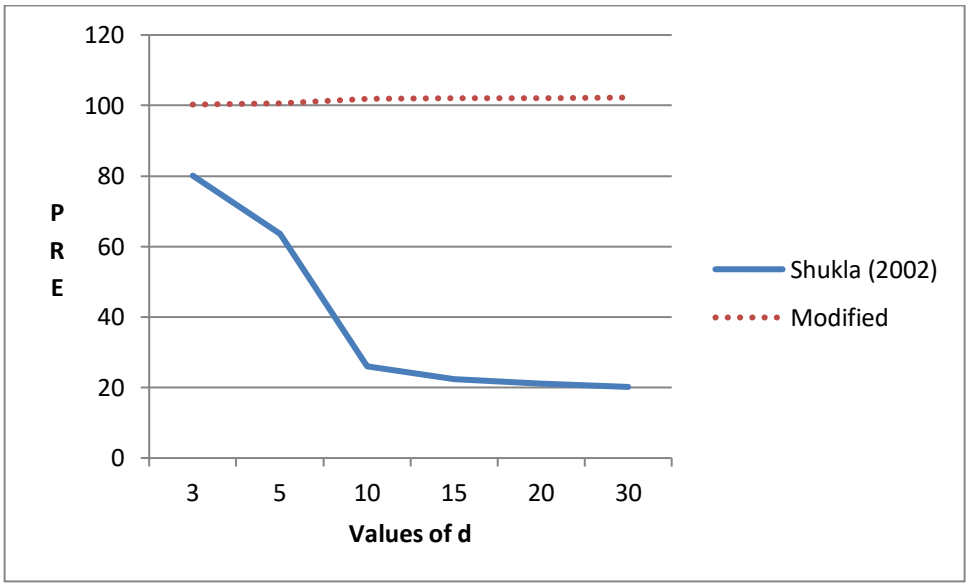


Figure 4.2.22B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when $\rho_{xy} = -0.8$

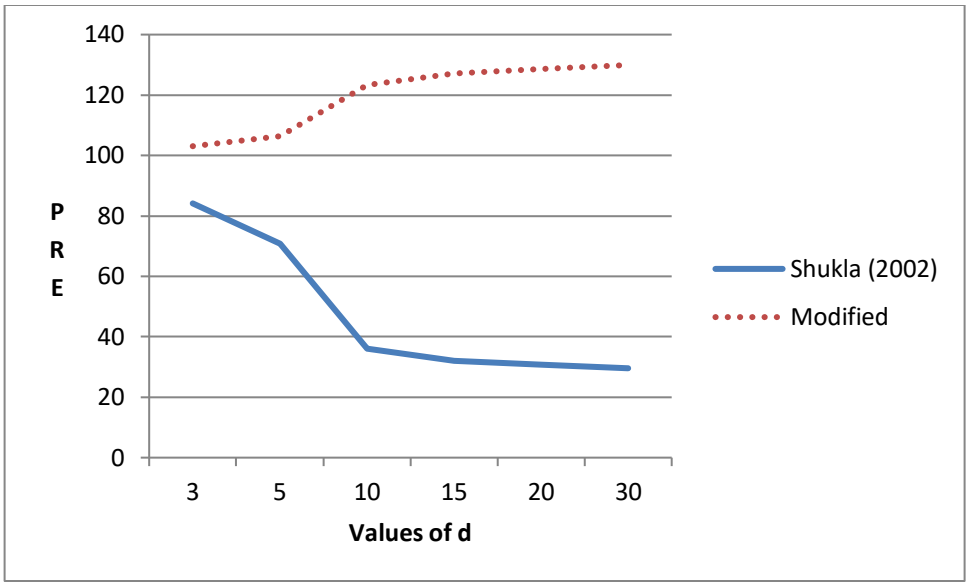


Figure 4.2.23B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case I when $\rho_{xy} = -0.99$

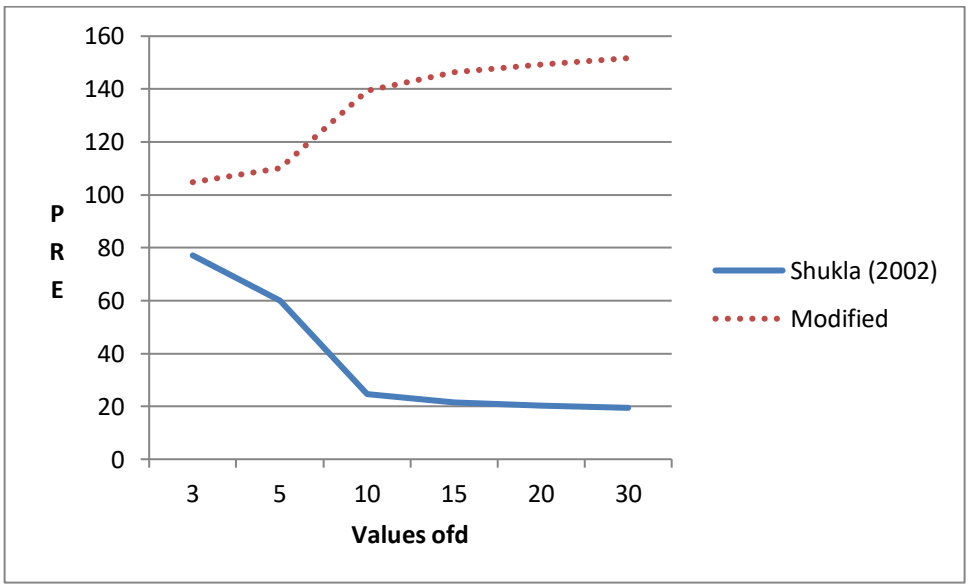


Figure 4.2.24B: Graph showing PRE of $\bar{y}_{FTd}$ and $\bar{y}_{FTAA}^{(d)}$ Estimators under case II when

$$\rho_{xy} = -0.99$$

COMMENT: Figures (4.2.17A – 4.2.24A), and (4.2.17B – 4.2.24B) show the behaviourial pattern of MSEs and PRE of estimators considered in Tables 4.25A and 4.25B respectively and the plots generally exhibit upward and downward movement respectively as the values

of d increases. This implies that the MSEs of the estimators increases as the values of d increases and consequently decrease in precision of the estimators.

4.2.3 Efficiency Comparison of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ with some existing related Estimators

In this section, relative efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ with respect to some existing related ratio and product estimators were investigated.

4.2.3.1 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Ratio Estimators

In this subsection, relative efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ with respect to some existing related product estimators for some correlation coefficients $\rho_{xy}\left(0.2, 0.5, 0.8, 0.99\right)$ was investigated and the results are presented in Tables 4.26A and 4.26B.

Table 4.26A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators when $\rho_{xy} = 0.5$ and $\rho_{xy} = 0.8$

Estimators	$\rho_{xy} = 0.2$				$\rho_{xy} = 0.5$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean	P<-0.3241	0	$11.38X10^{-4}$	100	P<-0.8103	0	$8.54X10^{-4}$	100
Choudhury and Singh (2012)	P<0.08174	$5.24X10^{-4}$	$10.93X10^{-4}$	104.17	P<0.20436	$15.4X10^{-4}$	$6.41X10^{-4}$	133.33
Chand (1975)	P<0.67709	$5.39X10^{-4}$	$10.54X10^{-4}$	105.10	P<0.19118	$8.67X10^{-4}$	$6.98X10^{-4}$	122.37
Singh and Upadhay (2001)	P<0.67745	$5.55X10^{-4}$	$10.97X10^{-4}$	103.75	P<0.19161	$4.72X10^{-4}$	$5.76X10^{-4}$	148.3
Singh et al. (2007)	P<0.68077	$8.27X10^{-4}$	$10.02X10^{-4}$	103.64	P<0.19561	$5.12X10^{-4}$	$6.20X10^{-4}$	137.83
Singh and Upadhay (2001)	P<0.71628	$6.20X10^{-4}$	$11.07X10^{-4}$	102.79	P<0.2377	$5.47X10^{-4}$	$6.48X10^{-4}$	131.8
Singh (2001)	P<0.68955	$16.8X10^{-4}$	$19.15X10^{-4}$	59.448	P<0.21098	$13.1X10^{-4}$	$9.78X10^{-4}$	87.352
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-2.7899	$6.92X10^{-4}$	$10.02X10^{-4}$	113.64	P=-6.992	$23.2X10^{-4}$	$5.41X10^{-4}$	157.83
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=-1.2382	$4.81X10^{-4}$	$10.75X10^{-4}$	107.14	P=-5.6609	$31.7X10^{-4}$	$4.87X10^{-4}$	175.19
CASE II								
Sample mean	P<-0.3241	0	$11.38X10^{-4}$	100	P<-0.8103	0	$8.54X10^{-4}$	100
Choudhury and Singh (2012)	P<0.40872	$7.52X10^{-4}$	$10.36X10^{-4}$	109.89	P<0.40872	$12.6X10^{-4}$	$6.83X10^{-4}$	125.1
Chand (1975)	P<0.67709	$5.31X10^{-4}$	$11.59X10^{-4}$	98.166	P<0.19118	$16.1X10^{-4}$	$7.59X10^{-4}$	112.54
Singh and Upadhay(2001)	P<0.67745	$11.1X10^{-4}$	$11.57X10^{-4}$	98.788	P<0.19161	$16.6X10^{-4}$	$7.31X10^{-4}$	116.90
Singh et al. (2007)	P<0.68077	$21.8X10^{-4}$	$10.98\ X10^{-4}$	103.64	P<0.19561	$12.9X10^{-4}$	$6.20X10^{-4}$	137.83
Singh and Upadhay(2001)	P<0.71628	$12.1X10^{-4}$	$9.09\ X10^{-4}$	125.18	P<0.2377	$15.2X10^{-4}$	$5.96X10^{-4}$	143.31
Singh (2001)	P<0.6896	$14.3X10^{-4}$	$21.94X10^{-4}$	51.897	P<0.21098	$24.1X10^{-4}$	$8.48X10^{-4}$	100.68
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-2.1437	$21.2X10^{-4}$	$8.52X10^{-4}$	133.68	P=-1.3639	$13.9\ X10^{-4}$	$4.52X10^{-4}$	188.96
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=-6.2751	$16.1X10^{-4}$	$8.41X10^{-4}$	135.25	P=-16.417	$17.2X10^{-4}$	$5.08X10^{-4}$	168.10

Table 4.26B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related estimators when $\rho_{xy} = 0.5$ and $\rho_{xy} = 0.8$

Estimators	$\rho_{xy} = 0.8$				$\rho_{xy} = 0.99$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean	P<-1.2464	0	$6.64X10^{-4}$	100	P<-1.6043	0	$6.64X10^{-4}$	100
Choudhury and Singh (2012)	P<0.32697	$23.4X10^{-4}$	$2.39X10^{-4}$	277.78	P<0.40463	$18.7X10^{-4}$	$1.92X10^{-4}$	345.03
Chand (1975)	P<-0.2819	$16.7X10^{-4}$	$2.85X10^{-4}$	232.69	P<-0.6110	$11.8X10^{-4}$	$6.62X10^{-4}$	100.31
Singh and Upadhaya(2001)	P<-0.2792	$14.2X10^{-4}$	$3.37X10^{-4}$	197.3	P<-0.612	$12.6X10^{-4}$	$4.04X10^{-4}$	163.91
Singh et al. (2007)	P<-0.2520	$19.2X10^{-4}$	$3.01X10^{-4}$	220.65	P<-0.638	$16.5X10^{-4}$	$1.31X10^{-4}$	506.87
Singh and Upadhay(2001)	P<-0.0912	$31.4X10^{-4}$	$3.77X10^{-4}$	176.1	P<-0.0912	$22.5X10^{-4}$	$5.51X10^{-4}$	120.48
Singh (2001)	P<-0.2182	$23.1X10^{-4}$	$3.73X10^{-4}$	178.27	P<-0.776	$30.1X10^{-4}$	$1.59X10^{-4}$	417.26
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-4.1937	$19.8X10^{-4}$	$2.07X10^{-4}$	320.65	P=-10.248	$28.9X10^{-4}$	$1.308X10^{-4}$	507.43
Modifie $\bar{y}_{FTAA}^{\beta_2(d)}$	P=-1.2505	$24.7X10^{-4}$	$1.99X10^{-4}$	333.24	P=-2.1055	$18.4X10^{-4}$	$1.23X10^{-4}$	524.10
CASE II								
Sample mean	P<-1.2964	0	$6.64X10^{-4}$	100	P<-1.604	$1.77X10^{-4}$	$6.64X10^{-4}$	100
Choudhury and Singh (2012)	P<0.40872	$6.04\ X10^{-4}$	$3.68X10^{-4}$	180.41	P<0.40871	$2.65X10^{-4}$	$1.92X10^{-4}$	345.04
Chand (1975)	P<-0.2819	$13.1X10^{-4}$	$6.12X10^{-4}$	108.57	P<-0.610	$3.04\ X10^{-4}$	$6.62X10^{-4}$	100.31
Singh and Upadhay(2001)	P<-0.2791	$7.97X10^{-4}$	$4.09X10^{-4}$	162.29	P<-0.6123	$3.40X10^{-4}$	$1.82X10^{-4}$	364.61
Singh et al. (2007)	P<-0.2520	$-11.2X10^{-4}$	$3.01X10^{-4}$	220.65	P<-0.6377	$-19.6X10^{-4}$	$2.16X10^{-4}$	307.43
Singh and Upadhay(2001)	P<-0.0912	$-8.02X10^{-4}$	$4.73X10^{-4}$	140.42	P<-0.0912	$-33.9X10^{-4}$	$1.69X10^{-4}$	393.36
Singh (2001)	P<-0.2182	$2.39\ X10^{-4}$	$4.46X10^{-4}$	148.79	P<-0.7761	$-10.6X10^{-4}$	$3.61X10^{-4}$	184.01
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=-1.5793	$3.82X10^{-4}$	$2.84X10^{-4}$	233.63	P=-1.7601	$-11.9X10^{-4}$	$1.58X10^{-4}$	421.13
Modifie $\bar{y}_{FTAA}^{\beta_2(d)}$	P=-13.478	$-8.42X10^{-4}$	$2.75X10^{-4}$	241.09	P=-1.6108	$1.77X10^{-4}$	$1.02X10^{-4}$	653.44

COMMENT: Tables 4.26A and 4.26B show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some related estimators under two-phase simple random sampling scheme when the study and auxiliary variables are positively correlated with 0.2, 0.5 and 0.8, 0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over others. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that the data satisfied the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over others when $P=-2.78991$, $P=-1.23822$, $P=-2.14373$, $P=-6.27512$, $P=-6.992$, $P=-5.6609$, $P=-1.3639$, $P=-16.417$, $P=-4.1937$, $P=-1.2505$, $P=-1.5793$, $P=-13.478$, $P=-10.248$, $P=-2.1055$, $P=-1.7601$ and $P=-1.6108$ and they have minimum MSEs and high PRE.

4.2.3.2 Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related Product Estimators

In this subsection, relative efficiency of $\bar{y}_{FTAA}^{\beta_1(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ with respect to some existing related product estimators for some correlation coefficients $\rho_{xy}\left(-0.2,-0.5,-0.8,-0.99\right)$ was investigated and the results are presented in Tables 4.27A and 4.27B.

Table 4.27A: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} = -0.2$ and $\rho_{xy} = -0.5$

Estimators	$\rho_{xy} = -0.2$				$\rho_{xy} = -0.5$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean	P<1.32410	$5.59X10^{-4}$	$11.38X10^{-4}$	100	P<1.81027	0	$8.54\ X10^{-4}$	100
Choudhury and Singh (2012)	P< 2.0817	$7.12X10^{-4}$	$10.93X10^{-4}$	104.17	P<2.2044	$15.4X10^{-4}$	$6.41X10^{-4}$	133.33
Chand (1975)	P<1.32524	$5.69X10^{-4}$	$10.95X10^{-4}$	103.9	P<2.07102	$8.67X10^{-4}$	$6.19X10^{-4}$	137.96
Singh and Upadhayaya(2001)	P<1.32558	$5.23X10^{-4}$	$10.81X10^{-4}$	105.23	P< 2.0652	$4.72X10^{-4}$	$5.91\ X10^{-4}$	144.39
Singh et al. (2007)	P<1.32874	$2.45X10^{-4}$	$9.83X10^{-4}$	115.97	P<2.02982	$5.12X10^{-4}$	$5.94X10^{-4}$	143.88
Singh and Upadhayaya(2001)	P<1.36252	$4.11X10^{-4}$	$9.73\ X10^{-4}$	116.94	P<1.90218	$5.47X10^{-4}$	$4.77\ X10^{-4}$	178.86
Singh (2001)	P<1.33547	$2.19X10^{-4}$	$10.01X10^{-4}$	113.65	P<2.02032	$13.1X10^{-4}$	$5.88\ X10^{-4}$	145.48
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=1.08838	$2.13X10^{-4}$	$9.03X10^{-4}$	125.98	P=1.79020	$23.2X10^{-4}$	$4.41X10^{-4}$	193.88
Modifie $\bar{y}_{FTAA}^{\beta_2(d)}$	P=0.26601	$2.34X10^{-4}$	$7.79\ X10^{-4}$	146.18	P=1.22439	$31.7X10^{-4}$	$4.23\ X10^{-4}$	202.03
CASE II								
Sample mean	P<0.32410	0	$11.38X10^{-4}$	100	P<0.81026	0	$8.54\ X10^{-4}$	100
Choudhury and Singh (2012)	P<0.40872	$5.54X10^{-4}$	$10.33X10^{-4}$	110.15	P<0.42872	$-6.5X10^{-4}$	$4.13X10^{-4}$	206.67
Chand (1975)	P<1.32524	$6.19X10^{-4}$	$8.25\ X10^{-4}$	137.94	P<2.07102	$13.1X10^{-4}$	$4.51X10^{-4}$	189.72
Singh and Upadhayaya(2001)	P<1.32558	$1.25X10^{-4}$	$8.05\ X10^{-4}$	141.43	P<2.06519	$1.62X10^{-4}$	$4.91X10^{-4}$	174.44
Singh et al. (2007)	P<1.32874	$1.55X10^{-4}$	$9.81\ X10^{-4}$	115.98	P<2.02982	$2.81X10^{-4}$	$5.93X10^{-4}$	143.88
Singh and Upadhayaya(2001)	P<1.36251	$2.06X10^{-4}$	$10.06X10^{-4}$	113.09	P<1.90218	$5.2X10^{-4}$	$7.45X10^{-4}$	114.59
Singh (2001)	P<1.33547	$2.12X10^{-4}$	$10.98X10^{-4}$	103.55	P<2.02032	$7.13X10^{-4}$	$7.06\ X10^{-4}$	120.94
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=0.14020	$2.24X10^{-4}$	$7.93\ X10^{-4}$	143.58	P=0.32281	$3.16X10^{-4}$	$3.84\ X10^{-4}$	224.88
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=0.27512	$2.28X10^{-4}$	$4.62\ X10^{-4}$	246.24	P=0.41708	$5.31X10^{-4}$	$2.34\ X10^{-4}$	249.28

Table 4.27B: Bias, MSE and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some existing related product estimators when $\rho_{xy} = -0.8$ and $\rho_{xy} = -0.99$

Estimators	$\rho_{xy} = -0.8$				$\rho_{xy} = -0.99$			
	Efficiency Conditions	Bias	MSE	PRE	Efficiency Conditions	Bias	MSE	PRE
CASE I								
Sample mean	P<1.29641	0	$9.81 X10^{-4}$	100	P<1.60431	0	$6.64X10^{-4}$	100
Choudhury and Singh (2012)	P< 2.3269	$32.5X10^{-4}$	$3.75X10^{-4}$	261.72	P<1.40463	$6.30X10^{-4}$	$1.89X10^{-4}$	351.13
Chand (1975)	P<2.30134	$61.6X10^{-4}$	$6.07X10^{-4}$	161.70	P<2.72054	$6.77X10^{-4}$	$3.38X10^{-4}$	196.74
Singh and Upadhayaya(2001)	P<2.30229	$41.3X10^{-4}$	$5.88 X10^{-4}$	166.83	P<2.71781	$7.18X10^{-4}$	$3.33X10^{-4}$	199.60
Singh et al. (2007)	P<2.29732	$59.3X10^{-4}$	$4.68 X10^{-4}$	209.53	P<2.69739	$60.7X10^{-4}$	$1.58X10^{-4}$	421.34
Singh and Upadhayaya(2001)	P<2.38309	$35.7X10^{-4}$	$6.07 X10^{-4}$	161.58	P<2.64408	$32.1X10^{-4}$	$3.09X10^{-4}$	215.06
Singh (2001)	P<2.31526	$42.5X10^{-4}$	$6.91X10^{-4}$	141.94	2.680445	$38.6X10^{-4}$	$3.16X10^{-4}$	209.88
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=1.1484	$37.4X10^{-4}$	$3.51X10^{-4}$	279.52	P=0.69523	$42.9X10^{-4}$	$1.58 X10^{-4}$	421.34
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=1.28067	$45.1X10^{-4}$	$3.33X10^{-4}$	294.79	P=1.31089	$45.4X10^{-4}$	$1.52 X10^{-4}$	438.23
CASE II								
Sample mean	P<1.29641	0	$9.81 X10^{-4}$	100	P<1.60431	$8.62X10^{-4}$	$6.64 X10^{-4}$	100
Choudhury and Singh (2012)	P<1.40872	$6.04X10^{-4}$	$6.13 X10^{-4}$	159.98	P<1.41726	$8.31X10^{-4}$	$3.75X10^{-4}$	176.69
Chand (1975)	P<2.30134	$13.1X10^{-4}$	$6.77X10^{-4}$	144.99	P<2.72054	$22.7X10^{-4}$	$4.87 X10^{-4}$	136.34
Singh and Upadhayaya(2001)	P<2.30229	$7.97X10^{-4}$	$9.21 X10^{-4}$	106.52	P<2.71781	$3.19X10^{-4}$	$4.22 X10^{-4}$	157.26
Singh et al. (2007)	P<2.29732	$-11X10^{-4}$	$4.68 X10^{-4}$	209.53	P<2.69739	$21.6X10^{-4}$	$2.99 X10^{-4}$	221.34
Singh and Upadhayaya(2001)	P<2.38309	$-8.1X10^{-4}$	$5.69X10^{-4}$	172.37	P<2.64408	$4.16X10^{-4}$	$3.77 X10^{-4}$	176.01
Singh (2001)	P<2.31526	$2.39X10^{-4}$	$5.85X10^{-4}$	167.75	P<2.68045	$6.34X10^{-4}$	$3.98 X10^{-4}$	166.73
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	P=0.56384	$3.82X10^{-4}$	$4.35X10^{-4}$	225.58	P=0.75205	$6.30X10^{-4}$	$2.85 X10^{-4}$	232.75
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	P=0.14986	$8.42X10^{-4}$	$4.47 X10^{-4}$	210.21	P=1.11080	$4.76X10^{-4}$	$2.53 X10^{-4}$	262.08

COMMENT: Tables 4.27A and 4.27B show the biases, MSEs and PRE of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ and some related estimators under two-phase simple random sampling scheme when the study and auxiliary variables are negatively correlated with -0.2, -0.5 and -0.8, -0.99 coefficients respectively as well as the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over others. These properties (Bias, MSE and PRE) were computed under cases I and II. The results of the analysis revealed that the data satisfied the efficiency conditions of $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ over others when $P=1.08838$, $P=0.26601$, $P=0.14020$, $P=0.27512$, $P=1.79020$, $P=1.22439$, $P=0.32281$, $P=0.41708$, $P=1.1484$, $P=1.28067$, $P=0.56384$, $P=0.14986$, $P=0.69528$, $P=1.31089$, $P=0.75205$ and $P=1.11080$ and they have minimum MSEs and high PRE.

4.2.4 Robustness under Non-Response Model

In this section, robustness of the modified estimators were investigated with respect to that of some existing related ratio, product, dual-to-ratio and factor-type estimators under non-response model defined in section 1.

The results of robustness of the modified estimators and some existing related estimators under non-response model for some values of correlation coefficients $\rho_{xy}\left(\pm 0.2, \pm 0.5, \pm 0.8, \pm 0.99\right)$ are presented in Tables 4.28-4.35. The robustness results of the considered estimators were obtained by averaging their expected MSEs for some proportions of non-response and ranked accordingly to their level of efficiency.

Table 4.28: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.2$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Singh and Upadhayaya (2001)1	0.000309	0.000244	0.000179	0.000113	0.000211	1st
Singh and Upadhayaya (2001)2	0.000311	0.000246	0.000181	0.000115	0.000213	2nd
Chand (1975)	0.000337	0.000272	0.000206	0.000141	0.000239	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.000449	0.000384	0.000318	0.000253	0.000351	4th
Singh et al. (2007)	0.001263	0.001198	0.001133	0.001067	0.001165	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.001263	0.001198	0.001133	0.001067	0.001165	6th
Choudhury and Singh (2012)	0.001354	0.001289	0.001224	0.001158	0.001256	7th
Singh and Shukla (1987, 1993)	0.001354	0.001289	0.001224	0.001158	0.001256	7th
Upadhyaya and Singh (1999) 1	0.001362	0.001297	0.001231	0.001166	0.001264	9th
Shukla (2002)	0.001366	0.0013	0.001235	0.00117	0.001268	10th
Srivenkataramana (1980)	0.001368	0.001302	0.001237	0.001172	0.00127	11th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.001394	0.001329	0.001263	0.001198	0.001296	12th
Modified $\bar{y}_{FTAA}$	0.001396	0.00133	0.001265	0.0012	0.001298	13th
Modified $\bar{y}_{FTAA}^{(d)}$	0.001397	0.001331	0.001266	0.001201	0.001299	14th
Sample mean	0.0014	0.001335	0.001269	0.001204	0.001302	15th
Singh et. al. (2004)	0.001416	0.00135	0.001285	0.00122	0.001318	16th
Shukla et. al. (2013)	0.001446	0.00138	0.001315	0.00125	0.001348	17th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.001512	0.001447	0.001381	0.001316	0.001414	18th
Thakur and Gupta (2013)	0.001578	0.001513	0.001448	0.001382	0.00148	19th
Malik and Tailor (2013)	0.002129	0.002064	0.001999	0.001933	0.002031	20th
Singh (2001)	0.002177	0.002111	0.002046	0.00198	0.002078	21st
Sukhatme (1962)	0.002279	0.002214	0.002148	0.002083	0.002181	22nd
Singh and Tailor (2003)	0.002326	0.002261	0.002195	0.00213	0.002228	23rd
Sisodia and Diwivedi (1981)	0.002331	0.002266	0.002201	0.002135	0.002233	24th
Upadhyaya and Singh (1999) 2	0.002501	0.002436	0.002371	0.002305	0.002403	25th
Cochran (1942)	0.002572	0.002507	0.002441	0.002376	0.002474	26th
Srivenkataramana (1988) two-phase	0.003122	0.003057	0.002991	0.002926	0.003024	27th

COMMENT: COMMENT: Table 4.28 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = 0.2$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked thirteenth, fourteenth, sixth and fourth respectively.

Table 4.29: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.5$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Singh and Upadhayaya (2001)	0.000231	0.000179	0.000128	0.000076	0.000153	1st
Singh and Upadhayaya (2001)	0.000233	0.000181	0.000129	0.000077	0.000155	2nd
Chand (1975)	0.000276	0.000224	0.000173	0.000121	0.000199	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.000436	0.000384	0.000332	0.000281	0.000358	4th
Singh et al. (2007)	0.000826	0.000774	0.000723	0.000671	0.000748	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.000826	0.000774	0.000723	0.000671	0.000748	6th
Choudhury and Singh (2012)	0.000847	0.000795	0.000744	0.000692	0.000769	7th
Singh and Shukla (1987, 1993)	0.000847	0.000795	0.000744	0.000692	0.000769	8th
Singh et. al. (2004)	0.000851	0.0008	0.000748	0.000696	0.000774	9th
Srivenkataramana (1980)	0.000854	0.000802	0.00075	0.000699	0.000776	10th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.000858	0.000806	0.000755	0.000703	0.00078	11th
Thakur and Gupta (2013)	0.000865	0.000813	0.000761	0.00071	0.000787	12th
Shukla (2002)	0.000894	0.000843	0.000791	0.000739	0.000817	13th
Upadhyaya and Singh (1999)	0.000973	0.000921	0.00087	0.000818	0.000895	14th
Modified $\bar{y}_{FTAA}$	0.001041	0.00099	0.000938	0.000886	0.000964	15th
Modified $\bar{y}_{FTAA}^{(d)}$	0.001046	0.000994	0.000942	0.000891	0.000968	16th
Singh and Tailor (2003)	0.001051	0.001	0.000948	0.000896	0.000974	17th
Sample mean	0.00106	0.001009	0.000957	0.000905	0.000983	18th
Malik and Tailor (2013)	0.001084	0.001032	0.000981	0.000929	0.001006	19th
Sisodia and Diwivedi (1981)	0.001181	0.001129	0.001078	0.001026	0.001103	20th
Singh (2001)	0.001184	0.001132	0.001081	0.001029	0.001106	21st
Sukhatme (1962)	0.001252	0.001201	0.001149	0.001097	0.001175	22nd
Upadhyaya and Singh (1999) 2	0.00127	0.001218	0.001166	0.001115	0.001192	23rd
Shukla et. al. (2013)	0.001275	0.001224	0.001172	0.00112	0.001198	24th
Cochran (1942)	0.001307	0.001256	0.001204	0.001152	0.00123	25th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.001586	0.001535	0.001483	0.001431	0.001509	26th
Srivenkataramana (1988) two-phase	0.002892	0.00284	0.002788	0.002737	0.002814	27th

COMMENT: Table 4.29 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = 0.5$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked fifteenth, sixteenth, sixth and fourth respectively.

Table 4.30: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.8$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Singh and Upadhayaya (2001)	0.000223	0.000169	0.000115	6.24E-05	0.000142	1st
Singh and Upadhayaya (2001)	0.000224	0.000171	0.000118	6.49E-05	0.000144	2nd
Chand (1975)	0.000266	0.000213	0.00016	0.000107	0.000186	3rd
Choudhury and Singh (2012)	0.000451	0.000398	0.000345	0.000292	0.000372	4th
Singh and Shukla (1987, 1993)	0.000451	0.000398	0.000345	0.000292	0.000372	5th
Singh and Tailor (2003)	0.000456	0.000403	0.00035	0.000297	0.000376	6th
Thakur and Gupta (2013)	0.000467	0.000414	0.000361	0.000308	0.000388	7th
Singh et. al. (2004)	0.000494	0.000441	0.000388	0.000335	0.000415	8th
Srivenkataramana (1980)	0.0005	0.000447	0.000394	0.000341	0.00042	9th
Singh et al. (2007)	0.000513	0.00046	0.000407	0.000354	0.000434	10th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.000513	0.00046	0.000407	0.000354	0.000434	11th
Sisodia and Diwivedi (1981)	0.000521	0.000468	0.000415	0.000362	0.000442	12th
Upadhyaya and Singh (1999)	0.000552	0.000499	0.000447	0.000393	0.000473	13th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.000565	0.000512	0.000459	0.000406	0.000485	14th
Cochran (1942)	0.000576	0.000523	0.00047	0.000417	0.000497	15th
Singh (2001)	0.000585	0.000532	0.000479	0.000426	0.000505	16th
Shukla (2002)	0.000633	0.00058	0.000527	0.000474	0.000554	17th
Malik and Tailor (2013)	0.000641	0.000588	0.000535	0.000482	0.000562	18th
Sukhatme (1962)	0.000705	0.000652	0.000599	0.000546	0.000625	19th
Upadhyaya and Singh (1999)	0.000719	0.000666	0.000613	0.00056	0.000639	20th
Modified $\bar{y}_{FTAA}$	0.000777	0.000724	0.000671	0.000618	0.000697	21st
Modified $\bar{y}_{FTAA}^{(d)}$	0.000819	0.000766	0.000713	0.00066	0.00074	22nd
Sample mean	0.000876	0.000823	0.00077	0.000717	0.000797	23rd
Shukla et. al. (2013)	0.001304	0.001251	0.001198	0.001145	0.001224	24th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.001923	0.00187	0.001817	0.001764	0.001844	25th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.001786	0.001839	0.001892	0.001945	0.001865	26th
Srivenkataramana (1988) two-phase	0.002203	0.00215	0.002097	0.002044	0.002124	27th

COMMENT: Table 4.30 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = 0.8$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked twenty-first, twenty-second, eleventh and twenty-sixth respectively.

Table 4.31: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = 0.99$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Singh and Upadhayaya (2001)	0.000183	0.000138	9.36E-05	0.000049	0.000116	1st
Singh and Upadhayaya (2001)	0.000185	0.00014	9.51E-05	0.00005	0.000118	2nd
Choudhury and Singh (2012)	0.000192	0.000148	0.000103	0.000058	0.000125	3rd
Singh and Shukla (1987, 1993)	0.000192	0.000148	0.000103	0.000058	0.000125	4th
Sisodia and Diwivedi (1981)	0.000205	0.00016	0.000115	0.000075	0.000139	5th
Singh and Tailor (2003)	0.00021	0.000165	0.00012	0.000075	0.000143	6th
Upadhyaya and Singh (1999)	0.000222	0.000178	0.000133	0.000088	0.000155	7th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.000172	0.000163	0.000161	0.000131	0.000157	8th
Cochran (1942)	0.000232	0.000187	0.000142	0.000098	0.000165	9th
Chand (1975)	0.000245	0.000201	0.000156	0.000111	0.000178	10th
Thakur and Gupta (2013)	0.000272	0.000227	0.000183	0.000138	0.000205	11th
Singh et al. (2007)	0.00031	0.000265	0.000221	0.000176	0.000243	12th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.00031	0.000265	0.000221	0.000176	0.000243	13th
Srivenkataramana (1980)	0.000334	0.000289	0.000244	0.000199	0.000267	14th
Singh (2001)	0.000338	0.000294	0.000249	0.000204	0.000271	15th
Shukla (2002)	0.000378	0.000334	0.000289	0.000244	0.000311	16th
Malik and Tailor (2013)	0.000383	0.000338	0.000293	0.000248	0.000316	17th
Singh et. al. (2004)	0.000394	0.000349	0.000305	0.00026	0.000327	18th
Sukhatme (1962)	0.000407	0.000362	0.000317	0.000272	0.00034	19th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.000434	0.00039	0.000345	0.0003	0.000367	20th
Upadhyaya and Singh (1999)	0.000695	0.00065	0.000606	0.000561	0.000628	21st
Modified $\bar{y}_{FTAA}$	0.000765	0.00072	0.000675	0.000631	0.000698	22nd
Modified $\bar{y}_{FTAA}^{(d)}$	0.000787	0.000743	0.000698	0.000653	0.00072	23rd
Sample mean	0.000843	0.000799	0.000754	0.000708	0.000776	24th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.002447	0.002402	0.002357	0.002312	0.00238	25th
Srivenkataramana (1988) two-phase	0.002725	0.00268	0.002635	0.002591	0.002658	26th
Shukla et. al. (2013)	0.02405	0.024006	0.023961	0.023916	0.023983	27th

COMMENT: Table 4.31 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = 0.99$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked twenty-second, twenty-third, thirteenth and eighth respectively.

Table 4.32: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.2$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Chand (1975)	0.000322	0.000255	0.000188	0.000121	0.000221	1st
Singh and Upadhayaya (2001)	0.000333	0.000266	0.000199	0.000132	0.000233	2nd
Singh and Upadhayaya (2001)	0.000336	0.000269	0.000202	0.000135	0.000235	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.001047	0.00098	0.000913	0.000846	0.000946	4th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.00125	0.001183	0.001116	0.001045	0.001148	5th
Singh et al. (2007)	0.00125	0.001183	0.001116	0.001049	0.001149	6th
Choudhury and Singh (2012)	0.001361	0.001294	0.001227	0.00116	0.001261	7th
Singh and Shukla (1987, 1993)	0.001361	0.001294	0.001227	0.00116	0.001261	8th
Shukla et. al. (2013)	0.001361	0.001294	0.001227	0.00116	0.001261	9th
Shukla (2002)	0.001375	0.001308	0.001241	0.001174	0.001274	10th
Modified $\bar{y}_{FTAA}$	0.001398	0.001331	0.001264	0.001197	0.001298	11th
Modified $\bar{y}_{FTAA}^{(d)}$	0.001401	0.001334	0.001267	0.0012	0.0013	12th
Sample mean	0.001407	0.00134	0.001273	0.001205	0.001306	13th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.001478	0.001411	0.001344	0.001277	0.001377	14th
Upadhyaya and Singh (1999)	0.001498	0.001431	0.001364	0.001297	0.001397	15th
Srivenkataramana (1980)	0.001656	0.001588	0.001521	0.001454	0.001555	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.001695	0.001628	0.001561	0.001494	0.001594	17th
Singh et. al. (2004)	0.001927	0.00186	0.001793	0.001726	0.001826	18th
Thakur and Gupta (2013)	0.002072	0.002005	0.001938	0.001871	0.001971	19th
Srivenkataramana (1988) two-phase	0.002221	0.002153	0.002086	0.002019	0.00212	20th
Singh (2001)	0.002876	0.002809	0.002742	0.002675	0.002775	21st
Sukhatme (1962)	0.003001	0.002934	0.002867	0.0028	0.002901	22nd
Malik and Tailor (2013)	0.003244	0.003177	0.00311	0.003043	0.003143	23rd
Sisodia and Diwivedi (1981)	0.003364	0.003297	0.00323	0.003163	0.003263	24th
Upadhyaya and Singh (1999)	0.003577	0.00351	0.003443	0.003376	0.003477	25th
Cochran (1942)	0.003703	0.003636	0.003569	0.003501	0.003602	26th
Singh and Tailor (2003)	0.004162	0.004095	0.004028	0.003961	0.004062	27th

COMMENT: Table 4.32 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = -0.2$. The results revealed that out of the twenty-seven

competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked eleventh, twelfth, fifth and fourth respectively.

Table 4.33: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.5$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.000167	0.000123	0.000079	0.000035	0.000101	1st
Chand (1975)	0.000199	0.000155	0.000111	0.000068	0.000133	2nd
Singh and Upadhayaya (2001)	0.000227	0.000183	0.000139	0.000095	0.000161	3rd
Singh and Upadhayaya (2001)	0.000235	0.000191	0.000147	0.000103	0.000169	4th
Singh et al. (2007)	0.000769	0.000725	0.000681	0.000637	0.000703	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.000769	0.000725	0.000681	0.000637	0.000703	6th
Choudhury and Singh (2012)	0.000816	0.000772	0.000728	0.000684	0.00075	7th
Singh and Shukla (1987, 1993)	0.000816	0.000772	0.000728	0.000684	0.00075	8th
Shukla (2002)	0.000922	0.000879	0.000835	0.000791	0.000857	9th
Sample mean	0.001029	0.000985	0.000941	0.000898	0.000963	10th
Modified $\bar{y}_{FTAA}^{(d)}$	0.001039	0.000996	0.000952	0.000908	0.000974	11th
Modified $\bar{y}_{FTAA}$	0.001049	0.001005	0.000962	0.000918	0.000984	12th
Srivenkataramana (1988) two-phase	0.001152	0.001109	0.001065	0.001021	0.001087	13th
Upadhyaya and Singh (1999)	0.001159	0.001116	0.001072	0.001028	0.001094	14th
Shukla et. al. (2013)	0.001244	0.0012	0.001156	0.001112	0.001178	15th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.001486	0.001443	0.001399	0.001355	0.001421	16th
Srivenkataramana (1980)	0.001525	0.001481	0.001437	0.001393	0.001459	17th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.001555	0.001511	0.001467	0.001424	0.001489	18th
Singh et. al. (2004)	0.001629	0.001585	0.001541	0.001497	0.001563	19th
Thakur and Gupta (2013)	0.001932	0.001888	0.001845	0.001801	0.001867	20th
Singh (2001)	0.002053	0.002009	0.001965	0.001921	0.001987	21st
Sukhatme (1962)	0.002206	0.002162	0.002119	0.002075	0.002141	22nd
Malik and Tailor (2013)	0.002681	0.002637	0.002593	0.002549	0.002615	23rd
Sisodia and Diwivedi (1981)	0.003072	0.003028	0.002984	0.00294	0.003006	24th
Upadhyaya and Singh (1999)	0.003278	0.003234	0.00319	0.003147	0.003212	25th
Cochran (1942)	0.003383	0.00334	0.003296	0.003252	0.003318	26th
Singh and Tailor (2003)	0.004726	0.004682	0.004639	0.004595	0.004661	27th

COMMENT: Table 4.33 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2,

0.25. 0.33 and 0.5) when $\rho_{xy} = -0.5$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked twelfth, eleventh, fifth and first positions respectively.

Table 4.34: MSE of $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, \bar{y}_{FTAA}^{\beta_1(d)}$ and $\bar{y}_{FTAA}^{\beta_2(d)}$ under Non-Response Model when $\rho_{xy} = -0.8$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Chand (1975)	0.000291	0.000225	0.000159	0.00009	0.000191	1st
Singh and Upadhayaya (2001)	0.000323	0.000257	0.000191	0.000125	0.000224	2nd
Singh and Upadhayaya (2001)	0.000327	0.000261	0.000195	0.000129	0.000228	3rd
Choudhury and Singh (2012)	0.000617	0.000551	0.000485	0.000419	0.000518	4th
Singh and Shukla (1987, 1993)	0.000617	0.000551	0.000485	0.000419	0.000518	5th
Singh et al. (2007)	0.000732	0.000666	0.0006	0.000534	0.000633	6th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.000732	0.000666	0.0006	0.000534	0.000633	7th
Shukla (2002)	0.000955	0.000889	0.000823	0.000757	0.000856	8th
Srivenkataramana (1988) two-phase	0.00104	0.000974	0.000908	0.000842	0.000941	9th
Sample mean	0.001244	0.001178	0.001112	0.001046	0.001145	10th
Modified $\bar{y}_{FTAA}^{(d)}$	0.001251	0.001185	0.001119	0.001053	0.001152	11th
Modified $\bar{y}_{FTAA}$	0.001259	0.001193	0.001127	0.001061	0.00116	12th
Upadhyaya and Singh (1999)	0.00146	0.001394	0.001328	0.001262	0.001361	13th
Shukla et. al. (2013)	0.001876	0.00181	0.001744	0.001678	0.001777	14th
Srivenkataramana (1980)	0.001932	0.001866	0.0018	0.001734	0.001833	15th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.001997	0.001931	0.001865	0.001799	0.001898	16th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.001925	0.001991	0.002056	0.002122	0.002024	17th
Singh et. al. (2004)	0.002169	0.002103	0.002037	0.001971	0.00207	18th
Singh (2001)	0.002601	0.002535	0.002469	0.002403	0.002502	19th
Thakur and Gupta (2013)	0.00266	0.002594	0.002528	0.002462	0.002561	20th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.00279	0.002724	0.002658	0.002592	0.002691	21st
Sukhatme (1962)	0.002824	0.002758	0.002692	0.002626	0.002725	22nd
Malik and Tailor (2013)	0.003976	0.00391	0.003844	0.003778	0.003877	23rd
Sisodia and Diwivedi (1981)	0.004251	0.004185	0.004119	0.004053	0.004152	24th
Upadhyaya and Singh (1999)	0.004539	0.004473	0.004407	0.004341	0.00444	25th
Cochran (1942)	0.004673	0.004607	0.004541	0.004475	0.004574	26th
Singh and Tailor (2003)	0.008543	0.008477	0.008411	0.008345	0.008444	27th

COMMENT: Table 4.34 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2,

0.25. 0.33 and 0.5) when $\rho_{xy} = -0.8$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$, and $\bar{y}^{\beta_2(d)}_{FTAA}$ are ranked twelfth, eleventh, sixth and seventeenth respectively.

Table 4.35: MSE of $\bar{y}_{FTAA}, \bar{y}^{(d)}_{FTAA}, \bar{y}^{\beta_1(d)}_{FTAA}$ and $\bar{y}^{\beta_2(d)}_{FTAA}$ under Non-Response Model when $\rho_{xy} = -0.99$

Estimators	$(1/k)$				Average MSE	Rank
	$(1/5)$	$(1/4)$	$(1/3)$	$(1/2)$		
Chand (1975)	0.000202	0.000153	0.000104	0.000055	0.000129	1st
Choudhury and Singh (2012)	0.000209	0.00016	0.000111	0.000062	0.000136	2nd
Singh and Shukla (1987, 1993)	0.000209	0.00016	0.000111	0.000062	0.000136	3rd
Singh and Upadhayaya (2001)	0.000255	0.000207	0.000158	0.000109	0.000182	4th
Singh and Upadhayaya (2001)	0.00026	0.000211	0.000162	0.000113	0.000187	5th
Singh et al. (2007)	0.000353	0.000304	0.000256	0.000207	0.00028	6th
Modified $\bar{y}^{\beta_1(d)}_{FTAA}$	0.000353	0.000304	0.000256	0.000207	0.00028	7th
Shukla (2002)	0.000437	0.000388	0.000339	0.00029	0.000364	8th
Srivenkataramana (1988) two-phase	0.000463	0.000414	0.000365	0.000316	0.00039	9th
Modified $\bar{y}^{\beta_2(d)}_{FTAA}$	0.000461	0.000466	0.000471	0.000475	0.000468	10th
Sample mean	0.00086	0.000811	0.000762	0.000713	0.000787	11th
Modified $\bar{y}^{(d)}_{FTAA}$	0.000975	0.000926	0.000878	0.000829	0.000902	12th
Modified $\bar{y}_{FTAA}$	0.001038	0.000989	0.00094	0.000891	0.000964	13th
Upadhyaya and Singh (1999)	0.001073	0.001024	0.000975	0.000926	0.000999	14th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.001467	0.001418	0.001369	0.00132	0.001394	15th
Singh et. al. (2004)	0.001709	0.00166	0.001611	0.001562	0.001636	16th
Srivenkataramana (1980)	0.001741	0.001692	0.001643	0.001594	0.001668	17th
Shukla et. al. (2013)	0.001783	0.001734	0.001685	0.001636	0.001709	18th
Thakur and Gupta (2013)	0.001981	0.001932	0.001883	0.001834	0.001908	19th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.002463	0.002414	0.002366	0.002317	0.00239	20th
Singh (2001)	0.002487	0.002438	0.002389	0.00234	0.002413	21st
Sukhatme (1962)	0.00257	0.002522	0.002473	0.002424	0.002497	22nd
Sisodia and Diwivedi (1981)	0.003181	0.003132	0.003083	0.003034	0.003108	23rd
Upadhyaya and Singh (1999)	0.003375	0.003326	0.003278	0.003229	0.003302	24th
Cochran (1942)	0.003494	0.003445	0.003396	0.003347	0.003421	25th
Malik and Tailor (2013)	0.004287	0.004238	0.004189	0.00414	0.004213	26th
Singh and Tailor (2003)	0.007933	0.007884	0.007835	0.007786	0.007859	27th

COMMENT: Table 4.35 shows the MSEs and average MSEs of the modified estimators and those of some existing related estimators for some proportion of non-response (0.2, 0.25. 0.33 and 0.5) when $\rho_{xy} = -0.99$. The results revealed that out of the twenty-seven competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, and $\bar{y}_{FTAA}^{\beta_2(d)}$ are ranked thirteenth, eleventh, sixth and tenth respectively.

4.2.5 Robustness under Super-Population Model

In this section, robustness of the modified estimators were investigated with respect to that of some existing related ratio, product, dual-to-ratio and factor-type estimators under super-population model defined in chapter three using simulated data.

Biases and MSEs of these modified estimators and some existing related estimators considered in the study are obtained as shown in Tables 4.36A-4.42B under linear (I), quadratic (II) and Cubic (III) Models respectively.

Table 4.36A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.2$

Estimators	Bias	MSE	Rank
Chand (1975)	1.453	2.11207	1st
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.457	2.12555	2nd
Shukla (2002)	1.458	2.1273	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.463	2.12918	4th
Malik and Tailor (2013)	1.458	2.12922	5th
Shukla et al. (2013)	1.46	2.13261	6th
Srivenkataramana (1980)	1.46	2.1329	7th
Singh and Shukla (1987, 1993)	1.46	2.13351	8th
Modified $\bar{y}_{FTAA}^{(d)}$	1.463	2.13355	9th
Modified $\bar{y}_{FTAA}$	1.463	2.13384	10th
Cochran (1942)	1.46	2.13397	11th
Upadhyaya and Singh (1999)	1.46	2.13408	12th
Sisodia and Diwivedi (1981)	1.46	2.13434	13th
Singh and Tailor (2003)	1.46	2.13435	14th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.461	2.13601	15th
Singh et. al. (2004)	1.461	2.13674	16th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.461	2.1369	17th
Upadhyaya and Singh (1999)	1.461	2.13784	18th
Sample mean	1.461	2.13825	19th
Upadhyaya and Singh (1999)	1.462	2.13867	20th
Singh (2003)	1.462	2.13963	21st
Singh et. al. (2004)	1.462	2.13977	22nd
Singh (2003)	1.462	2.14036	23rd
Singh (2003)	1.463	2.14189	24th
Singh et al. (2007)	1.463	2.14216	25th
Singh and Tailor (2003)	1.463	2.1422	26th
Pandey and Dubey (1988)	1.463	2.14221	27th
Singh (2003)	1.463	2.14238	28th
Thakur and Gupta (2013)	1.463	2.1424	29th
Upadhyaya and Singh (1999)	1.463	2.14248	30th
Product	1.463	2.14259	31st
Singh (2001)	1.463	2.14417	32nd
Singh and Upadhayaya (2001)	1.464	2.14503	33rd

Table 4.36B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.2$

Estimators	Bias	MSE	Rank
Singh and Upadhayaya (2001)	1.465	2.14803	1st
Singh (2001)	1.465	2.14891	2nd
Modified $\bar{y}_{FTAA}$	1.465	2.14924	3rd
Modified $\bar{y}_{FTAA}^{(d)}$	1.465	2.14953	4th
Product	1.466	2.15051	5th
Upadhyaya and Singh (1999)	1.467	2.15062	6th
Thakur and Gupta (2013)	1.466	2.15071	7th
Singh (2003)	1.466	2.15073	8th
Pandey and Dubey (1988)	1.466	2.1509	9th
Singh and Tailor (2003)	1.466	2.15091	10th
Singh et al. (2007)	1.466	2.15096	11th
Singh (2003)	1.466	2.15123	12th
Singh (2003)	1.466	2.1528	13th
Singh et. al. (2004)	1.467	2.15339	14th
Singh (2003)	1.467	2.15354	15th
Upadhyaya and Singh (1999)	1.467	2.15453	16th
Sample mean	1.467	2.15495	17th
Upadhyaya and Singh (1999)	1.467	2.15537	18th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.468	2.15633	19th
Singh et. al. (2004)	1.468	2.15649	20th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.468	2.15725	21st
Singh and Tailor (2003)	1.469	2.15894	22nd
Sisodia and Diwivedi (1981)	1.468	2.15896	23rd
Upadhyaya and Singh (1999)	1.469	2.15923	24th
Cochran (1942)	1.469	2.15934	25th
Singh and Shukla (1987, 1993)	1.469	2.15981	26th
Srivenkataramana (1980)	1.469	2.16043	27th
Shukla et al. (2013)	1.469	2.16073	28th
Malik and Tailor (2013)	1.47	2.16422	29th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.475	2.16426	30th
Shukla (2002)	1.471	2.1662	31st
Modified 3	1.472	2.1678	32nd
Chand (1975)	1.476	2.18195	33rd

Table 4.36C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.2$

Estimators	Bias	MSE	Rank
Chand (1975)	1.46	2.13349	1st
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.462	2.13933	2nd
Shukla (2002)	1.462	2.14009	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.462	2.14091	4th
Malik and Tailor (2013)	1.462	2.14093	5th
Shukla et al. (2013)	1.463	2.14239	6th
Srivenkataramana (1980)	1.463	2.14252	7th
Singh and Shukla (1987, 1993)	1.463	2.14278	8th
Cochran (1942)	1.463	2.14298	9th
Upadhyaya and Singh (1999)	1.463	2.14303	10th
Sisodia and Diwivedi (1981)	1.463	2.14314	11th
Singh and Tailor (2003)	1.463	2.14315	12th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.463	2.14386	13th
Singh et. al. (2004)	1.463	2.14418	14th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.463	2.14425	15th
Upadhyaya and Singh (1999)	1.464	2.14466	16th
Sample mean	1.464	2.14484	17th
Upadhyaya and Singh (1999)	1.464	2.14502	18th
Singh (2003)	1.464	2.14543	19th
Singh et. al. (2004)	1.464	2.14549	20th
Singh (2003)	1.464	2.14575	21st
Singh (2003)	1.464	2.14641	22nd
Singh et al. (2007)	1.464	2.14653	23rd
Pandey and Dubey (1988)	1.464	2.14655	24th
Singh and Tailor (2003)	1.464	2.14655	25th
Singh (2003)	1.464	2.14662	26th
Thakur and Gupta (2013)	1.464	2.14663	27th
Upadhyaya and Singh (1999)	1.464	2.14667	28th
Product	1.464	2.14671	29th
Modified $\bar{y}_{FTAA}^{(d)}$	1.464	2.14713	30th
Modified $\bar{y}_{FTAA}$	1.464	2.14725	31st
Singh (2001)	1.465	2.1474	32nd
Singh and Upadhayaya (2001)	1.465	2.14777	33rd

COMMENT: Tables 4.36A, 4.36B and 4.36C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = 0.2$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$ emerged second, fourth, nineth, tenth positions respectively in Table 4.36A, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged third, fourth, thirtieth, thirty-twoth positions respectively in Table 4.36B and $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta 2(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$ emerged second, fourth, thirtieth, thirty-oneth repectively in Table 4.36C.

From these results, it is deduced that:

i. The modified estimator $\bar{y}_{FTAA}$ is efficient when the relationship between the study and auxiliary variables is either linear or quadratic but less efficient when it is cubic in comparison with other estimators considered.

ii. The modified estimator $\bar{y}_{FTAA}^{(d)}$ is also efficient when the relationship between the study and auxiliary variables is either linear or quadratic but less efficient when it is cubic in comparison with other estimators considered.

iii. The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is efficient when the relationship between the study and auxiliary variables is either linear or cubic but less efficient when the relationship is quadratic in comparison with other estimators considered.

iv. The modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ is also efficient when the relationship between the study and auxiliary variables is either linear or cubic but less efficient when the relationship is quadratic in comparison with other estimators considered.

Table 4.37A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.0371	1.07683	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.0372	1.07699	2nd
Chand (1975)	1.0445	1.09226	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.0891	1.09626	4th
Malik and Tailor (2013)	1.0686	1.14319	5th
Srivenkataramana (1980)	1.0686	1.14335	6th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.0695	1.14511	7th
Cochran (1942)	1.07	1.14628	8th
Upadhyaya and Singh (1999)	1.0701	1.1465	9th
Sisodia and Diwivedi (1981)	1.0704	1.14706	10th
Singh and Tailor (2003)	1.0708	1.14802	11th
Shukla (2002)	1.0711	1.14857	12th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.072	1.15049	13th
Singh and Shukla (1987, 1993)	1.0723	1.15124	14th
Singh et. al. (2004)	1.0727	1.15203	15th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0728	1.15233	16th
Upadhyaya and Singh (1999)	1.0737	1.15428	17th
Sample mean	1.0741	1.15512	18th
Upadhyaya and Singh (1999)	1.0745	1.15596	19th
Shukla et al. (2013)	1.0752	1.15756	20th
Singh (2003)	1.0754	1.15794	21st
Singh et. al. (2004)	1.0755	1.15822	22nd
Thakur and Gupta (2013)	1.0757	1.15847	23rd
Singh (2003)	1.0761	1.15942	24th
Singh and Tailor (2003)	1.0774	1.16228	25th
Singh (2003)	1.0776	1.1626	26th
Pandey and Dubey (1988)	1.0779	1.16327	27th
Singh (2003)	1.078	1.16362	28th
Upadhyaya and Singh (1999)	1.0781	1.16383	29th
Product	1.0783	1.16406	30th
Singh et al. (2007)	1.0864	1.18165	31st
Singh (2001)	1.0891	1.1877	32nd
Singh and Upadhayaya (2001)	1.0903	1.19033	33rd

Table 4.37B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.0507	1.10516	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.0507	1.10525	2nd
Chand (1975)	1.055	1.11424	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.0988	1.11659	4th
Malik and Tailor (2013)	1.069	1.14405	5th
Srivenkataramana (1980)	1.069	1.14414	6th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.0695	1.14517	7th
Cochran (1942)	1.0698	1.14585	8th
Upadhyaya and Singh (1999)	1.0699	1.14598	9th
Sisodia and Diwivedi (1981)	1.07	1.1463	10th
Singh and Tailor (2003)	1.0703	1.14686	11th
Shukla (2002)	1.0704	1.14718	12th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.071	1.1483	13th
Singh and Shukla (1987, 1993)	1.0711	1.14874	14th
Singh et. al. (2004)	1.0714	1.14919	15th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0714	1.14937	16th
Upadhyaya and Singh (1999)	1.072	1.1505	17th
Sample mean	1.0722	1.15099	18th
Upadhyaya and Singh (1999)	1.0724	1.15148	19th
Shukla et al. (2013)	1.0728	1.15241	20th
Singh (2003)	1.0729	1.15263	21st
Singh et. al. (2004)	1.073	1.15279	22nd
Thakur and Gupta (2013)	1.0731	1.15294	23rd
Singh (2003)	1.0733	1.15349	24th
Singh and Tailor (2003)	1.0741	1.15515	25th
Singh (2003)	1.0742	1.15533	26th
Pandey and Dubey (1988)	1.0743	1.15572	27th
Singh (2003)	1.0745	1.15593	28th
Upadhyaya and Singh (1999)	1.0745	1.15605	29th
Product	1.0746	1.15618	30th
Singh et al. (2007)	1.0793	1.16638	31st
Singh (2001)	1.0809	1.16988	32nd
Singh and Upadhayaya (2001)	1.0816	1.17139	33rd

Table 4.37C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.0388	1.08027	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.0388	1.08041	2nd
Chand (1975)	1.0453	1.09398	3rd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.047	1.09753	4th
Malik and Tailor (2013)	1.0667	1.13916	5th
Srivenkataramana (1980)	1.0667	1.1393	6th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.0675	1.14086	7th
Cochran (1942)	1.068	1.1419	8th
Upadhyaya and Singh (1999)	1.068	1.14209	9th
Sisodia and Diwivedi (1981)	1.0683	1.14258	10th
Singh and Tailor (2003)	1.0687	1.14344	11th
Shukla (2002)	1.0689	1.14393	12th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.0697	1.14563	13th
Singh and Shukla (1987, 1993)	1.07	1.14629	14th
Singh et. al. (2004)	1.0703	1.14699	15th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0704	1.14726	16th
Upadhyaya and Singh (1999)	1.0712	1.14898	17th
Sample mean	1.0716	1.14972	18th
Upadhyaya and Singh (1999)	1.0719	1.15047	19th
Shukla et al. (2013)	1.0726	1.15189	20th
Singh (2003) 3	1.0726	1.15222	21st
Singh et. al. (2004)	1.0729	1.15247	22nd
Thakur and Gupta (2013)	1.073	1.1527	23rd
Singh (2003)	1.0734	1.15354	24th
Singh and Tailor (2003)	1.0745	1.15607	25th
Singh (2003)	1.0747	1.15635	26th
Pandey and Dubey (1988)	1.0749	1.15694	27th
Singh (2003)	1.0751	1.15725	28th
Upadhyaya and Singh (1999)	1.0752	1.15744	29th
Product	1.0753	1.15764	30th
Singh et al. (2007)	1.0825	1.17321	31st
Singh (2001)	1.0849	1.17856	32nd
Singh and Upadhayaya (2001)	1.086	1.18088	33rd

COMMENT: Tables 4.37A, 4.37B and 4.37C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = 0.5$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, , \bar{y}_{FTAA}^{\beta_2(d)}, , \bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, fourth, seventh positions respectively in in all the three Tables.

From these results, it is deduced that:

i. The four modified estimators $\bar{y}_{FTAA}, \bar{y}_{FTAA}^{(d)}, , \bar{y}_{FTAA}^{\beta_2(d)}, , \bar{y}_{FTAA}^{\beta_1(d)}$ are highly efficient when the relationship between the study and auxiliary variables is either linear or quadratic or cubic in comparison with other estimators considered.

ii. The modified estimator $\bar{y}_{FTAA}$ outperformed all other three modified estimators $\bar{y}_{FTAA}^{(d)}, , \bar{y}_{FTAA}^{\beta_2(d)}, , \bar{y}_{FTAA}^{\beta_1(d)}$ modified in this study.

Table 4.38A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = 0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{(d)}$	1.0212	1.04359	1st
Modified $\bar{y}_{FTAA}$	1.0218	1.04475	2nd
Singh and Upadhayaya (2001)	1.052	1.10715	3rd
Singh (2001)	1.0524	1.10805	4th
Singh et al. (2007)	1.0537	1.11072	5th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.0524	1.11436	6th
Product	1.0595	1.1231	7th
Upadhyaya and Singh (1999)	1.0596	1.12327	8th
Singh (2003)	1.0597	1.12343	9th
Pandey and Dubey (1988)	1.0597	1.12351	10th
Thakur and Gupta (2013)	1.0599	1.12375	11th
Singh (2003)	1.0599	1.12386	12th
Shukla (2002)	1.0603	1.12466	13th
Singh (2003)	1.0603	1.12473	14th
Singh (2003)	1.0607	1.12553	15th
Singh et. al. (2004)	1.0608	1.12571	16th
Upadhyaya and Singh (1999)	1.0614	1.12715	17th
Singh and Tailor (2003)	1.0619	1.12767	18th
Malik and Tailor (2013)	1.0617	1.1277	19th
Shukla et al. (2013)	1.0617	1.12777	20th
Sample mean	1.0617	1.12778	21st
Singh and Tailor (2003)	1.0618	1.12789	22nd
Upadhyaya and Singh (1999)	1.062	1.1284	23rd
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0624	1.12924	24th
Singh et. al. (2004)	1.0627	1.12984	25th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.0629	1.13021	26th
Sisodia and Diwivedi (1981)	1.0637	1.13201	27th
Upadhyaya and Singh (1999)	1.0638	1.13224	28th
Singh and Shukla (1987, 1993)	1.0639	1.1324	29th
Cochran (1942)	1.0639	1.13241	30th
Srivenkataramana (1980)	1.0649	1.1344	31st
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.0684	1.14188	32nd
Chand (1975)	1.0684	1.14194	33rd

Table 4.38B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = 0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.188	1.04121	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.1899	1.04167	2nd
Chand (1975)	1.0298	1.06104	3rd
Modified 3	1.0299	1.06122	4th
Srivenkataramana (1980)	1.0419	1.08594	5th
Cochran (1942)	1.045	1.09257	6th
Singh and Shukla (1987, 1993)	1.045	1.09261	7th
Upadhyaya and Singh (1999)	1.0453	1.09313	8th
Sisodia and Diwivedi (1981)	1.0457	1.09392	9th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.0486	1.09995	10th
Singh et. al. (2004)	1.0491	1.10119	11th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0501	1.1032	12th
Upadhyaya and Singh (1999)	1.0514	1.10601	13th
Singh and Tailor (2003)	1.0523	1.10775	14th
Sample mean	1.0524	1.10812	15th
Shukla et al. (2013)	1.0525	1.10816	16th
Malik and Tailor (2013)	1.0526	1.10837	17th
Singh and Tailor (2003)	1.0526	1.10849	18th
Upadhyaya and Singh (1999)	1.0534	1.11024	19th
Singh et. al. (2004)	1.0558	1.11512	20th
Singh (2003)	1.056	1.11573	21st
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	3.799	1.11614	22nd
Singh (2003)	1.0573	1.11843	23rd
Shukla (2002)	1.0574	1.11868	24th
Singh (2003)	1.0587	1.12139	25th
Thakur and Gupta (2013)	1.0589	1.12175	26th
Pandey and Dubey (1988)	1.0593	1.12258	27th
Singh (2003)	1.0594	1.12285	28th
Upadhyaya and Singh (1999)	1.0597	1.1234	29th
Product	1.0599	1.12398	30th
Singh et al. (2007)	1.0799	1.16662	31st
Singh (2001)	1.0842	1.17596	32nd
Singh and Upadhayaya (2001)	1.0856	1.17913	33rd

Table 4.38C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = 0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{(d)}$	1.1622	1.051536	1st
Chand (1975)	1.0381	1.078138	2nd
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.0382	1.078278	3rd
Srivenkataramana (1980)	1.0474	1.097582	4th
Cochran (1942)	1.0499	1.102749	5th
Singh and Shukla (1987, 1993)	1.0499	1.10278	6th
Upadhyaya and Singh (1999)	1.0501	1.103184	7th
Sisodia and Diwivedi (1981)	1.0504	1.1038	8th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.0526	1.108498	9th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	2.275	1.109255	10th
Singh et. al. (2004)	1.0531	1.109458	11th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.0538	1.111026	12th
Upadhyaya and Singh (1999)	1.0549	1.113213	13th
Singh and Tailor (2003)	1.0555	1.114569	14th
Sample mean	1.0556	1.114854	15th
Shukla et al. (2013)	1.0556	1.114882	16th
Malik and Tailor (2013)	1.0557	1.115047	17th
Singh and Tailor (2003)	1.0558	1.11514	18th
Upadhyaya and Singh (1999)	1.0564	1.116499	19th
Singh et. al. (2004)	1.0582	1.120294	20th
Singh (2003)	1.0584	1.120773	21st
Singh (2003)	1.0594	1.122869	22nd
Shukla (2002)	1.0595	1.123067	23rd
Singh (2003)	1.0605	1.12517	24th
Thakur and Gupta (2013)	1.0606	1.125446	25th
Pandey and Dubey (1988)	1.0609	1.126091	26th
Singh (2003)	1.061	1.126304	27th
Upadhyaya and Singh (1999)	1.0612	1.126728	28th
Product	1.0615	1.127179	29th
Modified $\bar{y}_{FTAA}$	1.1607	1.134806	30th
Singh et al. (2007)	1.0769	1.160228	31st
Singh (2001)	1.0802	1.167451	32nd
Singh and Upadhayaya (2001)	1.0814	1.169895	33rd

COMMENT: Tables 4.38A, 4.38B and 4.38C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = 0.8$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, sixth, thirty-twoth positions respectively in Table 4.38A, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, , $\bar{y}_{FTAA}^{\beta_4(d)}$ emerged first, second, fourth, twenty-twoth positions respectively in Table 4.38B and $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, , $\bar{y}_{FTAA}^{\beta_4(d)}$ $\bar{y}_{FTAA}$ emerged first, third, tenth, thirtieth positions respectively in Table 4.38C.

From these results, it is deduced that:

i. The modified estimator $\bar{y}_{FTAA}$ is highly efficient when the relationship between the study and auxiliary variables is either linear or quadratic but less efficient for cubic relationship in comparison with other estimators considered.

ii. The modified estimator $\bar{y}_{FTAA}^{(d)}$ is also highly efficient when the relationship between the study and auxiliary variables is either linear or quadratic or cubic in comparison with some estimators considered.

iii. The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ is efficient when the relationship between the study and auxiliary variables is either quadratic or cubic but less efficient when the relationship is linear in comparison with other estimators considered.

iv. The modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ is also efficient when the relationship between the study and auxiliary variables is either linear or cubic but less efficient when the relationship is quadratic in comparison with some estimators considered.

Table 4.39A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = 0.99$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	0.2491	0.012067	1st
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.2474	0.013394	2nd
Product	0.152	0.023132	3rd
Upadhyaya and Singh (1999)	0.1522	0.023204	4th
Singh (2003)	0.1524	0.023274	5th
Pandey and Dubey (1988)	0.1528	0.02338	6th
Thakur and Gupta (2013)	0.1529	0.023427	7th
Singh (2003)	0.1535	0.023593	8th
Singh and Tailor (2003)	0.155	0.024064	9th
Singh (2003)	0.1561	0.0244	10th
Singh (2003)	0.1577	0.024904	11th
Singh et. al. (2004)	0.1579	0.024976	12th
Upadhyaya and Singh (1999)	0.1603	0.025736	13th
Sample mean	0.1612	0.026024	14th
Upadhyaya and Singh (1999)	0.1621	0.026315	15th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.1641	0.026967	16th
Singh et. al. (2004)	0.1645	0.027096	17th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.166	0.027605	18th
Singh and Tailor (2003)	0.1674	0.028069	19th
Sisodia and Diwivedi (1981)	0.1697	0.028826	20th
Upadhyaya and Singh (1999)	0.1702	0.029024	21st
Cochran (1942)	0.1705	0.029106	22nd
Singh and Shukla (1987, 1993)	0.1708	0.029233	23rd
Malik and Tailor (2013)	0.171	0.029293	24th
Srivenkataramana (1980)	0.1728	0.029903	25th
Chand (1975)	0.102	0.030446	26th
Shukla (2002)	0.1765	0.031185	27th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.1876	0.035246	28th
Shukla et al. (2013)	0.2112	0.044672	29th
Singh et al. (2007)	0.235	0.055277	30th
Singh (2001)	0.2474	0.061271	31st
Modified $\bar{y}_{FTAA}^{(d)}$	0.2488	0.061922	32nd
Singh and Upadhayaya (2001)	0.2512	0.063167	33rd

Table 4.39B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = 0.99$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{(d)}$	0.1352	0.018294	1st
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.2671	0.018331	2nd
Modified $\bar{y}_{FTAA}$	0.1353	0.018352	3rd
Product	0.1626	0.026365	4th
Upadhyaya and Singh (1999)	0.1624	0.026426	5th
Singh (2003)	0.1626	0.026481	6th
Pandey and Dubey (1988)	0.1629	0.026565	7th
Thakur and Gupta (2013)	0.163	0.026603	8th
Singh (2003)	0.1634	0.026733	9th
Singh and Tailor (2003)	0.1645	0.027105	10th
Singh (2003)	0.1653	0.02737	11th
Singh (2003)	0.1665	0.027765	12th
Singh et. al. (2004)	0.1667	0.027822	13th
Upadhyaya and Singh (1999)	0.1684	0.028415	14th
Sample mean	0.1691	0.02864	15th
Upadhyaya and Singh (1999)	0.1698	0.028866	16th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.1713	0.029372	17th
Singh et. al. (2004)	0.1715	0.029476	18th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.1727	0.029866	19th
Singh and Tailor (2003)	0.1737	0.030224	20th
Sisodia and Diwivedi (1981)	0.1754	0.030806	21st
Upadhyaya and Singh (1999)	0.1758	0.030957	22nd
Cochran (1942)	0.176	0.03102	23rd
Singh and Shukla (1987, 1993)	0.1763	0.031118	24th
Malik and Tailor (2013)	0.1764	0.031164	25th
Srivenkataramana (1980)	0.1777	0.03163	26th
Shukla (2002)	0.1804	0.032605	27th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.1887	0.035661	28th
Chand (1975)	0.1252	0.035712	29th
Shukla et al. (2013)	0.2063	0.04259	30th
Singh et al. (2007)	0.2239	0.050176	31st
Singh (2001)	0.2331	0.054391	32nd

Singh and Upadhayaya (2001)	0.2359	0.055714	33rd

Table 4.39C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = 0.99$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.1248	0.015608	1st
Product	0.1551	0.02411	2nd
Upadhyaya and Singh (1999)	0.1553	0.024172	3rd
Singh (2003)	0.1555	0.024232	4th
Pandey and Dubey (1988)	0.1558	0.024321	5th
Thakur and Gupta (2013)	0.1559	0.024362	6th
Singh (2003)	0.1564	0.024503	7th
Singh and Tailor (2003)	0.1577	0.024903	8th
Singh (2003)	0.1586	0.025188	9th
Singh (2003)	0.1599	0.025615	10th
Singh et. al. (2004)	0.1601	0.025676	11th
Upadhyaya and Singh (1999)	0.1621	0.026318	12th
Sample mean	0.1628	0.026561	13th
Upadhyaya and Singh (1999)	0.1636	0.026806	14th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.1653	0.027355	15th
Singh et. al. (2004)	0.1656	0.027464	16th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.1669	0.027891	17th
Singh and Tailor (2003)	0.168	0.02828	18th
Sisodia and Diwivedi (1981)	0.1699	0.028913	19th
Upadhyaya and Singh (1999)	0.1704	0.029078	20th
Cochran (1942)	0.1706	0.029147	21st
Singh and Shukla (1987, 1993)	0.1709	0.029253	22nd
Malik and Tailor (2013)	0.1711	0.029303	23rd
Srivenkataramana (1980)	0.1725	0.029812	24th
Shukla (2002)	0.1756	0.030877	25th
Modified $\bar{y}_{FTAA}^{(d)}$	0.1792	0.032152	26th
Modified $\bar{y}_{FTAA}$	0.1795	0.032239	27th
Chand (1975)	0.1135	0.032914	28th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.1849	0.034233	29th
Shukla et al. (2013)	0.2046	0.041914	30th
Singh et al. (2007)	0.2244	0.050421	31st
Singh (2001)	0.2348	0.055183	32nd

Singh and Upadhayaya (2001)	0.238	0.056682	33rd

COMMENT: Tables 4.39A, 4.39B and 4.39C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = 0.99$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{(d)}$ emerged first, second, twenty-eighth, thirty-twoth positions respectively in Table 4.39A, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_4(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, twenty-eighth positions respectively in Table 4.39B and $\bar{y}_{FTAA}^{\beta_4(d)}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, twenty-sixth, twenty-seventh, twenty-nineth positions respectively in Table 4.39C.

From these results, it is deduced that:

i. The modified estimator $\bar{y}_{FTAA}$ is highly efficient when the relationship between the study and auxiliary variables is either linear or quadratic but less efficient for cubic relationship in comparison with other estimators considered.

ii. The modified estimator $\bar{y}_{FTAA}^{(d)}$ is highly efficient only when the relationship between the study and auxiliary variables is quadratic but less efficient when their relationship is either linear or cubic in comparison with some estimators considered.

iii. The modified estimator $\bar{y}_{FTAA}^{\beta_1(d)}$ performed poorly under all the three models (linear, quadratic and cubic).

iv. The modified estimator $\bar{y}_{FTAA}^{\beta_2(d)}$ is highly efficient when the relationship between the study and auxiliary variables is either linear or quadratic or cubic in comparison with some estimators considered.

Table 4.40A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = -0.2$

Estimators	Bias	MSE	Rank
Chand (1975)	0.042	0.0036	1st
Modified $\bar{y}_{FTAA}$	-2.349	0.05175	2nd
Shukla (2002)	0.237	0.0581	3rd
Malik and Tailor (2013)	0.243	0.0609	4th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.457	0.0674	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.261	0.0701	6th
Modified $\bar{y}_{FTAA}^{(d)}$	-2.36	0.0708	7th
Singh and Tailor (2003)	0.284	0.0831	8th
Product	0.29	0.0864	9th
Upadhyaya and Singh (1999)	0.292	0.0874	10th
Singh (2003)	0.294	0.0884	11th
Pandey and Dubey (1988)	0.295	0.0892	12th
Singh (2003)	0.299	0.0915	13th
Thakur and Gupta (2013)	0.306	0.096	14th
Singh (2003)	0.311	0.0991	15th
Singh (2003)	0.321	0.1049	16th
Singh et. al. (2004)	0.322	0.106	17th
Upadhyaya and Singh (1999)	0.338	0.1163	18th
Sample mean	0.344	0.1207	19th
Upadhyaya and Singh (1999)	0.351	0.1252	20th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.359	0.1312	21st
Singh et. al. (2004)	0.366	0.1365	22nd
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.37	0.1392	23rd
Sisodia and Diwivedi (1981)	0.394	0.1574	24th
Upadhyaya and Singh (1999)	0.397	0.1599	25th
Cochran (1942)	0.399	0.1613	26th
Shukla et al. (2013)	0.4	0.1618	27th
Singh and Tailor (2003)	0.405	0.16609	28th
Singh and Shukla (1987, 1993)	0.409	0.1692	29th
Srivenkataramana (1980)	0.412	0.1723	30th
Singh et al. (2007)	0.431	0.188	31st
Singh (2001)	0.457	0.2109	32nd

Singh and Upadhayaya (2001)	0.468	0.2213	33rd

Table 4.40B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = -0.2$

Estimators	Bias	MSE	Rank
Singh and Upadhayaya (2001)	0.248	0.0638	1st
Singh (2001)	0.25	0.065	2nd
Singh et al. (2007)	0.256	0.0679	3rd
Srivenkataramana (1980)	0.26	0.07	4th
Singh and Shukla (1987, 1993)	0.261	0.0704	5th
Singh and Tailor (2003)	0.262	0.0708	6th
Shukla et al. (2013)	0.263	0.0714	7th
Cochran (1942)	0.263	0.0715	8th
Upadhyaya and Singh (1999)	0.264	0.0717	9th
Sisodia and Diwivedi (1981)	0.264	0.0721	10th
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.27	0.0749	11th
Singh et. al. (2004)	0.27	0.0753	12th
Modified $\bar{y}_{FTAA}$	0.868	0.07545	13th
Modified $\bar{y}_{FTAA}^{(d)}$	0.871	0.07588	14th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.272	0.0762	15th
Upadhyaya and Singh (1999)	0.274	0.0772	16th
Sample mean	0.275	0.0779	17th
Upadhyaya and Singh (1999)	0.277	0.0787	18th
Singh et. al. (2004)	0.28	0.0806	19th
Singh (2003)	0.28	0.0808	20th
Singh (2003)	0.282	0.082	21st
Thakur and Gupta (2013)	0.284	0.0826	22nd
Singh (2003)	0.285	0.0835	23rd
Pandey and Dubey (1988)	0.286	0.084	24th
Singh (2003)	0.286	0.0842	25th
Upadhyaya and Singh (1999)	0.287	0.0844	26th
Product	0.287	0.0846	27th
Singh and Tailor (2003)	0.288	0.0853	28th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.294	0.0883	29th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.496	0.089	30th
Malik and Tailor (2013)	0.298	0.0907	31st
Shukla (2002)	0.299	0.0914	32nd

Chand (1975)	0.342	0.1188	33rd

Table 4.40C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = -0.2$

Estimators	Bias	MSE	Rank
Chand (1975)	0.217	0.0491	1st
Modified $\bar{y}_{FTAA}$	-0.254	0.0649	2nd
Modified $\bar{y}_{FTAA}^{(d)}$	-0.26	0.066	3rd
Shukla (2002)	0.256	0.0674	4th
Malik and Tailor (2013)	0.257	0.068	5th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	0.259	0.0694	6th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.26	0.0699	7th
Singh and Tailor (2003)	0.265	0.0724	8th
Product	0.266	0.073	9th
Upadhyaya and Singh (1999)	0.267	0.0732	10th
Singh (2003)	0.267	0.0733	11th
Pandey and Dubey (1988)	0.267	0.0735	12th
Singh (2003)	0.268	0.0739	13th
Thakur and Gupta (2013)	0.269	0.0747	14th
Singh (2003)	0.27	0.0752	15th
Singh (2003)	0.272	0.0762	16th
Singh et. al. (2004)	0.272	0.0764	17th
Upadhyaya and Singh (1999)	0.276	0.0781	18th
Sample mean	0.277	0.0788	19th
Upadhyaya and Singh (1999)	0.278	0.0795	20th
Jain and Shukla (2013) $\bar{y}_{FT1}$	0.28	0.0805	21st
Singh et. al. (2004)	0.281	0.0813	22nd
Jain and Shukla (2013) $\bar{y}_{FT2}$	0.282	0.0817	23rd
Sisodia and Diwivedi (1981)	0.287	0.0844	24th
Upadhyaya and Singh (1999)	0.287	0.0847	25th
Cochran (1942)	0.288	0.0849	26th
Shukla et al. (2013)	0.288	0.085	27th
Singh and Tailor (2003)	0.289	0.0856	28th
Singh and Shukla (1987, 1993)	0.289	0.086	29th
Srivenkataramana (1980)	0.29	0.0865	30th
Singh et al. (2007)	0.294	0.0886	31st

Singh (2001)	0.299	0.0917	32nd
Singh and Upadhayaya (2001)	0.301	0.093	33rd

COMMENT: Tables 4.40A, 4.40B and 4.40C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = -0.2$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{(d)}$ emerged second, fifth, sixth, seventh positions respectively in Table 4.40A, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged thirteenth, fourteenth, twenty-nineth, thirtieth positions respectively in Table 4.40B and $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged second, third, sixth, seventh positions respectively in Table 4.40C.

From these results, it is deduced that:

i. The four modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ are highly efficient when the relationship between the study and auxiliary variables is either linear or cubic in comparison with other estimators considered.

ii. The four modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ were highly challenged by some existing estimators considered in the study.

Table 4.41A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = -0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	0.4001	0.160547	1st
Modified $\bar{y}_{FTAA}^{(d)}$	0.4001	0.16055	2nd
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	-0.721	0.519913	3rd
Singh and Tailor (2003)	1.1038	1.219434	4th
Product	1.1234	1.262926	5th
Upadhyaya and Singh (1999)	1.1251	1.266811	6th
Thakur and Gupta (2013)	1.1251	1.266841	7th
Singh (2003)	1.1267	1.270364	8th
Pandey and Dubey (1988)	1.1286	1.274695	9th
Singh (2003)	1.1328	1.284301	10th
Singh et. al. (2004)	1.1598	1.346215	11th
Upadhyaya and Singh (1999)	1.1762	1.384559	12th
Sample mean	1.1827	1.399822	13th
Chand (1975)	1.1859	1.407417	14th
Upadhyaya and Singh (1999)	1.1892	1.41522	15th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.2015	1.44458	16th
Shukla et al. (2013)	1.2015	1.444764	17th
Singh et. al. (2004)	1.2058	1.455134	18th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.3707	1.456141	19th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.2142	1.475253	20th
Singh (2003)	1.2173	1.483001	21st
Sisodia and Diwivedi (1981)	1.2384	1.534648	22nd
Upadhyaya and Singh (1999)	1.2421	1.543846	23rd
Cochran (1942)	1.2439	1.548412	24th
Singh and Shukla (1987, 1993)	1.2522	1.56907	25th
Srivenkataramana (1980)	1.2643	1.599608	26th
Singh and Tailor (2003)	1.2649	1.601084	27th
Shukla (2002)	1.2928	1.672621	28th
Malik and Tailor (2013)	1.3127	1.724408	29th
Singh (2003)	1.3392	1.794766	30th
Singh et al. (2007)	1.3586	1.847066	31st

Singh (2001)	1.3707	1.880082	32nd
Singh and Upadhayaya (2001)	1.375	1.891945	33rd

Table 4.41B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = -0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	-8.293	1.395274	1st
Singh and Upadhayaya (2001)	1.2989	1.688392	2nd
Singh (2001)	1.2999	1.690953	3rd
Singh et al. (2007)	1.3026	1.698132	4th
Singh (2003)	1.3071	1.709671	5th
Malik and Tailor (2013)	1.3131	1.725526	6th
Shukla (2002)	1.3177	1.737452	7th
Singh and Tailor (2003)	1.3241	1.754305	8th
Srivenkataramana (1980)	1.3242	1.754658	9th
Singh and Shukla (1987, 1993)	1.327	1.761996	10th
Cochran (1942)	1.3289	1.76701	11th
Upadhyaya and Singh (1999)	1.3293	1.768124	12th
Sisodia and Diwivedi (1981)	1.3301	1.770373	13th
Singh (2003)	1.3349	1.783159	14th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.3357	1.7851	15th
Singh et. al. (2004)	1.3376	1.79017	16th
Shukla et al. (2013)	1.3386	1.7928	17th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.3386	1.792846	18th
Upadhyaya and Singh (1999)	1.3414	1.800354	19th
Chand (1975)	1.3421	1.802366	20th
Sample mean	1.3429	1.80433	21st
Upadhyaya and Singh (1999)	1.3443	1.808296	22nd
Singh et. al. (2004)	1.3481	1.818378	23rd
Singh (2003)	1.3543	1.835027	24th
Pandey and Dubey (1988)	1.3552	1.837653	25th
Singh (2003)	1.3557	1.83884	26th
Thakur and Gupta (2013)	1.356	1.839809	27th
Upadhyaya and Singh (1999)	1.356	1.839817	28th
Product	1.3564	1.840886	29th
Singh and Tailor (2003)	1.3609	1.852992	30th
Modified $\bar{y}_{FTAA}^{(d)}$	1.5218	2.316241	31st
Modified $\bar{y}_{FTAA}$	1.5218	2.316244	32nd

Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	1.7781	3.161828	33rd

Table 4.41C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = -0.5$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	0.9522	0.907033	1st
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	1.0864	1.197178	2nd
Modified $\bar{y}_{FTAA}$	1.1324	1.282712	3rd
Modified $\bar{y}_{FTAA}^{(d)}$	1.1324	1.282713	4th
Singh and Tailor (2003)	1.2455	1.552204	5th
Product	1.2486	1.560049	6th
Upadhyaya and Singh (1999)	1.2489	1.560744	7th
Thakur and Gupta (2013)	1.2489	1.560749	8th
Singh (2003)	1.2492	1.561379	9th
Pandey and Dubey (1988)	1.2495	1.562152	10th
Singh (2003)	1.2501	1.563862	11th
Singh et. al. (2004)	1.2545	1.574757	12th
Upadhyaya and Singh (1999)	1.2571	1.581398	13th
Sample mean	1.2582	1.58402	14th
Chand (1975)	1.2587	1.58532	15th
Upadhyaya and Singh (1999)	1.2592	1.586652	16th
Jain and Shukla (2013) $\bar{y}_{FT1}$	1.2612	1.591639	17th
Shukla et al. (2013)	1.2612	1.59167	18th
Singh et. al. (2004)	1.2619	1.59342	19th
Jain and Shukla (2013) $\bar{y}_{FT2}$	1.2632	1.596802	20th
Singh (2003)	1.2637	1.598099	21st
Sisodia and Diwivedi (1981)	1.2671	1.606674	22nd
Upadhyaya and Singh (1999)	1.2677	1.608189	23rd
Cochran (1942)	1.268	1.608939	24th
Singh and Shukla (1987, 1993)	1.2693	1.612322	25th
Srivenkataramana (1980)	1.2713	1.617289	26th
Singh and Tailor (2003)	1.2714	1.617528	27th
Shukla (2002)	1.2759	1.629006	28th
Malik and Tailor (2013)	1.2791	1.637188	29th
Singh (2003)	1.2833	1.648142	30th
Singh et al. (2007)	1.2864	1.656168	31st
Singh (2001)	1.2884	1.661187	32nd

Singh and Upadhayaya (2001)	1.2891	1.662982	33rd

COMMENT: Tables 4.41A, 4.41B and 4.41C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = -0.5$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$, $\bar{y}^{\beta_2(d)}_{FTAA}$ emerged first, second, third, nineteenth positions respectively in Table 4.41A, $\bar{y}^{\beta_2(d)}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$ emerged first, thirty-oneth, thirty-twoth, thirty-threeth positions respectively in Table 4.41B and $\bar{y}^{\beta_1(d)}_{FTAA}$, $\bar{y}^{\beta_2(d)}_{FTAA}$, $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$ emerged first, second, third, fourth positions respectively in Table 4.41C.

From these results, it is deduced that:

i. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$ were highly efficient but $\bar{y}^{\beta_2(d)}_{FTAA}$ was challeged when the relationship between the study and auxiliary variables is either linear in comparison with other estimators considered.

ii. The modified estimator $\bar{y}^{\beta_2(d)}_{FTAA}$ was highly efficient but $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$ were challenged by some existing estimators considered in the study when the relationship between the study and auxiliary variables is quadratic.

iii. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}^{(d)}_{FTAA}$, $\bar{y}^{\beta_1(d)}_{FTAA}$, $\bar{y}^{\beta_2(d)}_{FTAA}$ outperformed all the related existing estimators considered in the study when the relationship between the study and auxiliary variables is cubic.

Table 4.42A: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model I when $\rho_{xy} = -0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	0.0699	0.009276	1st
Modified $\bar{y}_{FTAA}^{(d)}$	0.0735	0.009789	2nd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	2.6993	4.936327	3rd
Singh et al. (2007)	2.6526	7.037202	4th
Singh and Upadhayaya (2001)	2.6942	7.259434	5th
Singh (2001)	2.6993	7.287138	6th
Singh and Tailor (2003)	2.7224	7.412366	7th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	2.9114	7.476627	8th
Product	2.7407	7.511967	9th
Upadhyaya and Singh (1999)	2.7414	7.516151	10th
Thakur and Gupta (2013)	2.7417	7.517738	11th
Singh (2003)	2.7422	7.520485	12th
Pandey and Dubey (1988)	2.7431	7.525389	13th
Singh (2003)	2.7453	7.537644	14th
Singh et. al. (2004)	2.7581	7.608081	15th
Upadhyaya and Singh (1999)	2.7656	7.649247	16th
Sample mean	2.7685	7.665138	17th
Upadhyaya and Singh (1999)	2.7713	7.681023	18th
Singh (2003) 3	2.7715	7.682025	19th
Shukla et al. (2013)	2.7745	7.698408	20th
Singh (2003) 2	2.7757	7.70542	21st
Jain and Shukla (2013) $\bar{y}_{FT1}$	2.7771	7.713188	22nd
Singh et. al. (2004)	2.7787	7.722111	23rd
Jain and Shukla (2013) $\bar{y}_{FT2}$	2.7829	7.745019	24th
Shukla (2002)	2.7883	7.775566	25th
Sisodia and Diwivedi (1981)	2.7935	7.804386	26th
Upadhyaya and Singh (1999)	2.7951	7.813556	27th
Cochran (1942)	2.79592	7.817708	28th
Malik and Tailor (2013)	2.7977	7.827962	29th
Singh and Shukla (1987, 1993)	2.799	7.83522	30th
Srivenkataramana (1980)	2.8039	7.862285	31st
Singh and Tailor (2003)	2.8135	7.916277	32nd

Chand (1975)	2.8766	8.275388	33rd

Table 4.42B: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model II when $\rho_{xy} = -0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.7372	3.022302	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.7386	3.027205	2nd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	3.0538	6.636739	3rd
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	2.8449	7.093904	4th
Singh et al. (2007)	2.744	7.530391	5th
Singh and Upadhayaya (2001)	2.7602	7.619552	6th
Singh (2001)	2.7622	7.630608	7th
Singh and Tailor (2003)	2.7712	7.680419	8th
Product	2.7783	7.719852	9th
Upadhyaya and Singh (1999)	2.7786	7.721505	10th
Thakur and Gupta (2013)	2.7787	7.722132	11th
Singh (2003)	2.7789	7.723217	12th
Pandey and Dubey (1988)	2.7793	7.725153	13th
Singh (2003)	2.7802	7.729992	14th
Singh et. al. (2004)	2.7851	7.757752	15th
Upadhyaya and Singh (1999)	2.788	7.77394	16th
Sample mean	2.7892	7.780181	17th
Upadhyaya and Singh (1999)	2.7903	7.786416	18th
Singh (2003)	2.7904	7.78681	19th
Shukla et al. (2013)	2.7915	7.793236	20th
Singh (2003) 2	2.792	7.795985	21st
Jain and Shukla (2013) $\bar{y}_{FT1}$	2.7925	7.79903	22nd
Singh et. al. (2004)	2.7932	7.802527	23rd
Jain and Shukla (2013) $\bar{y}_{FT2}$	2.7948	7.811497	24th
Shukla (2002)	2.7969	7.823446	25th
Sisodia and Diwivedi (1981)	2.7989	7.834707	26th
Upadhyaya and Singh (1999)	2.7996	7.838287	27th
Cochran (1942)	2.7999	7.839908	28th
Malik and Tailor (2013)	2.8006	7.843909	29th
Singh and Shukla (1987, 1993)	2.8011	7.84674	30th
Srivenkataramana (1980)	2.803	7.857291	31st

Singh and Tailor (2003)	2.8067	7.878305	32nd
Chand (1975)	2.8313	8.016987	33rd

Table 4.42C: Bias and MSE of Modified Estimators and some existing related Estimators under some Super-population model III when $\rho_{xy} = -0.8$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}$	1.0189	1.042605	1st
Modified $\bar{y}_{FTAA}^{(d)}$	1.0213	1.047403	2nd
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	2.4171	5.843618	3rd
Singh et al. (2007)	2.6973	7.275992	4th
Singh and Upadhayaya (2001)	2.7243	7.422412	5th
Singh (2001)	2.7276	7.440608	6th
Singh and Tailor (2003)	2.7426	7.522707	7th
Product	2.7545	7.587828	8th
Upadhyaya and Singh (1999)	2.755	7.59056	9th
Thakur and Gupta (2013)	2.7552	7.591597	10th
Singh (2003)	2.7555	7.59339	11th
Pandey and Dubey (1988)	2.7561	7.596592	12th
Singh (2003)	2.7575	7.604592	13th
Singh et. al. (2004)	2.7658	7.650526	14th
Upadhyaya and Singh (1999)	2.7707	7.677336	15th
Sample mean	2.7725	7.687679	16th
Upadhyaya and Singh (1999)	2.7744	7.698014	17th
Singh (2003)	2.7745	7.698666	18th
Shukla et al. (2013)	2.7764	7.709321	19th
Singh (2003)	2.7773	7.71388	20th
Jain and Shukla (2013) $\bar{y}_{FT1}$	2.7782	7.71893	21st
Singh et. al. (2004)	2.7792	7.724729	22nd
Jain and Shukla (2013) $\bar{y}_{FT2}$	2.7819	7.739613	23rd
Shukla (2002)	2.7855	7.759448	24th
Sisodia and Diwivedi (1981)	2.7888	7.77815	25th
Upadhyaya and Singh (1999)	2.7899	7.784098	26th
Cochran (1942)	2.7904	7.78679	27th
Malik and Tailor (2013)	2.7915	7.793439	28th
Singh and Shukla (1987, 1993)	2.7924	7.798144	29th
Srivenkataramana (1980)	2.795	7.815684	30th
Singh and Tailor (2003)	2.8018	7.850642	31st

Chand (1975)	2.8428	8.082122	32nd
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	7.8654	8.211082	33rd

COMMENT: Tables 4.42A, 4.42B and 4.42C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = -0.8$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, eighth positions respectively in Table 4.42A, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, fourth positions respectively in Table 4.42B and $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$ emerged first, second, third, thirty-threeth positions respectively in Table 4.42C.

From these results, it is deduced that:

i. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ were highly efficient and competitive when the relationship between the study and auxiliary variables is either linear or quadratic in comparison with other estimators considered.

ii. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ were highly efficient but $\bar{y}_{FTAA}^{\beta_1(d)}$ was challenged by some existing estimators considered in the study when the relationship between the study and auxiliary variables is cubic.

iii. The modified estimator $\bar{y}_{FTAA}$ outperformed all the remaining three modified estimators.

Table 4.43A: Bias and MSE of modified Estimators and some related Estimators under some Super-population model I when $\rho_{xy} = -0.99$

Estimators	Bias	MSE	Rank
Modified $\bar{y}_{FTAA}^{(d)}$	2.4953	6.226758	1st
Modified $\bar{y}_{FTAA}$	2.4984	6.242129	2nd
Singh and Upadhayaya (2001)	2.5842	6.677946	3rd
Singh (2001)	2.5857	6.685817	4th
Singh et al. (2007)	2.5907	6.711577	5th
Modified $\bar{y}_{FTAA}^{\beta_1(d)}$	2.6107	6.715576	6th
Modified $\bar{y}_{FTAA}^{\beta_2(d)}$	2.5857	6.716409	7th
Singh (2003)	2.593	6.723741	8th
Singh and Tailor (2003)	2.597	6.744636	9th
Srivenkataramana (1980)	2.5981	6.749983	10th
Singh and Shukla (1987, 1993)	2.5982	6.750823	11th
Cochran (1942)	2.5984	6.751852	12th
Upadhyaya and Singh (1999)	2.5985	6.752098	13th
Sisodia and Diwivedi (1981)	2.5986	6.752512	14th
Shukla et al. (2013)	2.599	6.754614	15th
Jain and Shukla (2013) $\bar{y}_{FT2}$	2.5991	6.755435	16th
Singh et. al. (2004)	2.5993	6.756226	17th
Jain and Shukla (2013) $\bar{y}_{FT1}$	2.5994	6.756993	18th
Upadhyaya and Singh (1999)	2.5997	6.758422	19th
Sample mean	2.5999	6.759331	20th
Upadhyaya and Singh (1999)	2.6	6.76024	21st
Singh et. al. (2004)	2.6005	6.762434	22nd
Singh (2003)	2.6011	6.765587	23rd
Pandey and Dubey (1988)	2.6012	6.766141	24th
Singh (2003)	2.6012	6.766322	25th
Upadhyaya and Singh (1999)	2.6013	6.766553	26th
Thakur and Gupta (2013)	2.6013	6.766643	27th
Product	2.6013	6.766798	28th
Singh and Tailor (2003)	2.6027	6.773979	29th
Singh (2003) 2	2.6035	6.778164	30th

Shukla (2002)	2.6163	6.845037	31st
Malik and Tailor (2013)	2.6283	6.908171	32nd
Chand (1975)	2.6439	6.990446	33rd

Table 4.43B: Bias and MSE of modified Estimators and some related Estimators under some Super-population model II when $\rho_{xy} = -0.99$

Estimators	Bias	MSE	Rank
Modified 2	2.3649	5.592595	1st
Modified 1	2.3713	5.623174	2nd
Singh and Upadhayaya (2001)	2.5512	6.508602	3rd
Singh (2001)	2.5544	6.524907	4th
Singh et al. (2007)	2.5648	6.578347	5th
Singh (2003) 3	2.5697	6.603622	6th
Singh and Tailor (2003)	2.5782	6.6471	7th
Srivenkataramana (1980)	2.5803	6.658236	8th
Singh and Shukla (1987, 1993)	2.5807	6.659986	9th
Cochran (1942)	2.5811	6.66213	10th
Upadhyaya and Singh (1999) 2	2.5812	6.662644	11th
Sisodia and Diwivedi (1981)	2.5814	6.663505	12th
Shukla et al. (2013)	2.5822	6.667887	13th
Jain and Shukla (2013) 2	2.5825	6.669597	14th
Singh et. al. (2004)	2.5829	6.671246	15th
Jain and Shukla (2013) 1	2.5832	6.672844	16th
Upadhyaya and Singh (1999) 1	2.5838	6.675824	17th
Sample mean	2.5841	6.67772	18th
Upadhyaya and Singh (1999) 3	2.5845	6.679616	19th
Singh et. al. (2004)	2.5854	6.68419	20th
Singh (2003) 1	2.5866	6.690767	21st
Pandey and Dubey (1988)	2.5869	6.691921	22nd
Singh (2003) 4	2.587	6.692299	23rd
Upadhyaya and Singh (1999) 4	2.587	6.692781	24th
Thakur and Gupta (2013)	2.5871	6.692969	25th
Product	2.5871	6.693293	26th
Singh and Tailor (2003)	2.59	6.708279	27th
Singh (2003) 2	2.5917	6.717016	28th
Modified 3	2.6068	6.79526	29th
Modified 4	2.582	6.797006	30th
Shukla (2002)	2.6186	6.857042	31st

Malik and Tailor (2013)	2.6438	6.989929	32nd
Chand (1975)	2.6766	7.164094	33rd

Table 4.43C: Bias and MSE of modified Estimators and some related Estimators under some Super-population model III when $\rho_{xy} = -0.99$

Estimators	Bias	MSE	Rank
Modified 2	2.4163	5.838393	1st
Modified 1	2.4214	5.863189	2nd
Singh and Upadhayaya (2001)	2.5642	6.575087	3rd
Singh (2001)	2.5667	6.588095	4th
Singh et al. (2007)	2.575	6.630705	5th
Singh (2003) 3	2.5789	6.650845	6th
Singh and Tailor (2003)	2.5856	6.685471	7th
Srivenkataramana (1980)	2.5873	6.694336	8th
Singh and Shukla (1987, 1993)	2.5876	6.695729	9th
Cochran (1942)	2.5879	6.697436	10th
Upadhyaya and Singh (1999) 2	2.588	6.697845	11th
Sisodia and Diwivedi (1981)	2.5881	6.69853	12th
Shukla et al. (2013)	2.5888	6.702018	13th
Modified 3	2.6083	6.70329	14th
Jain and Shukla (2013) 2	2.5891	6.703378	15th
Modified 4	2.6086	6.704677	16th
Singh et. al. (2004)	2.5893	6.704691	17th
Jain and Shukla (2013) 1	2.5896	6.705963	18th
Upadhyaya and Singh (1999) 1	2.59	6.708334	19th
Sample mean	2.5903	6.709843	20th
Upadhyaya and Singh (1999) 3	2.5906	6.711351	21st
Singh et. al. (2004)	2.5913	6.714991	22nd
Singh (2003) 1	2.5923	6.720224	23rd
Pandey and Dubey (1988)	2.5925	6.721142	24th
Singh (2003) 4	2.5926	6.721443	25th
Upadhyaya and Singh (1999) 4	2.5926	6.721826	26th
Thakur and Gupta (2013)	2.5927	6.721976	27th
Product	2.5927	6.722234	28th
Singh and Tailor (2003)	2.595	6.734154	29th
Singh (2003) 2	2.5964	6.741103	30th
Shukla (2002)	2.6177	6.852342	31st

Malik and Tailor (2013)	2.6377	6.957693	32nd
Chand (1975)	2.6637	7.095461	33rd

COMMENT: Tables 4.43A, 4.43B and 4.43C show the biases and MSEs of the modified factor-type estimators under linear, quadratic and cubic models respectively when $\rho_{xy} = -0.99$. The results revealed that out of the thirty-three competing estimators, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged first, second, sixth, seventh positions respectively in Table 4.43A, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged first, second, twenty-nineth, thirtieth positions respectively in Table 4.43B and $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ emerged first, second, fourteenth, sixteenth positions respectively in Table 4.43C.

From these results, it is deduced that:

i. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$, $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ were highly efficient and competitive when the relationship between the study and auxiliary variables is linear in comparison with other estimators considered.

ii. The modified estimators $\bar{y}_{FTAA}$, $\bar{y}_{FTAA}^{(d)}$ were highly efficient but $\bar{y}_{FTAA}^{\beta_1(d)}$, $\bar{y}_{FTAA}^{\beta_2(d)}$ were highly challenged by some existing estimators considered in the study when the relationship between the study and auxiliary variables is either quadratic or cudic.

iii. The modified estimator $\bar{y}_{FTAA}^{(d)}$ outperformed all the remaining three modified estimators.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.0 INTRODUCTION

In this chapter, the summary of the findings, recommendations based on the findings as well as some suggestions for further study were presented.

5.1 SUMMARY OF RESULTS

Efficiency of the modified factor-type estimators have been compared with some existing related factor-type and non-factor-type estimators reviewed in the literature. These were achieved by estimating the MSEs and PREs of the estimators using both real life and simulated data when correlation between the study and auxiliary variables is either weak, moderate or strong positively or negatively. The modified estimators demonstrated high level of efficiency over the existing related estimators considered in the study. Graphs were also plotted to depict the behaviour of the MSEs and PREs of the modified and existing related factor-type estimators for some arbitrary values of $d = 3, d = 5, d = 10, d = 15, d = 20$, $d = 30$. Also, robustness of the modified estimators and some existing related estimators under non-response and super-population (linear, quadratic and cubic) models were also investigated. The modified estimators display high level of robustness under both non-response and super-population models as they compete favourably with almost existing related estimators considered.

5.2 CONCLUSION

By comparison, the results of the empirical study using simulated data on the efficiency and robustness of modified factor-type estimators are in agreement with the results of empirical

study using real life data and therefore, it is inferred that the modified estimators suggested in this study demonstrated high relative efficiency over existing related factor-type estimators. Conclusively, the modified estimators are improved version of the existing ones.

5.3 GENERAL FINDINGS OF THIS RESEARCH WORK

The general findings of this research work are:

vi. four modified factor-type estimators for population mean were obtained;

vii. Biases and MSEs of these modified factor-type estimators were theoretically and empirically derived;

viii. conditions under which these modified estimators were more efficient were derived;

ix. using both real life and simulated data, the modified factor-type estimators were found to have minimum Mean Square Errors and highest relative efficiency, hence more efficient than the conventional ones.

x. the four modified estimators demonstrated high level of robustness under super-population and non-response models in comparison to the conventional ones.

5.4 CONTRIBUTION TO KNOWLEDGE

The present study has contributed to knowledge in the following directions:

i. identifying the weakness of some existing factor-type estimators.

ii. four new modified factor-type estimators for estimating population mean under single-phase and two-phase simple random sampling have been modified.

iii. properties of the modified estimators and their supremacy conditions over the existing estimators considered have been established and this makes further study and review of the modified estimators easier.

5.5 SUGGESTION FOR FURTHER STUDY

Based on the limitation of this study, the following are suggested for further study;

i. the modified estimators in this work can be further studied under other sampling scheme like stratified random sampling, systematic sampling, multistage sampling e.t.c.

ii. the modified estimators can further be modified to suit situation of missing values using imputation strategies.

iii. other improvement strategies like transformation using exponential function, use of unknown weight e.t.c. can be apply on the existing factor-type estimators.