

THE COMBINED EFFECTS OF UNIFORM PARTIALLY DISTRIBUTED MOVING MASS, CORIOLIS AND CENTRIPETAL FORCES ON THE RESPONSE OF EULER-BERNOULLI BEAMS

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ABSTRACT

In this paper, the dynamic response of a pinned-pinned (simply-supported) Euler-Bernoulli beam under the influence of a uniform partially distributed moving mass, whose velocity and magnitude are assumed constant, is studied. The influence of both Coriolis and centripetal forces, which are always assumed negligible, is then analyzed. The governing partial differential equation of the beam is first transformed into a set of ordinary differential equations using separation of variable technique. Next, Duhamel's integral solution technique is adopted for analyzing the influence of the uniform partially distributed moving mass, Coriolis and centripetal forces on the dynamic response of the beam. Various numerical results, which showed the individual as well as the combined influence of the distributed moving mass, Coriolis and centripetal forces on the response of the beam, are presented. Furthermore, concluding remarks, which are useful for the purpose of structural designs, are presented.

KEYWORDS: Coriolis and centripetal forces, uniform partially distributed moving mass, concentrated moving mass, moving force, Euler-Bernoulli beam.

INTRODUCTION

The multi-parameter problem of determining the dynamic response of moving load on elastic structure such as beams has been studied by several researchers [1-19]. The analysis of dynamic behaviour of the beams of finite length under moving loads was carried out by expanding the unknown deflection function as an infinite series [1]. A technique capable of improving both the bending moment and shear force responses of a uniform Euler-Bernoulli beam under a moving load was developed in [2]. This technique involved using modal analysis method. Achenbach and Sun [3] had earlier studied the steady-state response of a beam to a load moving at constant speed over the beam. Since the fast-moving load might excite high-frequency modes, they considered Timoshenko theory instead of the Euler-Bernoulli theory. Mei and Mace [4] studied the dynamic behaviour of beams subjected to moving loads. They obtained various mode shapes of the beam corresponding to various frequencies. A method different from that in [2], aimed at improving the response of an Euler-Bernoulli beam subjected to a moving load, was presented in [5]. Kargarnovin and Younesian [6] also studied

the problem of determining the response of Timoshenko beam with uniform cross-section and infinite length resting on a generalized Pasternak-type viscoelastic foundation under an arbitrary-distributed harmonic moving load using integral transform technique. Furthermore, Zehsaz *et al.* [7] presented the dynamic analysis of a uniform Timoshenko beam resting on the Pasternak viscoelastic foundation and subjected to a moving load. In particular, a new method, which is capable of capturing the discontinuities in the shear force and the bending moment in the moving load problem, was developed. Most recently, Oni and Awodola [11] studied the dynamic behaviour of an elastically supported uniform Euler-Bernoulli beam resting on an elastic foundation with variable stiffness under a concentrated moving mass. They observed that the response amplitudes of the beam decrease as the foundation moduli increase.

In [1, 9, 10], however, the moving load problems are said to be of three main types. The first is the problem involving negligible load inertia when compared to the beam inertia. In the second case, it is the beam inertia that is considered negligible in comparison to the load inertia. Both the beam inertia and the load inertia are considered comparable in magnitude with each other in the third type. Consequently, this last type of moving load problem involves taking into account the effects of both centripetal and Coriolis forces as these forces are of practical importance. The only work, to the best knowledge of the authors, which studied the effect of these forces on the response of a beam, is that of Michaltsos and Kounadis [19]. Meanwhile, most of the moving load problems have involved various forms (random, non-random and vibrational) of concentrated moving load with uniform or non-uniform velocity [1]. However, it is well known that concentrated moving load is not realistic. A realistic moving load is the distributive one [15, 18]. As a matter of fact, concentrated moving loads do not exist in reality. Furthermore, it has been confirmed [12-18] that consideration of an interval for the load distribution enhances the reality of the problem formulation and that the degree of influence of the parameter such as the length of the load distribution is of practical importance and should, therefore, be evaluated when considering moving load problems. We remark, at this juncture, that most of the above mentioned works (the exception being those in [12-18]) concern concentrated loads. In fact, the work in [19], in which the effect of Coriolis and centripetal forces are studied, involves only concentrated mass, while the works in [12-18], which considered distributed moving masses, do not involve the evaluation of the degree of influence of Coriolis and centripetal forces.

In this context, the main objective of the present paper is to study the combined effect of a uniform partially distributed moving load (as opposed to concentrated moving load), Coriolis and centripetal forces on the response of a pinned-pinned Euler-Bernoulli beam of finite length.

1. EQUATIONS OF MOTION

In this study, a beam with finite length, constant mass per unit length (m) and uniform cross section is considered. Figure 1 shows the coordinate system and significant dimensions. The x -axis runs along the axis of the beam. The following assumptions are adopted in this work to make the mathematical equations simple:

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- (i) The geometrical and mechanical parameters of the beam such as density, modulus of elasticity and moment of inertia are constant,
- (ii) The dynamic behaviour of the beam is described by Euler-Bernoulli theory,
- (iii) The material of the beam is isotropic and linear elastic,
- (iv) The inertia of the beam and the moving load are considered with the gravitational influence of the moving load,
- (v) The speed of the moving load is constant,
- (vi) Structural depreciation is negligible,
- (vii) The moving load always keeps contact with the beam, and
- (viii) The effects of longitudinal motion and damping are negligible.

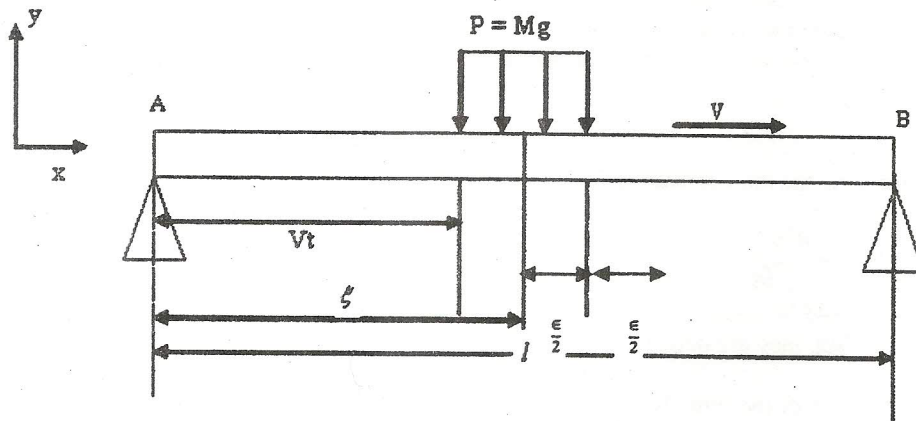


Figure 1. A pinned-pinned beam traversed by uniform partially distributed moving load

Referring to Figure 1 again, the beam AB of length l is traversed by a constant load P of mass M . The mass M is considered to be uniform partially distributed. The distribution is over a fixed length ϵ of the beam. It is assumed that the moving load enters at end A at time $t = 0$, with constant velocity V while the deflection of the beam before time $t = 0$ is zero. The instantaneous position of the load (ξ) from the end A at time t is given as $\xi = Vt + \epsilon/2$.

Based on these assumptions, the fundamental equations of elastodynamics and the consideration of the balance of all the forces (the applied force distribution, the shearing force and the inertia force) acting on an element of the beam, the governing partial differential equation describing the vibrational behaviour of the system (shown in Figure 1) is [1]

$$EI \frac{\partial^4 W(x, t)}{\partial x^4} + M \frac{\partial^2 W(x, t)}{\partial t^2} = F(x, t), \quad 0 \leq x \leq l \quad (1)$$

where $F(x, t)$ is the applied moving mass defined as

$$F(x, t) = \frac{1}{\varepsilon} \left[-Mg - M \frac{d^2 W(x, t)}{dt^2} \right] \left[H \left(x - \xi + \frac{\varepsilon}{2} \right) - H \left(x - \xi - \frac{\varepsilon}{2} \right) \right] \quad (2)$$

In Figure 1, Equation (1) and Equation (2), the following notations have been used:

- EI - flexural rigidity of the beam
- E - modulus of elasticity
- I - moment of inertia
- $W(x, t)$ - lateral deflection of the beam measured upwards from its equilibrium position when unloaded
- l - finite span of the beam
- x - axial coordinate
- g - acceleration due to gravity, and
- $H(x)$ - Heaviside unit function defined as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \quad (3)$$

Furthermore, the total derivative $\frac{d^2 W(x, t)}{dt^2}$ which appears in equation (2) is defined as

$$\frac{d^2 W(x, t)}{dt^2} = \frac{\partial^2 W(x, t)}{\partial t^2} + 2V \frac{\partial^2 W(x, t)}{\partial t \partial x} + V^2 \frac{\partial^2 W(x, t)}{\partial x^2} \quad (4)$$

Also it should be noted that Equation (4) involves the Coriolis and centripetal forces. As a matter of fact, they are defined as follows [19]:

$$\text{Centripetal force per unit mass } M \text{ of the load} = V^2 \frac{\partial^2 W(x, t)}{\partial x^2} \quad (5)$$

$$\text{Centripetal force per unit mass } M \text{ of the load} = 2V \frac{\partial^2 W(x, t)}{\partial t \partial x} \quad (6)$$

Substituting Equation (4) into Equation (2) and the resulting equation into Equation (1) we obtain

$$EI \frac{\partial^4 W(x, t)}{\partial x^4} + M \frac{\partial^2 W(x, t)}{\partial t^2} = \frac{1}{\varepsilon} \left[-Mg - M \left(\frac{d^2 W(x, t)}{dt^2} + 2V \frac{\partial^2 W(x, t)}{\partial t \partial x} + V^2 \frac{\partial^2 W(x, t)}{\partial x^2} \right) \right] \left[H \left(x - \xi + \frac{\varepsilon}{2} \right) - H \left(x - \xi - \frac{\varepsilon}{2} \right) \right] \quad (7)$$

The support conditions of the beam are as yet unspecified. In what follows, it is assumed that the beam is pinned-pinned (i.e. simply supported).

Mathematically, this means that we have

$$W(0, t) = W(l, t) = \frac{\partial^2 W(0, t)}{\partial x^2} = \frac{\partial^2 W(l, t)}{\partial x^2} = 0 \quad (8)$$

It is remarked, however, that any of the other classical boundary conditions can also be specified for the analysis and formulation, hereby presented. The associated initial conditions are

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$$W(x, 0) = \frac{\partial W(x, 0)}{\partial t} \quad (9)$$

Equations (7) – (9) constitute the initial-boundary-value problems (IBVP) describing the behaviour of the pinned–pinned Euler–Bernoulli beam under the uniform partially distributed moving load.

2. METHOD OF SOLUTION

To solve the above IBVP, the governing partial differential equation (PDE) is first reduced, in the next subsection, to a set of coupled ordinary differential equations (ODES) which is solved in section (2.2).

2.1. REDUCTION OF THE PDE TO ODES

The lateral deflection of the beam can be written, using separation of variable technique, in the following form;

$$W(x, t) = \sum_{n=1}^{\infty} X_n(x) Q_n(t) \quad (10)$$

In Equation (10),

$$X_n(x) = \sin \frac{n\pi x}{l} \quad (11)$$

is the normalised deflection for the n th mode of the freely vibrating pinned–pinned beam. $Q_n(t)$ is the unknown corresponding modal amplitude.

Substituting Equation (10) into Equation (7) one obtains

$$\begin{aligned} & \sum_n [EI X_n^{iv}(x) Q_n(t) + m X_n(x) \ddot{Q}_n(t)] \\ &= \left(\frac{1}{\varepsilon} \right) \left\{ -Mg - M \sum_n [X_n(x) \ddot{Q}_n(t) + V^2 X_n''(x) Q_n(t) + 2V X_n'(x) \dot{Q}_n(t)] \right\} X \left[H \left(x - \xi + \frac{\varepsilon}{2} \right) \right. \\ & \quad \left. - H \left(x - \xi - \frac{\varepsilon}{2} \right) \right] \end{aligned} \quad (12)$$

where $(\cdot)$ denotes differentiation with respect to x , while $(\ddot{\cdot})$ denotes differentiation with respect to time t .

But, for free vibration, we have

$$X_n^{iv}(x) = m \omega_n^2 X_n(x) \quad (13)$$

where

$$\omega_n^2 = \frac{n^4 \pi^4 EI}{ml^4} \quad (14)$$

and hence Equation (12) becomes

$$\begin{aligned} & \sum_{n=1}^{\infty} [m\omega_n^2 X_n(x) Q_n(t) + mX_n(x) \ddot{Q}_n(t)] \\ &= \left(\frac{1}{\varepsilon}\right) \left\{ -Mg - M \sum_{n=1}^{\infty} [X_n(x) \ddot{Q}_n(t) + V^2 X_n''(x) Q_n(t) + 2V X_n'(x) \dot{Q}_n(t)] \right\} \\ & \quad \cdot \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] \end{aligned} \quad (15)$$

Multiplying both sides of Equation (15) by $X_k(x)$ ($k \neq n$), we obtain

$$\begin{aligned} & m \sum_{n=1}^{\infty} [\omega_n^2 X_n(x) Q_n(t) + X_n(x) \ddot{Q}_n(t)] \\ &= \left(\frac{1}{\varepsilon}\right) \left\{ -Mg X_k(x) \right. \\ & \quad - M \sum_{n=1}^{\infty} [(X_n(x) X_k(x) \ddot{Q}_n(t) + V^2 X_n''(x) X_k(x) Q_n(t) \\ & \quad \left. + (2V X_n'(x) X_k(x) \dot{Q}_n(t))) \right] \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] \end{aligned} \quad (16)$$

Integrating both sides of Equation (16) over the interval $[0, l]$ with respect to x and using the following orthogonal properties

$$\int_0^l X_n(x) X_k(x) dx = \begin{cases} \lambda = \frac{l}{2}, & n = k \\ 0, & n \neq k \end{cases} \quad n, k = 1, 2, 3, \dots \quad (17)$$

yields

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = \left\{ \frac{-2Mg}{ml\varepsilon} A_{11} - \frac{-2M}{ml\varepsilon} \sum_{n=1}^{\infty} [\ddot{Q}_k(t) B_{11} + Q_k(t) V^2 C_{11} + 2V \dot{Q}_k(t) D_{11}] \right\} \quad (18)$$

where

$$A_{11} = \int_0^l X_n(x) \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx \quad (19a)$$

$$B_{11} = \int_0^l X_n(x) X_k(x) \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx \quad (19b)$$

$$C_{11} = \int_0^l X_n''(x) X_k(x) \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx \quad (19c)$$

$$D_{11} = \int_0^l X_n'(x) X_k(x) \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx \quad (19d)$$

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Next, we evaluate the integrals in Equations (19a-d). Consider, for instance, Equation (19a). In view of Equation (11), we have

$$A_{11} = \int_0^l \frac{\sin n\pi x}{l} \left[H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx \quad (20)$$

Using the method of integration by parts Equation (20) becomes

$$A_{11} = \left[\left(H\left(x - \xi + \frac{\varepsilon}{2}\right) - H\left(x - \xi - \frac{\varepsilon}{2}\right) \right) \int \frac{\sin n\pi x}{l} dx \right]_0^l - \int_0^l \left\{ \frac{\sin n\pi x}{l} \left[H'\left(x - \xi + \frac{\varepsilon}{2}\right) - H'\left(x - \xi - \frac{\varepsilon}{2}\right) \right] \right\} dx \quad (21)$$

$$= - \int_0^l \int \frac{\sin n\pi x}{l} \left[H'\left(x - \xi + \frac{\varepsilon}{2}\right) \right] dx dx + \int_0^l \int \frac{\sin n\pi x}{l} \left[H'\left(x - \xi - \frac{\varepsilon}{2}\right) \right] dx dx \quad (22)$$

where the definition of the Heaviside function has been used.

Now, set

$$F_\alpha(x) = \int \frac{\sin n\pi x}{l} dx, \quad (23)$$

and on noting that

$$H'\left(x - \xi + \frac{\varepsilon}{2}\right) - H'\left(x - \xi - \frac{\varepsilon}{2}\right) = \delta\left(x - \xi + \frac{\varepsilon}{2}\right) - \delta\left(x - \xi - \frac{\varepsilon}{2}\right),$$

Equation (22) becomes

$$A_{11} = - \int_0^l F_\alpha(x) \delta\left(x - \xi + \frac{\varepsilon}{2}\right) dx + \int_0^l F_\alpha(x) \delta\left(x - \xi - \frac{\varepsilon}{2}\right) dx$$

Using the property of Dirac delta function, we have

$$A_{11} = -F_\alpha\left(\xi - \frac{\varepsilon}{2}\right) + F_\alpha\left(\xi + \frac{\varepsilon}{2}\right)$$

i.e.

$$\begin{aligned} A_{11} &= - \int \frac{\sin n\pi x}{l} \left(\xi - \frac{\varepsilon}{2}\right) d\xi + \int \frac{\sin n\pi x}{l} \left(\xi + \frac{\varepsilon}{2}\right) d\xi \\ &= \frac{l}{n\pi} \cos \frac{n\pi}{l} \left(\xi - \frac{\varepsilon}{2}\right) - \frac{l}{n\pi} \cos \frac{n\pi}{l} \left(\xi + \frac{\varepsilon}{2}\right) \end{aligned} \quad (24)$$

But it is known that

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

Hence,

$$A_{11} = \frac{2l}{n\pi} \sin\left(\frac{n\pi\xi}{l}\right) \sin\left(\frac{n\pi\varepsilon}{2l}\right) \quad (25a)$$

Similarly,

$$B_{11} = \frac{l}{\pi(n-k)} \sin\left(\frac{\pi\varepsilon(n-k)}{2l}\right) \cos\left(\frac{\pi\xi(n-k)}{l}\right) - \frac{l}{\pi(n+k)} \sin\left(\frac{\pi\varepsilon(n+k)}{2l}\right) \cos\left(\frac{\pi\xi(n+k)}{l}\right) \quad (25b)$$

$$C_{11} = \left(\frac{n\pi}{l}\right)^2 \left\{ \frac{-l}{\pi(n-k)} \sin\frac{\pi\varepsilon}{2l}(n-k) \cos\frac{\pi\xi}{l}(n-k) + \frac{l}{\pi(n+k)} \sin\frac{\pi\varepsilon}{2l}(n+k) \cos\frac{\pi\xi}{l}(n+k) \right\} \quad (25c)$$

$$D_{11} = \left(\frac{n\pi}{l}\right) \left\{ \frac{l}{\pi(n+k)} \sin\frac{\pi\xi}{l}(n+k) \sin\frac{\pi\varepsilon}{2l}(n+k) - \frac{l}{\pi(n-k)} \sin\frac{\pi\xi}{l}(n-k) \sin\frac{\pi\varepsilon}{2l}(n-k) \right\} \quad (25d)$$

Substituting Equations (25a-d) into Equation (18), we finally obtain

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = -\frac{4Mg}{m\pi\epsilon l} l \sin\frac{n\pi\varepsilon}{2l} \sin\frac{n\pi\xi}{l} - R_1 \left\{ \sum_{k=1}^{\infty} \ddot{Q}_k(t) \{h_{nk1} - h_{nk2}\} + V^2 \left(\frac{n\pi}{l}\right)^2 \sum_{k=1}^{\infty} Q_k(t) \{h_{nk2} - h_{nk1}\} + 2V \left(\frac{n\pi}{l}\right) \sum_{k=1}^{\infty} \dot{Q}_k(t) \{h_{nk3} - h_{nk4}\} \right\} \quad (26)$$

where

$$R_1 = \frac{2M}{m\pi\epsilon l}$$

$$h_{nk1} = \frac{l}{(n-k)} \sin\frac{\pi\varepsilon}{2l}(n-k) \cos\frac{\pi\xi}{l}(n-k); \quad h_{nk2} = \frac{l}{(n+k)} \sin\frac{\pi\varepsilon}{2l}(n+k) \cos\frac{\pi\xi}{l}(n+k)$$

$$h_{nk3} = \frac{l}{(n-k)} \sin\frac{\pi\varepsilon}{2l}(n-k) \cos\frac{\pi\xi}{l}(n-k); \quad h_{nk4} = \frac{l}{(n+k)} \sin\frac{\pi\varepsilon}{2l}(n+k) \cos\frac{\pi\xi}{l}(n+k) \quad (27)$$

Hence, the problem now reduces to that of solving the set of coupled ordinary differential equations (26) subject to the initial conditions (9). This is carried out in the next subsection.

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2.2. SOLUTION OF THE REDUCED GOVERNING EQUATION

The complex nature of the right hand side of Equations (26) precludes its exact solution, hence, in what follows we seek its approximate solutions. In particular, we start by considering in subsection 2.2.1 a special case of Equations (26) which yields the first approximate solution.

2.2.1. EULER-BERNOULLI BEAM TRAVERSED BY A DISTRIBUTIVE MOVING FORCE - FIRST APPROXIMATE SOLUTION

The differential equation describing the response of a Euler-Bernoulli beam under a moving force may be obtained from Equations (26) by setting $R_1 = 0$.

Thus, we have,

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = -\frac{4Mg}{mn\pi\epsilon l} \sin \frac{n\pi\epsilon}{2l} \sin \frac{n\pi\xi}{l} \quad (28)$$

But

$$\sin \frac{n\pi\epsilon}{2l} \sin \frac{n\pi\xi}{l} = \sin \frac{n\pi\epsilon}{2l} \sin \frac{n\pi}{l} \left(Vt + \frac{\epsilon}{2} \right) = \sin \alpha_n \sin(\Omega_n t + \alpha_n)$$

$$\text{where } \alpha_n = \frac{n\pi\epsilon}{2l} \quad \text{and} \quad \Omega_n = \frac{n\pi V}{l}$$

so that Equation (28) becomes

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = -\frac{4Mgl}{mn\pi\epsilon l} \sin \alpha_n \sin(\Omega_n t + \alpha_n) \quad (29)$$

Solving equation (29), we obtain

$$Q_n(t) = A \cos \omega_n t + B \sin \omega_n t - \frac{4Mgl \sin \alpha_n}{mn\pi\epsilon l(\omega_n^2 - \Omega_n^2)} \sin(\Omega_n t + \alpha_n) \quad (30)$$

where A and B are arbitrary constants. Applying the initial conditions,

$$Q(0) = \dot{Q}(0) = 0 \quad (31)$$

we have

$$A = \frac{2Mgl \Omega_n \sin \alpha_n}{mn\pi\epsilon l(\omega_n^2 - \Omega_n^2)}, \quad B = \frac{4Mgl \sin^2 \alpha_n}{mn\pi\epsilon l \omega_n(\omega_n^2 - \Omega_n^2)}$$

so that Equation (30) becomes

$$Q_n(t) = \frac{4Mgl \sin \alpha_n}{mn\pi\epsilon l(\omega_n^2 - \Omega_n^2)} \left[\frac{\Omega_n}{\omega_n} \cos \alpha_n \sin \omega_n t + \sin \alpha_n \cos \omega_n t - \sin(\Omega_n t + \alpha_n) \right] \quad (32)$$

We remark that the first approximate solution (32) does not involve the effect of Coriolis and centripetal forces. It, however, includes the effect of the distributed moving mass. This approximate solution in Equation (32) is compared, in Table 1, to the corresponding solution in [19] (see Equation (12) in [19]) involving concentrated moving load.

2.2.2. EULER-BERNOULLI BEAM TRAVERSED BY A DISTRIBUTIVE MOVING MASS—SECOND APPROXIMATE SOLUTION

In this subsection, we take into account the whole second term, on the right hand side of Equation (26) neglected in subsection (2.2.1). To solve the differential equation, we follow Kounadis [20], Michaltsos and Kounadis [19], and first substitute equation (32) into this second term.

Thus, we have

$$\begin{aligned} \ddot{Q}_n(t) + \omega_n^2 Q_n(t) = & -2 \left(\frac{2M}{m\pi l} \right) \frac{gl}{n\epsilon} \sin \alpha_n \sin(\Omega_n t + \alpha_n) - 2 \left(\frac{2M}{m\pi l} \right) \frac{gl^2}{\epsilon^2} \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} \\ & \cdot [-\Omega_k \omega_k \cos \alpha_k \sin \omega_k t - \omega_k^2 \sin \alpha_k \cos \omega_k t + \Omega_k^2 \sin(\Omega_k t + \alpha_k)] \\ & \cdot \left[\frac{1}{(n-k)} \sin \alpha_{(n-k)} \cos(\Omega_{(n-k)} t + \alpha_{(n-k)}) \right. \\ & \quad \left. - \frac{1}{(n+k)} \sin \alpha_{(n+k)} \cos(\Omega_{(n+k)} t + \alpha_{(n+k)}) \right] \\ & - 2 \left(\frac{2M}{m\pi l} \right)^2 \frac{gl^2}{\epsilon^2} \Omega_n^2 C_0 \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} \left[\frac{\Omega_k}{\omega_k} \cos \alpha_k \sin \omega_k t \right. \\ & \quad \left. + \sin \alpha_k \cos \omega_k t - \sin(\Omega_k t + \alpha_k) \right] \left[-\frac{1}{(n-k)} \sin \alpha_{(n-k)} \cos(\Omega_{(n-k)} t + \alpha_{(n-k)}) \right. \\ & \quad \left. - \frac{1}{(n+k)} \sin \alpha_{(n+k)} \cos(\Omega_{(n+k)} t + \alpha_{(n+k)}) \right] \\ & - 4 \left(\frac{2M}{m\pi l} \right)^2 \frac{gl^2}{\epsilon^2} \Omega_n C_1 \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} [\Omega_k \cos \alpha_k \cos \omega_k t \\ & \quad - \omega_k \sin \alpha_k \sin \omega_k t \\ & \quad - \Omega_k \cos(\Omega_k t + \alpha_k)] \left[\frac{1}{(n-k)} \sin \alpha_{(n-k)} \sin(\Omega_{(n-k)} t + \alpha_{(n-k)}) \right. \\ & \quad \left. - \frac{1}{(n+k)} \sin \alpha_{(n+k)} \sin(\Omega_{(n+k)} t + \alpha_{(n+k)}) \right], \end{aligned} \quad (33)$$

where two constants C_0 and C_1 have been introduced. C_0 represents the unit effect of the centripetal force while C_1 denotes the unit effect of the Coriolis force. Each of these constants assumes value 0 or 1 depending on whether the effect of the corresponding force is neglected or considered.

Putting

$$\frac{2M}{m\pi l} = A; \quad \frac{l}{\epsilon} = A_1; \quad (\Omega t + \alpha) = E; \quad (\Omega_k t + \alpha_k) = E_k \quad \text{and} \quad (\Omega_n t + \alpha_n) = H_n, \quad (34)$$

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Equation (33) becomes

$$\begin{aligned}
 \ddot{Q}_n(t) + \omega_n^2 Q_n(t) = & -2A \frac{g}{n} A_1 \sin \alpha_n \sin H_n \\
 & -2A^2 \frac{g}{n} A_1^2 \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} [-\Omega_k \omega_k \cos \alpha_k \sin \omega_k t - \omega_k^2 \sin \alpha_k \cos \omega_k t + \Omega_k^2 \sin E_k] \\
 & \left[\frac{1}{(n-k)} \sin \alpha_{(n-k)} \cos E_{(n-k)} - \frac{1}{(n+k)} \sin \alpha_{(n+k)} \cos E_{(n+k)} \right] \\
 & -2A^2 g A_1^2 \Omega_n^2 C_0 \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} \left[\frac{\Omega_k}{\omega_k} \cos \alpha_k \sin \omega_k t + \sin \alpha_k \cos \omega_k t - \sin E_k \right] \\
 & \left[-\frac{1}{(n-k)} \sin \alpha_{(n-k)} \cos E_{(n-k)} + \frac{1}{(n+k)} \sin \alpha_{(n+k)} \cos E_{(n+k)} \right] \\
 & -4A^2 g A_1^2 \Omega_n C_1 \sum_{k=1}^{\infty} \frac{\sin \alpha_k}{k(\omega_k^2 - \Omega_k^2)} [\Omega_k \cos \alpha_k \cos \omega_k t - \omega_k \sin \alpha_k \sin \omega_k t - \Omega_k \cos E_k] \\
 & \left[\frac{1}{(n+k)} \sin \alpha_{(n+k)} \sin E_{(n+k)} - \frac{1}{(n-k)} \sin \alpha_{(n-k)} \sin E_{(n-k)} \right] \quad (34)
 \end{aligned}$$

When Equation (34) is further simplified and rearranged, one obtains

$$\begin{aligned}
 \ddot{Q}_n(t) + \omega_n^2 Q_n = & A A_1 \frac{g}{n} [-\cos \Omega_n t + \cos(\Omega_n t + 2\alpha_n)] \\
 & -A^2 A_1 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n-k)} t - 2\alpha_{(n-k)}) - \omega_k t] \\
 & + \sin[2\alpha_k - (\Omega_{(n-k)} t + 2\alpha_{(n-k)}) - \omega_k t] + \sin[2\alpha_k + \Omega_{(n-k)} t - \omega_k t] \\
 & - \sin[\Omega_{(n-k)} t + \omega_k t] - \sin[2\alpha_k + (\Omega_{(n-k)} t + 2\alpha_{(n-k)}) - \omega_k t] \\
 & + \sin[(\Omega_{(n-k)} t + 2\alpha_{(n-k)}) + \omega_k t] - \sin[2\alpha_k - \Omega_{(n-k)} t - \omega_k t] \\
 & - \sin[\Omega_{(n-k)} t - \omega_k t] \} \\
 & + A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n-k)} t + 2\alpha_{(n-k)}) + \omega_k t] \\
 & + \sin[2\alpha_k - (\Omega_{(n-k)} t + 2\alpha_{(n-k)}) + \omega_k t] + \sin[2\alpha_k + \Omega_{(n-k)} t + \omega_k t] \\
 & - \sin[\Omega_{(n-k)} t - \omega_k t] - \sin[2\alpha_k + (\Omega_{(n-k)} t + 2\alpha_{(n-k)}) + \omega_k t] \\
 & + \sin[(\Omega_{(n-k)} t + 2\alpha_{(n-k)}) - \omega_k t] - \sin[2\alpha_k - \Omega_{(n-k)} t + \omega_k t] \\
 & - \sin[\Omega_{(n-k)} t + \omega_k t] \} \\
 & + A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n+k)} t + 2\alpha_{(n+k)}) - \omega_k t] \\
 & + \sin[2\alpha_k - (\Omega_{(n+k)} t + 2\alpha_{(n+k)}) - \omega_k t] + \sin[2\alpha_k + \Omega_{(n+k)} t - \omega_k t] \\
 & - \sin[\Omega_{(n+k)} t + \omega_k t] - \sin[2\alpha_k + (\Omega_{(n+k)} t + 2\alpha_{(n+k)}) - \omega_k t] \\
 & + \sin[(\Omega_{(n+k)} t + 2\alpha_{(n+k)}) + \omega_k t] - \sin[2\alpha_k - \Omega_{(n+k)} t - \omega_k t] \\
 & - \sin[\Omega_{(n+k)} t - \omega_k t] \}
 \end{aligned}$$

$$\begin{aligned}
 & -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
 & \quad + \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] + \sin[2\alpha_k + \Omega_{(n+k)}t + \omega_k t] \\
 & \quad - \sin[\Omega_{(n+k)}t - \omega_k t] - \sin[2\alpha_k + (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
 & \quad + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] - \sin[2\alpha_k - \Omega_{(n+k)}t + \omega_k t] \\
 & \quad - \sin[\Omega_{(n+k)}t + \omega_k t] \} \\
 & + A^2 A_1^2 g C_1 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n}{4(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \\
 & \quad - \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] + \sin[2\alpha_k + \Omega_{(n+k)}t - \omega_k t] \\
 & \quad + \sin[\Omega_{(n+k)}t + \omega_k t] + \sin[2\alpha_k - \Omega_{(n+k)}t - \omega_k t] - \sin[\Omega_{(n+k)}t - \omega_k t] \\
 & \quad - \sin[2\alpha_k + (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \} \\
 & - A^2 A_1^2 g C_1 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n}{4(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \\
 & \quad - \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] + \sin[2\alpha_k + \Omega_{(n-k)}t - \omega_k t] \\
 & \quad + \sin[\Omega_{(n-k)}t + \omega_k t] + \sin[2\alpha_k - \Omega_{(n-k)}t - \omega_k t] - \sin[\Omega_{(n-k)}t - \omega_k t] \\
 & \quad - \sin[2\alpha_k + (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \} \\
 & + A^2 A_1^2 g C_1 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n}{4(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \\
 & \quad - \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] + \sin[2\alpha_k + \Omega_{(n-k)}t + \omega_k t] \\
 & \quad + \sin[\Omega_{(n-k)}t - \omega_k t] + \sin[2\alpha_k - \Omega_{(n-k)}t + \omega_k t] - \sin[\Omega_{(n-k)}t + \omega_k t] \\
 & \quad - \sin[2\alpha_k + (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \} \\
 & - A^2 A_1^2 g C_1 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n}{4(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
 & \quad - \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] + \sin[2\alpha_k + \Omega_{(n+k)}t + \omega_k t] \\
 & \quad + \sin[\Omega_{(n+k)}t - \omega_k t] + \sin[2\alpha_k - \Omega_{(n+k)}t + \omega_k t] - \sin[\Omega_{(n+k)}t + \omega_k t] \\
 & \quad - \sin[2\alpha_k + (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \} \\
 & - A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k + C_0 \Omega_n)^2}{8\Omega_k(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \Omega_k t] \\
 & \quad + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - (\Omega_k t + 2\alpha_k)] + \sin[\Omega_{(n+k)}t + (\Omega_k t + 2\alpha_k)] \\
 & \quad + \sin[\Omega_{(n+k)}t - \Omega_k t] - \sin[\Omega_{(n+k)}t - (\Omega_k t + 2\alpha_k)] - \sin[\Omega_{(n+k)}t + \Omega_k t] \\
 & \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + (\Omega_k t + 2\alpha_k)] - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \Omega_k t] \} \\
 & - 2A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k + \Omega_k)} \{ -\sin[\Omega_{(n+k)}t - \Omega_k t] \\
 & \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - (\Omega_k t + 2\alpha_k)] + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \Omega_k t] \\
 & \quad + \sin[\Omega_{(n+k)}t - (\Omega_k t + 2\alpha_k)] \}
 \end{aligned}$$

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$$\begin{aligned}
& +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k + C_0 \Omega_n)^2}{8\Omega_k(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \Omega_k t] \\
& \quad + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - (\Omega_k t + 2\alpha_k)] + \sin[\Omega_{(n+k)}t + (\Omega_k t + 2\alpha_k)] \\
& \quad + \sin[\Omega_{(n+k)}t - \Omega_k t] - \sin[\Omega_{(n+k)}t - (\Omega_k t + 2\alpha_k)] - \sin[\Omega_{(n+k)}t + \Omega_k t] \\
& \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + (\Omega_k t + 2\alpha_k)] - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \Omega_k t] \} \\
& +2A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k - \Omega_k)} \{ -\sin[\Omega_{(n+k)}t - \Omega_k t] \\
& \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - (\Omega_k t + 2\alpha_k)] + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \Omega_k t] \\
& \quad + \sin[\Omega_{(n+k)}t - (\Omega_k t + 2\alpha_k)] \} \\
& -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k + C_0 \Omega_n)^2}{8\Omega_k(\omega_k - \Omega_k)} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \Omega_k t] \\
& \quad + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - (\Omega_k t + 2\alpha_k)] + \sin[\Omega_{(n-k)}t + (\Omega_k t + 2\alpha_k)] \\
& \quad + \sin[\Omega_{(n-k)}t - \Omega_k t] - \sin[\Omega_{(n-k)}t - (\Omega_k t + 2\alpha_k)] - \sin[\Omega_{(n-k)}t + \Omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + (\Omega_k t + 2\alpha_k)] - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \Omega_k t] \} \\
& -2A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k - \Omega_k)} \{ -\sin[\Omega_{(n-k)}t - \Omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - (\Omega_k t + 2\alpha_k)] + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \Omega_k t] \\
& \quad + \sin[\Omega_{(n-k)}t - (\Omega_k t + 2\alpha_k)] \} \\
& +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k + C_0 \Omega_n)^2}{8\Omega_k(\omega_k + \Omega_k)} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \Omega_k t] \\
& \quad + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - (\Omega_k t + 2\alpha_k)] + \sin[\Omega_{(n-k)}t + (\Omega_k t + 2\alpha_k)] \\
& \quad + \sin[\Omega_{(n-k)}t - \Omega_k t] - \sin[\Omega_{(n-k)}t - (\Omega_k t + 2\alpha_k)] - \sin[\Omega_{(n-k)}t + \Omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + (\Omega_k t + 2\alpha_k)] - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \Omega_k t] \} \\
& +2A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{(\Omega_k^2 + C_0 \Omega_n^2)}{8\Omega_k(\omega_k + \Omega_k)} \{ -\sin[\Omega_{(n-k)}t - \Omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - (\Omega_k t + 2\alpha_k)] + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \Omega_k t] \\
& \quad + \sin[\Omega_{(n-k)}t - (\Omega_k t + 2\alpha_k)] \} \\
& -A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] - \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \\
& \quad + \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \} \\
& -A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[2\alpha_k + \Omega_{(n-k)}t + \omega_k t] - \sin[2\alpha_k + \Omega_{(n-k)}t - \omega_k t] \\
& \quad + \sin[\Omega_{(n-k)}t + \omega_k t] - \sin[\Omega_{(n-k)}t - \omega_k t] \}
\end{aligned}$$

$$\begin{aligned}
& +A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + 2\alpha_k + \omega_k t] \\
& \quad - \sin[2\alpha_k(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \\
& \quad - \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \} \\
& +A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[2\alpha_k - \Omega_{(n-k)}t + \omega_k t] - \sin[2\alpha_k - \Omega_{(n-k)}t - \omega_k t] \\
& \quad - \sin[\Omega_{(n-k)}t - \omega_k t] + \sin[\Omega_{(n-k)}t + \omega_k t] \} \\
& +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{1}{8} \{ \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] + \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \\
& \quad + \sin[2\alpha_k - (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] + \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \} \\
& +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k + \Omega_{(n-k)}t + \omega_k t] - \sin[\Omega_{(n-k)}t + \omega_k t] - \sin[\Omega_{(n-k)}t - \omega_k t] \\
& \quad + \sin[2\alpha_k + \Omega_{(n-k)}t - \omega_k t] \} \\
& -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k + (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) + \omega_k t] \\
& \quad - \sin[(\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] + \sin[2\alpha_k + (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) - \omega_k t] \} \\
& -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k - \Omega_{(n-k)}t + \omega_k t] + \sin[\Omega_{(n-k)}t - \omega_k t] + \sin[\Omega_{(n-k)}t - \omega_k t] \\
& \quad + \sin[2\alpha_k - \Omega_{(n-k)}t - \omega_k t] \} \\
& +A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
& \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] - \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \\
& \quad + \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \} \\
& +A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[2\alpha_k + \Omega_{(n+k)}t + \omega_k t] - \sin[2\alpha_k + \Omega_{(n+k)}t - \omega_k t] \\
& \quad + \sin[\Omega_{(n+k)}t + \omega_k t] - \sin[\Omega_{(n+k)}t - \omega_k t] \} \\
& -A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + 2\alpha_k + \omega_k t] \\
& \quad - \sin[2\alpha_k(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
& \quad - \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \} \\
& -A^2 A_1^2 g C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{\Omega_n^2}{8\omega_k \Omega_k} \{ \sin[2\alpha_k - \Omega_{(n+k)}t + \omega_k t] - \sin[2\alpha_k - \Omega_{(n+k)}t - \omega_k t] \\
& \quad - \sin[\Omega_{(n+k)}t - \omega_k t] + \sin[\Omega_{(n+k)}t + \omega_k t] \}
\end{aligned}$$

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$$\begin{aligned}
 & -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{1}{8} \{ \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] + \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \\
 & \quad + \sin[2\alpha_k - (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] + \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \} \\
 & -A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k + \Omega_{(n+k)}t + \omega_k t] - \sin[\Omega_{(n+k)}t + \omega_k t] - \sin[\Omega_{(n+k)}t - \omega_k t] \\
 & \quad + \sin[2\alpha_k + \Omega_{(n+k)}t - \omega_k t] \} \\
 & +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k + (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) + \omega_k t] \\
 & \quad - \sin[(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] + \sin[2\alpha_k + (\Omega_{(n+k)}t + 2\alpha_{(n+k)}) - \omega_k t] \} \\
 & +A^2 A_1^2 g \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \frac{1}{8} \{ \sin[2\alpha_k - \Omega_{(n+k)}t + \omega_k t] + \sin[\Omega_{(n+k)}t - \omega_k t] + \sin[\Omega_{(n+k)}t + \omega_k t] \\
 & \quad + \sin[2\alpha_k - \Omega_{(n+k)}t - \omega_k t] \} ,
 \end{aligned}$$

where the following trigonometric identities

$$\begin{aligned}
 \sin x \sin y \sin z &= \frac{1}{4} [\sin(x+y-z) + \sin(x+z-y) + \sin(y+z-x) - \sin(x+y+z)] \\
 \cos x \sin y \sin z &= \frac{1}{4} [\cos(x+y-z) + \cos(x+z-y) - \cos(x+y+z) - \cos(y+z-x)] \\
 \sin x \cos y \cos z &= \frac{1}{4} [\sin(x+y+z) - \sin(y+z-x) + \sin(x+y-z) + \sin(x+z-y)] \\
 \cos x \cos y \cos z &= \frac{1}{4} [\cos(x+y-z) + \cos(y+z-x) + \cos(z+x-y) + \cos(x+y+z)] \\
 \sin x \cos y &= \frac{1}{2} [\sin(x+y) + \sin(x-y)] \text{ and} \\
 \cos x \cos y &= \frac{1}{2} [\sin(x+y) - \sin(x-y)]
 \end{aligned}$$

have been used.

2.2.3. APPLICATION OF DUHAMEL'S INTEGRAL

In order to solve Equation (35), we use (following Michaltsos and Kounadis [19]) Duhamel's integral. To this end, the desired solution of the Euler-Bernoulli beam traversed by a distributive moving mass (second approximate solution) using the initial conditions stated in Equation (9) is given by Duhamel's integral as follows.

$$\bar{w}_n(\bar{x}_2, \tau) = \frac{\sin(n\pi\bar{x}_2)}{2n^2} \int_0^\tau Q_n(\tau^*) \cdot \sin[2n^2\pi(\tau - \tau^*)] d\tau^* , \quad (36)$$

where $\bar{w}_n(\bar{x}_2, \tau)$ is the non-dimensional n th mode displacement of the Euler-Bernoulli beam and

$$\begin{aligned}
 Q_n(\tau^*) &= A'A_1 \frac{1}{n} [-\cos D' + \cos J'] \\
 &- A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \beta_1' [\sin X_1' + \sin X_2' + \sin X_3' + \sin X_6' - \sin X_4' - \sin X_5' - \sin X_7' - \sin X_8'] \\
 &+ A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \beta_2' [\sin X_6' + \sin X_9' + \sin X_{10}' + \sin X_{11}' - \sin X_4' - \sin X_8' - \sin X_{12}'] \\
 &+ A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \beta_1' [\sin P_1' + \sin P_2' + \sin P_3' + \sin P_6' - \sin P_4' - \sin P_5' - \sin P_7' - \sin P_8'] \\
 &- A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \beta_2' [\sin P_6' + \sin P_9' + \sin P_{10}' + \sin P_{11}' - \sin P_4' - \sin P_8' - \sin P_{12}'] \\
 &+ A^2 A_1^2 C_1 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \mu_2' [\sin P_1' + \sin P_3' + \sin P_4' + \sin P_7' - \sin P_2' - \sin P_5' - \sin P_6' - \sin P_8'] \\
 &- A^2 A_1^2 C_1 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \mu_2' [\sin X_1' + \sin X_3' + \sin X_4' + \sin X_7' - \sin X_2' - \sin X_5' - \sin X_6' - \sin X_8'] \\
 &+ A^2 A_1^2 C_1 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \mu_1' [\sin X_6' + \sin X_8' + \sin X_{10}' + \sin X_{12}' - \sin X_1' - \sin X_4' - \sin X_9' - \sin X_{11}'] \\
 &- A^2 A_1^2 C_1 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \mu_1' [\sin P_6' + \sin P_8' + \sin P_{10}' + \sin P_{12}' - \sin P_1' - \sin P_4' - \sin P_7' - \sin P_{11}'] \\
 &- A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \gamma_2' [\sin Z_1' + \sin Z_2' + \sin Z_3' + \sin Z_4' - \sin Z_5' - \sin Z_6' - \sin Z_7' - \sin Z_8'] \\
 &- 2A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \beta_1' [\sin Z_5' + \sin Z_8' - \sin Z_2' - \sin Z_4'] \\
 &+ A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \gamma_1' [\sin Z_1' + \sin Z_2' + \sin Z_3' + \sin Z_4' - \sin Z_5' - \sin Z_6' - \sin Z_7' - \sin Z_8'] \\
 &+ 2A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \beta_2' [\sin Z_5' + \sin Z_8' - \sin Z_2' - \sin Z_4'] \\
 &- A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \gamma_1' [\sin Y_1' + \sin Y_2' + \sin Y_3' + \sin Y_4' - \sin Y_5' - \sin Y_6' - \sin Y_7' - \sin Y_8']
 \end{aligned}$$

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$$\begin{aligned}
& -2A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \beta_2' [\sin Y_5' + \sin Y_8' - \sin Y_2' - \sin Y_4'] \\
& + A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \gamma_2' [\sin Y_1' + \sin Y_2' + \sin Y_3' + \sin Y_4' - \sin Y_5' - \sin Y_6' - \sin Y_7' - \sin Y_8'] \\
& + 2A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \beta_1' [\sin Y_5' + \sin Y_8' - \sin Y_2' - \sin Y_4'] \\
& - A^2A_1^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \Gamma [\sin X_6' + \sin X_9' - \sin X_1' - \sin X_2'] \\
& - A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \Gamma [\sin X_4' + \sin X_{10}' - \sin X_3' - \sin X_8'] \\
& + A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \Gamma [\sin X_6' + \sin X_{11}' - \sin X_1' - \sin X_5'] \\
& + A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n-k)} \cdot \Gamma [\sin X_4' + \sin X_{12}' - \sin X_7' - \sin X_8'] \\
& + A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n-k)} \cdot [\sin X_1' + \sin X_2' - \sin X_6' - \sin X_9'] \\
& + A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n-k)} \cdot [\sin X_3' + \sin X_{10}' - \sin X_4' - \sin X_8'] \\
& - A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n-k)} \cdot [\sin X_5' + \sin X_{11}' - \sin X_1' - \sin X_6'] \\
& - A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n-k)} \cdot [\sin X_4' + \sin X_7' - \sin X_8' - \sin X_{12}'] \\
& + A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \Gamma [\sin P_6' + \sin P_9' - \sin P_1' - \sin P_2'] \\
& + A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \Gamma [\sin P_4' + \sin P_{10}' - \sin P_3' - \sin P_8'] \\
& - A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \Gamma [\sin P_6' + \sin P_{11}' - \sin P_1' - \sin P_5'] \\
& - A_1^2A^2C_0 \sum_{k=1}^{\infty} \frac{1}{k(n+k)} \cdot \Gamma [\sin P_4' + \sin P_{12}' - \sin P_7' - \sin P_8'] \\
& - A^2A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n+k)} \cdot [\sin P_1' + \sin P_2' - \sin P_6' - \sin P_9']
\end{aligned}$$

$$\begin{aligned}
 & -A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n+k)} [\sin P_3' + \sin P_{10}' - \sin P_4' - \sin P_8'] \\
 & + A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n+k)} [\sin P_5' + \sin P_{11}' - \sin P_1' - \sin P_6'] \\
 & + A^2 A_1^2 \sum_{k=1}^{\infty} \frac{1}{8k(n+k)} [\sin P_4' + \sin P_7' - \sin P_9' - \sin P_{12}']
 \end{aligned} \quad (37)$$

such that

$$\begin{aligned}
 \beta_1' &= \frac{\bar{V}(C_0 n^2 + k^2)}{8k^2(2k + \bar{V})}, & \beta_2' &= \frac{\bar{V}(C_0 n^2 + k^2)}{8k^2(2k - \bar{V})}, & \gamma_1' &= \frac{\bar{V}(C_0 n + k)^2}{8k^2(2k - \bar{V})} \\
 \gamma_2' &= \frac{\bar{V}(C_0 n + k)^2}{8k^2(2k + \bar{V})}, & \mu_1' &= \frac{n\bar{V}}{4k(2k - \bar{V})}, & \mu_2' &= \frac{n\bar{V}}{4k(2k + \bar{V})} \\
 \Gamma &= \frac{n^2 \bar{V}}{16k^3}, & A' &= \frac{2M}{\pi}
 \end{aligned}$$

$$\begin{aligned}
 X_1' &= [(n-k)(\pi \bar{V} \tau^* + \pi \bar{x}) - 2k^2 \pi \tau^*] & X_2' &= [(2k-n)\pi \bar{x} - (n-k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] \\
 X_3' &= [k\pi \bar{x} + (n-k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & X_4' &= [(n-k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 X_5' &= [n\pi \bar{x} + (n-k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & X_6' &= [(\pi \bar{V} \tau^* + \pi \bar{x})(n-k) + 2k^2 \pi \tau^*] \\
 X_7' &= [k\pi \bar{x} - (n-k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & X_8' &= [(n-k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] \\
 X_9' &= [(2k-n)\pi \bar{x} - (n-k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] & X_{10}' &= [k\pi \bar{x} + (n-k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 X_{11}' &= [n\pi \bar{x} + (n-k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] & X_{12}' &= [k\pi \bar{x} - (n-k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 P_1' &= [(n+k)(\pi \bar{V} \tau^* + \pi \bar{x}) - 2k^2 \pi \tau^*] & P_2' &= [-n\pi \bar{x} - (n+k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] \\
 P_3' &= [k\pi \bar{x} + (n+k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & P_4' &= [(n+k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 P_5' &= [(2k+n)\pi \bar{x} + (n+k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & P_6' &= [(\pi \bar{V} \tau^* + \pi \bar{x})(n+k) + 2k^2 \pi \tau^*] \\
 P_7' &= [k\pi \bar{x} - (n+k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] & P_8' &= [(n+k)\pi \bar{V} \tau^* - 2k^2 \pi \tau^*] \\
 P_9' &= [-n\pi \bar{x} - (n+k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] & P_{10}' &= [k\pi \bar{x} + (n+k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 P_{11}' &= [(2k+n)\pi \bar{x} + (n+k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] & P_{12}' &= [k\pi \bar{x} - (n+k)\pi \bar{V} \tau^* + 2k^2 \pi \tau^*] \\
 Y_1' &= [n\pi \bar{V} \tau^* + (n-k)\pi \bar{x}] & Y_2' &= [(n-2k)\pi \bar{V} \tau^* + (n-2k)\pi \bar{x}] \\
 Y_3' &= [n\pi \bar{V} \tau^* + k\pi \bar{x}] & Y_4' &= [(n-2k)\pi \bar{V} \tau^*] \\
 Y_5' &= [(n-2k)\pi \bar{V} \tau^* - k\pi \bar{x}] & Y_6' &= [n\pi \bar{V} \tau^*] \\
 Y_7' &= [n\pi \bar{V} \tau^* + n\pi \bar{x}] & Y_8' &= [(n-2k)\pi \bar{V} \tau^* + (n-k)\pi \bar{x}] \\
 Z_1' &= [(n+2k)\pi \bar{V} \tau^* + (n+k)\pi \bar{x}] & Z_2' &= [n\pi \bar{V} \tau^* + n\pi \bar{x}] \\
 Z_3' &= [(n+2k)\pi \bar{V} \tau^* + k\pi \bar{x}] & Z_4' &= [n\pi \bar{V} \tau^*] \\
 Z_5' &= [n\pi \bar{V} \tau^* - k\pi \bar{x}] & Z_6' &= [(n+2k)\pi \bar{V} \tau^*] \\
 Z_7' &= [(n+2k)\pi \bar{V} \tau^* + (n+2k)\pi \bar{x}] & Z_8' &= [n\pi \bar{V} \tau^* + (n+k)\pi \bar{x}] \\
 D' &= [n\pi \bar{V} \tau^*] & J' &= [n\pi \bar{V} \tau^* + n\pi \bar{x}]
 \end{aligned}$$

Note that in getting equations (36) and (37) the following dimensionless parameters

$$\bar{x} = \frac{\varepsilon}{l}, \quad \bar{x}_2 = \frac{X}{l}, \quad \bar{V} = \left(\frac{2}{\pi} \sqrt{\frac{ml^2}{EI}} \right) V, \quad A_1 = \frac{l}{\varepsilon}, \quad \tau = \frac{t}{T_1}, \quad T = \frac{2}{\pi} \sqrt{\frac{ml^4}{EI}}, \quad \bar{M} = \frac{M}{ml}$$

have been introduced and hence we also have

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$$\Omega_n t = n\pi\bar{V}\tau, \quad \omega_n t = 2n^2\pi\tau, \quad \Omega_n = \frac{n\pi\bar{V}}{T_1}, \quad \omega_n = \frac{2n^2\pi}{T_1}, \quad dt = T_1 d\tau$$

$$\alpha_n = \frac{n\pi\bar{x}}{2l}, \quad (\Omega_{(n-k)}t + 2\alpha_{(n-k)}) = (n-k)(\pi\bar{V}\tau + \pi\bar{x}), \quad \Omega_{(n-k)}t = (n-k)\pi\bar{V}\tau = \frac{n\pi\bar{x}}{2}$$

$$(\Omega_{(n+k)}t + 2\alpha_{(n+k)}) = (n+k)(\pi\bar{V}\tau + \pi\bar{x}), \quad \Omega_{(n+k)}t = (n+k)\pi\bar{V}\tau$$

and $2\alpha_n = n\pi\bar{x}$
 $2\alpha_k = k\pi\bar{x}.$

3. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

For the numerical work, the integral Equation (36) using the expression for $Q(\tau^*)$ as defined in Equation (37) was evaluated. MATLAB was used for this purpose. The calculations were carried out for the data considered in reference [19], in which the concentrated mass case was considered. The following values were used: $l = 50\text{m}$, $\bar{V} = 0.05, 0.10, 0.15, 0.20, 0.25, 0.30$ and 0.35 respectively, $\bar{M} = 0.10, 0.20, 0.30, 0.40, 0.50, 1.00$ and 1.50 . The weight per unit length of the beam is 15KN/m .

Table 1 compares the modal amplitude of the moving force problem (first approximate solution) for a Euler-Bernoulli beam subjected to (i) concentrated moving load ($T(t)$) and (ii) uniform partially distributed moving load ($Q(t)$). It is observed that the modal amplitude of the moving force problem under concentrated load is smaller than that under uniform partially distributed load.

S/N	$T(\text{sec})$	$T(t)$ (see Equation (12) of [19])	$Q(t)$ (see Equation (32))
1	0	0	0
2	0.1	8.72731E-11	1.19475E-07
3	0.2	2.26252E-07	2.37788 E-07
4	0.3	2.3509E-09	3.53798 E-07
5	0.4	5.56115E-09	4.66376 E-07
6	0.5	1.08332E-08	5.74411 E-07
7	0.6	1.866 E-08	6.76821 E-07
8	0.7	2.95194 E-08	7.72556 E-07
9	0.8	4.38719 E-08	8.60603 E-07
10	0.9	6.21576 E-08	9.39997 E-07
11	1.0	8.47935 E-08	1.00982 E-06
12	1.1	1.12172 E-07	1.06921 E-06
13	1.2	1.44656 E-07	1.11737 E-06
14	1.3	1.82582 E-07	1.15356 E-06
15	1.4	2.26252 E-07	1.17711E-06

Table 1. The modal amplitude (first approximate solution) of Euler-Bernoulli beam under (i) concentrated moving load ($T(t)$) (see Equation (12) in [19]) and (ii) uniform partially distributed moving load ($Q(t)$)

		$\bar{V}=0.050$	$\bar{V}=0.100$	$\bar{V}=0.150$	$\bar{V}=0.200$	$\bar{V}=0.250$	$\bar{V}=0.300$	$\bar{V}=0.350$
$\bar{M}=0.10$	W_0	0.0086	0.0168	0.0227	0.0175	0.0121	0.0153	0.0259
	W_c	0.0086	0.0165	0.0222	0.0173	0.0112	0.0134	0.0247
	R%	0.0000	-1.7857	-2.2026	-3.3520	-7.4380	-12.4183	-4.6332
$\bar{M}=0.20$	W_0	0.0510	0.0888	0.1048	0.0907	0.0757	0.0818	0.1070
	W_c	0.0506	0.0877	0.1027	0.0881	0.0711	0.0740	0.1021
	R%	-0.7843	-1.2387	-2.0038	-2.9512	-6.0766	-9.5355	-4.5794
$\bar{M}=0.30$	W_0	0.1271	0.2160	0.2461	0.2183	0.1915	0.1995	0.2435
	W_c	0.1262	0.2135	0.2414	0.2124	0.1805	0.1819	0.2324
	R%	-0.7081	-1.1574	-1.9098	-2.7027	-5.7441	-8.8221	-4.5585
$\bar{M}=0.40$	W_0	0.2367	0.3984	0.4468	0.4008	0.3591	0.3683	0.4353
	W_c	0.2352	0.3940	0.4384	0.3093	0.3386	0.3370	0.4160
	R%	-0.6337	-1.1044	-1.8800	-2.6198	-5.7087	-8.4985	-4.4337
$\bar{M}=0.50$	W_0	0.3801	0.6360	0.7068	0.6381	0.5808	0.5882	0.6823
	W_c	0.3778	0.6290	0.6937	0.6217	0.5488	0.5394	0.6523
	R%	-0.6051	-1.1006	-1.8534	-2.5701	-5.5096	-8.2965	-4.3969
$\bar{M}=1.00$	W_0	1.6023	2.6518	2.8964	2.6475	2.4805	2.4554	2.7470
	W_c	1.5931	2.6238	2.8438	2.5820	2.3466	2.2601	2.6268
	R%	-0.5742	-1.0559	-1.8160	-2.4740	-5.3981	-7.9539	-4.3757
$\bar{M}=1.50$	W_0	3.6666	6.0473	6.5689	6.0281	5.6992	5.6014	6.1941
	W_c	3.6460	5.9844	6.4505	5.8809	5.3979	5.1621	5.9235
	R%	-0.5618	-1.0401	-1.8024	-2.4419	-5.2867	-7.8427	-4.3687

Table 2. Percentile comparison of non-dimensional deflections of the Euler-Bernoulli beam between cases that include (a) centripetal and Coriolis forces (W_c) (b) without those forces (W_0).

In Table 2 are shown the values of the non-dimensional deflections at the middle of the beam with (W_c) and without (W_0) the effect of centripetal and Coriolis forces, respectively, for various values of $\bar{M}$ and $\bar{V}$. From the table, it can be seen that, for a given non-dimensional velocity $\bar{V}$, the value of the non-dimensional deflections at the middle of the beam involving centripetal and Coriolis forces (W_c) is smaller than that without the forces (W_0) for various values of $\bar{M}$ considered. This behaviour holds haphazardly (for particular values of $\bar{M}$ and $\bar{V}$) for $\bar{V} > 0.2$ while the opposite holds, in general, for $\bar{V} \leq 0.2$ for the case in reference [19] involving concentrated mass. Also, as the deflection ratio (R%) defined as $\frac{W_c - W_0}{W_0} \%$ decreases the non-dimensional velocity $\bar{V}$ increases except for $\bar{V} = 0.350$.

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The percentile comparison of non-dimensional deflections at the mid-span of the beam between the cases involving (i) centripetal and Coriolis forces (W_c) and (ii) without the forces (W_0) for various values of the length of the load (ϵ) and non-dimensional load mass ($\bar{M}$) and a given value of $\bar{V}$ is shown in Table 3. It can be seen that as ϵ increases the values of W_c are smaller than those of W_0 for a given $\bar{M}$ and $\bar{V} = 0.30$. The deflection ratio R also decreases as ϵ increases for $\bar{V}=0.30$ and a prescribed value of $\bar{M}$. Furthermore, the deflection ratio decreases as $\bar{M}$ increases for $\bar{V}=0.30$ and a given value of ϵ .

		$\epsilon=0.20$	$\epsilon=0.40$	$\epsilon=0.60$	$\epsilon=0.80$	$\epsilon=1.00$
$\bar{M}=0.10$	W_0	0.1515	0.0298	0.0075	0.0034	0.0011
	W_c	0.1394	0.0268	0.0065	0.0029	0.0009
	R%	-7.9868	-10.0671	-13.333	-14.706	-18.182
$\bar{M}=0.20$	W_0	0.6267	0.1400	0.0503	0.0190	0.0050
	W_c	0.5782	0.1278	0.0449	0.0165	0.0042
	R%	-7.7390	-8.7143	-10.736	-13.158	-16.000
$\bar{M}=0.30$	W_0	1.4256	0.3304	0.1285	0.0584	0.0262
	W_c	1.3166	0.3030	0.1163	0.0514	0.0225
	R%	-7.6459	-8.2930	-9.4942	-11.986	-14.122
$\bar{M}=0.40$	W_0	2.5483	0.6011	0.2421	0.1172	0.0598
	W_c	2.3546	0.5524	0.2204	0.1049	0.0519
	R%	-7.6011	-8.1018	-8.9632	-10.495	-13.211
$\bar{M}=0.50$	W_0	3.9947	0.9520	0.3910	0.1957	0.1059
	W_c	3.6920	0.8760	0.3589	0.1765	0.0935
	R%	-7.5775	-7.9832	-8.2097	-9.8109	-11.709
$\bar{M}=1.00$	W_0	16.0829	3.9112	1.6661	0.8837	0.5234
	W_c	14.8722	3.6070	1.5302	0.8068	0.4739
	R%	-7.5259	-7.7981	-8.1568	-8.7020	-9.4574

Table 3. Percentile comparison of non-dimensional deflections of the Euler-Bernoulli beam between the cases that include (a) centripetal and Coriolis forces (W_c) (b) without those forces (W_0) for various values of the length of the load (ϵ) and non-dimensional load ($\bar{M}$) at a fixed value of speed ($\bar{V}=0.30$).

The numerical results in Table 4 concern the values of W_c and W_0 for various values of ϵ and $\bar{V}$ for a given $\bar{M}$. It should be noted that the deflection ratio, R , decreases as $\bar{V}$ increases for a fixed value of ϵ and $\bar{M} = 0.50$. As ϵ increases for a fixed $\bar{V}$, both W_c and W_0 decrease while the value of W_0 is greater than the corresponding value of W_c .

		$\varepsilon=0.20$	$\varepsilon=0.40$	$\varepsilon=0.60$	$\varepsilon=0.80$	$\varepsilon=1.00$
$\bar{V}=0.050$	W_0	2.7182	0.6276	0.2467	0.1156	0.0560
	W_c	2.7033	0.6240	0.2451	0.1148	0.0555
	R%	-0.5482	-0.5736	-0.6486	-0.6920	-0.8929
$\bar{V}=0.100$	W_0	4.2968	1.0277	0.4241	0.2139	0.1170
	W_c	4.2479	1.0159	0.4192	0.2111	0.1153
	R%	-1.1381	-1.1482	-1.1554	-1.3090	-1.4530
$\bar{V}=0.150$	W_0	4.6309	1.1269	0.4787	0.2519	0.1469
	W_c	4.5482	1.1063	0.4696	0.2468	0.1436
	R%	-1.7858	-1.8280	-1.9010	-2.0246	-2.2464
$\bar{V}=0.200$	W_0	4.1573	1.0173	0.4313	0.2247	0.1282
	W_c	4.0562	0.9918	0.4199	0.2182	0.1241
	R%	-2.4319	-2.5066	-2.6432	-2.8927	-3.1982
$\bar{V}=0.250$	W_0	4.0535	0.9527	0.3787	0.1864	0.0995
	W_c	3.8399	0.9000	0.3547	0.1742	0.0929
	R%	-5.2695	-5.5316	-6.3375	-6.5451	-6.6332
$\bar{V}=0.300$	W_0	3.9947	0.9520	0.3910	0.1957	0.1059
	W_c	3.6920	0.8760	0.3571	0.1765	0.0945
	R%	-7.5775	-7.9832	-8.6701	-9.8109	-10.7649

Table 4. Percentile comparison of non-dimensional deflections of the Euler-Bernoulli beam between cases that include (a) centripetal and Coriolis forces (W_c) (b) without those forces (W_0) for various values of the length of the load (ε) and non-dimensional velocity ($\bar{V}$) at a fixed load-mass ($\bar{M}=0.50$)

Figure 2 depicts curves describing the variation of the deflection ratio R against various dimensionless velocities $\bar{V}$. It is seen that all the curves exhibit maximum amplitude at $\bar{V}=0.30$ for various values of dimensionless masses. Also, an increase in the dimensionless mass $\bar{M}$ yields an increase in the deflection ratio R .

Figures 3-5 are the response curves at the middle of the beam due to a uniform partially distributed moving load with and without the influence of the Coriolis and centripetal forces, respectively, for $\bar{M}=0.30$ and $\bar{V}=0.25, 0.5$ and 0.75 . The graphs, in each of the three figures, show that the response maximum amplitude when the Coriolis and centripetal forces are neglected is greater than the corresponding one involving the forces.

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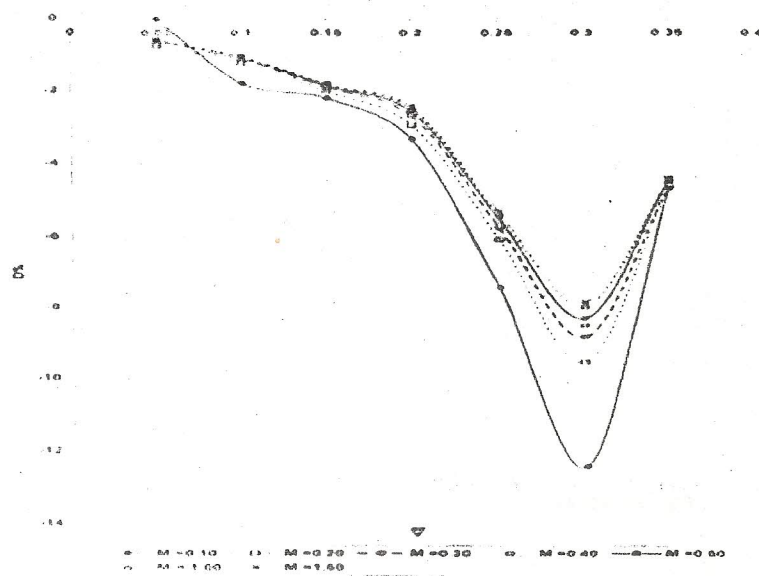


Figure 2. The graph of the ratio R versus the non-dimensional velocity V for various values of the non-dimensional load-mass M

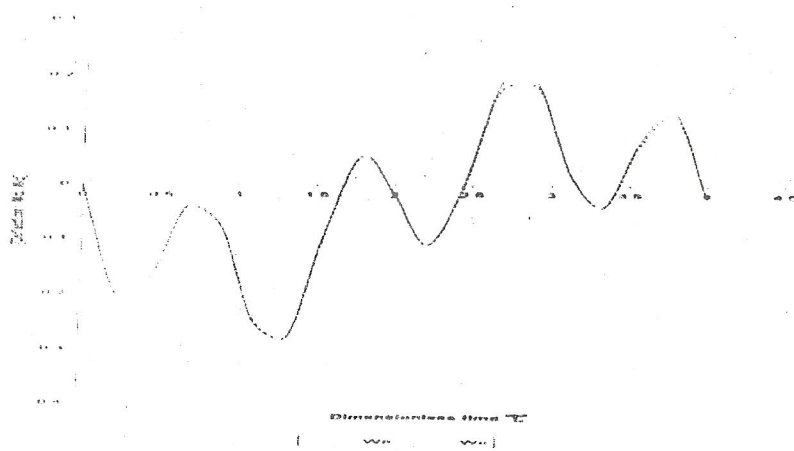


Figure 3. Middle-span deflection of beam for $M = 0.30$ and $V = 0.25$

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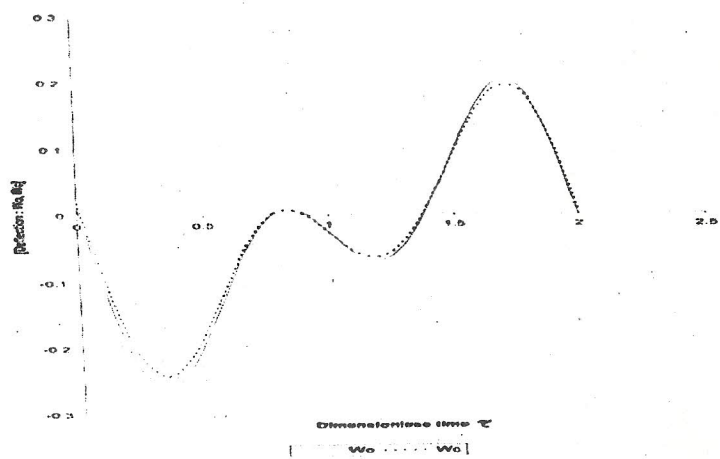


Figure 4. Middle-span deflection of beam for $\bar{M} = 0.30$ and $\bar{V} = 0.5$

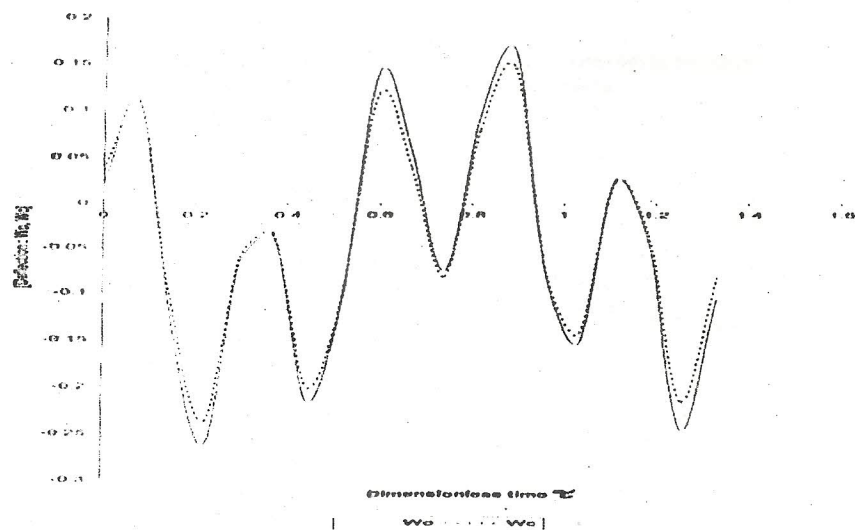


Figure 5. Middle-span deflection of beam for $M=0.30$ and $V=0.75$

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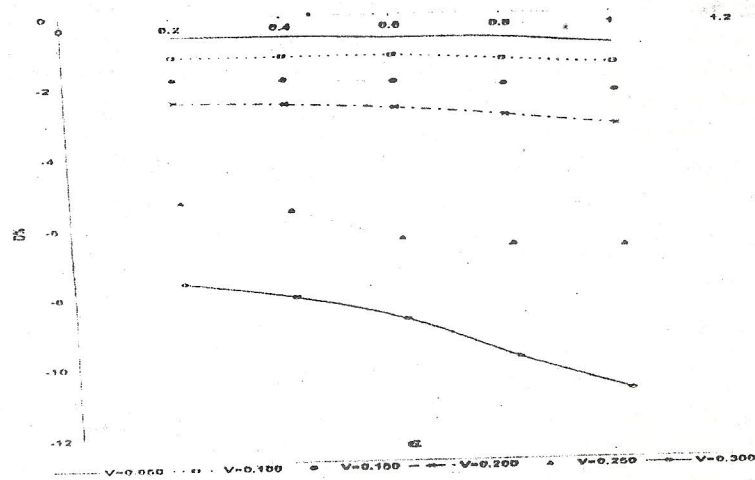


Figure 6. The graph of the ratio R versus the non-dimensional velocity $\bar{V}$ for various values of the non-dimensional load-mass $\bar{M}$

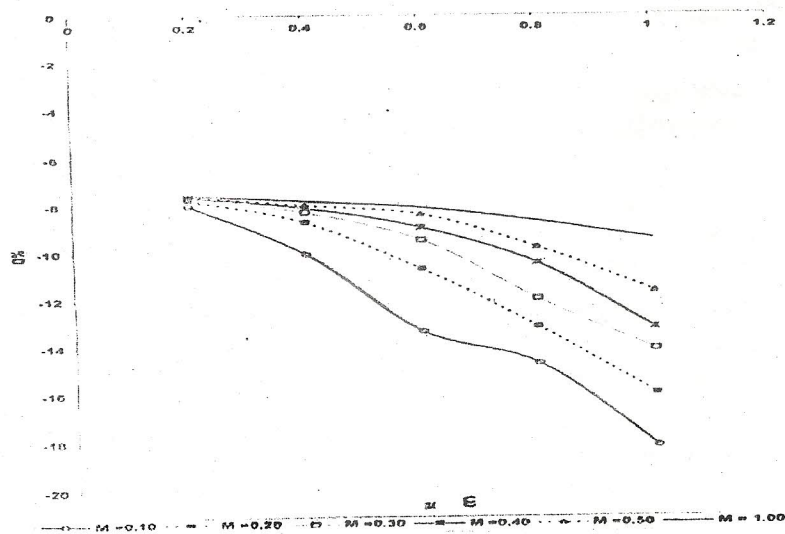


Figure 7. The graph of the ratio R versus the length of the load ϵ for various values of the non-dimensional velocity $\bar{V}$ and a fixed load-mass $\bar{M}$

Figures 6 and 7 are the profiles dealing with the relationship between the deflection ratio R and the load's length (ε). The curves in Figure 6 are for various values of the non-dimensional masses $\bar{M}$ and at a fixed value of dimensionless speed $\bar{V}$. It is observed that as the length of the load (ε) decreases, the effect of the ratio R increases. This effect becomes more pronounced as the non-dimensional mass $\bar{M}$ increases. The curves in figure 7 are for various values of the non-dimensional velocity $\bar{V}$ and a fixed value of the load mass $\bar{M}$. Clearly from the curves, the length of the load (ε) decreases with an increase in the effect of the Coriolis and centripetal forces.

CONCLUSIONS

The dynamics of Euler-Bernoulli beam under the influence of uniform partially distributed moving masses of constant magnitude and velocity including the effect of the Coriolis and centripetal forces has been studied. The separation of variable technique is used to transform the governing partial differential equation into a set of ordinary differential equations. For simplicity in analysis, the beam is assumed to be pinned-pinned and a closed form solution of the moving force problem has been obtained. This was compared in Table 1 with the corresponding solution obtained in [19], involving concentrated moving force. The solution of the modified moving mass problem involving the uniform partially distributed moving mass has been obtained using Duhamel's integral.

From the numerical analysis and results presented in this paper, it could be concluded that:

- (i) The modal amplitude of the Euler-Bernoulli beam under the uniform partially distributed moving force is greater than that under the concentrated moving force.
- (ii) The effect of Coriolis and centripetal forces on the non-dimensional midspan deflection decreases with the decrease of the non-dimensional mass $\bar{M}$ of the moving load for various values of the length of the load (ε). The decrease may be up to 18% as ε increases.
- (iii) The effect of the forces on the non-dimensional mid-span deflection increases for various values of ε . Such a decrease becomes more pronounced for $\bar{V} > 0.70$ and $\varepsilon > 0.6$ and may reach about 11% (for $\varepsilon = 1.0$).
- (iv) The deflection ratio R which measures the influence of the centripetal and Coriolis forces on the non-dimensional midspan deflection increases while it decreases as the non-dimensional velocity increases (except for $\bar{V} = 0.350$) for a fixed value of ε . All the curves of R against $\bar{V}$, for various values of non-dimensional mass exhibit response amplitudes at $\bar{V} = 0.30$ for a given value of ε . The maximum amplitude occurs when $\bar{M} = 0.10$.

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