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THIRD HANKEL DETERMINANT FOR α-STARLIKE FUNCTIONS

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Abstract. In this paper we investigate the third Hankel determinant, $H_3(1)$, for normalized univalent functions $f(z) = z + a_2z^2 + \cdots$ belonging to the class of α-starlike functions denoted by M_α. This class includes two important subclasses of the family of univalent functions - starlike and convex functions denoted by S^* and C. Our results therefore includes the special cases of the third Hankel determinants for the two classes of functions.

1. Introduction and preliminaries

Let S denote the class of normalized analytic univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

where $z \in D = \{z : |z| < 1\}$. The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & & a_{n+2(q-1)} \end{vmatrix}$$

This determinant has been considered by several authors. For example, Noor in [12] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary rotation. In particular, sharp bounds on $H_2(2)$ were obtained by the authors of articles [1, 5, 6, 7, 8, 13, 14] for the different classes of functions.

Easily, one can observe that $H_2(1) = |a_3 - a_2^2|$ is a special case of the well known Fekete and Szegö functional $|a_3 - \mu a_2^2|$ where μ is real. In this paper, we consider the Hankel determinant in the case $q = 3$ and $n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

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For $f \in S$, $a_1 = 1$ so that,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.2}$$

The class M_α is defined in [3] as follows:

Definition 1.1. Let f be given by (1.1) and $f(z)f'(z) \neq 0$ in $0 < |z| < 1$ and suppose α is real. Then $f \in M_\alpha$ if and only if, for $0 \leq \alpha \leq 1$

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0,\ z \in D. \tag{1.3}$$

Functions in the class M_α are called α-starlike. For $\alpha = 0$ we have the well known class S^* of starlike functions, that is functions satisfying

$$\text{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in D$$

while for $\alpha = 1$ we also have the well known class C of convex functions satisfying

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ z \in D.$$

For the class M_α, we study the determinant given by (1.2). We shall make use of the known sharp bound, 1 for $\alpha = 0$, and $2(1 + 8\alpha + 3\alpha^2)/(1 + 3\alpha)(2 + 15\alpha + 24\alpha^2 + 7\alpha^3)$ for $0 < \alpha \leq 1$ respectively, for the functional $|H_2(2)| = |a_2a_4 - a_3^2|$ (see [15]) together with other functionals, we will determine shortly in Section 2, so that our result for $H_3(1)$ follows as a simple corollary by sum.

Let P be the family of all functions p analytic in D for which Re $p(z) > 0$ and

$$p(z) = 1 + c_1z + c_2z^2 + \cdots$$

for $z \in D$.

Lemma 1.2. [4] If $p \in P$ then $|c_k| \leq 2$ for each $k \in N$.

Lemma 1.3. [10, 11] Let $p \in P$, then

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.4}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{1.5}$$

for some value of x, z such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.4. [1] Let $p \in P$. Then

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| = \begin{cases} 2(1-\sigma) & if \quad \sigma \leq 0, \\ 2 & if \quad 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & if \quad \sigma \geq 2. \end{cases}$$

2. Main results

Theorem 2.1. *Let $f \in M_\alpha$. Then we have the sharp inequalities:*

$$|a_2| \leq \frac{2}{1+\alpha},$$

$$|a_3| \leq \frac{\alpha^2+8\alpha+3}{(1+\alpha)^2(1+2\alpha)},$$

$$|a_4| \leq \frac{4(\alpha^4+11\alpha^3+38\alpha^2+19\alpha+3)}{3(1+\alpha)^3(1+2\alpha)(1+3\alpha)},$$

and

$$|a_5| \leq \frac{18\alpha^7+244\alpha^6+1319\alpha^5+3193\alpha^4+2642\alpha^3+1012\alpha^2+197\alpha+15}{3(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}.$$

Proof. Since $f \in M_\alpha$. Then there exists a $p \in P$ such that

$$(1-\alpha)z[f'(z)]^2+\alpha f(z)[f'(z)+zf''(z)] = f(z)f'(z)p(z) \tag{2.1}$$

for some $z \in D$. Equating coefficients in (2.1) yields

$$a_2 = \frac{c_1}{1+\alpha} \tag{2.2}$$

$$a_3 = \frac{c_2}{2(1+2\alpha)} + \frac{(1+3\alpha)c_1^2}{2(1+\alpha)^2(1+2\alpha)} \tag{2.3}$$

$$a_4 = \frac{c_3}{3(1+3\alpha)} + \frac{(1+5\alpha)c_1c_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(17\alpha^2+6\alpha+1)c_1^3}{6(1+\alpha)^3(1+2\alpha)(1+3\alpha)} \tag{2.4}$$

$$\begin{aligned} a_5 = {} & \frac{c_4}{4(1+4\alpha)} + \frac{(1+7\alpha)c_1c_3}{3(1+\alpha)(1+3\alpha)(1+4\alpha)} \\ & + \frac{(80\alpha^3+53\alpha^2+10\alpha+1)c_1^2c_2}{4(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} + \frac{(1+8\alpha)c_2^2}{8(1+2\alpha)^2(1+4\alpha)} \\ & + \frac{(304\alpha^4+201\alpha^3+55\alpha^2+15\alpha+1)c_1^4}{24(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} \end{aligned} \tag{2.5}$$

and the results follow by triangle inequality and using Lemma (1.2). □

Theorem 2.2. *Let $f \in M_\alpha$. Then we have the sharp inequalities:*

$$|a_2a_3 - a_4| \leq \begin{cases} 2 & \text{if} \quad \alpha = 0, \\ \frac{2(1+5\alpha+2\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)}\sqrt{\frac{1+6\alpha+7\alpha^2+2\alpha^3}{2\alpha(1+4\alpha+\alpha^2)}} & \text{if} \quad 0 < \alpha \leq 1. \end{cases}$$

Proof. From (2.2) to (2.4) we find that

$$\begin{aligned} |a_2a_3 - a_4| = & \left| \frac{-c_1c_2\alpha}{(1+\alpha)(1+2\alpha)(1+3\alpha)} \right. \\ & \left. + \frac{c_1^3(1+5\alpha)}{3(1+\alpha)^2(1+2\alpha)(1+3\alpha)} - \frac{c_3}{3(1+3\alpha)} \right| \end{aligned} \tag{2.6}$$

Substituting for c_2 and c_3 from equations (1.4) and (1.5) and leting $c_1 = c$ we get

$$|a_2a_3 - a_4| = \left|\frac{-\alpha c(c^2 + x(4-c^2))}{2(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(1+5\alpha)c^3}{3(1+\alpha)^2(1+2\alpha)(1+3\alpha)} - \frac{c^3 + 2xc(4-c^2) - x^2c(4-c^2) + 2z(1-|x|^2)(4-c^2)}{12(1+3\alpha)}\right|$$

which gives

$$|a_2a_3 - a_4| = \left|\frac{(3+10\alpha-11\alpha^2-2\alpha^3)c^3}{12(1+\alpha)^2(1+2\alpha)(1+3\alpha)} - \frac{(1+6\alpha+2\alpha^2)cx(4-c^2)}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{cx^2(4-c^2)}{12(1+3\alpha)} - \frac{2z(1-|x|^2)(4-c^2)}{12(1+3\alpha)}\right|.$$

Since $|c| = |c_1| \leq 2$ by using the Lemma (1.2), we may assume without restriction $c \in [0,2]$. Then using the triangle inequality, with $\rho = |x|$ and noting that $3 + 10\alpha - 11\alpha^2 - 2\alpha^3 \geq 0$ since $0 \leq \alpha \leq 1$ by definition, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{(3+10\alpha-11\alpha^2-2\alpha^3)c^3}{12(1+\alpha)^2(1+2\alpha)(1+3\alpha)} + \frac{(1+6\alpha+2\alpha^2)c(4-c^2)\rho}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} \\ &\quad + \frac{4-c^2}{6(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho^2}{12(1+3\alpha)} \\ &= F(\rho). \end{aligned}$$

Then

$$F'(\rho) = \frac{(1+6\alpha+2\alpha^2)c(4-c^2)}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho}{6(1+3\alpha)}.$$

Note also that $F'(\rho) \geq F'(1) > 0$. Then there exist $c^* \in [0,2]$ such that $F'(\rho) > 0$ for $c \in (c^*, 2]$ and $F'(\rho) \leq 0$ otherwise. Then for $c \in (c^*, 2]$, $F(\rho) \leq F(1)$, that is:

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1+5\alpha+2\alpha^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}c - \frac{2\alpha(1+4\alpha+\alpha^2)}{3(1+\alpha)^2(1+2\alpha)(1+3\alpha)}c^3 \\ &= G(c) \end{aligned}$$

If $\alpha = 0$, we have $G(c) = c \leq 2$. Otherwise, by elementary calculus $G(c)$ is maximum at $c = \sqrt{(1+6\alpha+7\alpha^2+2\alpha^3)/(2\alpha+8\alpha^2+2\alpha^3)}$ and is given by

$$G(c) \leq \frac{2(1+5\alpha+2\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)}\sqrt{\frac{1+6\alpha+7\alpha^2+2\alpha^3}{2\alpha(1+4\alpha+\alpha^2)}}.$$

Now suppose $c \in [0, c^*]$, then $F(\rho) \leq F(0)$, that is,

$$\begin{aligned} F(\rho) &\leq \frac{(3+10\alpha-11\alpha^2-2\alpha^3)c^3}{12(1+\alpha)^2(1+2\alpha)(1+3\alpha)} + \frac{4-c^2}{6(1+3\alpha)} \\ &= G(c) \end{aligned}$$

which implies that $G(c)$ turns at $c = 0$ and $c = 4(1+\alpha)^2(1+2\alpha)/[3(3+10\alpha - 11\alpha^2 - 2\alpha^3)]$ with its maximum at $c = 0$. That is

$$G(c) \leq \frac{2}{3(1+3\alpha)}.$$

Thus for all admissible $c \in [0,2]$, the maximum of the functional $|a_2a_3 - a_4|$ are given by the inequalities of the theorem. This completes the proof. $\square$

For $\alpha = 0$, we have $|a_2a_3 - a_4| \leq 2$, while for $\alpha = 1$ we have $|a_2a_3 - a_4| \leq 4/[9\sqrt{3}]$. These results are sharp and agree with those obtained by Babalola in [2].

For $\alpha > 1$, the analytic functions defined by the geometric condition (1.3) are not known to consist of univalent functions. For those functions we can derive the following theorem from the proof of the above result.

Theorem 2.3. *Suppose the normalized analytic function $f(z) = z + a_2z^2 + \cdots$ satisfy (1.3) for $\alpha \geq 1$, then*

$$|a_2a_3 - a_4| \leq \frac{2(1+5\alpha+2\alpha^2)}{3(1+\alpha)(1+2\alpha)(1+3\alpha)}\sqrt{\frac{1+6\alpha+7\alpha^2+2\alpha^3}{3+14\alpha+5\alpha^2+2\alpha^3}}.$$

Proof. From the proof of Theorem (2.2) we have

$$\begin{aligned}|a_2a_3 - a_4| = \Bigg| &\frac{2z(1-|x|^2)(4-c^2)}{12(1+3\alpha)} - \frac{cx^2(4-c^2)}{12(1+3\alpha)} \\ &+ \frac{(1+6\alpha+2\alpha^2)cx(4-c^2)}{6(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(2\alpha^3+11\alpha^2-10\alpha-3)c^3}{12(1+\alpha)^2(1+2\alpha)(1+3\alpha)}\Bigg|\end{aligned}$$

where $2\alpha^3 + 11\alpha^2 - 10\alpha - 3 \geq 0$ for $\alpha \geq 1$. The rest of the proof follows as in Theorem (2.2). $\square$

Theorem 2.4. *Let $f \in M_\alpha$. Then we have the best possible bound*

$$|a_3 - a_2^2| \leq \frac{1}{1+2\alpha}.$$

Proof. Since $f \in M_\alpha$, then using equation (2.2) and (2.3) we find that

$$\begin{aligned}|a_3 - a_2^2| &= \left|\frac{c_2}{2(1+2\alpha)} - \frac{c_1^2}{2(1+\alpha)(1+2\alpha)}\right| \\ &= \frac{1}{2(1+2\alpha)}\left|c_2 - \frac{2}{(1+\alpha)}\frac{c_1^2}{2}\right|.\end{aligned}$$

Now, using the Lemma (1.4), with $0 \leq \sigma = 2/(1+\alpha) \leq 2$, we have

$$|a_3 - a_2^2| \leq \frac{1}{1+2\alpha}$$

as desired. $\square$

Again for $\alpha = 0$, we have $|a_3 - a_2^2| \leq 1$, while for $\alpha = 1$ we have $|a_3 - a_2^2| \leq \frac{1}{3}$. These sharp results also agree with those obtained by Keogh [9].

Now using the bounds obtained in Theorems (2.1), (2.2) and (2.4) together with the known bound $|a_2a_4 - a_3^2| = 1$ for $\alpha = 0$, and $2(1+8\alpha+3\alpha^2)/(1+3\alpha)(2+15\alpha+24\alpha^2+7\alpha^3)$ for $0 < \alpha \leq 1$ respectively, [15] in (1.2), we have the the following best possible bound for $|H_3(1)|$ for the class M_α of α-starlike functions.

Corollary 2.5. *Let $f \in M_\alpha$. Then we have the best possible inequalities:*

$$|H_3(1)| \leq \begin{cases} 16 & if \quad \alpha = 0, \\ \frac{M_1 + M_2\sqrt{2\alpha(1+10\alpha+32\alpha^2+36\alpha^3+15\alpha^4+2\alpha^5)}}{M_3} & if \quad 0 < \alpha \leq 1. \end{cases}$$

where

$$M_1 = 3\alpha(48 + 1327\alpha + 15930\alpha^2 + 109795\alpha^3 + 482338\alpha^4 + 1411420\alpha^5 + 2780596\alpha^6 + 3638314\alpha^7 + 3060628\alpha^8 + 1588795\alpha^9 + 496722\alpha^{10} + 90547\alpha^{11} + 8922\alpha^{12} + 378\alpha^{13}).$$

$$M_2 = 4(6 + 149\alpha + 1586\alpha^2 + 9464\alpha^3 + 34703\alpha^4 + 80481\alpha^5 + 117092\alpha^6 + 103046\alpha^7 + 51849\alpha^8 + 14252\alpha^9 + 1980\alpha^{10} + 112\alpha^{11}).$$

$$M_3 = 9\alpha(1+\alpha)^4(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)(2+23\alpha+86\alpha^2+118\alpha^3+52\alpha^4+7\alpha^5).$$

That the above inequalities for $H_3(1)$ for the class M_α of α-starlike functions are the best possible bound follows from the fact that each of the component functionals in (1.2) is sharp.

For $\alpha = 0$, we have $|H_3(1)| \leq 16$, while for $\alpha = 1$ we have $|H_3(1)| \leq 0.714933452973167$. These sharp results also agree with those obtained by Babalola [2].

References

1. K. O. Babalola and T. O. Opoola, *On the coefficients of certain analytic and univalent functions*, Advances in Inequalities for Series, (Edited by S. S. Dragomir and A. Sofo) Nova Science Publishers, (2006), 5-17.
2. K.O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequality Theory and Applications, **6** (2010), 1-7.
3. H. B. Coonce and S. S. Miller, *Distortion properties of p-fold symmetric alpha-startlike functions*, Proc. Amer. Math. Soc., **44** (2), (1974).
4. P.L. Duren, *Univalent functions*, Springer Verlag, New York Inc., (1983).
5. T. Hayami and S. Owa, *Generalized Hankel determinant for certain classes*, Int. Journal of Math. Analysis, **4** (52) (2010), 2573-2585.
6. A. Janteng, *Hankel Determinant for Starlike and Convex Functions*, Int. Journal of Math. Analysis, **1** (13) (2007), 619-625.
7. A. Janteng, S. A. Halim and M. Darus, *Estimate on the second Hankel functional for functions whose derivative has a positive real part*, Journal of Quality Measurement and Analysis JQMA, **4** (1) (2008), 189-195.
8. N. Kaharudin, A. Akbarally, D. Mohamad and S. C. Soh, *The second Hankel determinant for the class of close-to-convex functions*, European Journal of Scientific Research ISSN 1450-216X, **66** (3) (2011), 421-427.
9. F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Sco., **20**, (1969), 8-12.
10. R. J. Libera and E. J. Zlotkiewicz, *Early coefficient of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85** (2) (1982), 225-230.
11. R. J. Libera and E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in P*, Proc. Amer. Math. Soc., **87** (2) (1983), 251-257.

12. K. I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roum. Math. Pures Et Appl., **28** (8) (1983), 731-739.
13. C. Selvaraj and N. Vasanthi, *Coefficient bounds for certain subclasses of close-to-convex functions*, Int. Journal of Math. Analysis, **4** (37) (2010), 1807-1814.
14. G. Shanmugam, B. Adolf Stephen and K. G. Subramanian, *Second Hankel determinant forcertain classes of analytic functions*, Bonfring International Journal of Data Mining., **2** (2), (2012), 57-60.
15. G. Shanmugam and B. Adolf Stephen, *Second Hankel determinant for α-starlike functions*, International J. of Math. Sci. and Engg. Appls., **6**, (5) (2012), 157-161.

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