



**National Mathematical Centre (NMC), Abuja,
Nigeria
&
Commission for Science and Technology for
Sustainable Development in the South
(COMSATS), Pakistan**



[May 2009]



**THE
PROCEEDINGS
OF**

**INTERNATIONAL CONFERENCE ON
MATHEMATICAL MODELLING OF
SOME GLOBAL CHALLENGING
PROBLEMS IN THE 21ST CENTURY
(26 – 30 NOVEMBER, 2008), Abuja, Nigeria**



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THE PROCEEDINGS OF

THE FIRST INTERNATIONAL CONFERENCE ON MATHEMATICAL MODELLING OF SOME
GLOBAL CHALLENGING PROBLEMS IN THE 21ST CENTURY

Organized by:

National Mathematical Centre (NMC), Abuja, Nigeria

And

Commission for Science and Technology for Sustainable Development
in the South (COMSATS), Pakistan

*Held at the National Mathematical Centre, Abuja, Nigeria
(26 – 30 NOVEMBER, 2008)*

Editors

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Preambles

In the 21st Century the world has so many challenging problems amongst are the global warming, occurrence of ecological problems, outbreak of diseases these include HIV/AIDS, bird flu and genetic related diseases like cancer, sickle cell anemia and thalassaemia. Also included in the list are security problem, food and energy crises and high raise in the price of crude oil. The unemployment problem and collapse of many financial giant institutions and the list can go on and on.

In order to understand complex situations surrounding human being there is the need to develop models. Models are mathematical structure for managing real life situations, to animate, visualize or interact with complex structures and processes.

The goal of the conference was to bring together experts working on Modelling, Scientists and Students interested in Modelling/Simulation together to interact and present their findings to the public ;suggest to policy makers how to improve the quality of life of people and our environment ; how to predict natural disaster using mathematical models and simulation techniques.

The Conference also accepted papers for presentation in the following sub-themes:

SUB-THEMES

Mathematical Modelling/Simulations in the following areas:

1. Ecology: Climatic problems;
2. Medicine & Pharmacology: Infection diseases and Genetic related ones and Drug administration;
3. Engineering and Science;
4. Fossil and Renewable Energy;
5. Security systems: Food security, Computer security, Encryption and it uses in financial house especially ATM machine;
6. Financial Investment : Portfolio management, Market volatility, Exchange rates and term structure;
7. Nanotechnology;
8. Equipment for processing raw materials e.g. equipment for production of biodiesel, fuel ethanol, production of drugs and in the manufacturing plants etc.
9. Telecommunication and Communication
Network: Performance models, Quality of Service (QOS) in a Network, Protocol and Topology in a Network.
10. Culture, Education and Social Science.

Host

Professor Sam O. Ale mni, OFR
The Director- General
National Mathematical Centre, Abuja,Nigeria

Chairmen Local Organizing Committee

1. Prof. P. Onumanyi, Deputy Director-General

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National Mathematical Centre, Abuja, Nigeria
2. Associate Professor B. O. Oyelami,

Coordinator Mathematics Programme,
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Prof. R.O. Ayeni (Nigeria)
Prof. Femi Taiwo (Nigeria)
Prof.J.A.Ogidi (Nigeria)
Prof. Walford Chukwu (Nigeria)

Keynote Speakers

The Prospects of Quantitative analysis and financial engineering in the 21 Century Banking and financial market - Professor Chukwuma Soludo, The Governor of the Central Bank of Nigeria, Abuja

The role of Space Technology in addressing Global Climate Change Problems

-Professor Ajayi Boroface the Director General Obasanjo Space Centre, Abuja, Nigeria.

The use of mathematical modeling and simulation in solving the 21st Century Energy problems- Professor Abubakar Sani Sambo, The Director-General, Energy Commission of Nigeria, Abuja, Nigeria.

R.O. Ayeni, the President Nigerian Mathematical Society-Velocity profile of a deoxyhemoglobins blood whose viscosity is unsteady.

Energy Crisis in the 21st in the Century – Prof. J. S. A. Adelabu, National Mathematical Centre, Abuja, Nigeria.

The Meta-Heuristics of Global Financial Risk Management in the Eyes of the Credit Squeeze: Any Lessons for Modelling Emerging Financial Markets? Patrick Oseloka EZEPUE, Research Coordinator of the Business Intelligence & Quantitative Modelling Research Group, Sheffield Hallam University, UK

On Steady Flow and Heat transfer in a pipe with temperature dependent viscosity and convective cooling - O. D. Makinde, Faculty of Engineering, Cape Peninsula University of Technology, Bellville, South Africa.

The outcome of the Conference

The conference was advertised within short notice and many people responded. The advertisements were made on the Websites of National Mathematical Centre, the University of Benin, Nigeria and AIMs South Africa. The Pan African Statistical Society used its mailman service to pass the message to its registered members. About 50 abstracts were received and about 40 papers were presented in various areas on modeling as related to global challenges in 21st Century. Prof. S. O. Ale presented a paper on MATHEMATICAL MODELLING POTENT TOOL FOR SOLVING THE 21ST CENTURY GLOBAL CHALLENGING PROBLEMS

Prof. Ajayi Boroface from the National Space Centre, Abuja sent a representative to present paper on the role of Space Technology in addressing Global Climate Change Problems.

Prof. Patrick Ezepue Sheffield Hallam University Sheffield United Kingdom who was invited guest speaker at the Opening ceremony of the Conference. Prof. Sani Sambo was on official assignment to Germany but send a representative and Prof. R. O. Ayeni was also on official assignment but send a representative to present his paper.

Other papers presented at the meeting were:

3. A Random Walk through Mathematical Sciences with Some Hints on a Model-Based Approach to Capacity Building in Developing Economies – Par II Discussions-Patrick Oseloka EZEPUE
4. How to appropriately manage mathematical model parameters for accuracy and reliability: A case of monitoring levels of particulate emissions in ecological systems- *Kassim Smwitondi. and Patrick Ezepue.*
5. Mathematical Modeling of Mammalian Blood Count- ADEWOLE J. K. and Osunleke A. S.
6. A Random Walk through Mathematical Sciences with Some Hints on a Model-Based Approach to Capacity Building in Developing Economies: Part I Theoretics-Patrick Oseloka EZEPUE.
7. The Meta-Heuristics of Global Financial Risk Management in the Eyes of the Credit Squeeze: Any Lessons for Modelling Emerging Financial Markets? Patrick Oseloka EZEPUE and Adewale R T SOLARIN.
8. Model simulation for bioavailability and biodegradation: A mathematical approach to the bioremediation of polycyclic aromatic hydrocarbon contaminated site - C. N. Owabor, S. E. Ogbeide and A. A. Susu
9. Model simulation for bioavailability and biodegradation: A mathematical approach to the bioremediation of polycyclic aromatic hydrocarbon contaminated site - C. N. Owabor, S. E. Ogbeide and A. A. Susu.
10. Empirical Mathematical Model for Palm Kernel Oil expression using Screw Press- IGBEKA, J. C., RAJI, A. O. and R. AKINOSO.
11. A Mathematical Model to assess the impact of Counseling and Antiretroviral Therapy on the spread of HIV/AIDS- A.R. KIMBIR AND H. K. ODUWOLE.
12. A note on generalized model for estimating life expectancy of populations- O. A. Adekola.
13. Volatility Modelling For Stock Prices- Akinlawon, O.J.

14. Preventive Repair Policy And Overhaul Policy Of Repairable System- Walford I.E.Chukwu and Nwosu C.
15. A new family of A-stable Four-step method on GAM –Reverse GAM pair for Stiff Initial Value Problem- Ajie I.J. and Onumanyi P.
16. Role of Engineering and Science in Sustainable Development in the 21st Century- Abdulkarrem Ozi Aliyu and Abdulkabir Aliyu.
17. The Fluid Mechanics of the Cochlea due to Noise- O .H. Adagba
18. A Mathematical Model to determine the growth of Investments using Share Prices- Stephen E. Onah
19. An Instability patterns in a Mathematical Model for Tumour Development
- Atabong, T. A., Oyesanya M.O.and Gideon A. N.
20. Comparative methods of determination of Demand responses based on the Sales of Grains in Bauchi metropolis, Nigeria-Adamu, M. M., Garba, E. J. D. and Hamidu, B. M.
21. Velocity profile of a Deoxyhemoglobins Blood whose Viscosity is Unsteady-R. O Ayeni, A. O Oyebanjo, L. M. Erinle and T. O. Oluyo.
22. A model of prey-predator with third order interaction- E.A Bakare.
23. Application of the B-transform to some Impulsive Control Models- Sam O. Ale, Benjamin Oyediran Oyelami and Jonathan A. Ogidi.
24. A model for classifying incidence rate of filariasis in a habitat
Oyelami B.O, Ale S.O. Ogidi J.A.
25. Effect of Noise on the level of Hearing- ADAGBA O. H.
26. Some Malaria Models Treating both Sensitive and Resistant Strains in Single and Multigroup Populations -Sylvanus Aneke
27. Simulation of temperature distribution and solidification fronts in
Squeeze cast commercially pure aluminum- J. O. Aweda
28. On the Modelling and representation of DNA codes- ‘DELE OLUWADE.
29. A Mathematical Model for Lassa fever Pandemic- N. I. Akinwande & S. Abdurrahman.
30. A Mathematical Model of HIV and the Immune System- Sirajo Abdurrahman
31. Mathematical Model of HIV/AIDS Pandemic with the Effect of Drug Application- Sirajo Abdurrahman
32. On Algae Population Dynamics on a Water Body Using an Exponential Mathematical Model- N. I. Akinwande , N. Abdurrahman & S. Abdurrahman.

33. A mathematical model for a Reacting Rayleigh-Stokes problem for Non-Newtonian Medium with Memory- Mary Durojaiye and Ayeni R.O.
34. A model of prey-predator with third order interaction-E.A.Bakare and R.O.Ayeni.
35. A Two-dimension Unsteady model for Sub-retina Fluid Drainage in a Detached Retina – J.N.Adam,J.R.Blake and E.J.D.Garba.
36. Relationship between Continuity and Momentum Equations in Two Dimension Flows- Idowu,I.A.,Olayiwola,M.O. and Gbolagade A.W.

We also received submissions from some Postgraduate students from AIMs South Africa and Botswana and Ethiopia who could not attend the conference because of financial problems but their papers were included in the book of abstracts of the conference.

Good will messages

We received good will messages from international scholars as such as
Prof. Tijjamu Hussein, the former Executive Director of COMSATS, Pakistan

Prof. J. Neito, Spain

Prof. Julio Dix the Managing Editor, Electronic Journal of Differential equations, Texas State University San Marcos USA.

Dr. Anders Wandahl, the Coordinator e-math for Africa, Sweden.

Prof. Ogana, University of Kenya and the International Coordinator of AMS

Prof. Francis Alloytey, Ghana

Preface

The 21st Century was ushered in with several challenging problems amongst is the medical problems like HIV/AIDs, Bird flu and Swine flu and Sars. There are other environmental and financial problems which are everywhere present and there is indeed the need for world scientists, engineers and educators to develop capacity to tackle these global problems.

The proceedings for the *International Conference on Mathematical Modelling of some Global Challenging Problems in the 21st Century* is the showcase of novel report of findings of some Scientists, engineers and educators who have love and passion for solving humanity problems Through modelling and simulation were assembled together at the conference.

The outcome of research findings of the Presenters after a peer review process are published in this Volume.

The participants in the conference advocated that this conference should made to be an annual Event with possibility of forming the Society for Mathematical Modelling and Simulation Future. It also suggested that some of the findings and recommendation should be adopted by National Planners and Policy Makers in their nations.

Finally, before I end this preface I wish to acknowledge the support and encouragement the National Mathematical Centre, Abuja, Nigeria received from the COMSATS National Secretariat and our dear member states of COMSATS for the mandate given us to develop capacity on mathematical modelling for the sub-region of Africa. We shall keep on carry on the mandate to ensure that everybody is mathematical modelling literate in the sub-region and to encourage policy makers that no decision should be carryout without being thoroughly modeled or simulated on the computer.

Professor Sam O. Ale mni,OFR
The Director-General,
National Mathematical Centre
Abuja, Nigeria

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Mathematical modelling potent tool for solving the 21st Century global challenging problems

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Centre, Abuja, Nigeria

This being the keynote address presented at the NMC-COMSATS International Conference on Mathematical Modelling Held on 26-30 November 2008, Abuja, Nigeria.

Mathematics Subject Classification 2000:00-02,01A67 & 93B10.

1 Research and its challenges in the 21st Century

The 20th Century Research brought in credible advances in our understanding of the universe, solar system, and genetics and so on. In the 21st Century it is envisaged that Research will play a major role in contributing to new understanding of our surrounding, our history, our science, our culture and also contribute to our well being in no small measure [3 & 4].

Science and Technology(ST) researches conducted in academic institutions, have succeeded in raising our standard of living ,created job opportunities, have shed more light on some natural phenomena. But as international economic competition intensifies in the years ahead, research will be even important in meeting national objectives [4].

The outcome of research in the 21st century will have impact in the formulation of national policy. The USA had long realized the importance of investment in Science and Technology (ST) and spends about 284 billion dollars annually on Research and Development (RD). These include combine research funding from the public and the private sector research development investments. This investment in ST made USA to have leading role in R&D globally today.US has a well articulated policy on commercialization of scientific advances and had help to nurture entrepreneurship and dissemination of information on new technologies[4].

The future development of a nation depends in no small measure on how grounded the higher education is developed and by extension on what kind of research culture the nation is pursuing. The success story of a nation can best be told if the nation's Research policy accommodates effective applications of research to solve societal problems and also provides the education of her citizens to provide the contextual knowledge and specific skills that will make them effective thinkers and leaders and also informed decision makers and responsible citizens.

Technology and the nation's future are intertwined, if the government has a policy on technology development that facilitate the translation of new knowledge to new capacity then the future will be very bright otherwise the future has nothing to offer to that nation.

2 Global challenges in the 21st Century

The most challenging issues facing humanity have central social and individual dimensions – environmental change, population problems, national and religious conflict, the social, cognitive and neural aspects of aging, and others.

The 20th Century witnessed many landmark achievements in Research and Development but many problems were carried over to the 21st Century. In the 21st Century, therefore, researchers must spend sleepless night to proffer solutions to humanity's problem in this century. We will look at the 21st Century Global Challenging Problems as originated from following disciplines:

2.1 Science and Technology

Majority of the topmost problems are in the medicine. The occurrences of diseases and syndromes like HIV/AIDS, Malaria and Tuberculosis, Sars, Bird flu, Swine flu Cancers, Parkinson diseases and so on. There are also genetic related diseases, haemoglobinopathies like sickle cell anemia and thallassaemia [1].

The World Health Organization (WHO) has classified some of those diseases and syndromes mentioned above as being contagious to man. WHO and its agencies like USAID, Global fund and other multi lateral organizations are funding research on these diseases and syndromes. They also provide education to people on the spread of the diseases and syndromes, chemotherapy, development of vaccines and other related support services ([3] & [6]).

Most challenging medical problems are on how to develop potent vaccines for HIV, malaria and Tuberculosis (HMT). Although, there are many vaccines undergoing clinical trials, but up till now, we can say there is no most potent HMT vaccine. For HIV/AIDS drugs- Antiretroviral (AR) and highly active antiretroviral (HAAR) drugs have many complications when patients apply them for long period, the patients often suffer cardiovascular problems. Research on potent vaccines and Drugs that would cure HMT diseases and syndrome without notable medical side effects are needed in the 21st century.

WHO seems not to classify genetic related diseases (GRD) among most dangerous global diseases to mankind but GRD is most endemic among the black races and the people of Mediterranean. Other medical problems are on Genome and ethical issue and this will dominate research in the 21st Century in medicine ([1]).

Furthermore, we also expect that there would be some engineering challenges like the use of engineering to better medicine vis-a-vis development of medical equipment for medical imaging and treatment of diseases which would differ completely from the traditional radiotherapy. More problems are also centered on medical informatics. Intensified researches are expected to be on cancer, psychiatric disorders, diabetes, and Alzheimer's disease [1].

HIV and AIDs are multifaceted global challenging problems in which mathematical modelling can be use to solve in the following ways:

Use the existing models and even develop epidemiological, demographic, chemotherapeutic, nutritional as well as sexual behavioral models, intervention strategy models, and models to make projection into the future. The projection will take into consideration all the socioeconomic, socio-cultural and educational factors that enhance the spread of HIV/AIDS; other factors to be considered are the sexual behavioral pattern and contribution of the other co-factors e.g. the sexual transmitted diseases (STD) and Tuberculosis in the spread of HIV.

Mathematical models have general utility and let us identify some fundamental applications in medicine.

Models can be use to:

- Determine the role of condom in reducing HIV and STD infections;
- Determine the viability of the Condom option;
- Study the growth of Population of a country accordance to age structure and compare with the result with that of National population Commission report and World Bank estimates. The population can be obtained in absence and the presence of HIV/AIDS pandemic. This will enable us to determine risk level of the affected individuals as overall percentage of the population for various parameters and timelines;
- Monitor the disease growth;
- Determine infectiousness for example heterosexual situation measure, the level of promiscuity and patronage of commercial sexual workers (CSW);
- Compute the contribution of nonsexual transmission factors to overall HIV/AIDS scourge;
- Assess the Demographic structure of the country.

- Design the geographic information system (GIS) with view to identify/classify areas prone to risk according to being hyper-endemic, hypo-endemic or meso-endemic areas;
- Determine the cost analysis for anti-retroviral drugs and highly active antiretroviral therapy for channeling of drugs;
- Estimate the most effective way of challenging the resources to fight diseases and syndromes;
- Determine how factors like socio-cultural practice, believe system, and gender sexuality contribute to the growth of disease such as HIV/AIDs;
- Determine the efficacy of ARD and HAART drugs.
- Design and calibrate medical equipment.
- Design most effective medical imaging equipment and for simulation surgery.

2.2 Information Technology

Developments of satellite and related wireless technology have revolutionized information technology. Computer found many applications in every Human Endeavour. The introduction wireless cell phones, internet services and ATM machines made life easy for people. Internet services and ATM machines have made business transaction easier these days. The topmost problem in the information technology is that of cyber security. This is one of the major issues and typical research challenging problem of the 21 century arise in telecommunication and communication. There are many mathematical models and algorithms that are useful in designing random generated codes for encrypting for computer security purposes.

Many mathematical models are being used for Quality of Service (QoS) in telecommunication network and for traffic control. These models make use of some telecommunication calculus for determining the optimal resource to be put into the system whenever there are traffic congestions. This kind arrangement helps in managing congestions at the peak (Busy-Hours) periods for the network and also help the telecommunication network not crash.

There other models using fuzzy logic that are used to improve quality of sound and images in the telecommunication network. These models have reduced echoes and call drops these days in cell phones. There are also good models for generating random numbers which are used as secrete codes for recharge cards.

Most mathematical modelling /Simulation researches in the industry of this century would be targeted at productions of products that give value for money, researches that explore the possibility of producing high quality products and with beautiful packages should be encouraged.

2.3 Environmental issues

Global warming is topmost in the list of environmental problems. Continuous emission of carbon dioxide in the atmosphere by burning of fossil oil had enormous rise in the tidal level of the ocean which had led to flood in many places globally. The occurrence of earthquake, tornadoes and tsunamis are constantly threatening the existence of life in the earth planet. We need models to forecast the occurrence of the natural disasters in order to avoid them. Models can be used to design warning alarm system.

Managing of nitrogen and avoiding dangerous interference with nitrogen cycle are issues of concern. The uses of fertilizers to increase available nitrogen on the planet, which led to global warming and ecological imbalance. The recycling of products especially biodegradable products like plastic materials is another environmental problem. Pollutions of land and drinkable water with poisonous substances such as heavy metals and crude oil arising from spillages constitute another problem. Models are useful in analyzing the extent of the pollution in environment especially when carrying out impact assessment.

Current and future problem must be on how to provide clean and safe water to drink and how to reduce air pollution especially in the cities inhabited by human. Other issue that may likely dominate the 21st century research is the issue of alternative energy to fossil

energy which is expensive and some scientists say is not environmentally friendly. The price of crude oil is soaring in the international market and the West is seeking development of alternative energy in form of renewable energy. Research on the use of fuel ethanol and biodiesel fuel is being explored as alternate sources to the fossil fuel. The conversion of food produce to fuel ethanol and biodiesel had led to food crisis in the recent times.

Simulation of cars using optimization algorithms has led to development of low fuel consuming cars. Models are now being used to design low fuel consuming carburetors and injectors for cars. The end of the first decade 21st century will gradually usher in electric cars, hybrid cars using fossil oil or biofuel to save energy costs. The green car project is all about production of cars with low fuel consumption and low gas emissions hence producing 21st century environmental friendly cars. Mathematical modellers are hereby encouraged to go into researches on combustion problems and the design of equipment for processing of biofuel and seeking our environment, the most abundant, non-food produce that can be used as raw materials for production of biofuel.

Nigeria needs to develop high level human capacity in renewable energy researches especially modeling and simulation in this field.

2.4 Economics and Finance Related Research

In recent years research on Ecophysics is being intensified wherein principles or theories in physics are applied to economic problems. The state of economy of a nation can be modeled using some ecophysical principles. The researches in the mathematics of finance require high quality of Mathematics background and garnished with Computer Science and Statistics, there will be further work here also.

Many financial transactions need sophisticated quantitative techniques to understand, hence many financial and mortgage houses spend several billion of dollars on researches on financial mathematics. Problems that are of interest are on hedging, portfolio management, interest rates and market volatility. Pricing options both for vanilla and exotic ones is another interesting one.

Furthermore, other economic research of interest is prediction of the state of the economy of a nation in future which includes simulation of gross domestic product (GDP), internationalization of trade to boost the foreign reserve, by increasing per capita income of the nation. In Nigeria two contending research problems are on exchange rates for foreign currency and re-decimalization of naira.

There are several mathematical models being used for taking core decisions in modern finance these days and Nigeria needs to develop capacity in quantitative analysis and financial engineering.

Many giant financial institutions world-wide are collapsing because of liquidity problem. Many might be due to inaccurate application of mathematical models for which faulty decisions are taken from. Another reason may be due to operational characteristics of variables in the model that does not properly capture the real financial scenario it modeled.

2.5 Researches in Social Sciences

Modern approaches to important research topics increasingly reflect multi-disciplinary research capacity and research teams. Currently, several areas of research in social sciences create significant extramural funding opportunities: cognitive science, brain imaging, sociology, and some economic specialties including transportation, development, and health economics. Increasing grants in these areas, along with new and emerging areas, will contribute to major research programmes, but also generate the need for added infrastructure support in grants management, grants writing, and technology management [3,4 &5].

Other issues of research interest are on inflation, population with associated immigration, emigrations, refugees and displacement of persons'. R&D programmes on conflict management, sociology, psychology and other forms of anthropological studies are being promoted and supported in many countries.

Many models are available that should be introduced to our young social scientists and such models can also be simulated on the computer and outcome can be used for effective decision making.

2.6 Education and Human Capital Development

Education plays major role in human development in equipping people with basic skill to read and write. The United Nation has developed several strategies to put the Dakar Framework for Action on Education for All (EFA) into practice ([5]).

In order to meet-up with EFA developmental goals, many countries introduced Skill acquisition programmes into their educational plans. Many research problems are on skill acquisition to inculcate the people with the basic skills that would make them relevant in the main stream of workforce of the nation. Unemployment and underemployment crises are abounding because of the fact that people do not have basic skills that would make them employable or even earn good wages.

Furthermore, most of 21st century problems require sophisticated skills and research tools to study them. Many Academic Institutions and Research Centres worldwide are introducing Graduate and Research Programmes to develop high level manpower to solve the societal problems. This trend will continue even beyond the 21st Century. Modern society must be composed of people who are mathematical modelling and simulation literate or leaders who will not take decision which is not based on the outcome of modelling or simulation.

3 Other useful areas for Mathematical modelling

There are several areas where Mathematical modelling can be useful, here are some of them:

- Development of nuclear weapons or use in simulation nuclear plant for civic purposes.
- Biotechnology: in the research on genetic modified food, genome and forensic research.
- Space exploration and analysis satellite image system to study the climatic changes and topography of places, to study the movement of insects, birds, and animals' encroachments.
- Can used to forecast the movement in earth crust or geo-seismic analysis.

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The role of Space Technology in addressing Global Climate Change Problems

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1. Introduction

For many centuries, humans have tried to change the weather for one reason or the other. For example, people have used conventional and non-conventional means to: seek for more rains, stop rains (e.g. the 2008 Olympic Games in China), reduce heat intensity, and to warm up things when the weather gets too cold. Rarely have we tried to deliberately change our climate, but we have unintentionally changed our climate over time. Our planet is unique and different from other planets because it provides suitable environment for human habitation. This is possible since the earth's atmosphere keeps the planet warm. Without the warming cover of natural greenhouse gases (primarily carbon dioxide- CO_2 , and water vapour), life cannot exist on Earth. However, through the anthropogenic driven release of greenhouse gases such as CO_2 , methane, Chlorofluorocarbon (CFCs) and Nitrous Oxide (N_2O) (which account for almost 99% of the total greenhouse gases responsible for climate change) our climate has continue to change.

Climate change is a change in the "average weather" that is experienced over time. Average weather includes average temperature, precipitation and wind patterns. Climate change involves changes in the variability or average state of the atmosphere over durations ranging from decades to millions of years. These changes can be caused by natural events driven by dynamic processes on Earth or, external forces including variations in sunlight intensity.

Changes in climate can also be attributed to human activities. Most of the discussions on climate change are focused on the anthropogenic drivers rather than the natural causes. Nevertheless, how fast and where exactly is climate changing is still controversial. But there is consensus in the scientific community that the consequences now and in the future may be serious. For example, the expected rise in sea levels may threaten islands and nations with low coastlines such as the Nigerian Coastal area; changes in rainfall levels and patterns may affect natural vegetation, agriculture and forestry.

The loss of biodiversity may be accelerated if climate zones move so fast that species (e.g. in rain forests) cannot follow them; weather anomalies such as hurricanes may occur more frequently, causing immense damage to humans and their property, and to nature. Yet, all the possible consequences of climate change are not fully understood. For instance, it is uncertain to what extent greenhouse gas-induced disturbances of the ocean-atmosphere equilibrium contribute to altered global circulation patterns such as the *El Niño* phenomenon; second, it is not clear whether the gulf stream, Europe's central heating system, could change its direction and/or intensity, thus leading to a drastic cooling of Europe's climate-a phenomenon known as abrupt climate change.

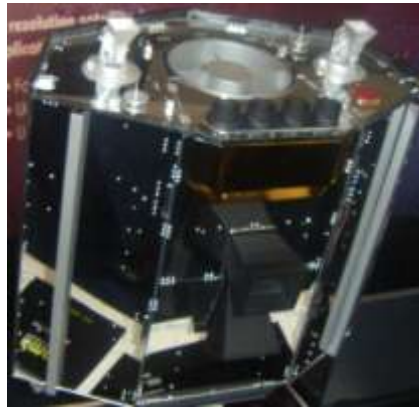
Nonetheless, for a comprehensive understanding of the complex inter-related components of climate change, access to reliable data and their transformation into information are essential. To achieve this goal, space technology has provided cost effective and reliable approaches. What makes artificial satellites developed from space technology so valuable in this context is their unsurpassed coverage and homogeneity of observations across the globe. As a result, earth observing satellites have been orbiting the globe and taking the climate pulse of the planet for over three decades. The measurement of such environmental variable is necessary if the human-induced climate change problem plaguing our planet is to be sustainably addressed. Consequently, the role of space technology in addressing climate change problems cannot be over emphasized and they include: Climate Modeling, Air Pollution, Ozone layer depletion, and Forest Resource depletion.

2 Space Technology and Climate Models

Our climate is a result of the complex interactions between the atmospheres, cryosphere (ice), Hydrosphere (oceans), lithosphere (land), and biosphere (life), driven by the non-uniform spatial distribution of incoming solar radiation (Stute et al. 2001). Consequently, Climate models are systems of differential equations that are based on the integration of a variety of fluid dynamical, chemical, and sometimes biological equations. Climate models estimates the various components of climate. For example, atmospheric models calculate winds, heat transfer, radiation, relative humidity, and surface hydrology within predefined grid and evaluate interactions with neighboring points in the grids. Our knowledge of climate change is based on results from climate models. Climate models retrace our Climate back several decades ago. They also predict what our climate will be in the future.

There are three groups of climate related models. They range from zero-dimension models to multi dimension models. Zero-dimension models include simple models for estimating climate related variables. Examples of such models are models for estimating the radioactive equilibrium of the Earth; and models used to estimate the effective earth emissivity of long wave radiation emitted to space. A second group of climate models are the one-dimension models. One of the commonly used one-dimension model is the Energy Balance Model (with latitude as the dimension). Most energy balance models are not global models, but zonal, or latitudinal, models. The third group of models is the higher dimension models, including Earth-system Models such as the Global Climate Models or General Circulation Models. The role of space technology in climate models involves the acquisition of spatial datasets required for the numerical estimation of the models. Satellite data provide the much needed datasets for estimating climate models. Data from satellites are particularly attractive when global coverage datasets are required. It is economically challenging to acquire such datasets using terrestrial approaches.

A typical example of satellite application to climate models is the Satellite Application Facility on Climate Monitoring (CM-SAF) programme (Schulz et al. 2005). CM-SAF is a joint project of the meteorological services of Belgium, Finland, the Netherlands, Sweden, Switzerland, and Germany. One of the major goals of CM-SAF is to support the climate modelling communities by the provision of satellite-derived geophysical parameter data sets. CM-SAF provides data sets of several cloud parameters, surface albedo, radiation fluxes at the top of the atmosphere and at the surface, atmospheric temperature and water vapour profiles as well as vertically integrated water vapour (total, layered integrated) (Kaspar 2008). The datasets produced by CM-SAF are derived from measurements of the SEVIRI and GERB instruments on the geostationary Meteosat Second Generation satellites as well as from AVHRR, ATOVS and SSM/I instruments on the polar orbiting NOAA and DMSP platforms, respectively. The products from Meteosat cover the full Earth disk that extends from South America to the Middle East, with Africa below the satellite and Europe to the top. AVHRR derived products cover Europe and the East Atlantic. SSM/I (over ocean only) and ATOVS products offer global coverage. The data sets cover different time periods depending on the availability of the individual sensors utilized.



- Fig.1: Proposed launch of high-resolution Earth Observation satellite (NigeriaSat-2 and NigeriaSat-X) in 2009, with capacity building and knowledge acquisition by 25 Nigerian Engineers and Scientists.

3 Air Pollution

The term "air pollution" is used to describe substances that are artificially introduced into the air. Air pollution is the result of gases and airborne particles which, in excess, are harmful to human health, buildings and ecosystems. Within the context of climate change, the concept of greenhouse gases was presented in the introductory section of this paper. The gases are pollutants contributing to climate change. This is because the earth's atmosphere acts much like a giant greenhouse. The gases allow solar radiation (heat) to pass through the atmosphere but, after it is absorbed and re-radiated by the earth, the gases prevent this heat from escaping back into space.

Under natural circumstances the phenomenon of greenhouse keeps the earth warm enough to support life (as mentioned in the introduction). However, current conditions are far from natural. It is becoming ever clearer that average temperatures and sea levels are rising and climate changes are occurring as a result of the global warming induced by greenhouse gases such as carbon dioxide and methane emitted into the atmosphere through the activities of humans. It has been pointed out that if the situation remains as it is, extreme climate change is likely to occur within a few centuries. Since the beginning of the Industrial Revolution, when humans began burning fossil fuels on an unprecedented scale, greenhouse gases have steadily been piling up in the atmosphere. Many of these gases last far longer than a century.

Current carbon dioxide (CO₂) concentrations are now 35.4% higher than pre-industrial levels and growing rapidly. They are now far above any level in the past 650,000 years. Likewise, methane (CH₄) concentrations have more than doubled too far above anything seen in the past 650,000 years. Global emissions of all greenhouse gases have increased by 70% between 1970 and 2004. The consequence of all this is that more and more heat is being trapped in our atmosphere, leading to an "enhanced greenhouse effect". Satellite technology has been deployed to monitor and observe the concentration and increase/decrease in greenhouse gases at various locations throughout the world.

The major strength of the satellite program is that it overcomes the lack of sufficient greenhouse gases ground observation stations which are strongly geographically biased. An example of such satellite program is the Japanese Japan Aerospace Exploration Agency's (JAXA) Greenhouse Gases Observing Satellite (GOSAT) (to be launched in January 2009). GOSAT is an artificial satellite that observes the concentration distribution of greenhouse gases from outer space, and its purpose is to contribute to the international effort towards the prevention of warming, including monitoring the greenhouse gas absorption and emission state. Similarly, scientists have for the first time detected regionally elevated atmospheric carbon originating from manmade emissions using data from the SCIAMACHY instrument aboard European Space Agency's Envisat environmental satellite (Science Daily 2008). The findings show an extended plume over Europe's most populated area, the region from Amsterdam in the Netherlands to Frankfurt, Germany. Nevertheless, significant gaps remain in the knowledge of carbon

dioxide's sources, such as fires, volcanic activity and the respiration of living organisms, and its natural sinks, such as the land and ocean. It is believed that satellite technology will play a key role in filling these gaps in the very near future.



Fig.2: How Does Greenhouse gases cause Global Warming?

Current carbon dioxide (CO₂) concentrations are now 35.4% higher than pre-industrial levels and growing rapidly.

-Increased by 80% between 1970 and 2004.

4 Ozone Layer Depletion

The ozone layer is a concentration of ozone molecules in the stratosphere. About 90% of the planet's Ozone is in the ozone layer. Stratospheric ozone is a naturally-occurring gas that filters the sun's ultraviolet (UV) radiation. A diminished ozone layer allows more radiation to reach the Earth's surface. For humans, the overexposure to UV rays can lead to skin cancer, cataracts, and weakened immune systems. Depleted ozone layer can increased the earth surface exposure to Ultra Violet radiation which can results in reduced crop yield, disruptions in the marine food chain and increase in global surface temperature. There is increasing evidence that elevated UV radiation has significant effects on the terrestrial biosphere with important implications for the cycling of carbon, nitrogen and other elements. Increased UV has been shown to contribute to climate change by inducing carbon monoxide production from dead plant matter in terrestrial ecosystems, nitrogen oxide production from Arctic and Antarctic snow-packs, and halogenated substances from several terrestrial ecosystems (Zepp et al. 2003). The depletion of the ozone layer has been monitored using satellite technology for over two decades. Satellites measure ozone over the entire globe every day, providing comprehensive time series data. In-orbit satellites are capable of observing the atmosphere in all types of weather, and over the most remote regions on the Earth surface. They are capable of measuring total ozone levels, ozone profiles, and elements of atmospheric chemistry. For instance, the Total Ozone Mapping Spectrometer (TOMS) aboard Nimbus-7 and Meteor-3 satellites provided global measurements of total column ozone on a daily basis. Datasets from the two satellites have provided a complete data set of daily ozone from November 1978 - December 1994. After an eighteen month period when the program had no on-orbit capability, ADEOS TOMS was launched on August 17, 1996 and provided data until June 29, 1997. Earth Probe TOMS was launched on July 2, 1996 to provide supplemental measurements, but was boosted to a higher orbit to replace the failed ADEOS. Earth Probe continues to provide near real-time ozone data.

5 Monitoring Deforestation

The forests are very important sink for carbon. They absorb CO₂ from the atmosphere and thus regulate

The existence of the major greenhouse gas (CO₂). As a result, deforestation exacerbates the climate change crises. However, space technology has been very useful in assessing global deforestation. The concept of mapping deforestation with the use of satellite data has improved our knowledge of the rate of deforestation in different parts of the world. Tropical forests (such as the forests in Nigeria) have large leaf area and very dense canopy. On the other hand, non forest areas have greater visible reflectance mainly due to the spectral contribution of the soil. The characteristic spectral response of the tropical forest cover enables its separation with other land use classes using optical data. Classification techniques are used in satellite images, which are acquired on different dates, in order to identify changes and to map the deforestation.

The most broadly used sensors for tropical deforestation mapping at local and regional scales are NigeriaSat-1, Landsat MSS, TM, ETM+ and SPOT. Similarly, for continental studies, coarse resolution data are often used (e.g. NOAA AVHRR). Medium spatial resolution data are capable of producing locally accurate results, but with challenges in quality due to cloud cover. On the other hand, coarse resolution data with high temporal resolution can produce relatively cloud-free scenes, albeit with limited accuracy in local level. Thus, to achieve a better classification, medium resolution data can be used for correction and validation of coarse resolution data. This methodology is preferred in global scale inventories. For example, the Tropical Ecosystem Environment observation by Satellite (TREES) project set up by the European Commission and the European Space Agency in 1990, used NOAA AVHRR 1km data. A sample of selected Landsat TM data was used to correct and validate the AVHRR data classifications. However, optical images are limited by cloud cover, which is a ubiquitous problem in the tropics and causes satellite image data gaps. These gaps can be filled by using SAR images; Radar signals can penetrate the clouds. As a result, the National Space Research and Development Agency (NASRDA) is in the process of acquiring radar images for research studies in the Niger Delta.

6 Conclusion

The role of space technology in addressing the global climate crisis cannot be over-emphasized. Our understanding of the global climate crisis has been largely driven by the availability of climate models. These models require substantial datasets that are seldom acquired using terrestrial approaches. As a result, satellite based methods have made immense contribution to data challenge, in particular, at the global scale. In view of the great role that satellite technology plays in addressing global climate issues, the National Space Research and development Agency launched the Nigeria Sat-1 environment and hazards satellite. The data from the satellite has been used to address some climate related problems in the country such as Deforestation, Air Pollution modeling, Ecosystem change assessment, and early warning system for desertification.

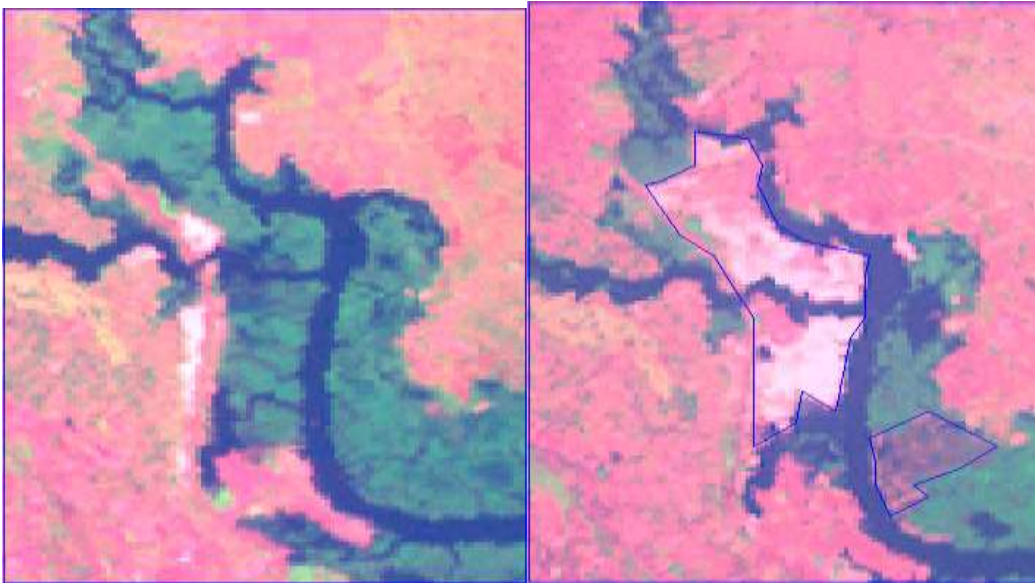


Fig 3: 1986(Baseline)

Fig 4: 2003 (Assessment)

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A Random Walk through Mathematical Sciences with Some Hints on a Model-Based Approach to Capacity Building in Developing Economies – Par II Discussions

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Abstract

In this paper we re-examine the *paradoxes* in single versus multidiscipline-based academic career development and the implications of the CA model explored in part I for *continual productivity* in career life-spans of academics and knowledge workers. The thrust of the discussions is on how the interactions among the PRD, PAD and GCL domains of work in *academic business* engender career impacts and possibilities that contuse to a mass professionalization of mathematical sciences in knowledge work, enhanced capacities for knowledge transfer, entrepreneurial education, creation of enabling technologies, discourse, pedagogy and practice of mathematical modelling, research directions and action plans linking these understandings to *capacity building* for *economic development* of Nigeria and similar developing countries, especially in Sub-Sahara Africa.

Key words:

Mathematical modelling, optimization of human potential, academic entrepreneurship, capacity building, economic development.

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1 Introduction

We expatiate on the ideas surrounding most of the key concepts encountered in Part I of this paper system. We do this in such a way that the concepts not only explain the deep interactions among the three corporate academic (CA) domains in the production of academic knowledge, but also how this enhanced productivity contributes to capacity building and economic development in Nigeria and sub-Sahara Africa, for example. Recall that the central use of the CA model is that it enables senior academics and higher educational institutions to train up beginning academics in the *art* and *science* of *academic business*. This training facilitates their career progress.

The novelty of this research is that whilst some form of academic mentorship clearly happens in higher educational institutions, the approach is rather informal and does not use a transactional mechanism that connects all aspects of the diverse range of activities that

academics undertake in modern university settings. For example, PhD supervision of young would-be academics necessarily focuses on getting them to acquire the PhD, not to excel in all the activity domains which the CA model talks to. This is why using the model as a basis for formal training of people in knowledge work is especially important in developing countries, as CA players learn the skills requisite for fast creation of wealth in those economies. We can recall the statement which echoes these ideas from the lived experiences of the case scientists studied in Part I as follows:

*'Most species of animals have developed procedures aimed at teaching its young the secrets of survival, a sine qua none of 'success'. How well do we do this in research? Do we explore the approaches to knowledge production with our students, the nature of meaningful knowledge and how best we succeed in all these? Is it not more the case than not that we operate an inner wheel of success and sometimes leave observers of our progress bewildered at our 'prodigious' rate of production of ideas, papers, products and services? The CA model is an attempt to unearth those processes and offer **a** way to enable new CAs understand what successful others typically do. We say **a** not **the** way because success strategies necessarily differ from player to player, but there should be some common standards that define success in **academic business**, somewhat. Exploring such standards is a useful way to start conversations about what success is like in knowledge work'.*

The rest of the paper is as follows. Section 2 offers a concept-based understanding of why the model enhances career progress of beginning and mature academics. The other sections also adopt a **concept-based approach**: section 3 on how the model facilitates a mass professionalization of the mathematical sciences; section 4 on enhanced capacities for knowledge transfer; section 5 links the model and entrepreneurial education; section 6 on how the model leads to the creation of an enabling technology that is suitable for training knowledge workers; section 7 on using the model to facilitate the discourse, pedagogy and practice of mathematical modelling; section 8 on further research directions. Section 9 concludes the paper with emphases on action plans linking the model to capacity building in developing economies.

2 A concept-based understanding of why the model enhances career progress of beginning and mature academics

The model demystifies the **paradox** of single discipline focus in academic careers in favour of a measured cross- or multi-disciplinary approach, structured around the three domains. We have seen that the **career scripts** of proven academics in the case stories covered in Part I support this thesis. The benefits of this reconceptualization of the true nature of value-adding academic work include: the ability of academics to multi-task key wealth creating activities at an early stage in their careers; to be **continually productive** by exploiting the **interactions** among the domains; getting their learning and teaching, if based on a CA model-driven curriculum, to talk to societal needs as they blend **deep theory** with **deep practice**; hence, enabling a locust effect in socio-economic development of a country to happen, through the intellectual efforts of so many graduates who imbibe entrepreneurial skills innate to the model.

The fact that the model expressly motivates CA players to see their academic work in literally business terms and hence use **business modelling**, **strategic planning** and **excellence frameworks** such as the balanced scorecard to create success plans is simple but revolutionary, Ezepue (2005, 2006). This is because people usually do not apply the same rigour in managing a formal organization as they do in their personal life-worlds; the CA model therefore ensures that such rigour is exercised in the away academics (and other knowledge workers) perform their roles.

Hence, a CA player creates and maintains **academic identity** in a balanced PRD-PAD specialist area of work, sharpens her **attitudes** and **productivities**, exploits domain interactions, with affordances of **nonlinearities** and **increasing returns** on academic effort, and skillfully professionalizes her academic knowledge, through effective creation of products and services (**CA artifacts**) of interest to society.

An example of profoundly enhanced academic productivity facilitated by the model is the fact that a CA player through the **CA diary system** is in general far more reflexive about the contingencies around career progress than one who does not use the model. The player, therefore, easily creates **topic banks** for her personal research and that of her research students and **project banks**, with projects or proposals that translate technical knowledge gained in the specialist areas of work to results and interventions that are valued by client publics. Some of the CA players that have used the model have succeeded in redefining academic work through their ability to present **paper systems** which exploit technical results more deeply than traditional academic papers.

Clearly, such types of rather more complete works would seem to promote the socio-economic development efforts of a country better than the traditional papers mainly used in disseminating basic theoretical research. We know that the wide range of journals in the world offer some scope for publishing this genre of papers, but they are not common. We are therefore tempted to suggest that the community of academics and professionals in developing countries found a journal devoted mainly (but not exclusively) to publishing complete works that offer more scope for deeper discussions of the implications of research results for resolving identified problems of societal significance. This will be complemented by longer review papers that address those problems. An example worth considering is *Journal of Interdisciplinary Research and Innovation in Mathematical Sciences (JIRIMS)*, which can be hosted by the National Mathematical Centre, Abuja, Nigeria.

On the **transactional nature** of the CA model and its capacity to prompt **reflexive thinking** about opportunities, consider Figure 3.1 in Part I of the paper system. A CA player can use such visualizations of career success to interrogate her performances along a number of key performance indicators. For example, only two such indicators are shown for simplicity – numbers of journal and conference papers produced in a year – you can see that the number of journal papers is flat while that of conference papers is steeply rising. The CA player thus recognizes that action must be instituted in that year or the following year to convert the conference papers to high-impact journal papers. In a more realistic visualization of the work effort, we can juxtapose quite a richer variety of indicators, also carefully targeted in the model to cover all key aspects of a stellar academic work e.g. quality of stakeholder services (including lecture notes and teaching styles, with students conceived as internal stakeholders).

A good approach would be to indicate on the plot estimates of all the intensity parameters associated with the model e.g. those for citations, PG supervision, ambition, happiness and networking as well as the year-to-year changes in the parameters. For example, seeing that in a previous year a CA player's intensity of useful networking was **0.40 (40%)** compared to **0.15 (15%)** the year before shows a remarkable improvement. Tracking the changes in these parameters (in this case **25%**) reveals to force or momentum of improvement in academic business.

We can see that getting the promotion ideals to mirror this sense of overall quality and balance in all types of academic activities is necessary to avoid such situations whereby some academics could sweat the PRD activities on their way to a professorship at the expense of their teaching quality, because of the current over-emphasis on research output in academic career success.

The transactional nature of the model facilitates **extended intuition** around the work flow via **richer metrication** as discussed here and also metaphors and analogies drawn from similar high performance in other fields e.g. sports, banking, executive management, and birds foraging for more abundant sources of food, etc. The model induces in a player a crafting of collaborative networks with other players in the three domains, that is intelligent and value-seeking, see Ezepue (2007).

3 How the CA model facilitates a mass professionalization of mathematical sciences (in developing economies)

We can see that getting academics trained up in CA work planning and execution is akin to a train the trainer initiative since the academics propagate the benefits of the model to their students. This is in our view the starting point of action planning for spreading best practices mandated by the model across all the higher educational institutions in Nigeria and other developing countries. We may call the **project** something like:

New Approaches to Professionalization of Mathematical Sciences: Using a Corporate Academic Model to Train Nigerian Academics in Knowledge Work

Managing, Measuring and Improving Performance in Academic Business: Using a Corporate Academic Model to Train (Nigerian) Academics in Knowledge Work (Project MaMeIPAB)

As in the second title above, this project can be adapted to other disciplinary clusters e.g. social sciences, humanities, environmental sciences, and to particular fields within these clusters e.g. history, physics, mathematics, sociology, economics, real estate management,

business and management. One of the author's PhD students is currently applying the model to real estate management; the research topic is *The Pedagogy and Practice of Real Estate Management: Entrepreneurial Perspectives* and could well have been framed as *The Pedagogy, Practice and Entrepreneurship of Real Estate Management: Implications of a Corporate Academic Model*.

In addition to training Nigerian academics as argued above, the model can be used as a framework for embedding related skills in the curricula of higher educational institutions in Nigeria. Indeed, the model links the **core curriculum** of a discipline to other subjects depicted as electives, in a way that enables the learner to see how these fields actually combine to generate new knowledge – the kind of knowledge that adds real value to organizations and wider society.

For example, the core curriculum in statistics as stipulated by Nigeria's National University Commission (NUC) Minimum Guidelines involves compulsory courses in statistics and mathematics taken in the first two years, with the third and fourth years devoted to upper undergraduate statistics courses such as Statistical Inference, Bayesian Inference and Nonparametric, Sampling Theory, Experimental Design, Operational Research, Time Series Analysis, Multivariate Statistical Methods, Design and Analysis of Experiments, etc. There are electives in Economics (up to the second year) and traditionally the core sciences e.g. Physics, Chemistry, Biology (typically in the first year), with compulsory general studies elements e.g. Social Sciences and Humanities. There are academics appointed to help new students as academic advisers who use their experiences on the technical contents of these courses and their potential career implications to guide students in choosing the electives.

It is clear from this paper that what these advisers do by intuitive know-how could far more effectively be achieved on the basis of the systemic links among the various subjects, if they are roughly grouped in the three CA domains. For example, all core statistics and mathematics in the PRD, the key electives e.g. economics in the PRD and the GS courses (plus other subjects that the student could be advised to read up in self-learning) in the GCL. The interactions among these domains as explored in the model will yield deeper insights about the choices open to the student. The student that chooses economics then knows that a career blending statistics and economics could be sustained; this motivates the choice of even more economics electives in second year instead of a scatter-gun approach to electives, determined mainly by perceptions about the relative ease of excelling in the examinations on the subjects. A national project that can be created to achieve this aim could be something like:

Curriculum Renewal of the Mathematical Sciences in Nigeria Using Insights from the Corporate Academic Model (Project CUREMAS)

An example of this project is currently being implemented in the Mathematics, Statistics and Operational Research (MSOR) subject areas by the author.

Finally, it will be useful to disseminate the CA model ideas and how they inform production of all kinds of **disciplinary knowledge** in a series of national conferences, seminars and training workshops, as argued in Ezepe (2008). Indicative titles for these activities are listed below:

- (Inter)national conferences, seminars and training workshops on model-based human potential improvement projects that will enable Nigerian and African academics, students, researchers and knowledge workers to achieve excellence and academic entrepreneurship in research, teaching and consulting, at individual staff, research teams, research institutes and overall institutional levels (**Project CA-CON**). Some topics that could be covered in these activities include: *Producing Mathematical Sciences Knowledge: Using the CA Model to Leverage Mathematical Sciences Knowledge in Economic Development*. Similar topics can be explored for business and management, social sciences, humanities, engineering and environmental sciences, for example. Ezepe (2008) suggests that a five-part framework could be used to draw contributions from different disciplines into such integrated conference and training activities. For statistics as a subject area, the five-part framework is as follows:
 - a. The statistical researcher as producer of statistical knowledge
 - b. Approaches to knowledge production using insights from the CA model
 - c. From data to theory or data and/or in theory
 - d. Co-producing statistical and management knowledge (biological, economic, social science) knowledge – in which case statistical science is a PRD and these other fields are PADs
 - e. Statisticians as academic entrepreneurs – theory, success strategies and cases.

- (Inter) national intervention programme on functional education in (mathematical sciences) different disciplinary clusters via Curriculum Realignment, Institutional and Structural Invigoration and Sanitization (CRISIS) solutions in (African)Nigerian higher educational institutions (HEI) to be based on the CA Model (**Project CRISIS**).
- National programme for producing a new genre of CA-inspired **super books** which embed African identity and consciousness in new national education models, with strong emphasis on socio-economic development and cultures of sub-Sahara African countries (**Project SUPER-BOOKS**). A super book is a book or research monograph that is for good reason unorthodox in the sense that it is far more encyclopedic (around a well defined subject anyway) than a traditional text. Such a text aims to develop an understanding of **deep praxis** – a term we use to refer to attempts to explain the full theoretical ideas underpinning a subject, discuss fully the practical applications of the ideas, illustrate those applications in cogent problem contexts, using a blend of case studies, vignettes, street stories and conversations that document the lived experiences of recognized experts and practitioners in the field, and outline the emerging research directions in the field. The aim is to equip the learner with the skills that enable her to hit the ground running as far as work in the subject area is concerned. A super book is intrinsically multidisciplinary, since inculcating the art, science and skills of/related to a subject in learners involves an exploration of the connections between the subject and cognate subjects or skills. It is clear that developing super books is not a primary interest of traditional publishing firms, since the books are too big to be produced at profit. We suggest therefore a national programme for producing such texts in Nigeria and developing countries, complemented by funding from foundations, charities and private sector stakeholders. An example of a super book that comes to mind is Penrose (2005). We do not at the moment know whether a text planned to be developed using papers from this international conference on mathematical modelling of global challenging problems as hub (titled *The New Mathematical Modelling for the 21st Century: Global Challenging Problems and Development Perspectives*) will turn out to be a super book, but feel that the diverse range of topics covered and their relevance to developing economies are vital ingredients for sculpting such a book.

The nature of skills development that can be embedded in a super book is further explored in Ezepue (2006 and 2008) and Ezepue & Mwitondi (2008).

4 The CA model and enhanced capacities for knowledge transfer

We are satisfied that the CA model facilitates effective **knowledge transfer** among international institutions and Nigerian institutions, and particularly links Nigerians in Diaspora with their home-based professional colleagues. For instance, it is useful to structure knowledge transfer programmes in **transnational education** and **research** around the strategic economic goals engendered by the CA model, and using the range of concepts native to the model.

As an example, the author uses the CA model as a framework for implementing projects deemed suitable for enhancing the pedagogy and practice of financial mathematics at the National Mathematical Centre, Abuja, Nigeria. These include curriculum design and implementation of a **world-leading integrated programme** of training of middle to senior level quantitative finance academics and professionals in high finance, via MSc/specialist MBA/PhD programmes (**Project FINE** in Ezepue & Solarin 2008).

Using the CA model as a hub in project origination, design, execution and effectiveness in all these strands is reinforced by surrounding this hub with tested (project) management models e.g. a model used in the University of Warwick (and popularized by the IMD Business School), which considers three key pillars of actions and ideas – the structural aspects of the problem or engagement involved, understanding alternative partners that could contribute to the programme success and the ways they do things e.g. organizational culture and capabilities map, and the most effective strategy and tactics for programme success. We can create a portable tool that accommodates all these ideas for speedy completion of partnership projects in Nigeria and sub-Sahara Africa, a **tool** than can be called

5 An Integrated Model-Based Project Management Model (for Managing Complex Multi-Objective Transnational Education Projects in sub-Sahara Africa) (Project IMP)

Training **CA champions** in Nigeria and sub-Sahara Africa as discussed in the above projects foregrounds an understanding of this tool and helps to produce illustrative full case studies of IMP projects, which are mapped to key economic development goals of the continent e.g. Millennium Development Goals (and Nigerian NEEDS), including wealth creation, poverty alleviation, gender emancipation (female empowerment), etc

A similar reason that the model facilitates knowledge transfer at individual staff levels in organizations is the fact that a coherent model for structuring (doctoral) research projects can be built around the model. This model ensures effective on-time supervision of the topics, with deeper controls on topic origination and development of proposals, such that sufficient depth ensues with clear links among research outcomes, objectives and questions, which make most of the projects theoretically and practically value-adding.

The author of this paper has trialed such a model in up to eight PhD research topics and suggests the need for further work to turn this experience into a tool for facilitating modern research supervision, to be called the Corporate Academic Research Structuring Model (**Project CARESS**), which can also be software enabled with the same name for the software.

The importance of having a unified model that successfully manages the arduous task of producing capable researchers in good time in Nigeria is revealed by experiences that show PhD students spending up to ten years before completing their work in some cases. In another case an MSc student in mathematics spent six years. We have held intimate discussions with senior academics which show that some of them struggle with issues of **depth**, **originality** and **significance** in thinking about and outlining doctoral work, issues which are rather easily resolved by the CARESS framework. As a result of these experiences we suggest the convening of a national conference on modern research supervision in Nigeria with indicative title

National Conference on Masters and Doctorates in the 21st Century: Training Research Supervisors in Various Disciplines (e.g. Mathematical Sciences, Finance & Economics, etc) (Project 21st CENTURY RESEARCH SUPERVISION).

6 The links between the CA model and entrepreneurial education

We argue in this section that the CA model is foundational for a reconceptualization of **entrepreneurship education** in such a way that both the formal and informal aspects are promoted, with profound implications for national development. We have structured a doctoral research work on these ideas titled

Formal and Informal Models of Entrepreneurship Education: Implications for Socio-economic Development of Nigeria.

7 Brief summary of the work entailments

Much more focus has been placed on formal entrepreneurship education of the types taught in business schools, which emphasize skills underpinning business start ups, for example. It is, however, clear that entrepreneurial skills and mindsets can manifest informally in situations when the skilled workforce of a nation are intrapreneurial within their employer organizations e.g. private sector firms or government departments. The study of these informal aspects of entrepreneurship is now beginning to attract the interests of entrepreneurship scholars and organizations in the UK and developed world in general, but this is not the case in Nigeria and most developing countries of Sub-Sahara Africa.

It is ironically this form of entrepreneurship that, if well grounded in the *curricula* of the higher education institutions, *across disciplines* other than just business and management fields, will lead to a *mass skilling up* of the workforce of these nations. The result will be an improved capacity of individuals *at work* or in *self employment* to succeed, create wealth or add value in their various life-worlds.

This research will, therefore, examine the *formal* and *informal* models of entrepreneurship education in *Nigeria*, draw insights on how these models can potentially be improved to achieve the aim of up-skilling Nigeria's citizens and workforce, and particularly develop the informal aspects which could be accessed by a wider segment of the population e.g. recipients of the microfinance funding introduced in the country, academics in different disciplines, professionals in banks, communications industry, etc.

7.1 Indicative plan of study

- a. Initial consolidation of current knowledge in entrepreneurship, especially on the formal aspects, systematization of the entire work, with emphasis on the informal aspects, and production of a final layout of the thesis; this stages uses the Corporate Academic Research Structuring Model (*CARESS*).
- b. Implement a *master literature review* on all aspects of the work, mainly a critical re-appraisal of existing literature and new developments in entrepreneurship thinking with regards to their affects on socio-economic development, and emphasizing the *informal (societal) entrepreneurship* frameworks. Examples include works on a template of (formal) entrepreneurial outcomes drawn up by the UK National Council for Graduate Entrepreneurship (NCGE) – entrepreneurial behaviour, attitude and skill development, creating empathy with the entrepreneurial life world, key entrepreneurial values, motivation to entrepreneurial career, understanding of processes of business entry and tasks, generic entrepreneurship skills, key minimum business how to's, and managing relationships – plus work on alternative (informal) models of entrepreneurship, which emphasize such dimensions as – personal capacity to behave entrepreneurially in different settings, understanding the globalized life world of greater uncertainty and complexity, and coping entrepreneurially with this flux in the contexts of working flexibly and creating wealth in small organizations, working similarly in larger organizations, navigating a fast changing labour market, achieving a fulfilling personal, family and social lifestyle, possibly setting up own business.
- c. Work on *selected cases* with (international/UK comparators) relevant to Nigeria's quest for socio-economic development e.g. *academic entrepreneurship* (informed by the CA model), which will map the entrepreneurship skills and behaviours of Nigerian academics to the models under research, to their capacities therefore to effectively develop similar skills in their students, and to the various national development agenda e.g. the Millennium Development Goals (MDGs), the Nigerian NEEDS (a national policy document for achieving the MDGs), the President Yar Adua's 7-Point Agenda, etc. The cases will also cover *professionals in practice* e.g. bankers, middle to senior administrative staff in government departments, *university students* and would be graduates in selected disciplines, and very *informal categories* e.g. moderately literate recipients of the microfinance funds. The comparator cases will inform a sense of what gaps exist in the Nigerian cases.
- d. Of particular interest in this research is the trialing of the *Corporate Academic Model* on a select sample of Nigerian academics in order to assess its *efficacy* in producing entrepreneurial academics.
- e. Write-up of the thesis chapters along the above lines and emphasizing *key themes (objectives)* of the research such as:
 - developing entrepreneurial mindsets in people in different life worlds;
 - developing entrepreneurial academics and graduates;
 - developing entrepreneurial universities and curricula;
 - linking all these to informal (societal) models of entrepreneurship, especially;
 - mapping these goals to the socio-economic development needs of Nigeria, including growth in wealth creation capacities and competitiveness of the workforce;
 - exploring in sufficient details the issues involved in so extending entrepreneurship education across and beyond universities; and
 - Considering the role of all stakeholders to the nation's development agenda in promoting these objectives.

7.2 *Benefits to Nigeria*

As detailed above, the benefits include a progressive extension of entrepreneurship education and capacity building of the nation's workforce and citizenry in order to accelerate **wealth creation**, **poverty alleviation** and **overall economic development** by a far larger proportion of the population than could be reached formally, and involving a wider range of stakeholders e.g. government agencies, universities and other educational institutions, private sector, civil society organizations and foundations.

8 **Future career of the research students (mostly existing academics in Nigerian universities)**

The novelty and topicality of these dimensions of entrepreneurship education considered in the research will prepare the researchers for interestingly useful careers in the area, both within the university, as with scholarly publications and projects, and in the society as with capacity building work, train the trainer initiatives to help develop other staff, and continuing collaborations with the author in maintaining cutting-edge understanding of the changing landscape of entrepreneurship education and practice, relevant to Nigeria's future economic development.

It is intended that textbooks on the topics will be developed towards effective dissemination of the research ideas. Memberships of such organizations as the NCGE, UKSEC, etc and equivalent Nigerian organizations, some of which will be formed to advance specialist aspects of the field considered in the research, will help the researchers to sustain a long-term career in entrepreneurship education and leadership in Nigeria and Sub-Sahara Africa.

9 **The CA model and the creation of an enabling technology for training knowledge workers in Nigeria (The Corporate Academic Career Optimization software)**

All we can say in this connection is that the model facilitates current work aimed at creating an **integrated software** that will enhance the way knowledge workers could be **trained**, **mentored** (or **coached**) to become hyper-productive in virtually all areas of **knowledge work**. Hints of the nature of such a software are provided in Ezepue (2007), but we are foreclosed by intellectual property rules from describing the detailed design considerations in an academic conference, publishing or public forums, for now. Versions or suits of the software will be created to handle other aspects of the model entailments e.g. the **research structuring** and **project management** tools. The import of this technology enablement in this paper is that it justifies the **originality** and **significance** of the CA model, in addition to all other gains articulated above. It also plays well with the suggestion in Part I of the paper system that, in situations in which this is possible, it is admirable to get our research to produce enabling technologies. It is clear that this competence is native to the CA model.

9.1 *The CA model and the pedagogy and practice of mathematical modelling*

As explored in Ezepue & Mwitondi (2008) it is particularly effective in teaching mathematical modelling (indeed every other subject), to be able to use a concept-based approach, built around the fundamental principles of the subject. See also Ezepue & Udoh (2006a & b) for more details on these ideas, including the use of **case studies**, **problem-based** and **project-based learning** (PBL) in achieving deep learning of a subject.

In this section we outline ideas on how this paper system can be used in a **concept-based learning** approach to teaching mathematical modelling to students of different levels of educational attainment – beginning undergraduate, senior undergraduate and masters students. For this, we get students to read the paper, produce a **concept map** (or **mind map**) which connects the concepts, according to their own understanding of **meaningful links** immanent in the highlighted concepts in the paper. It is advisable to let students do this as a **group activity**.

The lecturer also produces her own version of the concept map. **Conversations** are generated around issues revealed by the concept maps e.g. why the various groups of students produce the maps the way do, what they perceive as the key relationships in the map, the implications for learning about and understanding mathematical modelling, the links between **modelling** and **development**, their perceptions about the relevance of the CA model to their education, and so on. The lecturer takes a back seat in these conversations, which will naturally follow a **group presentation** of the maps produced by the groups. The idea is that the lecturer **facilitates** learning and serves as a clerk who records the insights that emerge from the students' discussions.

Following these exercises the lecturer (may be in another lecture) sums up the arguments and insights and expands on these insights with a **deeper reflection** on the labyrinths of mathematical modelling, the habituations of mathematical scientists who use models to mirror reality, philosophical underpinnings of the modelling approaches, the traditional progression of mathematical thinking, as revealed in the **lived experiences** of mathematical scientists discussed in Part I of the paper system, how the students can work to internalize the **best practices** e.g. the formulation of strong forms of a model and its **extension** to wider categories of phenomena via, say, a relaxation of the **model assumptions**, and other perspectives that combination of insights from the students' and lecturer's work suggest as important.

It is also useful to get the students to develop **individual reports** that summarize their understanding of the entire exercise. This forces a deeper reflection on the modelling process. These reports will be part of the **course work** in the course, which should typically account for a significant proportion of final marks awarded to the students, compared to examinations, in order to command students' attention. In some cases, especially in work-based learning situations such as distance learning graduate programmes, it is desirable to assess real learning through **100% course works**. For senior undergraduate and graduate students, it is apposite to extend the individual report to include related literature on the **philosophy of science**, and issues in learning, teaching and assessment (LTA) of mathematical sciences (or modelling specifically).

This summary of the ways the paper can serve as a **case study** on teaching mathematical modelling shows that a deeper approach to LTA work in mathematical sciences is quite feasible and desirable, but requires a re-skilling of the academics themselves in these matters. We have highlighted in bold italics the concepts that underpin such an attempt to train the academics and hence propagate this **constructivist approach** to the pedagogy of mathematical sciences – constructivist in the sense that the learners learn by constructing own meanings as well as discussing **fundamental concepts** of the subject, while the lecturer refrains from being the **sage on the stage** in favour of serving as the **guide by the side**.

We can see that this deeper approach to teaching mathematical sciences provides affordances of generic skills (presentations, collaborative working, reflective thinking, etc) to the students, and enables them to develop a fuller understanding of the subject matter. Getting these practices embedded in a **new national education programme** is vital to producing a creative and **productive workforce**, a goal that impacts the **socio-economic development needs** of Nigeria and developing economies, especially.

10 Further work

There are a number of directions in which we seek to extend the research on the CA model and its entailments for career progress in knowledge work. We summarize some of these research themes as follows:

Work on the research theme *Model-Based Optimization of Human and Academic Potential: Interfaces among Individual Performance, Organizational Excellence and Decision Making (Project MOHAP)*: this research explores ideas useful for 'problematizing' and modelling human and academic performance, for different niches of knowledge work; understanding the constraints that limit career progress and productivity and mitigating them via the CA model or accounting for the constraints in the model; understanding the social psychology of work and human performance and hence improving available range of career counseling tools suitable for high-performance career settings; using the implications of the belief, desire and intention (*BDI*) theory of social action to improved the model's predictive power and relevance in training knowledge workers; using *career theory* as appropriate (see Arthur et al eds 1996) to improve the career optimization model; and mapping links among evolutionary psychology, cultural intelligence and integrative learning and CA model-enabled improvement of human and academic performance.

In addition to focused research papers on these dimensions, the following related papers are planned to flow from this theme:

- a) *On the algebra and geometry of idea generation, problem origination, structuring and solutions: insights from the CA model;*
- b) *Achieving intelligence in mathematical research via blended modelling: heuristics, problem spaces, solitons, interestment and identification of feasible solutions.*

All these ideas will be pursued as part of project MOHAP for which funding will be solicited from appropriate sources, under the auspices of the National Mathematical Centre, Nigeria.

Related work on *New Directions in Academic Innovation, Creativity and Entrepreneurship (Project NeDAICE)*: this study aims to research the theoretical connections amongst academic entrepreneurship (individual, team and organizational) innovation, creativity, human resource management and analytics and career theory, based on the CA model. The objectives include:

- a) Improving the innovation capacities of innovation entities in a society and the links to economic growth and competitiveness of Nigeria and developing economies;
- b) Developing new options for public policy via entrepreneurial capacities of knowledge workers at individual, team and organizational levels;
- c) Producing new methodologies and metrics for modelling, measuring and improving emergent innovation activities and systems in knowledge work;
- d) Contributing to the understanding, literature and praxis of global innovation management among individuals, teams and organizations in developing economies, by a deepening of CA research.

The key outputs for this research include:

- a) A novel workforce scorecard that places individual knowledge workers at the core of organizational performance management process;
- b) producing versions of the scorecard that are faithful to the peculiarities of different industry sectors, even as they share a common background of performance modelling;
- c) Linking key performance indicators in the sectors to competencies of knowledge workers and the effects on the four objectives stated above;
- d) An indication in the frameworks of the ways *self learning* and *knowledge management* could happen at the level of individuals, teams and organizations;
- e) Developing a framework for change managing *transitions* from the traditional ways of managing innovation to new ways arising from the research.

Emphases on these works will be placed on how the results could inform *best practices* and *institutional excellence* in Nigerian higher educational institutions, especially in teaching, learning, income generation, HEI-industry linkages and research excellence. The key foci of these efforts will be on *education* and *finance*. In order to foreground these works, we plan to deliver a number of related workshops at the National Mathematical Centre and internationally:

- Workshop on functional education in mathematical sciences and their links to national and regional economic development (**WoFED**)
- Workshop on Dynamic Data-Driven Applications in Financial Modelling (**DDAFiM**)
- Workshop on computational finance and business intelligence, with emphasis on emerging markets (**ComFIBIEM**)
- Workshop on model-based optimization of human and academic potential: interfaces among individual, team and organizational excellence and performance (**MOHAP**)
- Workshop on Data Mining in Finance and Investing (**DaMIFI**).

In order to use the model to motivate more women to take up careers in mathematical sciences and other male-dominated areas of knowledge work, we will enrich the model with additional insights from Vinnicombe & Bank eds. (2003) *Women with Attitude: Lessons for Career Management* and supplementary CA model-focused interviews of such high-achieving Nigerian women as Professor Dora Akunyili, Mrs. Oby Ezekewisili, Dr Okonjo Iweala, notable Nigerian female professors, bankers, etc.

A number of core references are deemed useful in finessing the measurement theory and metrics employed in the CA model and also in linking the model to simulation, wisdom-of-crowds type of evidence based-thinking, emergence and universality, adaptive thinking, the psychology of human performance and the future of work, Penrose (2005), Hand (2005), Ferguson (2001), Surowiecki (2005), Casti (1997), Greenfield (2006), Holland (2000), Ward (2002), Ball (2004) and Gigerenzer (2002) and Robbins (2001).

11 Summary and conclusion

We conclude this paper by reiterating the fact that the CA model has far-reaching implications for helping communities of academics and professionals in Nigeria and developing economies evolve new ways of working smartly in own or employer organizations, in order attain diverse individual and societal economic development goals. We have noted a number of projects that could be implemented with insights from the model and list out the projects below by way of summary:

1. Creation of an NMC *Journal of Interdisciplinary Research and Innovation in Mathematical Sciences* devoted to publishing innovative papers that are linked to significant disciplinary and societal problems (**Project JIRIMS**)
2. Managing, Measuring and Improving Performance in Academic Business: Using a Corporate Academic Model to Train (Nigerian) Academics in Knowledge Work (**Project MaMeIPAB**)
3. Curriculum Renewal of the Mathematical Sciences in Nigeria Using Insights from the Corporate Academic Model (**Project CUREMAS**)
4. (Inter)national intervention programme on functional education in (mathematical sciences) different disciplinary clusters via Curriculum Realignment, Institutional and Structural Invigoration and Sanitization (CRISIS) solutions in (African)Nigerian higher educational institutions (HEI) to be based on the CA Model (**Project CRISIS**)
5. National programme for producing a new genre of CA-inspired **super books** which embed African identity and consciousness in new national education models, with strong emphasis on socio-economic development and cultures of sub-Sahara African countries (**Project SUPERBOOKS**)
6. An Integrated Model-Based Project Management Model (for Managing Complex Multi-Objective Transnational Education Projects in sub-Sahara Africa) (**Project IMP**)
7. Further work to turn this experience into a tool for facilitating modern research supervision to be called the Corporate Academic Research Structuring Model (**Project CARESS**), which can also be software enabled with the same name for the software
8. National Conference on Masters and Doctorates in the 21st Century: Training Research Supervisors in Various Disciplines (e.g. Mathematical Sciences, Finance & Economics, etc) (**Project 21st CENTURY RESEARCH SUPERVISION**)
9. Work on the research theme *Model-Based Optimization of Human and Academic Potential: Interfaces among Individual Performance, Organizational Excellence and Decision Making* (**Project MOHAP**)
10. Related work on *New Directions in Academic Innovation, Creativity and Entrepreneurship* (**Project NeDAICE**).

We are satisfied that the above pool of five workshops and ten projects of strategic national importance are useful engines that will enable us to drive related **capacity building** and **socio-economic development** work in Nigeria, on the basis of continuing work on the CA model.

12 Acknowledgements

We are again grateful to Sheffield Hallam University (SHU), United Kingdom, which noticed the potential of this model for a **mass professionalization** of academic work in virtually all disciplines of study and consequently provided the initial seed grant of some £8000 (=N=2m) towards its development; to SHU colleagues and international research collaborators (too numerous to mention here) who urged us to deepen the model building work; and very importantly, to the National Mathematical Centre, for the opportunity to present the paper(s) at the *2008 International Conference on Mathematical Modelling of Global Challenging Problems*, and to co-develop some of the above mentioned projects with academics and professionals in Nigeria through an enabling appointment of the author as a Professor of Stochastic Modelling in Finance and Business.

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How to appropriately manage mathematical model parameters for accuracy and reliability: A case of monitoring levels of particulate emissions in ecological systems

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Abstract

In the last few years there have been various initiatives aimed at reaching a global consensus on carbon emission limitations. While this goal may finally be achieved, the fast evolving ways of data collection, analysis and dissemination pose new challenges to the accuracy and reliability of the way the stipulated limitations are measured. This paper proposes a novel methodological approach to learning rules from data using particulate emissions data from a UK industrial firm which is legally required to meet well-specified standards. The data were collected by continuously monitoring levels of emission from a coal-fired boiler using a particulate monitoring system over regular intervals and the government-imposed requirements were assessed hourly over each 24 hour period. Parameters in a uniform and mixture of two normal models are used to highlight potential technical loopholes that may cause the firm to either "pass" or "fail" the particulate emission limitation test. The main idea of the paper is attaining perfection in measurements and predictions by building sharable environmental conditions for scientific decision making. Despite being based on ecological data, the findings of the paper cut across disciplines and hence it makes recommendations for a unified modeling process that would capture not only the envisioned environmental goals globally but also other phenomena which may be subjected to parameter-dependent learning algorithms.

Key words:

Data mining, data recycling, data sharing, learning algorithms, measuring for perfection, model accuracy, model reliability, over-fitting and particulate emissions.

Mathematics Subject Classification 2000:03C30 &03C52.

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1. Introduction and study motivation

This paper is motivated primarily by the fact that the African continent, like other continents, generates massive volumes of data which could be utilised by physical and social scientists across the globe for the benefit of the research community in particular and the human race in general. Further, over the years, the continent has had excessively many discussions and reports with most debates being centred on the role of Science Technology and Innovation (STI) in bringing about the much desired changes such as poverty alleviation. The key question is therefore how best to achieve those goals.

We set off from the premise that lack of informed decisions has complicated and continues to complicate the way the continent faces its challenges. The paper proposes a novel methodological approach to learning rules from data using as a caveat a practical illustration of a typical approach to measuring carbon emissions in the western world, based on a model applied to an industrial plant in northern England, which is legally required to meet well-specified standards. It focuses on how the African continent can positively adapt the existing methodologies in addressing some of the fundamental issues inherent in the continent's approach to decision making using available modeling skills.

The Italian scientist and philosopher Galileo Galilei (1564-1642) preferred mathematical to rhetorical-driven arguments (Drake, 1995) and one of his many pieces of advice to mankind was to "...measure what is measurable, and make measurable what is not so...". In most real life phenomena, being able to measure and manage the underlying parameters driving a particular phenomenon spells the difference between success and failure. Our ultimate goal is therefore to build foundations for versatile models capable of *measuring for perfection* and *recycling* data and information. In order to achieve that, there is an obvious and immediate need to transform Africa's highly fragmented data into coherent data sources to be shared by researchers across the continent and beyond.

Consider, for instance, the all too common example of economic growth and globalization (Dicken, 2007). Which basic parameters should be considered in order to determine the real impact of globalization on an economy? The ultimate goal (the impact) is obviously a function of some measurement process - referred to above as *measuring for perfection*. For instance, adding an Information and Communication Technology (ICT) variable to the two variables above almost certainly enhances the variable portfolio but generates another set of questions such as those highlighted by Kelles-Viitanen (2003, p84) who asked "...whether economic growth and globalisation the nexus of which ICT is expected to strengthen alone will reduce poverty...". According to the United Kingdom Department for International Development (DFID, 2007) one country, Tanzania, "...has little firm data to demonstrate how growth has fed through into poverty reduction. Despite this absence of data, DFID adopts 5% as the likely GDP growth over the period 1998 to 2002 with a corresponding 39% to 34% poverty reduction rate.

As another example, consider the fading biodiversity in, say, Lake Victoria in East Africa and all the initiatives for its restoration. Numerous approaches to measuring biological diversity have been published with good examples from Magurran (2003) who proposes statistical methods on comparing assemblage composition and Wilcox *et al.*, (2002) who developed biological indicators for detecting wetland degradation of the Great Lakes in America. Yet, despite these well-documented rigorous metrics, a simple computation of the impact of biodiversity restoration initiatives in, say, Lake Victoria, may prove difficult. To illustrate this, let us assume that we can work out the rate at which biodiversity in the lake has been dwindling and that we are able to quantify the restoration rate and denote the two parameters by Ω and Ψ respectively, measuring on the same scale. One typical issue here is that different studies - on different samples, methodologies or both may not necessarily agree on the arithmetic sign of $\Psi - \Omega = \Delta$. Such a disagreement is awful - with potential consequences including further degradation of the lake, wastefulness of resources or both - yet a more awful thing is its cause which, at first glance, may seem to be the inability to measure $|\Psi - \Omega| = \Delta$.

Many data-related questions can be raised. For instance, why is it that while a number of studies show that some African economies have steadily grown over the last few years as a consequence of liberalization (Mattoo *et al.*, 2006) the driving factors have not resulted into significant enhancement in home-grown technical skills in, say, the application of Science, Technology and Innovation (STI) in manufacturing.

The paper proposes a novel approach towards the provision of lasting solutions to the foregoing issues and it is organised as follows. Section 1 provides a practical illustration of measuring natural phenomena using particulate emission data from an industrial plant in northern England, Section 4 introduces the key building blocks as well as the prototype of the proposed methodology and discussions and concluding remarks in Section 5.

1. Measuring particulate emission - a practical illustration

In the last few years there have been various initiatives aimed at reaching a global consensus on carbon emission limitations. While this goal may finally be achieved, the fast evolving ways of data collection, analysis and dissemination pose new challenges to the accuracy and reliability of the way the stipulated limitations are measured. This section looks at an empirical illustration for measuring particulate emissions data from a UK industrial firm which is legally required to meet government-imposed requirements monitored via a Programmable Logic Controller (PLC) installed for automatic monitoring and controlling the industrial process. The government-imposed emission requirements are complied with if 95% of hourly average emission readings for each rolling 24 hours do not exceed mg/m^3 300 and the peak hourly average value does not exceed 1.5 times the emission limit value per any representative spot sample.

2. Data description

The data were collected by continuously monitoring levels of emission from a coal-fired boiler using a particulate monitoring system over regular intervals. The PLC captures emission readings in an interval of 6 minutes - this interval can be altered. There are also regular sessions of soot-blowing which typically last 30-40 minutes. Compliance or non-compliance was assessed on the basis of an index on the basis of two random samples extracted from the industrial process. Each of the two samples contained 240 cases and yielded the parameters described in Table 1.

VARIABLE	DESCRIPTION
SAMPLE1	Set one of twenty four hour-readings sample data provided by the client
SAMPLE2	Set two of twenty four hour-readings sample data provided by the client
UNI-MEANS1	Uniform distribution means for set one sample data
UNI-MEANS2	Uniform distribution means for set two sample data
UNI-VAR-COEFF1	Uniform distribution variances for set one sample data
UNI-VAR-COEFF2	Uniform distribution variances for set two sample data
HOURLY-MEANS1	Set one sample data averages over an hour
HOURLY-MEANS2	Set two sample data averages over an hour
CHECK1-300	Sample one 24-hour average limit compliance/non-compliance
CHECK2-300	Sample two 24-hour average limit compliance/non-compliance
VERDICT1	Sample one overall check on compliance/non-compliance
VERDICT2	Sample two overall check on compliance/non-compliance

Table 1: Description of variables

3. Data distribution and methodology

It was therefore reasonable to adopt a uniform distribution on the basis of its basic properties - that is, having a finite range and being typically uni-modal - except for the 4-hourly peaks which were legitimately treated as outliers. A clearer view of the two samples readings is given in 2-D plot in Figure 1. Note that over 90% of the data lie in the south-west corner between mg/m^3 100 and mg/m^3 200 with about 5% of the raw readings lying above mg/m^3 300 - i.e., the outliers mentioned above.

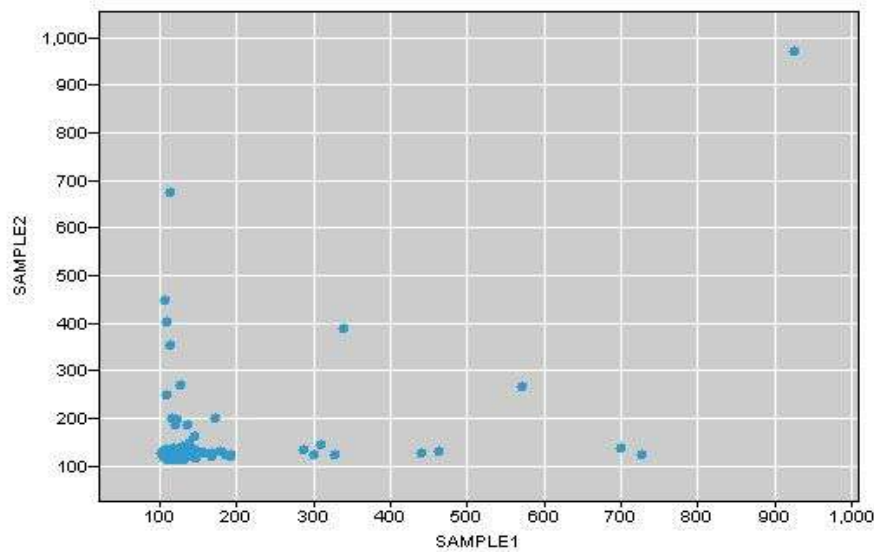


Figure 1: A 2-D illustration of the two samples

As noted above, the particulate emissions are complied with if 95% of hourly average emission readings for each rolling 24 hours do not exceed mg/m^3 300 and the peak hourly average value does not exceed 1.5 times the emission limit value for any representative spot sample. That is, hourly average emissions should fall below the 95th percentile and none of the hourly averages should exceed mg/m^3 450.

The computations in Table 2 track the magnitude of particulate emission given the threshold and the form of distribution. They were computed using the 6-minute readings from the two different data series each covering a 24-hour period based on the above formula for the uniform distribution. The computational process used was similar to the standard moving average – chosen to try and smoothen out fluctuations and highlight possible trends, while keeping within the distributional requirements.

MEASURE	SAMPLE 1	SAMPLE 2
Percentage of average hourly emission readings below the mg/m^3 300 limit (95 th Percentile Compliance)	100%	100%
Percentage of peak hourly averages below mg/m^3 450	100%	100%
COEFFICIENT OF VARIATION	3.76%	2.82%

Table 2: Computational summary based on two samples

Note, in particular, that the two coefficients of variation are fairly similar. The measure is often useful in comparing the rate of variation between the two samples even when the two sample means may look drastically different from each other. The two constraining measures – that is, the levels of particulate emission as measured against the set limit of 95% of all hourly reading averages falling below mg/m^3 300 and the peak hourly averages not exceeding 1.5 times the emission limit are given in the second and third rows of Table 2 respectively for both samples. Note that in all cases the “*means of the uniform distribution means*” have been computed on the assumption that in accordance with the Central Limit Theorem, they will follow a normal distribution. In fact, there would be no noticeable difference if they were computed as uniformly distributed means.

a. The computational algorithm

The algorithm for monitoring the emission levels is summarised below. The algorithm has two main criteria for deciding on the compliance or non-compliance of emission limit. These are the 95th percentile requirement for the hourly averages to fall below mg/m³ 300, which applies to any arbitrary 24-hour period and the hourly average value not exceeding mg/m³ 450. Note that the algorithmic steps below are based on one vector of readings spanning beyond 24 hours making it workable for any number of samples.

BEGIN

Initialise a vector M of length L to store sequence of means from the emission readings.

Denote the values in M by R_{li} for $i = 1, 2, 3, \dots, L$.

Initialise $i = 0$.

Initialise a new vector H in which to store hourly averages over 24 hours.

Initialise a new vector V to store readings exceeding the limit

Do While $i \leq L$.

$i = i + 1$

Store $\frac{R_{li} + R_{li+1}}{2}$ into M (the index grows by the reading interval)

End Do

Do while $j \leq l$, the counter j is initialized at an arbitrary sampling point and l is the length of the vector over a 24-hour period (depends on reading intervals).

Store Hourly Averages into H

IF $H(j) \geq 300$

APPEND $V(j)$ by $H(j)$

END IF

End Do

IF $\frac{LENGTH(V(j))}{LENGTH(H(j))} \geq 0.95$ AND $M(j) \leq 300 * 1.5$

*THEN **EMISSION ACCEPTABLE***

ELSE

EMISSION UNACCEPTABLE

END IF

END.

The 95th percentile of readings below the set emission limit of mg/m³ 300 applies to any arbitrary 24-hour period. The process is compliant if the ratio of the count of readings below the limit to the total number of the readings over the sampled 24-hour period is greater of equal to 95% and none of the hourly average value exceeds mg/m³ 450. As it stands, the process is nowhere near breaching the environmental requirements.

However, it is important to note that modeling of the process was based on a 6-minute reading intervals after exploratory data analysis of the two samples indicated that there could only be a marginal difference. The most influential aspect of the process appears to be "the moment of the peak", which massively inflates the level of particulate emission. For instance, readings taken every 3 minutes will increase the hourly frequency from 6 to 12 – making the process “more continuous” than it currently is and may potentially push up the 24-hourly averages. In other words, there are potential technical loopholes that may cause the firm to either "pass" or "fail" the particulate emission limitation test.

One major flaw of the model used here to measure the level of particulate emission is that it is data dependent in many ways. Although five simulations of new samples based on the same distribution passed the test, the overall model, taken as a function of time, data and concepts (environmental), cannot be guaranteed to pass especially given the versatility of reading intervals (currently 6 minutes). In the

next exposition, we propose a rigorous alternative to modelling not only environmental but also many other natural and social process which generate massive volumes of data.

4. Key building blocks for the proposed methodology

Decision making is characterized by many uncertainties. The recent financial turbulence across the world, particularly in the United States and the United Kingdom speaks a lot about what may happen if predictions of future trends aren't accurately captured in available data and information. The proposed methodology derives from two fundamental approaches to knowledge discovery from data - data clustering and classification which we briefly introduce in the following exposition.

a. Detecting naturally arising structures in data

Detecting naturally arising groups in data, also commonly referred to as data clustering (Kogan, 2007), arises in the absence of a priori information. Under this approach we seek to detect and define data groups as graphically illustrated in Figure 2 in which the problem is to estimate the densities $G_i(X)$ $i = 1, 2, 3, 4$. Obviously, getting the optimal estimates of the four densities is major challenge but a more practical problem arises from their overlapping. Hence, accepting the four modes as representing true groups in X requires resolving the problem of membership for all intersection values. On the other hand, the superimposed dotted line detects a three-modal structure by combining the heavily overlapping $G_2(X)$ and $G_3(X)$ and hence minimizing the ambiguity of the membership of any intersecting values between the groups. While the multi-modal pattern may clearly outline group heterogeneity, it is likely to lead to the detection of spurious clusters – a situation described above as over-fitting while reducing the number of modes may lead to loss of potentially useful information – also referred to as masking.

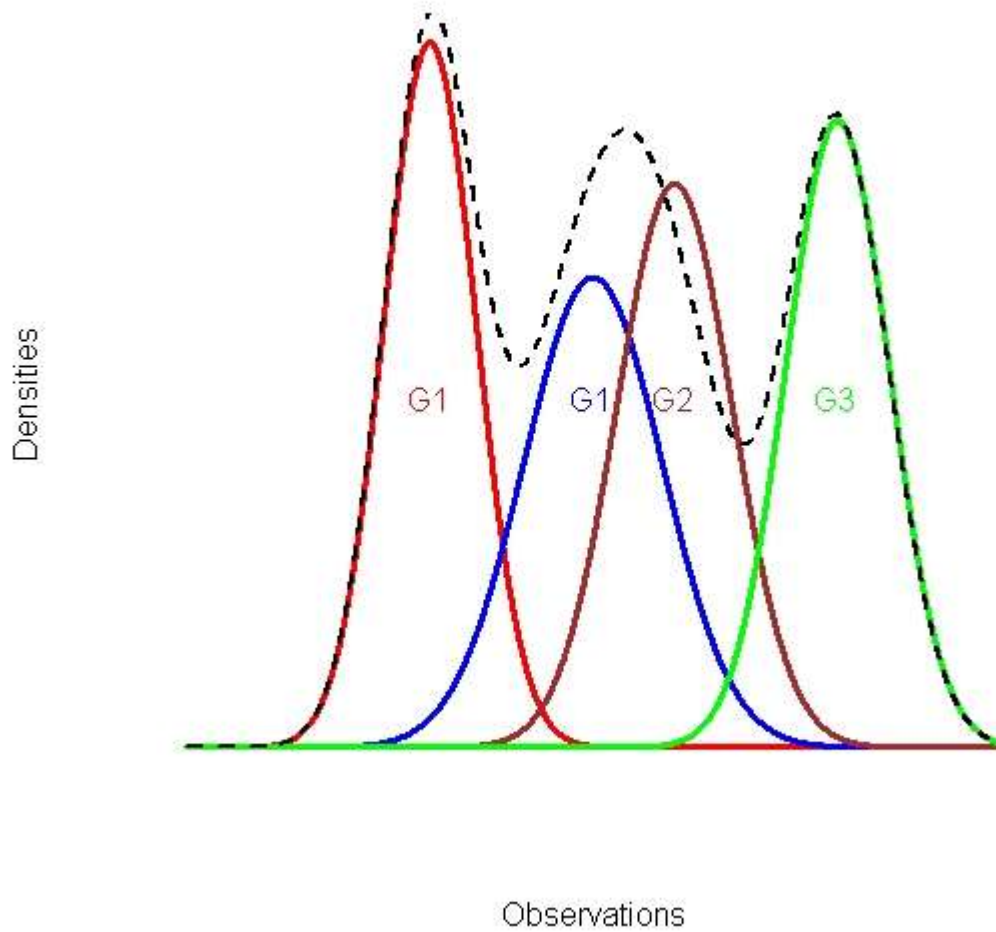


Figure 2: A graphical illustration of the data clustering challenge

Various data clustering algorithms have been studied including the K-Means (MacQueen, 1967) and Kohonen Self-Organising Maps (SOM) due to Kohonen (1995). Taylor and Mwitondi (2001) report that despite their rigorous nature, almost all these algorithms are associated with the issue of determining “what is interesting” such as deciding on the optimal number of groups in **Error! Reference source not found..** In this paper we adopt the variable kernel method (Silverman, 1986) for estimating densities as a building block for the measuring for perfection methodology. The kernel, defined in Equation 1, has a varying scaling parameter from one data point to another.

$$\theta(\lambda) = \frac{1}{n} \sum_{i=1}^n \frac{1}{bd_{i,k}} K\left(\frac{\lambda - x_i}{bd_{i,k}}\right)$$

Equation 1

The function in Equation 1 is defined for a total number of n observations where $K(.)$ is the kernel function, k is one of pre-defined finite points (number of groups), $d_{i,k}$ is the distance from one observation to its nearest neighbour and $|\lambda - x_i|$ denotes each of the distances between data points and it is proportional to $d_{i,k}$, making the kernel flatter where the data are sparse and bulging with data concentration. Consequently, once the initial number of groups in the data is defined, the smoothing parameter b usually referred to as the bandwidth, can be used as a tuning parameter to alter data patterns.

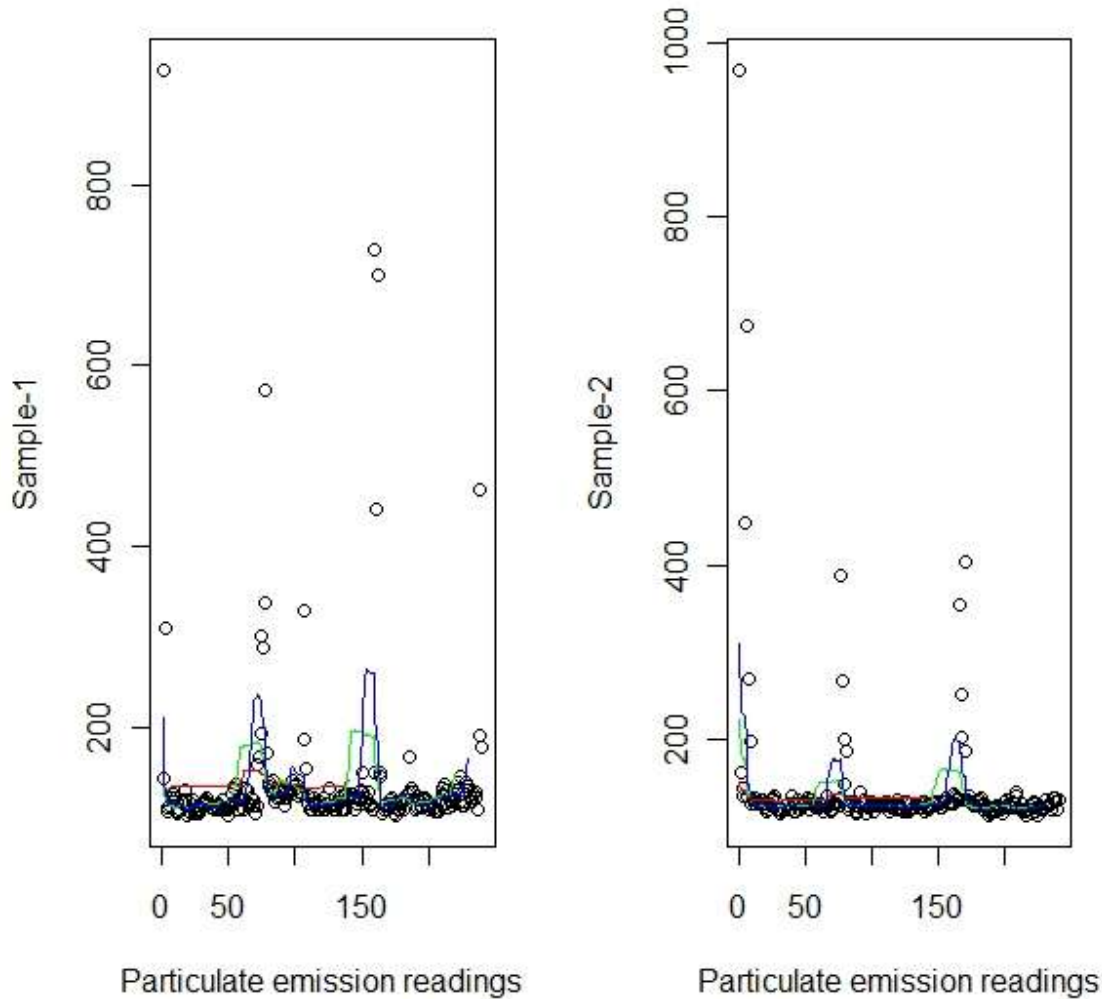


Figure 3: Three levels of bandwidth used in estimating the densities from the two samples

The graphical illustrations in Figure 3 correspond to the kernel estimates for the two samples as generated by the **kernel (.)** function in **R**. The three lines (blue, red and green) are equivalent to lower, mid and high bandwidth in Equation 1 – with the number of peaks inversely related to the smoothing parameter. Clearly, despite the rigour of the method, the question of “what is interesting” remains. Its answer is often data-specific and typically depends on the intervention of expert knowledge especially when it comes to the definition of the initial number of groups, k . the next section focuses on classification - another building block for the measuring for perfection methodology.

b. Classification

That is, for a two-group case ($k = 1, 2$), the maximum likelihood rule will allocate a new case to group one if $\pi_1 f_1(x) > \pi_2 f_2(x)$ and to group two otherwise - which only arises if

$$\frac{f_1(x)}{f_2(x)} > \frac{\pi_2}{\pi_1}, \text{ that is, } \frac{\sigma_2}{\sigma_1} e^{-\frac{1}{2} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{(x-\mu_2)^2}{\sigma_2^2} \right]} > \frac{\pi_2}{\pi_1}$$

Equation 2

Taking logarithms on both sides of Equation 2, multiplying by 2 and re-arranging yields

$$2 \ln \frac{\pi_1 \sigma_2}{\pi_2 \sigma_1} > x^2 \left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right] - 2x \left[\frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \right] + \left[\frac{\mu_1^2}{\sigma_1^2} - \frac{\mu_2^2}{\sigma_2^2} \right]$$

Equation 3

As discrimination rule given the parameters. If $\sigma_1 > \sigma_2$ then the coefficient of x^2 is negative and the data points for which Equation 3 holds will fall into two different regions of low and high values. If $\sigma_1 = \sigma_2$ the quadratic component in Equation 3 disappears and we obtain a discriminant rule by which $f_1(x) > f_2(x)$ if $|x - \mu_2| > |x - \mu_1|$ which, assuming that $\mu_2 > \mu_1$, yields the rule in **Error! Reference source not found.**

Indeed, if the parameters and densities in **Error! Reference source not found.** And **Error! Reference source not found.** Are known, the computation of the posteriors would be straightforward. In practice, however, these parameters have to be estimated from data. It is easy to see that the universality of the rule in **Error! Reference source not found.** Through Equation 3 strongly depends on the parameters π_k and σ_k both of which are data-dependent. Typically, these parameters are replaced by their sample estimates, $\bar{\pi}_k$ and S_k respectively. Consequently, the foregoing population-based critical point transforms into the sample-based version, $\mu_s = \frac{\bar{\mu}_1 + \bar{\mu}_2}{2}$ which leaves both model accuracy and reliability depending on data samples.

Our proposed measuring for perfection methodology, seeks to address the issues raised in the foregoing expositions via building a regularly updatable and sharable data and information sources. The methodology prototype, outlined below, seeks to minimise the impact of randomness by generating, updating and verifying data and decision rules across applications.

c. Prototype of measuring for perfection

The aims and objectives of measuring for perfection are embedded in the need for sharing and recycling data and information. The idea was motivated by the apparent need to build an environment upon which African scientists could be brought close together to share resources, concepts, tools, techniques and skills. It focuses on all sorts of data-related challenges relating to the processes of data collection, storage, sharing, analysis, dissemination of results. The unified process, implemented in an iterative way, forms what we call “data recycling”. We perceive data recycling as a novel process by which the arbitrary interval (a, b) in **Error! Reference source not found.** can be applied across an infinite length of time and across physical locations in order to arrive at a consensus on, say, the number of optimal groups in a dataset of breast cancer sufferers accessible by multiple researchers using a variety of methods at different points in time and locations. In other words, running data testing back and forth along the time domain enables detection of potentials shifts in concepts, definitions and the overall baseline behaviour of data. This type of approach has not used before, but it is needed for modelling perfection.

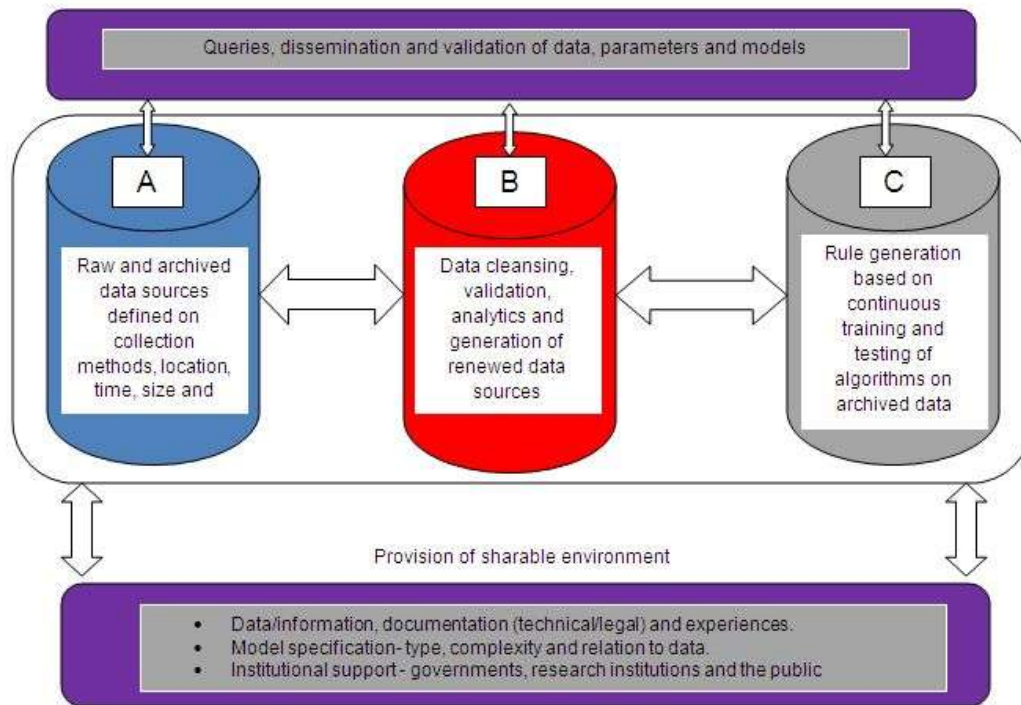


Figure 4: Measuring for perfection methodology

The model, graphically illustrated in Figure 4 seeks to build data and knowledge repositories that would, in the long run, play the role of data and parameter generators for researchers and modeling techniques of various complexities. As suggested by Juma and Yee-Cheong (2005) the perceived information and data sources could also be used to harness the massive informal bases of knowledge and technology. The proposed methodology prototype is characterised by built-in innovative and robust methods of data analysis, it is embedded with capabilities of providing instant performance measurement in all processes that can be subjected to mathematical modelling. Its main purpose is to initiate a data and information sharable environment leading to the continent's ultimate goal - poverty alleviation. For instance, in the case of measuring Lake Victoria's restoration of biodiversity discussed in Section 1, the quantities Ψ and Ω are initially generated at A in multiple versions. The quantities build up from work carried out from various locations around the lake, at different periods of time and by different researchers using a variety of methods. The queries, dissemination and validation level enables users to request or submit data and/or information via A, B or C. The interactions between A and B, B and C and A and A and C (via B) ensures that knowledge bases are continuously checked and validated for use and that new data archives and model parameters are continuously generated. Previous studies, models, parameters and data are retained within the system to provide comparative bases for current and new studies.

5. Discussions and concluding remarks

The practical illustration in Section 1 and the theoretical discussions in Section 4 confirmed that mathematical modelling of any phenomenon requires a balance between model accuracy and reliability across applications. When the underlying parameters and densities are estimated from data, the desire to get accurate results often leads to making the models data-specific – a problem commonly referred to in the literature as data-over-fitting. With model reliability typically entailing more modelling challenges than accuracy, the paper has highlighted the crucial role played by parameter identification and data behaviour in mathematical modelling. Implicitly, being averse to high prediction errors is more common than not - which makes generalising models hard to find. This issue is compounded by the use of different modelling techniques with different levels of complexity. The proposed measuring for perfection methodology seeks to address some of these issues by providing an accessible forum on which modelling techniques, data and ideas constantly flow in and out of modelling entities.

The proposed methodology seeks to identify a whole range of strongly interacting factors in all what is going on around the continent and beyond. Identifying these factors and their specificity to the continent will help avoid "all size-fits all" solutions. For instance, we all tend to believe that the African Diaspora - businessmen, academicians and other professionals - have a great role to play in turning the African wheel. They are believed to have been instrumental in revolutionalising some of the Asian economies (Smart and Hsu, 2004) - yet they haven't had such an impact on the economies of countries such as Jamaica - a Caribbean country with a huge population overseas - and Nigeria. May be we need to stop and ask ourselves why they have not had an impact there as the bottom-line may well be that we have not been able to measure up that impact. The continent needs a unified initiative not only on the role of the Diaspora but also on the consequences of increasing the number of engineers, inviting and encouraging foreign investment.

In the case of Lake Victoria example above, monitoring, documenting and measuring human migration, industrial and commercial activities around the lake as well as the impact these activities have on the overall biodiversity will obviously help plan for lake's future strategic plans. Reliability of the water quality and biodiversity measurements will only be possible if the data instantly and continuously inform planned clean-up regimes for more accurate environmental impact assessment. For instance, an environmental/pollution index can be set for use in measuring the lake's level of pollution and variations in the lake's biodiversity. The envisioned data-based system will not only be informative but could also possess built-in diagnostic power to help in highlighting some of the weakest links - which could then be timely rectified.

Despite being illustrated on ecological data, the findings of the paper cut across disciplines and hence it makes recommendations for a unified modeling process that would capture not only the envisioned environmental goals globally but also other phenomena which may be subjected to parameter-dependent learning algorithms. Similar ideas may be extended to manufacturing and design, education, health and other social services. Comparative studies in biomedical sciences may positively contribute towards understanding the clinical and/or pathogenic features of diseases particularly those known to be associated with genotypic and ethnic attributes (Gad, *et. al*, 2003).

As implied throughout this paper, measuring for perfection seeks to establish full-fledged data repository schemes across Africa for use by interested researchers and decision makers across the globe. For these schemes to succeed, African statistical and information institutions and governments should come up in full support which may prove to be an issue given the heterogeneous nature of the continent - technologically, economically, politically and culturally. It would therefore be reasonable to channel the initiatives through existing multilateral institutions such as the African Union, NEPAD (2005) and ICSU-ROA (2006/7). Key features of the schemes should be, *inter alia*:

- Commitment to the universality of science for development.
- Active engagement by African researchers both home and abroad.
- Commitment by Governments and research bodies.
- Continuity and sustainability through funding, support and scientific regulation.
- Multi-disciplinarily - the need to share skills and experiences.
- Usability.

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VELOCITY PROFILE OF A DEOXYHEMOGLOBINS BLOOD WHOSE VISCOSITY IS UNSTEADY

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ABSTRACT

We revisit the kinetics of solution-gel transformation of deoxyhemoglobin S. Particular interest is the dependence of viscosity on time. We show that the velocity decreases as time increases.

KEYWORDS:

Navier-Stokes equations, unsteady viscosity, deoxyhemoglobin S aggregation, Sick cell anaemia
Mathematics Subject Classification 2000:92E20,92C35 & 92C50.

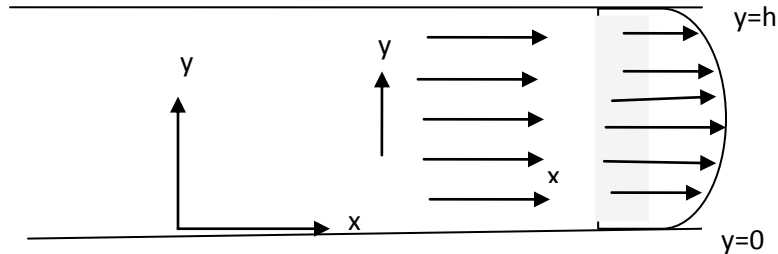
1. INTRODUCTION

Extensive research has accumulated rich amount of information on the molecular and cellular properties of sickle cell hemoglobin (Hbs) as well as its polymerization [1-7]. Hbs is a genetic variant of the human normal hemoglobin (HbA) in which the valyl residue at the β -positon replaces the normally occurring glutamyl residue. In the deoxygenated state, and under certain experimented conditions, Hbs molecules can polymerize in solution as well as inside the red blood cell. It is known [1-7] that there is a rapid ascent of viscosity and a fall after the overshoot and subsequent dependence of velocity on time has not been documented.

Apart from time, the viscosity depends on the volume function, the temperature, the deoxy-Hbs concentration and shear rate. The objective of the present study is to evaluate the effect of the above parameters on the velocity file. But because some of the parameters also depend on time, we shall first investigate the unsteady effect of viscosity on the blood flow.

2. MATHEMATICAL FORMULATION

We consider flow between two parallel plates.



The aggregation $A(t)$ Hbs molecules is given by [3] as

$$A(t) = A_{\infty} [1 + l e^{-\omega t}]^{-1/\gamma} \quad (1)$$

Where

$A(t)$ is aggregation at time t

A_{∞} is aggregation as $t \rightarrow \infty$

$$l = (A_{\infty}/A_0)^{\gamma} - 1$$

A_0 is aggregation at $t = 0$

γ is a positive number $0 < \gamma < 1$

ω is the relaxation frequency

The Navier-Stokes equations are

$$\frac{\partial u}{\partial x} = 0 \quad (1)$$

Continuity

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (2)$$

Momentum equation

$$u(x, 0) = f(y), \quad u(-h, t) = u(h, t) = 0, \quad (3)$$

$$\mu = k \frac{1 + a \exp(-\omega t)}{[1 + b \exp(-\omega t)]^n},$$

Where

$\omega = u(y, t)$ is the velocity

p = pressure

μ = viscosity

$a = \varepsilon \exp(\omega t_c)$

$\varepsilon = l(1 + \frac{\omega t_c}{\gamma})$

$$b = \text{le} \exp w t_d$$

$$t_c = \text{equilibrium time}$$

$$k = \mu \text{ as } t \rightarrow \infty$$

$$n = 1 + \frac{1}{v}$$

3. METHOD OF SOLUTION

The viscosity suggests that we take a solution of the form (when $\frac{\partial p}{\partial x} = 0$)

$$u = Y(t)T(t)$$

This leads to

$$\rho Y T' = k \frac{1 + a \exp(-wt)}{[1 + b \exp(-wt)]^n} Y'' T$$

Hence

$$\frac{\rho T'}{T k \frac{1 + a \exp(-wt)}{[1 + b \exp(-wt)]^n}} = \frac{Y''}{Y} = -\lambda^2 \quad (\text{say})$$

Hence $Y_n = \sin \frac{n\pi y}{h}$

$$T_n = \exp - \left\{ \frac{n^2 \pi^2}{\rho h^2} k \int_0^t \frac{1 + a \exp(-ws)}{[1 + b \exp(-ws)]^n} ds \right\}$$

$$u(y, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{h} \exp - \left\{ \frac{n^2 \pi^2}{\rho h^2} k \int_0^t \frac{1 + a \exp(-ws)}{[1 + b \exp(-ws)]^n} ds \right\}$$

$$b_n = \frac{2}{h} \int_0^t f(y) \sin \frac{n\pi y}{h} dy$$

1. SINUSOIDAL FLOW

Let $f(y) = \sin y$, $h = \pi$

Then $u = \sin y \exp - \left\{ \frac{k}{\rho} \int_0^t \frac{1 + a \exp(-ws)}{[1 + b \exp(-ws)]^n} ds \right\}$

This shows that $u(0, t)$ decreases as t increases.

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Mathematical Modeling of Mammalian Blood Count

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ABSTRACT

Simple mathematical models were developed and validated using standards from literature and a matlab program. The standard shows that the ratio of the quantity of RBCs (rc) to WBCs (wc) to Platelets (pc) is 800: 1: 30 for the lower limit and 600:1: 50 for the upper limit. The results obtained for a 70 kg male is 700:1:41, which is clearly between the standard ranges.

Mathematics Subject Classification 2000:92C10& 92C40.

1.0 INTRODUCTION

Test of a patient's blood, when used along with physical examination and medical history data, are major source of information needed for diagnosis and treatment of disease. The tests may be broadly grouped as chemical, immunological, hematological, and microbiological or immunohematological, according to the kinds of analysis performed. The most common procedure is to perform a complete blood count (CBC) of the number of red blood cells (RBCs), white blood cells (WBCs) and platelets per unit volume of blood along with a microscopic examination of the cells.

The chemical engineer (though not trained to carry out all these tests) by their highly developed analytical problem solving skills can predict the results of these tests prior to their performance. The knowledge of Mathematical Modeling can be applied successfully on the field of Medicine and Biotechnology.

The complete blood count (CBC) is a useful screening and diagnostic test that is often done as part of a routine physical examination. It can provide valuable information about the blood and blood forming tissues (especially the bone marrow) as well as body systems. Abnormal results can indicate the presence of a variety of conditions –including leukemia, anemia, and infections-sometimes before the patient experience symptoms of the disease (Karen, 2002). Blood count values can vary by sex, age physiological state and general health. The normal red blood cell count range from 4.2 – 5.4 million RBCs per micro liter for men and 3.6 – 5.0 million for women. Hemoglobin values range from 14 – 18 grams per deciliter of blood for men and 12 – 16 grams for women.

The normal number of WBCs for both men and women is approximately 4,000 –10,000 WBCs per micro liter of blood. Abnormal blood results are seen in some conditions. One of the most common is anemia, which are characterized by low RBCs count, hemoglobin and hemotocrits. Infections and leukemia are associated with increase in the number of WBCs. Leukocytosis (a white count increase to over 10,000/microliter) is seen in bacterial infections, inflammation, leukemia, trauma and stress. Leucopenia (a white count decrease to less than 40,000/microlitre) in some viral infection or bacterial infections and condition that affect the bone marrow such as dietary deficiencies, chemotherapy, radiation therapy autoimmune diseases.

This paper contains a set of mathematical models that were developed to predict the volume of blood; the quantity of: red blood cells, white blood cells, platelets and plasma contained in human body. The models are so simple that once you know your body weight then you can find these parameters.

1.1 MATHEMATICAL MODELING

Mathematical model (MM) is widely used in physical and social sciences and engineering fields. More recently, interest has shifted to biological, medical, natural and energy resources problems and urban development. Most astonishingly is the high rise in developing software in those areas. Chemical Engineering discipline, like others, explores this tool in a great deal. Hardly could one attend any conference in any of these disciplines today without finding one or two papers presented in this area. Though this is an area very peculiar to physical sciences and engineering, many researchers these days now see it as an all embracing approach of solving many physical problems. Note that researchers today have not yet done away with experimentation, no; the fact is that the latter will still be required in validating the results obtained from mathematical modeling. This is because MM gives prediction of behavior of systems represented by models in a more general sense.

Talking about MM in Chemical Engineering, it's an all-inclusive art as it occupies a central position to all its areas of specialization such as biochemical, separation processes, environmental, energy and process integration, petroleum and petrochemical, and systems engineering. Though the unit that anchors this art is the systems engineering, but all the units interact in intra-disciplinary research activities.

Chemical Engineering as it is; is all about development and design of processes from the very basis of harnessing resources in their natural form to the stage of marketing the product for public use. It may interest you that, the Chemical engineering Department Obafemi Awolowo University (OAU) has been collaboration in a multi-disciplinary research with other units in non related fields. Such research effort that has been done in this area with Advanced Research Laboratory, University college Hospital (UCH), Ibadan has to do with mathematical modeling of malaria parasite growth in patient with hepatomegaly (liver enlargement).

This is an interesting work. It means that through mathematical modeling, a lot of clinical observations could be easily predicted. That is, at the first clinical presentation, with the availability of a mathematical model, physician could predict the extent of parasitic load in the organ – liver and subsequently tell the minimum amount that may result into hepatomegaly state. The model developed can be used to predict the number of parasites present in a patient at a particular point in time without a laboratory test. It can be used to predict liver size and the days the patient is likely to kick the bucket if unattended to. MM has the potential to make significant contribution to social, socio-economic and behavioral sciences. We have proposed a mathematical model to company, which deals with production of servicing oil.

This work has to do with mathematical models for obtaining values for some rheological parameters which hitherto were determined by performing laboratory tests and measurements. The model if implemented will lead to considerable compression in flow time. MM has found increasing number of applications in design and planning optimization.

Conclusively, in the world today where we have in abundance tools for technical computing such as: Matlab, Mathematica, Mathcad, Maple (the list could be endless); one will be doing himself a lot of harm not availing the great opportunities at his disposal by still rigmarolling the old way of "slide-rule" while in the IT age.

1.2 Mammalian Blood

Blood is the river of life that flows through the human body. Mammalian blood consists of **plasma** as well as the **red (erythrocytes)** and **white cells (leukocytes)** and the **platelets (thrombocytes)**. In man, plasma chief component are water (90% - 92%) and protein (6 % - 8%). It also contains dissolved substances, including salts, nutrients (glucose, fats, and amino acids), carbon dioxide, nitrogen wastes, and hormones.

The plasma serves as transport system and medium for nutrients, wastes product and blood cells. It helps to maintain blood pressure and distribution of heat through the body. It is also involve in keeping the steady acid – base balance in blood stream and body. One of the main functions of plasma is to prevent excess fluid loss from the capillaries. Protein provides osmotic pressure that prevents the leakage of fluids.

Fibrinogen, another mammalian blood protein, plays a vital role in the process by which bleeding is halted by the formation of a clot. A third major class of plasma protein is the globulins. The gamma globulins are the antibodies, substances that protect the body

microorganisms and toxins. Alpha and beta globulins are molecules that specialize in transport of lipids (e.g. cholesterol), steroids, sugars, iron, copper and other minerals and free hemoglobin.

The red cells of mammals contain hemoglobin, which are oxygen carriers in the blood. The quantity of red cells in normal human being varies with age and sex as well as with external conditions, but there can be a pathological increase in the number of cells, called polycythemia or pathological decrease, called anemia. Anemia is not a disease but symptom of disease.

The white cells function primarily in body defense and repairs. There are three main types of white cells : granulocytes, which engulf and digest microorganisms ;monocytes which also digest microorganism as well as cellular debris; and lymphocytes which provide immunity to diseases by the production of antibodies and sensitized cells .The condition in which white cells are greater in number than normal is termed leukocytosis. It indicates the presence of an infection. An increase in the number of lymphocytes characterizes a viral infection seen frequently in young persons. It is accompanied by fever, sore throat, enlargement of the spleen, etc.

Platelets are vital in coagulation. The platelets release chemicals that initiate clot formation and the platelets themselves become an important part of the clot mesh work. Absolute reduction in number of platelets is termed thrombocytopenia. It is characterized by bleeding in to the skin and from the mucous membranes that line the digestive system and genitourinary tracts. Differences in any of the clotting factors results in hemorrhages following minor injuries.

1.3 Human Liver

The liver tissues consist of a mass of cells tunnels through with bile ducts and blood vessels. A group of liver cells called the Kupffer cells line the smallest channels of the livers vascular system and play a role in blood formation, antibody production and ingestion of foreign particles and cellular debris. The functions of liver are so enormous that they cannot all be mentioned in this paper. Two common liver diseases are the HEPTITIS (inflammation of the lobules) and CHIRROSIS or scarring of the lobules. A common sign of impaired liver is jaundice; a yellowness of the eyes and skin arising from excessive bilirubin in the blood. Jaundice can result from an abnormal level of red blood cell destruction (hemolytic jaundice), defense uptake or transport of bilirubin by the hepatic cells (hepatocellular jaundice) or a blockage in the bile duct system (obstructive jaundice).Because of the diversity of liver function and the varied and complicated metabolic process that may be affected by disease state. More than 100 tests have been devised to test the liver function.

2.0 MATERIAL AND METHODS

2.1 Model Formulation

According to the World Book (2001), the amount of blood in human body depends on his size and the altitude at which he lives. The New Encyclopedia Britannica (2005) states that the amount of blood varies with sex, age, weight, body build, and other factor but rough average figure for adult is about 7 to 8% of body weight. In The Encyclopedia Americana (1981), it was stated that ‘in human beings the volume of blood is 70ml for each kg of body weight.

Thus

$$Q_B = f(u, v, w, x, y, z) \quad 2.1$$

Where

Q_B = Quantity of blood

u = age of the person

v =body build

w =body weight

x =size of the body of a person

y =sex

z =altitude

Assume that u , have the same effect as x

So that

$$Q_B = f(u, w, x, y, z) \quad 2.2$$

2.2 Dimensional Analysis

The principle of dimensional homogeneity which was enunciated in 1922 by Fourier (Ogboja, 1996) was used to perform a dimensional analysis on the variables obtained in equation 2.2. Since y has no dimension, it can be excluded and 2.2 can be rewritten as

$$Q_B = f(u, w, x, z) \quad 2.2a$$

The dimensions of the variable in 2.2a equation are:

$$Q_B = L^3$$

$$u = T$$

$$w = M$$

$$x = L^2$$

$$z = L$$

Variable z can be rendered dimensionless in L using z to divide the equation as follows: $\frac{Q_B}{z} = f(u, w, \frac{x}{z})^3$

2.2b

The dimension of the variables in 2.2b is

$$Q_B = \frac{L^3}{L}$$

$$u = T$$

$$w = M$$

$$\frac{x}{z} = L$$

Variable $\frac{x}{z}$ and u can be rendered dimensionless by dividing through by z and u respectively. The resulting equation is thus

$$\frac{Q_B}{z^2 u} = f(w, \frac{x}{z^2}) \quad 2.2c$$

2.2c can be rewritten as

$$\frac{Q_B}{z^2 u} = f(w) \cdot f(\frac{x}{z^2}) = f(k) \cdot f(w) \text{ or}$$

$$Q_B = z^2 u f(k) \cdot f(w_B) = \alpha f(w_B) \quad 2.3$$

Where α is a constant to be determined.

2.3 Parameter Estimation and Model Development

2.4 Model Validation

The World Book (2001) states that:

An adult who weighs 73kg has about 4.7 Liters of blood

A 36 kg child has about half that amount

³ Note that $\frac{z}{z}=1$, **an absolute values have been used.

A 3.6kg infant has about 250ml

According to The Lexicon Universal Encyclopedia (1989), a normal 76.5kg man has about 5litres of blood in his body containing more than 25 trillion (25×10^{12}) red cells.

An adult of 70 kg has about 5000ml (5liters) of blood (The Encyclopedia Americana, 1981).Quoting from

The New Encyclopædia Britannica (1993), a microliter of blood normally contains about

4 – 6 millions red blood cells (rc)

5000 – 10 000 white blood cells (wc)

150 000 - 500 000 platelets (pc)

These values were used to formulate upper limit and lower limit ratios.

	Rc	Wc	Pc
Lower limit	800.00	1.00	30.00
Upper limit	600.00	1.00	50.00

In doing the validation, the mathematical models earlier derived were mat lab-programmed and the results obtained were compared with the standards.

3.0 RESULTS

Table 2.1 Comparison of Quantity of Blood Calculated with Standards from Literature

Weight W_B (kg)	Quantity of Blood Q_B (Lt)	Standards from Literature (Lt)
3.6	0.252	0.250
36.0	2.520	2.350
70.0	4.900	5.000 ***
73.0	5.110	4.700
76.5	5.360	5.000 ***

***These values were taken from two different literature sources.

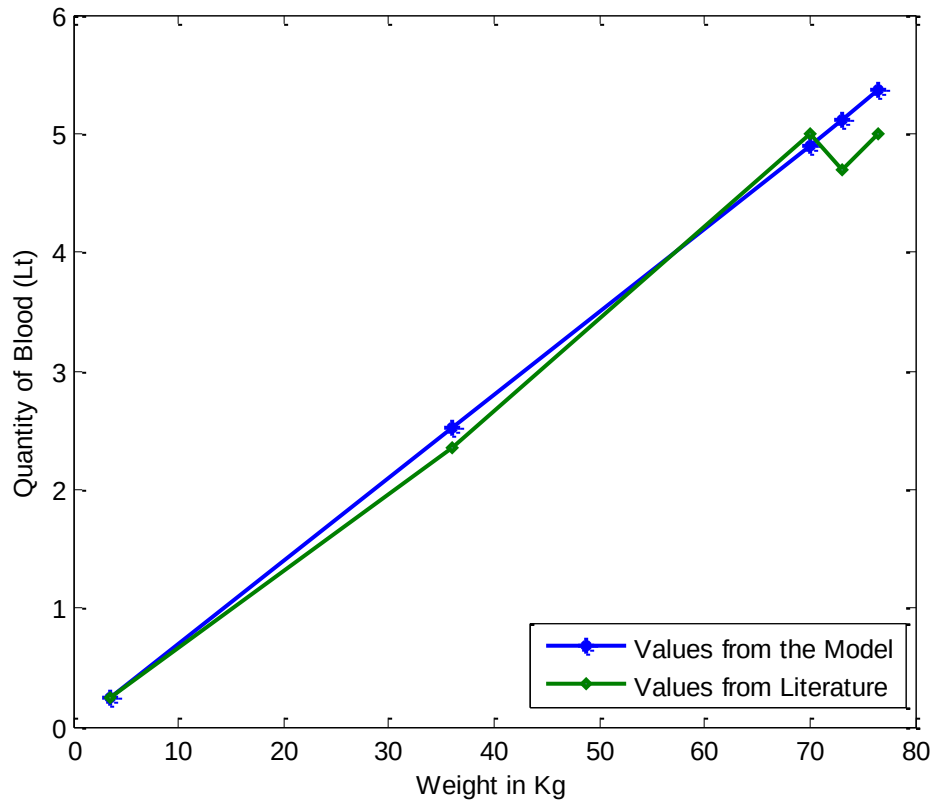


Fig 1: Variation of Body Weight with Quantity of Blood

Table 2.2 Results Obtained from the Models

Weight W_B (kg)	Q_r (Lt)	Q_w (Lt)	Q_{pe} (Lt)	Q_{pm} (Lt)	Q_{fe} (Lt)
3.6	0.108	0.0006012	0.018	0.162	0.090
36.0	1.080	0.0060120	0.180	1.620	0.900
70.0	2.100	0.0116900	0.350	3.150	1.750
73.0	2.190	0.0121910	0.365	3.285	1.825
76.5	2.295	0.0127755	0.383	3.443	1.913

8. CONCLUSION AND RECOMMENDATION

The results as shown in tables 2.1, 2.2 and figure 1 clearly indicate that it is quite possible to predict the amount (in liters) of mammalian blood and its components prior to results from laboratory tests. It is therefore recommended that encouragement and supports should be provided for further work to be done on this research so that the model can be modified to predict the number of cells for individual

blood component. This will then pave way for the models to be employed in designing digital equipment that can be used at homes and hospitals.

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10.APPENDIX

11.Sample Results Obtained from Mat lab Program

EDU» eko20051

what is your body weight 3.6

The Total Amount of Blood in Your Body is 0.252 Lt

The Total Amount of Red Blood Cells in Your Body is 0.108 Lt

The Total Amount of White Blood Cells in Your Body is 0.0006012 Lt

The Total Amount of Platelets in Your Body is 0.018 Lt

The Total Amount of Plasma in Your Body is 0.162 Lt

The Total Amount of Formed Element in Your Body is 0.09 Lt

The Sum Formed Elements and Plasma is 0.252 Lt



A New Family of A-Stable Block Methods Based on GAMS-REVERSE GAMS Pairs for Stiff IVP

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Abstract

The paper describes a new approach by A-stable block methods to the integration of stiff ordinary differential equations by initial value method. No boundary value method application of GAM is used as advocated in the past yet it sustains good stability property. The A-stable k-step block methods are accurate of order $k+1$ for $k=2,4,6,8,\dots,30$.

Key words

K-step GAMS-RGAMS pairs,A-stable block ivm and Stiff ivp.

Mathematics Subject Classification 2000: 65L05

1. Introduction

A commonly used nonlinear mathematical model for the stiff initial value problem (stiff ivp) is the Van der Pol's equation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^1 = \begin{pmatrix} y_2 \\ -y_1 + \lambda y_2(1 - y_1^2) \end{pmatrix}, \quad (1.1)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad x \in [0, x_N]; \quad (1.2)$$

where $\lambda \geq 20$ (stiffness parameter). The equation first appeared in 1935 by Van der Pol in the study of electrical circuit oscillations. It has appeared in many other areas like chemical processes, stability problems of structures, economics and mathematical biology.

Since 1952 one of the most studied phenomena in numerical analysis has been the problem of stiffness for ordinary differential equations (odes). The reason is due to a basic difficulty that arises in the numerical treatment (situation) of the stiff problem which is that of numerical stability due to the presence of fast and slow time scales in the solution of the problem. I will illustrate this important point as follows, when we apply the order 11 ($k = 10$) to the Van der Pol's equations (1.1)–(1.2), we plot the computed discrete solution in the phase space (see fig. 1.1 below, $\lambda = 10$). The method reproduces the limit circle while in fig. 1.2, we plot the solution of the problem (1.1)–(1.2), $\lambda = 10$ computed with the GAM of order 11 (No known analytic solution of (1.1)–(1.2)).

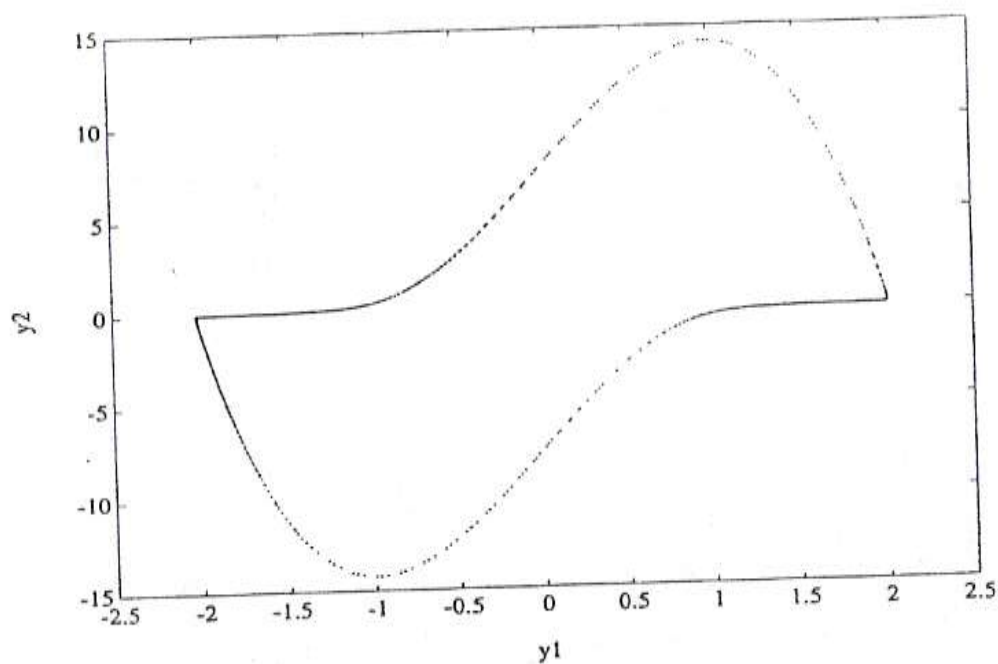


Fig. 1.1

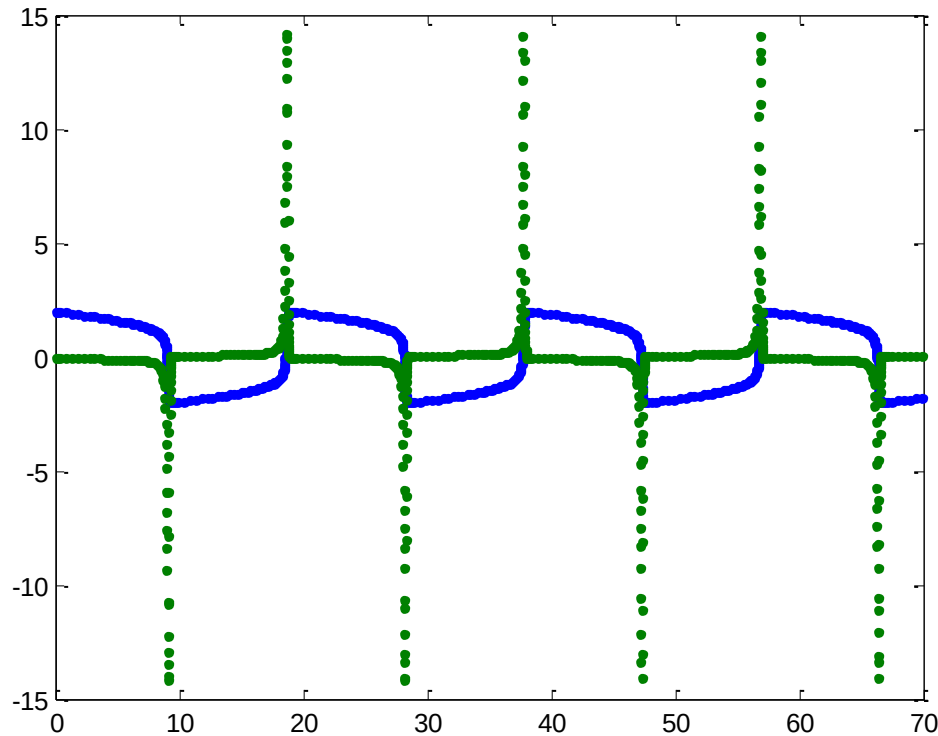


Fig. 1.2

In applying numerical methods to (1.1) – (1.2), it is important to use appropriate ones for two main reasons: first to generate solutions having the same qualitative behaviour as the continuous problems (1.1) – (1.2) as shown in fig. 1.1, and fig 1.2 by very high order methods with unbounded regions of absolute stability (RAS) combined with variable step-size $h_i = x_{i+1} - x_i$; $i = 0, 1, 2, \dots$. Second for efficiency in the choice of h_i because the implicit method is mandatory in each step of the integration of the stiff ivp, the linear multistep method (lmm) almost always requires nonlinear algebraic equations resulting at each step. Moreso that it is quite common in real life problems, such as in semi-discretization of PDEs by means of method of lines, to deal with systems of general nonlinear problems:

$$\underline{y}' = \underline{f}(x, \underline{y}), \quad x \in [x_0, x_N] \quad (1.3)$$

$$\underline{y}(x_0) = \underline{y}_0 \quad (1.4)$$

where $\underline{y}, \underline{f}, \underline{y}_0 \in R^m$, $m \geq 100$. The given function $\underline{f}$ is in general nonlinear and satisfies a Lipschitz condition for a unique solution.

2. The Linear Multistep Methods and Stability

For purpose of efficiency we shall be concerned here with the multistep methods

$$\sum_{r=0}^k \alpha_r y_{n+r} = h_i \sum_{r=0}^k \beta_r f(x_{n+r}, y_{n+r}) \quad (2.1)$$

of step number $k > 0$, $h_i = x_{i+1} - x_i$ is a variable step length. When $\beta_k \neq 0$ we call (2.1) implicit otherwise explicit. Formula (2.1) can be represented by two polynomials

$$\rho(z) = \sum_{r=0}^k \alpha_r z^r, \sigma(z) = \sum_{r=0}^k \beta_r z^r \quad (2.2)$$

with a unique stability polynomial

$$\rho(z) - \lambda h_i \sigma(z) = 0 \quad (2.3)$$

When (2.1) is applied to the test scalar equation $y' = \lambda y$, $\text{Re } \lambda < 0$, λ complex.

Definition 2.1: Region of absolute stability

Consider (2.3) with all the roots satisfying

$$0 < |z_r(\lambda h_i)| \leq 1. \quad (2.4)$$

A lmm (2.1) is said to be absolutely stable for a given value of λh_i , if each root z_r of (2.3) satisfies the inequality (2.4).

Our aim is, therefore, to single out these values of λh_i , for which the lmm (2.1) is absolutely stable (the region of absolute stability RAS). The RAS, ideally should admit all values of λ_i , $\text{Re}(\lambda) < 0$, so as to ensure that there is no limitation on the size of h_i , however large $|\lambda|$ may be. This leads us to the next definition for (2.1) to be useful in practice.

Definition 2.2: A-stability

A lmm is said to be A-stable if its region of absolute stability (RAS) contains the negative (left) complex half-plane $\mathbb{C}^-$.

The traditionally successful methods such as the Adams Moulton (AM) and the explicit Runge-Kutta (ERK) suffer step size constant imposed by stability when applied to stiff problems. Thus, to overcome this stability restriction on the step-size numerical methods that posses unbounded RAS (A-stable or stiffly stable) have been recommended for the solution of stiff ivp. Unfortunately, the condition of A-stability is extremely demanding. Dahlquist [1963] has shown the following results which are collectively known as his second Barrier theorem.

Theorem 2.1

- (i) No explicit lmm is A-stable ($\alpha_k \neq 0$);
- (ii) No A-stable lmm can have order greater than two;
- (iii) The second order A-stable lmm with the smallest error constant is the trapezium rule method.

3. The new block methods

In this paper, we consider a particular class of (2.1) having $\rho(z)$ in its simplest form,

$\rho(z) = z^{j-1}(z-1)$ $j = 1, 2, \dots, k$. These methods can be written as

$$y_{n+j} - y_{n+j-1} = h \sum_{i=0}^k \beta_i f_{n+i} \quad (3.1)$$

Putting $j = k$ gives Adams Moulton formula

Putting $j = 1$ and multiplying by -1 gives Reverse Adams Moulton formula

Putting $j = v$

$$\text{Where } v = \begin{cases} (k+1)/2 & \text{for odd } k \\ k/2 & \text{even } k \end{cases}$$

gives Generalized Adams Moulton (GAM) formula.

Rewriting GAM formula in form $y_{n+j} - y_{n+j-1} = h \sum_{i=0}^k \beta_i f_{n+k-i}$ and multiplying by -1 gives Reverse Generalized Adams Moulton

(RGAM) formula.

Our new family of GAM-RGAM based block methods are obtained by shifting the k -step pair of GAM and RGAM formulae along the grid points. They have $2(k-1)$ equations and the resulting block methods from the shifting are $2(k-1)$ step initial value methods (IVMs). Now we construct the new block methods as follows:

Case $k = 2$

$$y_{n+1} - y_n = \frac{h_n}{12} (5f_n + 8f_{n+1} - f_{n+2})$$

$$y_{n+1} - y_{n+2} = \frac{h_n}{12} (f_n - 8f_{n+1} - 5f_{n+2})$$

Case $k = 4$

$$y_{n+2} - y_{n+1} = \frac{h_n}{720} (-19f_n + 346f_{n+1} + 456f_{n+2} - 74f_{n+3} + 11f_{n+4})$$

$$y_{n+2} - y_{n+3} = \frac{h_n}{720} (-11f_n + 74f_{n+1} - 456f_{n+2} - 346f_{n+3} + 19f_{n+4})$$

$$y_{n+3} - y_{n+2} = \frac{h_{n+1}}{720} (-19f_{n+1} + 346f_{n+2} + 456f_{n+3} - 74f_{n+4} + 11f_{n+5})$$

$$y_{n+3} - y_{n+4} = \frac{h_{n+1}}{720} (-11f_{n+1} + 74f_{n+2} - 456f_{n+3} - 346f_{n+4} + 19f_{n+5})$$

$$y_{n+4} - y_{n+3} = \frac{h_{n+2}}{720} (-19f_{n+2} + 346f_{n+3} + 456f_{n+4} - 74f_{n+5} + 11f_{n+6})$$

$$y_{n+4} - y_{n+5} = \frac{h_{n+2}}{720} (-11f_{n+2} + 74f_{n+3} - 456f_{n+4} - 346f_{n+5} + 19f_{n+6})$$

Case k = 6

$$y_{n+3} - y_{n+2} = \frac{h_n}{60480} (271f_n - 2760f_{n+1} + 30819f_{n+2} + 37504f_{n+3} - 6771f_{n+4} + 1608f_{n+5} - 191f_{n+6})$$

$$y_{n+3} - y_{n+4} = \frac{h_n}{60480} (191f_n - 1608f_{n+1} + 6771f_{n+2} - 37504f_{n+3} - 30819f_{n+4} + 2760f_{n+5} - 271f_{n+6})$$

$$y_{n+4} - y_{n+3} = \frac{h_{n+1}}{60480} (271f_{n+1} - 2760f_{n+2} + 30819f_{n+3} + 37504f_{n+4} - 6771f_{n+5} + 1608f_{n+6} - 191f_{n+7})$$

$$y_{n+4} - y_{n+5} = \frac{h_{n+1}}{60480} (191f_{n+1} - 1608f_{n+2} + 6771f_{n+3} - 37504f_{n+4} - 30819f_{n+5} + 2760f_{n+6} - 271f_{n+7})$$

$$y_{n+5} - y_{n+4} = \frac{h_{n+2}}{60480} (271f_{n+2} - 2760f_{n+3} + 30819f_{n+4} + 37504f_{n+5} - 6771f_{n+6} + 1608f_{n+7} - 191f_{n+8})$$

$$y_{n+5} - y_{n+6} = \frac{h_{n+2}}{60480} (191f_{n+2} - 1608f_{n+3} + 6771f_{n+4} - 37504f_{n+5} - 30819f_{n+6} + 2760f_{n+7} - 271f_{n+8})$$

$$y_{n+6} - y_{n+5} = \frac{h_{n+3}}{60480} (271f_{n+3} - 2760f_{n+4} + 30819f_{n+5} + 37504f_{n+6} - 6771f_{n+7} + 1608f_{n+8} - 191f_{n+9})$$

$$y_{n+6} - y_{n+7} = \frac{h_{n+3}}{60480} (191f_{n+3} - 1608f_{n+4} + 6771f_{n+5} - 37504f_{n+6} - 30819f_{n+7} + 2760f_{n+8} - 271f_{n+9})$$

$$y_{n+7} - y_{n+6} = \frac{h_{n+4}}{60480} (271f_{n+4} - 2760f_{n+5} + 30819f_{n+6} + 37504f_{n+7} - 6771f_{n+8} + 1608f_{n+9} - 191f_{n+10})$$

$$y_{n+7} - y_{n+8} = \frac{h_{n+4}}{60480} (191f_{n+4} - 1608f_{n+5} + 6771f_{n+6} - 37504f_{n+7} - 30819f_{n+8} + 2760f_{n+9} - 271f_{n+10})$$

Case k = 8

$$y_{n+4} - y_{n+3} = \frac{h_n}{3628800} (-3233f_n + 36394f_{n+1} - 216014f_{n+2} + 1909858f_{n+3} + 2224480f_{n+4} - 425762f_{n+5} + 126286f_{n+6} - 25706f_{n+7} + 2497f_{n+8})$$

$$y_{n+4} - y_{n+5} = \frac{h_n}{3628800} (-2497f_n + 25706f_{n+1} - 126286f_{n+2} + 425762f_{n+3} - 2224480f_{n+4} - 1909858f_{n+5} + 216014f_{n+6} - 36394f_{n+7} + 3233f_{n+8})$$

$$y_{n+5} - y_{n+4} = \frac{h_{n+1}}{3628800} (-3233f_{n+1} + 36394f_{n+2} - 216014f_{n+3} + 1909858f_{n+4} + 2224480f_{n+5} - 425762f_{n+6} + 126286f_{n+7} - 25706f_{n+8} + 2497f_{n+9})$$

$$y_{n+5} - y_{n+6} = \frac{h_{n+1}}{3628800} (-2497f_{n+1} + 25706f_{n+2} - 126286f_{n+3} + 425762f_{n+4} - 2224480f_{n+5} - 1909858f_{n+6} + 216014f_{n+7} - 36394f_{n+8} + 3233f_{n+9})$$

$$y_{n+6} - y_{n+5} = \frac{h_{n+2}}{3628800} (-3233f_{n+2} + 36394f_{n+3} - 216014f_{n+4} + 1909858f_{n+5} + 2224480f_{n+6} - 425762f_{n+7} + 126286f_{n+8} - 25706f_{n+9} + 2497f_{n+10})$$

$$y_{n+6} - y_{n+7} = \frac{h_{n+2}}{3628800} (-2497f_{n+2} + 25706f_{n+3} - 126286f_{n+4} + 425762f_{n+5} - 2224480f_{n+6} - 1909858f_{n+7} + 216014f_{n+8} - 36394f_{n+9} + 3233f_{n+10})$$

$$y_{n+7} - y_{n+6} = \frac{h_{n+3}}{3628800} (-3233f_{n+3} + 36394f_{n+4} - 216014f_{n+5} + 1909858f_{n+6} + 2224480f_{n+7} - 425762f_{n+8} + 126286f_{n+9} - 25706f_{n+10} + 2497f_{n+11})$$

$$y_{n+7} - y_{n+8} = \frac{h_{n+3}}{3628800} (-2497 f_{n+3} + 25706 f_{n+4} - 126286 f_{n+5} + 425762 f_{n+6} - 2224480 f_{n+7} - 1909858 f_{n+8} \\ + 216014 f_{n+9} - 36394 f_{n+10} + 3233 f_{n+11})$$

$$y_{n+8} - y_{n+7} = \frac{h_{n+4}}{3628800} (-3233 f_{n+4} + 36394 f_{n+5} - 216014 f_{n+6} + 1909858 f_{n+7} + 2224480 f_{n+8} - 425762 f_{n+9} \\ + 126286 f_{n+10} - 25706 f_{n+11} + 2497 f_{n+12})$$

$$y_{n+8} - y_{n+9} = \frac{h_{n+4}}{3628800} (-2497 f_{n+4} + 25706 f_{n+5} - 126286 f_{n+6} + 425762 f_{n+7} - 2224480 f_{n+8} - 1909858 f_{n+9} \\ + 216014 f_{n+10} - 36394 f_{n+11} + 3233 f_{n+12})$$

$$y_{n+9} - y_{n+8} = \frac{h_{n+5}}{3628800} (-3233 f_{n+5} + 36394 f_{n+6} - 216014 f_{n+7} + 1909858 f_{n+8} + 2224480 f_{n+9} - 425762 f_{n+10} \\ + 126286 f_{n+11} - 25706 f_{n+12} + 2497 f_{n+13})$$

$$y_{n+9} - y_{n+10} = \frac{h_{n+5}}{3628800} (-2497 f_{n+5} + 25706 f_{n+6} - 126286 f_{n+7} + 425762 f_{n+8} - 2224480 f_{n+9} - 1909858 f_{n+10} \\ + 216014 f_{n+11} - 36394 f_{n+12} + 3233 f_{n+13})$$

$$y_{n+10} - y_{n+9} = \frac{h_{n+6}}{3628800} (-3233 f_{n+6} + 36394 f_{n+7} - 216014 f_{n+8} + 1909858 f_{n+9} + 2224480 f_{n+10} - 425762 f_{n+11} \\ + 126286 f_{n+12} - 25706 f_{n+13} + 2497 f_{n+14})$$

$$y_{n+10} - y_{n+11} = \frac{h_{n+6}}{3628800} (-2497 f_{n+6} + 25706 f_{n+7} - 126286 f_{n+8} + 425762 f_{n+9} - 2224480 f_{n+10} - 1909858 f_{n+11} \\ + 216014 f_{n+12} - 36394 f_{n+13} + 3233 f_{n+14})$$

The formulas are of order $k+1$. They have region of absolute stability (RAS) as shown in figures 1 and 2 and hence adequate for solving stiff ivp. They are A-stable for $k = 2, 4, 6, 8, \dots, 30$.

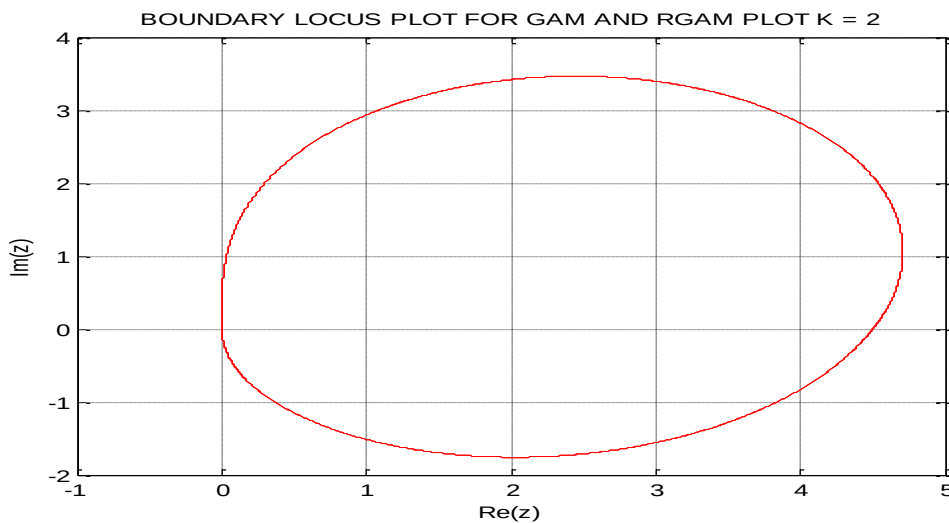


Fig. 1

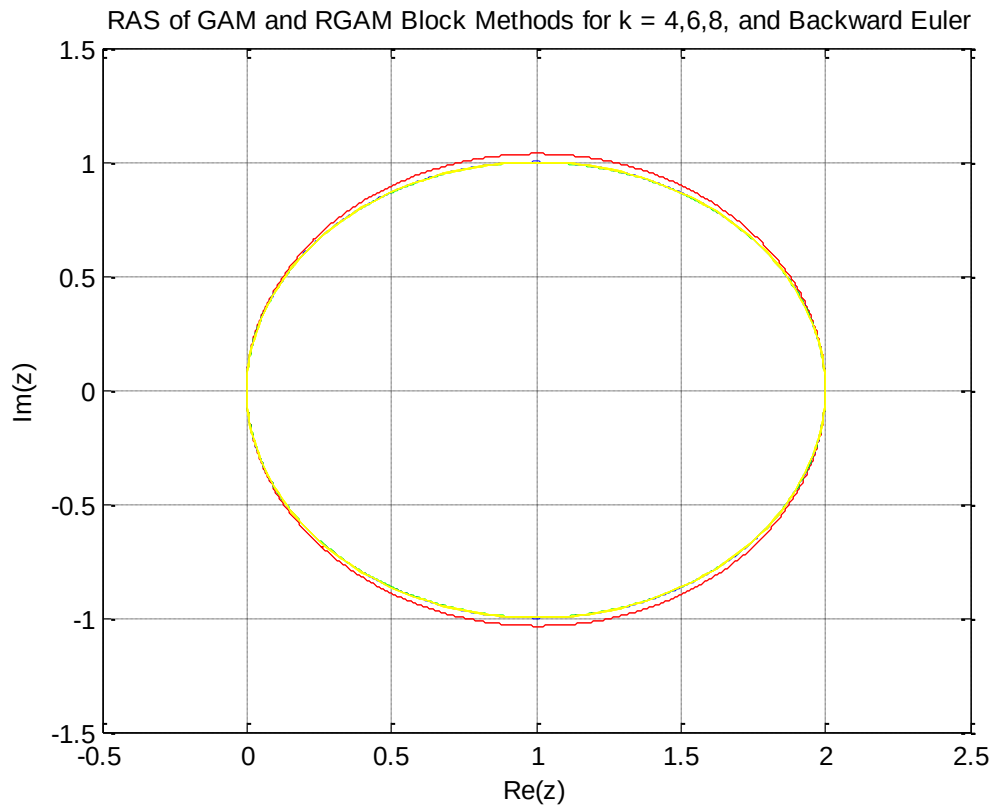


Fig. 2

5. Conclusion

The Generalized Adams Moulton (GAM) methods have been extended to Reverse Generalized Adams Moulton (RGAM) methods and from them (GAM/RGAM pairs) a new family of $2(k-1)$ step block methods has been developed for stiff initial value problems (ivp). These block methods are very fast, accurate of order $(k+1)$ and are A-stable for $k = 2, 4, 6, 8, \dots, 30$. The boundary locus plot of the region of absolute stability produces a unit circle for $k = 6, 8, \dots, 30$ which all coincide with that of Backward Euler (indicated by yellow, green,) lines in fig 2. This is a desirable stability property feature of the new block methods

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A model for classifying incidence rate of filariasis in a habitat

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Abstract

A new model for classifying the incidence rate of filariasis in the given habitat is developed base upon some inequalities, field experience, rapid test and transmission factors; the model was used for classifying the endemicity of the diseases in the habitat into meso-endemic, hypo-endemic and hyper-endemic. We applied the model to the data obtained from field study of 20 villages through Rapid Assessment Test (RAT). The present work has potential usefulness in priotising and mobilizing resources for combating filariasis in a given habitat.

Keywords and Phrases

Disease control, model, endemicity, filariasis, inequalities and transmission.

Mathematics Subject Classification 2000:15A29, 92C60 & 03C30.

1. Introduction

Filariasis is the general name used for diseases caused by threadlike nematodes (worms) and the most notable of this class is the one responsible for the river blindness (onchocerciasis) which is a disease of the tropical Africa and Central America. The vector that transmits the disease is the black flies and cause by infection with filaria form of nematodes of genus onchoceria, especially onchoceria volvulus Browne (1960) and characterized by nodular swelling on the skin and lesions of the eyes. The infection with the slender threadlike roundworm (filaria) deposited under the skin by the black flies leads to lesions in the skin 'leopard skin' as it is often called and those deposited in the eyes lead to visual impairment that lead to blindness if not treated (Cao et al. 1997; Edungbola and Asaola, 1984; Ogidi, 2000).

Filariasis is a blackly or mosquito borne disease predominantly found in rural areas and places with non-portable water supply. It is found in areas where filarial worms and the black fly/mosquito are common. The disease is one of the World Health Organization (WHO) diseases and could be very endemic in some areas (Burton, 1960; Cao et al. 1997; WHO, 1994; Freedman, et al. 1994).

Endemicity is all about measuring the degree of severity of the disease in the community. Whenever the severity of the disease is extremely high it said to be hyper-endemic; while milder form is called meso-endemic. Hypo-endemicity is the disease severity that is extremely low (Ogidi, 2000; Oyelami et al. 2001). We will discuss more on criteria for classifying the disease endemicity later.

The present paper is conceived from our interest to develop a model which could be used for classifying the endemicity of the disease according to various characterization parameters. Method of rapid assessment test (RAT) is often used for selecting at random some members of the community and simple medical observations of the disease symptoms on the patients are made. The results are then used for classifying the endemicity of the disease. There is the onchosim package which is often used for the classifications.

Onchosim is a simulation package developed in collaboration with the onchocerciasis Control Programme in West Africa (OCP) and is used in evaluation and planning of Control Operation in West Africa in evaluation and planning of Control Operation (Plaiser et.al.,1990) the underlying stochastic model used in onchosim involved detail life history of the parasite-onchocerca volvulus as transmitted to the victims via simulum flies (Gerard et.al.,2003)

The filaria model we will consider has many in-built mechanism for forecasting the population of people with worm, noodles and visual impairment from time to time using some equations .We can also use the model to obtain the situation analysis (incidence rate) or endemicity of the disease at given point in time. The model has the component for simulation to obtain the prevalence rate or the endemicity profile using the baseline data obtained from the incidence rate.

Furthermore, our ultimate goal is to develop a model which can be used to study spatial distribution of filariasis in a given habit. The model can be used to obtain the management information system on the spread of the filariasis in as integrated community like a country or region in a continent so long as there are information on worm and noodle presence, and visual impairments. The model can be applied by dividing the country or the region in to blocks, apply the model to each block to study the endemicity of the disease block wise. Moreover, we can use the model to develop a geomap or filarial map of the endemicity to capture the spread of the filariasis in the community under consideration. This work is still in the cradle of development and implementation. We hope to further develop the model to incorporate new trends that would further enhance wider utility of the model.

In this paper, our interest is to develop a functional equation kind of model couple with the use of elementary property of optimization and our experiences with the vector responsible for the disease, experimental and occupation factors to develop the model which can be used to measure the endemicity of the disease in a habitat. In order to develop the model, certain factors are considered base on our field experience with the occurrence of filariasis. These are field experience, vector and transmission factors and are considered as follows:

1.1 Field Experimental Factors

From experience, we know that intervention with mectrizan had caused change in most of the symptoms after five to twelve years of treatment. After application of mectrizan to filarial patients it was observed that:

1. Noodle carries were freed of most of the noodles;
2. There is a considerable skin improvement (improvement in the Leopard skin);
3. There is improvement in visual impairment if it has not reached stage of blindness.

1.2 Vector Factors

The model can only be applied to habitats where the vector can breed effectively these include foci, e.g., spillage or canals.

1.3 Transmission Factors

The following factors are responsible for the transmission of the disease:

- (1) The vector availability;
- (2) Proximity or suitability of the community to vector breeding site;
- (3) Occupational hazard, i.e., enhancing man-vector contact;
- (4) There must be a worm reservoir.

2. The Model

The filariasis model is developed under the assumptions that:

- (1) No treatment or intervention with anti-filarial drugs e.g., ivermectine (mectizan) takes places;
- (2) There is appropriate human vector availability.

Now consider the filariasis model

$$X = (W, N, L_s, B) = (\alpha_1 s_1 + \alpha_2 s_2)(W + N + L_s + B)^2 - f(N, L_s) - g(N, B) - H(L_s, B) - J(W, N) \quad (1)$$

$$W = \frac{k_1}{\lambda_1 + 1} X^\alpha$$

$$N = \lambda_2 X^\alpha W = \frac{k_1 \alpha_2}{\lambda_1 + 1} X^{\alpha+s}$$

$$L_s = p_1 X W = \frac{p_1 k_1}{\lambda_1 + 1} X^{\alpha+1}$$

$$B = \gamma X^{-c} W = \frac{\gamma k_1}{\lambda_1 + 1} X^{\alpha-c}$$

Where

X is the population of the people in the community showing at least one of the filarial symptoms, these can be divided into four population sub-groups namely:

X is the population of the people in the community showing at least one of the filarial symptoms, these can be divided into four population sub-groups namely:

W people infected with filarial worm

N people with nodules

L_s people with leopard skin

B people with visual impairment.

Furthermore, since there is some degree of overlapping of the subgroups, e.g. there are people in the community who have nodules as well as visual impairments or Leopard skin and nodules and so on. The population of the people with both nodules and visual impairments is what we have denoted by g(N,B) which is often unknown and can be estimated by some formulae or other means (e.g. By experimental means). Similarly, the population of people with leopard skins and nodules is denoted by f(N,L_s). Finally, the population of the people with leopard skins and visual impairments is denoted by H(L_s,B).

The quadratic dependence of the population of X on W, N, L_s and B was made rather than linear dependence this is because of two reasons. The first reason is due to the rapid test assessment for gathering data in which only people with at least one symptom of filarial disease were considered. Obviously, there are people within the population without any symptom of the disease. To develop an accurate model, we must therefore find a way of coming up with a model which is realistic and accurate in spite of this problem.

Furthermore, the second reasons was due to the fact that we are interested in a nonlinear kind of population dependence, which will perhaps, give more realistic estimation of the population, even though, most non-linear models often have many complexities and phenomena associated with them. The term

(W+N+ L_s +B)² can as well be replaced by (W+N+ L_s +B)^α where α is any real number or even may be a rational number. The choice of α will give rise to some family of equations for modelling filariasis. The question one needs to ask is which of the family will give the best situation analysis for filariasis? This is an open problem.

2.1 The Parameters used are:

k₁ the rate constant for people with filarial worm

λ₁ the proximity to river constant

p₁ depigmentation constant which is the measure of the appearance of leopard skin

a is the occupation type of people in the habitat eg. Farmers, hunters, woodcutters, fishermen, Cattle rarer etc., for less prone people, it is such that 0 < α ≤ 1.

We take

$a > 1$ for occupations that enhance man-vector interactions
 b patient's sensitivity parameter

λ_2 fly biting rate (the number of bit received from the flies)

γ is the visual coefficient which is the measure of visual impairment

c is a biological rate constant

α_i characterization of the village on the scale of $i=1$ if a village with high rate $i=2$ secondary village and $i=3$ alternative village s_i is sex parameter $i=1$ for male and $i=2$ for female.

3. Classification of a Village According to filaria endemicity

We classify a village according to types of endemicity and the data used was gathered from Rapid Assessment Test (RAT) which involve selecting at random some members of the community. Simple medical examination such as visual observation of the skin condition to determining depigmentation, detect presence of nodules and leopard skin. The classification is made by slightly modifying the WHO Expert Committee on Onchocerciasis in the Third report of 1987, see Table 1 below:

Type	Worm load	Criterion	Nodule presence	Visual impairment
Hypo-endemic	$W \leq 0.40X$	$N \geq 0.40X$		$B \geq 0.1X$
Meso-endemic	$0.4X < W < 0.7X$	$0.1X \leq N$		$B \leq 0.2X$
Hyper-endemic	$W > 0.76X$	$0.10X \leq N \leq 0.39X$		$B \leq 0.2X$

Table 1: The table showing the classification metric for classifying endemicity of worm, nodule and visual impairment

We need to develop a utility function from $X = X(W, N, L_s, B)$ by minimizing it with respect to N, W and L_s respectively.

$$\text{From (1) } \frac{\partial X}{\partial N} = 0, \frac{\partial X}{\partial W} = 0, \frac{\partial X}{\partial L_s} = 0$$

It implies that

$$\begin{aligned} \frac{\partial J}{\partial W} &= 2P(W + N + L_s + B) \\ \frac{\partial(f + J + H)}{\partial N} &= 2P(W + N + L_s + B) \\ \frac{\partial f + H}{\partial L_s} &= 2P(W + N + L_s + B) \\ \text{where } P &= \alpha_1 s_1 + \alpha_2 s_2 \end{aligned}$$

Integrations of above equations yield

$$J = 2P[(W+N+L_s+B)W+g_1(N, L_s, B)]$$

$$f+H = 2P[(W+N+L_s+B)L_s+g_2(N, W, B)]$$

For sake simplicity and situation analysis we consider $g_i, i=1,2$.

$$g_1 = a_0L_s^2 + a_1BL_s + a_2BW + a_3BN + a_4W + a_5L_s + a_6$$

$$g_2 = b_0W^2 + b_1BN + b_2NW + b_3BN + b_4N^2 + b_5W + b_7$$

Where a_i and b_i are constants.

There are several degree of freedom for choosing $g_i, i=1,2$ we decided to choose $g_1 = L_s^2 + B L_s + BW - BN$ and $g_2 = W^2 - N^2 + BN$ as particular members of the family. If the sum of the coefficients of g_1 is $\sum_1$ and that of g_2 is $\sum_2$ then we say the filarial model is $(\sum_1, \sum_2)$ filarial model. In this particular model, we use (2,1) filarial model for the analysis. We can show that

$$g_1 = 2P[L_s^2 + BW + B L_s - BN] = 2PU$$

$$g_2 = 2P[W^2 - N^2 + BN] = 2PV$$

Therefore

$$J = 2(\alpha_1 s_1 + \alpha_2 s_2)((W+N+L_s+B)W + L_s^2 + BW + B L_s - BN)$$

$$f+H = 2(\alpha_1 s_1 + \alpha_2 s_2)((W+N+L_s+B)L_s + W^2 - N^2 + BN)$$

Now let

$$S = W + N + L_s + B$$

$$U = L_s^2 + BW + B L_s - BN$$

$$V = W^2 - N^2 + BN$$

$$X = X(S, U, V) = PS^2 - 2P(SW + U) - 2P(S L_s + V)$$

3. Methodology

In order to apply our model, a Rapid Assessment Test (RAT) was conducted to obtain real life data to test the model. See Table 1 for the RAT result hereby called the infection matrix. The community where RAT was administered contains 20 villages and we divided the 20 villages into 4 blocks, namely, A, B, C, D and E. We applied the model to each block and study the endemicity of the disease in each block. Using Table 2 and the Matlab software, we simulated equations (19–22) and obtain the result in Table 3 and furthermore,

determine the absolute values of $\frac{W}{X}$, $\frac{N}{X}$ and $\frac{B}{X}$ as in Table 3.

We consider the following infection matrix:

Village	W	N	L_s	B	Total	s_1	s_2
1	10	0	0	10	20	15	05
2	02	08	10	10	30	20	10
3	03	16	01	00	20	08	12
4	03	01	03	07	14	06	08
5	06	02	03	01	12	06	06
6	05	05	00	00	10	04	06
7	07	01	00	00	08	08	08

8	00	00	01	06	07	03	04
9	03	03	02	00	08	07	01
10	15	01	08	01	25	13	12
11	05	07	06	05	23	13	10
12	07	08	10	15	40	29	11
13	09	10	11	12	42	30	12
14	16	15	20	01	42	31	11
15	07	15	02	06	30	15	15
16	17	15	12	20	64	33	31
17	30	50	10	20	110	67	43
18	20	15	30	40	105	60	45
19	19	30	15	20	84	67	17
20	30	20	15	04	69	42	27

Table 2: Result obtained from rapid test in the field for the community with 20 villages.

5. Results and Discussion

The information in the infection matrix in Table 2 was simulated and we obtained the following result:

Block	W	L_s	S	U	V	P	$ \frac{W}{X} $	$ \frac{N}{X} $
A	18	14	84	574	78.5	12.94	0.008530	0.001186
B	18	4	37	114	316	11.38	0.002400	0.003400
C	30	26	74	1453	938	6.75	0.009400	0.005900
D	45	45	184	3660	1145	6.00	0.000134	0.001657
E	99	70	368	11046	4626	24.25	0.000405	0.000470

Table 3: The showing the values of the parameters used for the model

Block	$ \frac{B}{X} $
A	0.001280
B	0.000940
C	0.000660
D	0.000029
E	0.000344

Table 4: Computation of basic parameters for the endemicity classification

Base upon the result in the Table 3 and using the classification criteria in the Table 1 , we have the following analysis :

Block	W-endemicity	N-endemicity	B-endemicity
A	hypo-endemic	no classification	hypo-endemic
B	hypo-endemic	no classification	hypo-endemic
C	hypo-endemic	no classification	hypo-endemic
D	hypo-endemic	no classification	hypo-endemic
E	hypo-endemic	no classification	hypo-endemic

Table 5: W, N and B endemicity classification in the blocks.

From the analysis of the result obtained in the Table 5 , the prevalence of the disease is classified according to three most vital parameters: The presence of worm, noddles and the visual impairment of the people selected in the 20 villages used for the (RAT).

The analysis of the results show that for presence of worms and visual impairments in each block are hypo-endemic in the five blocks (A-E) and the model gives no classification using the presence of noddles. What is the implication of this? And is the absence of information of endemicity of the noddle going to hamper our judgment about the endemicity of the disease in the given community? The presence of information on worm endemicity and visual impairment endemicity has not underscore the integrity of our conclusion on the endemicity of the disease in the community under study. In fact, the information on the worm distribution and visual impairment are logically enough for us to talk about the endemicity of the disease.

Hypo-endemicity of the worm presence suggests the fact that large substantial number of people has worms and could be dangerous as time goes on if not treated. Hypo-endemicity of visual impairment, on the other hand, suggests that majority of people in the community are now tending toward blindness. Therefore, an intervention is needed to eliminate the worms and to restore vision through the use of drugs like mectrizan.

The classification of the filarial presence according to worm, noddles and visual impairment is all about having comprehensive knowledge of the stage of the disease in the community. The initial stage information will involve the number of people who have the worm. The intermediate stage is dissipated by the presence of noddles and advanced stage is connoted by the visual impairment scenario. The endemicity of visual impairment is all about having comprehensive knowledge about people almost blind or totally blind.

Furthermore, hyper-endemic is the severity of highest order of the disease in the give community. Hypo-endemicity and meso-endemicity of the disease is all about moderate and mild presence of the disease respectively in the community. From the data analysis of the disease filarial worm is presence with some mild form (hypo-endemic) and hypo-endemic nature of the visual impairment is suggests that most victims are almost blind and it calls for rapid intervention vis-a-vis mobilization of resources material and human to control the disease in the community.

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Appendix



Figure 1: The 20 villages divided into Block A,...E

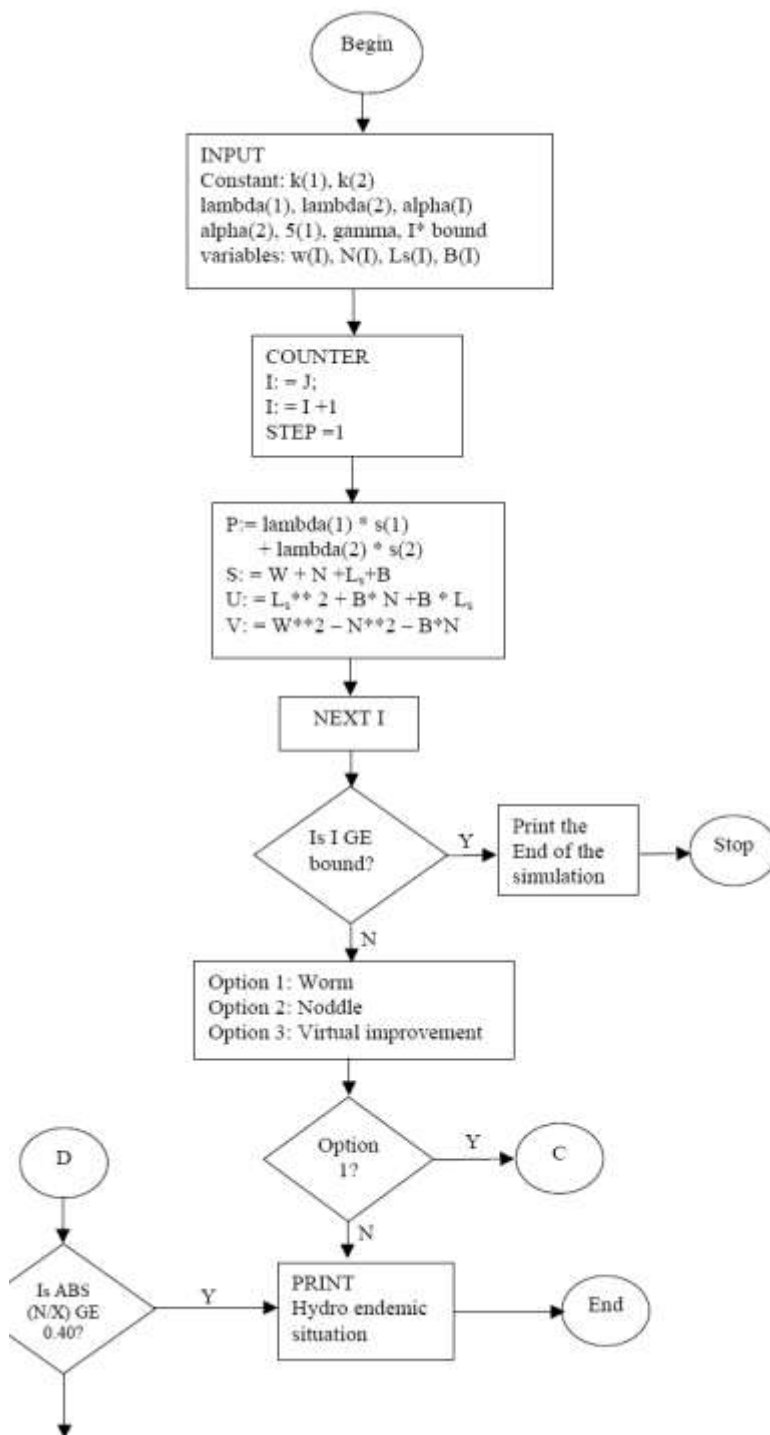


Figure 2a: The flow chat used for the simulation

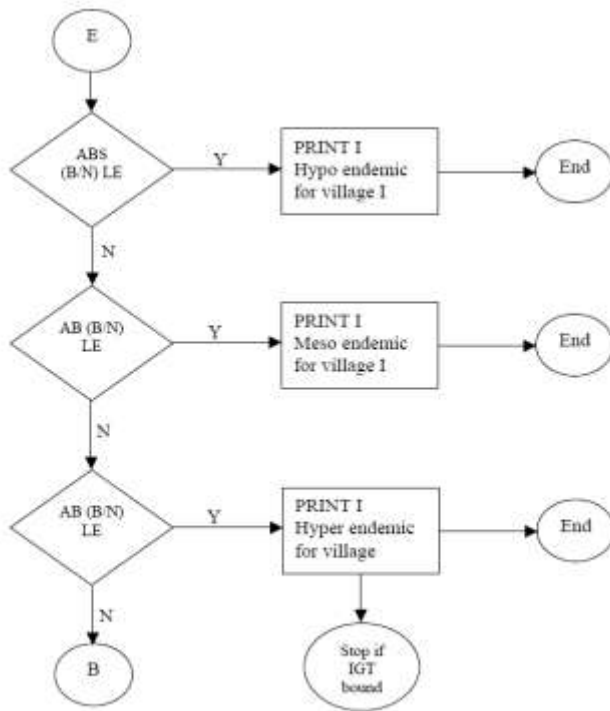


Figure 2b: The flow chat used for the simulation



APPLICATION OF THE B-TRANSFORM TO SOME IMPULSIVE CONTROL MODELS

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Abstract

In this paper the concept of the B-Stability is introduced to compliment the B-transform developed in [11, 13 – 15]. The results obtained are applied to some impulsive control problems. It is found that B-Stability it is easy to handle than the stability in the Lyapunov's sense for the impulsive control systems(ICS). Our interest is to study the stability of ICS via the B-transform. For this reason, a new stability concept is initiated, that is, the B-Stability. The B-Stability is used to investigate the stability property of some biological and physiological problems. The problem solved are: on the drug compartment model which is the impulsive delay type of the Kruger-Thermer's model (see ([5]&[8])); problem of implantation of the hydrocephalus valves in man and non-linear population stratified into groups.

It is established that if the zero solution of the ICS is B-Stable then it is stable in the Lyapunov's sense too.

The solution of the Kruger-Thermer's model is B-Asymptotically stable, the hydrocephalus model is B-Stable and we obtain result on B-Stability for the impulsive non-linear epidemiological models.

Keywords

B-Stability, B-Transform, impulsive control systems, models, biological and physiological problems.

1 INTRODUCTION

Impulsive differential equations are equations found to be exhibiting some "jumps", "shocks" etc. during the process of evolution. The solutions of impulsive differential equations (IDEs) possess discontinuities, and the impulsive moments describing them not only depend on some impulsive sets but also on the dynamics of the evolutionary process characterizing them ([15],[17]). The IDEs, of late are now constituting the core of many investigations in the differential equations and the control systems ([9],[18-19]).

There are a lot of potentialities on the study of IDEs and the knowledge of its will shed more light on the understanding and diagnoses of psychological disorders and brain diseases such as epilepsy, nacolety and convulsions. The theory of neural network has no doubt offer a lot of solutions to some challenging medical problems which remained insurmountable several years ago. Impulsive system theory has offered alternative approach to understand and to proffer solutions to many medical problems (see [2] & [15]).

The Control system is an area where IDEs have applications ([9],[18],[19-22]), even though, this area is gradually experiencing growth of late as per impulsive system theory is concerned. In control system, the interest is that of studying systems whose behavioral activities are governed by some equations, for which the fundamental problem is to regulate the systems in a prescribed manner. In the recent times it has been discovered that the state of the control systems may be impulsive and even, in some cases, the control variables that regulate them may also exhibit some impulsive tendency during the evolutionary process ([18-19]). In the switching ICS the impulse may generated from switching and are applied in the electrical engineering.

B-transform provides a formalism to obtain closed solution form to the fixed moments IDEs. The B-transform was developed to obtain the closed solution IDEs by considering the dynamics and the impulsive moments of the systems. Most existing operational calculus methods like Fourier, Laplace etc. are not known to be applicable to IDEs directly ([13],[14]&[15]). Moreover, in the literature, we observed that many publications were made on IDEs but very few are solely devoted to impulsive models. It is interesting to note that many publications on IDEs justified the existence of impulses in several real life processes, but few were devoted to the construction of impulsive models in a concrete term. There is large potential of applications of IDEs to real life processes which are yet to be explored.

In this paper we introduce some impulsive models with view to solve some biological and physiological problems. Another novelty of this paper is the introduction of the B-Stability. In general, to obtain the inverse B-transform it is always very difficult. The B-Stability concept was introduced to provide a

qualitative method for investigating the stability of IDEs without the need to find the inverse of the B-transform. We will also establish that if an IDEs has a B-Stable solution then solution is also stable in the Lyapunov sense but the converse may not be so.

The B-stability will be used to study some real life control problems that are of biological and physiological origins.

The Kruger-Thiermer process([8]&[17]) is used in a broader sense than in the one in the literature by introducing delay,control and impulses into the model in a justifiable way .This will offer us a broad platform to understand several real life problems that can be modeled in using ICS.We also consider a drug compartment model and the B-Stability will be used to analyze the model and the reasons why some people have physiological problems like kidney and liver failures are to be investigated using the model.

Furthermore,we will also consider the hydrocephalus model which is the modification of a typical non-impulsive hydrocephalus model ([7]).The result obtained through the B-Stability suggests that hydrocephalus valve can be constructed to ameliorate the problem of children with swollen heads because of cerebrospinal fluid draining into the bloodstreams. The construction of the hydrocephalus valve can be made by maintaining balance or equilibrium between the fluid in the head and the spinal column.

Population is noted to be impulsive (see [1],[5][15]&[17])and not continuous as most investigations on ODEs often assumed. Most epidemiological problems are typical population problems and hence it is reasonable to study them from the impulsive point of view ([1],[13]&[17]).Therefore the epidemiological model considered here is on impulsive population stratified into distinct homogenous groups which contain non-linear control functions which may have direct or indirect influence on the populations. The model seeks unifying several epidemiological models considering the impulsive nature of population,the non-linear biological control on the population with time lag and the use the B-stability to analyze the model.

Finally, this paper is organized in the following ways: section one gives the introduction; Section two introduce the notations ,statement of the problem, B-transform and the proposed B-stability along with some basic definitions. Main results are found in section three. In section four,the results obtained are applied to Kruger-Thiermer, hydrocephalus and epidemiological models to illustrate the real life applications of B-stability.

2 Notations and Statement of the Problem

We will denotes by $M_n(R^+)$ the set of $n \times n$ matrices on $R^+ = [0, +\infty)$ and $|x|$ will represent the norm of vector x which belongs to some subset of R^n . If $A \in M_n(R^+)$ then the norm of A being

$$|A| = \max \left(\sum_{i,j=1}^n |a_{ij}| \right).$$

$PC(D, R^+)$ is the set of functions $\Psi : R^+ \rightarrow D$ which are continuous for $t \in R^+, t \neq t_k$, having discontinuities of first type at the points t_k and are left continuous.

C is the space for which the control variables belong.

Finally, $H(q)$ will represent the transfer function for the given impulsive control system.

2.0.1 B-Transform and B-Stability

Consider impulsive non-linear control systems(ICS)

$$(1) \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), t \neq t_k, k = 0, 1, 2, 3... \\ \Delta x(t_k) = I(x(t_k)) \\ 0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = +\infty \end{cases}$$

$$f : R^+ \times R^n \times C \rightarrow R^n$$

$$I : R^n \rightarrow R^n$$

f and I satisfy some regularity conditions such that f is continuous and Lipschitz with respect to the second and the third variables and I is continuous and lipschitz in its argument. These conditions allow the solution $x = x(t)$ of eq.(1) to exist and to be uniquely determined in $PC(D, R^n)$ for any given $u(t) \in C$.

In order to set the stage for the application of the proposed B-stability, we linearize eq.(1) and obtain

$$(2) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), t \neq t_k, k = 0, 1, 2, \dots \\ \Delta x(t_k) = I(x(t_k)) \\ 0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = +\infty \end{cases}$$

Where

$$A = \left(\frac{\partial f}{\partial x} \right)_{x=x_0, u=u_0} \text{ and } B = \left(\frac{\partial f}{\partial u} \right)_{x=x_0, u=u_0}$$

$$A \in M_n(R^+), B \in M_n(R^+) \text{ and } u \in C.$$

Fundamental to our study are the following definitions:

Definition 1([5],[17])

$x(t) = x(t, t_0, x_0, u)$ will be said to be the solution of eq. (1) i.e. the solution of (ICS) existing for $t \geq t_0$ if the following conditions are satisfied:

(i) $x(t) \in PC(D, R^n)$ such that

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ for } t \neq t_k, k = 0, 1, 2, 3, \dots$$

$$x(t_k) + \Delta x(t_k) = x(t_k) + \beta_k x(t_k) = (1 + \beta_k)x(t_k) = I(x(t_k)).$$

(ii) For $t = t_k, k = 0, 1, 2, \dots, x(t)$ instantaneously jump from t_k to new positions

$$x + \Delta x = x(t_k) + \beta_k x(t_k) = (1 + \beta_k)x(t) = I(x(t_k))$$

and satisfies the following equation

$$\begin{cases} \frac{dz(t)}{dt} = f(t, z(t), u(t)), t \neq t_k, k = 0, 1, 2, \dots \\ \Delta z(t = t_k) = I(z(t_k)) \\ 0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = +\infty \end{cases}$$

Where

$z = x(t) \in PC(D, R^n)$ for $u(t) \in C$.

Definition 2 ([5])

We will say that the zero solution $x(t) = 0$ of (ICS) is (i) Stable, if for each $\epsilon > 0, \eta > 0$ and $t_0 \in R^+$ there exists $\delta = \delta(t_0, \epsilon, \eta) > 0$ such that $|x_0| < \delta$ implies $|x(t)| < \epsilon$ for $t \geq t_0$ and $|t - t_k| > \eta, k = 0, 1, 2, \dots$;

(ii) Uniformly Stable

If δ in (i) independent of t_0 ;

(iii) Asymptotically Stable

If it is stable and every for $\epsilon > 0$ and $t_0 \in R^+$ there exist $\delta_0 = \delta(t_0) > 0$ and $\tau = \tau(t_0, \epsilon, \eta) > 0$ such that $|x_0| < \delta$ implies $|x(t)| < \epsilon$ for $t \geq t_0 + \tau$ and $|t - t_k| > \eta$.

Definition 3 ([11], [13])

We define the B-transform of $x(t) \in PC(D, R^n)$ as

$$(4) \quad B_n' x(t) = \bar{x}(q) = L_c x(t) + L_I x(t)$$

Where

$$(5) \quad L_c x(t) = \int_0^\infty e^{-t/q^n} x(t) dt$$

And

$$(6) \quad L_I x(t) = \sum_{t_0 < t_k < t} e^{-t/q^{n'}} I(x(t_k))$$

Where $n' = 1, 2, 3, \dots$ is the order of the transform and $L_I x(t)$ gives the transform of the impulsive function, $I(x(t))$ at the points $\{t_k\}, k = 0, 1, 2, \dots$ (see Theorem 1 in [13]) where we showed that the inverse B-transform for $\bar{x}(q)$ was

$$(7) \quad x_c(t) = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} x_c(q) e^{tq} dq$$

and

$$x_I(t) = \sum_{t_0 < t_k < t} \phi(t_k, t) I(x(t_k))$$

Where

$$(9) \quad \phi(t_k, t) = \int_{v-i\infty}^{v+i\infty} dq e^{t_k/q^{n/} + tq}, k = 0, 1, 2, \dots$$

C is any arbitrary contour in the complex domain where the given integral exists.

Then

$$(10) \quad |x(t)| \leq \frac{1}{2\pi i} \int_C |x(q)| e^{tq} dq + \sum_{t_0 < t_k < t} |\phi(t_k, t)| |I(x(t_k))|$$

where

$$\phi(t_k, t) = \int_C dq e^{t_k/q^{n/} + tq}$$

Next, we use the B-transform to obtain the transfer function and the solution of the ICS.

Definition4

Consider the (ICS) in eq.(1) and the corresponding linearize form in eq.(2). We define the transfer function for eq.(2) as

$$(11) \quad H(q) = (A - \frac{1}{q}E)^{-1}, E \in M_n(R^+) \text{ is the identity matrix in } R^+.$$

We can now define the B-stability for (ICS) as follows:

Definition 5

Let $r > 0$ be a constant, the zero solution $x(t) = 0$ of eq.(1) is said to be

A. B-Stable: If for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0) > 0$ and a continuous function $\varphi(t, r)$ such that if $|x_0| < \delta$ then $|H(q)| < r$ for $\varphi(\epsilon, r) > \delta$.

B. B-Uniformly Stable: If $\delta = \delta(\epsilon)$ in (i) above

C. B- Asymptotically stable if it is B - Stable and there exists a $T = T(t_0, \epsilon, \eta)$ such that $|x_0| < \delta_0$ implies $|H(q)| < r$ for $q \geq q_0 + T$, $|q - t_k| > \eta$ and q_0 is an arbitrary real number. Let us now pose a question, what relationship exists between the normal stability (Stability in Lyapunov sense) and the B-stability concept? B-stability implies the regular stability concept defined in definition 1 whereas the converse is not unnecessarily true.

3

MAIN RESULTS

Theorem1

Suppose that the following conditions are satisfied:

H_1 : There exists a sequence of real numbers $\{r_k\}$ such that $r_k > 0, k = 1, 2, \dots$ such that $|I(x(t_k))| < r_k$ and the series $\sum_{t_0 < t_k < t} r_k e^{t_k/q}$ is convergent.

H_2 : There exist constants $m_1 > 0$ and $m_2 > 0$ such that $\int_0^\infty e^{t/q} |x(t)| dt < m_1$

$$\text{and } \sum_{t_0 < t_k < t} e^{-t/q} |I(x(t_k))| < m_2.$$

H_3 : There exists constant $r_0 > 0$ and the control function $u(t) \in C$ such that if $|u(q)| \leq r_0$.

Then if the zero solution of (ICS) is B- Stable then it is also stable in the sense of Lyapunov sense.

Proof

We note that H_2 above stipulates the condition for the existence of the B-transform for the problem under consideration (see [13]). Assume that the zero solution of the ICS is B-Stable and the application of L_c transform to eq(2) gives

$$x(q) = H(q)[x_0 + Bu(q)]$$

From which the solution to the eq.(1) is found to be

$$(12) \quad x(t) = \frac{1}{2\pi i} \int_C x(q) e^{tq} dq + \sum_{t_0 < t_k < t} \phi(t_k, t) I(x(t_k))$$

Where

$$(13) \quad \phi(t_k, t) = \int_C dq e^{t_k/q + tq}$$

C is any arbitrary contour in the complex domain containing all the singularities of $x(q)$ for which the given integral exists.

Therefore

$$(14) \quad |x(t)| \leq \frac{1}{2\pi} \int_C |x(q)| e^{tq} dq + \sum_{t_0 < t_k < t} |\phi(t_k, t)| |I(x(t_k))|$$

Since eq.(1) is B-Stable by the hypothesis, the given $\bar{\epsilon} > 0$ and let there exists $\delta_0 > 0$ and choose $\varphi(\bar{\epsilon}, r)$ such that

$$(15) \quad \varphi(\bar{\epsilon}, r) = \frac{2\pi\bar{\epsilon}}{r \left(1 + \sum_{t_0 < t_k < t} r_k e^{t_k/q} \int_C dq e^{tq} \right)} - |B|r_0 > \delta_0$$

Then

$|H(q)| < r$ and $\varphi(\bar{\epsilon}, r) > \delta$ if $|x_0| < \delta$. Therefore, from eq.(14) we have

$$|x(q)| \leq |H(q)|[|x_0| + |B|u(q)] \leq r[\delta_0 + |B|r_0].$$

Therefore, from eq(14) we have

$$|x(t)| < \frac{r}{2\pi} \left\{ \frac{2\pi\bar{\epsilon}}{r \left(1 + \sum_{t_0 < t_k < t} r_k e^{t_k/q} \int_C dq e^{tq} \right)} - |B|r_0 + |B|r_0 \right\} \\ \times (1 + \sum_{t_0 < t_k < t} r_k e^{t_k/q}) \int_C dq e^{tq}$$

$= \frac{r}{2\pi} \frac{2\pi\bar{\epsilon}}{r} = \bar{\epsilon}$ Therefore, the (ICS) is stable in the Lyapunov sense, in this case, the stability is uniform. However, it must be emphasized that the choice of $\varphi(\epsilon, r)$ will be valid, if the choice of the contour of integration for $\int_C e^{\frac{t_k}{q} + tq} dq$ is such that the integral does not vanish. Asymptotic expansion of $e^{\frac{t_k}{q}}$ for order 5 using Maple 11 software we have

$$e^{\frac{t_k}{q}} \approx 1 + \frac{t_k}{q} + \frac{t_k^2}{q^2} + \frac{t_k^3}{6q^3} + \frac{t_k^4}{24q^4} + \frac{t_k^5}{120q^5} + O\left(\frac{t_k^6}{q^6}\right)$$

$\int_C e^{\frac{t_k}{q} + tq} dq$ can be evaluated from the asymptotic expansion and the residue theorem. From the asymptotic expansion q has a pole of order 5 thus

$$\int_C e^{\frac{t_k}{q} + tq} dq \approx \int_C \left[1 + \frac{t_k}{q} + \frac{t_k^2}{q^2} + \frac{t_k^3}{6q^3} + \frac{t_k^4}{24q^4} + \frac{t_k^5}{120q^5} + O\left(\frac{t_k^6}{q^6}\right) \right] e^{tq} dq.$$

Where the integration is taken over a contour C containing the pole of order $k = 6$. By the use of Cauchy-Goursat theorem, the integral can be reduced to only an integration around a small circle about the pole of order $k = 6$. The value for

$\int_C \frac{e^{tq}}{q^k} dq$ can be obtained by the use of residue theorem as

$$\int_C \frac{e^{tq}}{q^k} dq = 2\pi i \operatorname{Res} s(e^{tq})|_{q=0} = 2\pi i \frac{1}{(k-1)!} \frac{d^{(k-1)}}{dx^{(k-1)}}|_{q=0} = 2\pi i \frac{t^k}{(k-1)!}$$

Therefore $\int_C \frac{t_k^i e^{tq}}{q^k} dq = 2\pi i \sum_{i=0}^5 \sum_{0 < t_0 < t_k < t} \left(\frac{t^i t_k^k}{(i-1)!} \right) + O\left(\frac{t^6 t_k^k}{5!}\right).$

4 Applications

There are several impulsive control systems with constant delay found in real life. The state variable quantifying such system may be lagged, that is, it is predetermined by the prehistory (delay) of the process governing such processes. Sometime, the control variable may even be lagged either by a constant delay or by a variable. The solutions to delay impulsive systems are much more difficult to obtain than that non-delay ones. Here, we will consider a simple application

to systems with constant delays, we will revisit the system in form of the classical Kruger-Thiemer process (CKTP) ([5]) which is slightly broader in sense than the one presented in ([5]&[8]).

5 Impulsive Delay Kruger-Thiemer's Model

Consider the impulsive delay Kruger-Thiemer's (IDKT) model

$$(18) \quad \begin{aligned} \dot{x}_1(t) &= x_1(t-a) + 2x_2(t-b) + \beta_1 u_1(t-a) & t \neq t_k, k=0,1,2,\dots \\ \dot{x}_2(t) &= x_2(t-2a) + 3x_2(t-2b) + \beta_2 u_2(t-3a) & t \neq t_k, k=0,1,2,\dots \\ \Delta x_i(t=t_k) &= I_i(x_i(t_k)), i=1,2. \end{aligned}$$

For strictly increasing sequence of times such that

$$0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = +\infty$$

Where

$$x_1, x_2 \in PC(D, R^2); u_1, u_2 \in C \text{ and } \beta_1, \beta_2 \in R = (-\infty, +\infty).$$

The above equation can be transformed to the following equivalent equation

$$(19) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & t \neq t_k, k=0,1,2,\dots \\ \Delta x(t=t_k) = I(x(t_k)) \end{cases}$$

For strictly increasing sequence of times such that

$$0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = \infty$$

Where

$$A = \begin{pmatrix} \delta_1 & \delta_2 \\ 2\delta_1 & 2\delta_2 \end{pmatrix}, B = \begin{pmatrix} \beta_1 a \\ 3\beta_2 a \end{pmatrix}, \delta_i \text{ and } \beta_i, i=1,2 \text{ are some shift functions.}$$

Therefore,

application of the B-transform to the problem yields

$$L_c(\dot{x}(t)) = -x_0 + 1/qx_c(q)$$

$$L_c(x(t-a)) = e^{a/q} \left[\int_a^0 e^{-s/q} x(s) ds + \int_0^\infty e^{-s/q} x(s) ds \right]$$

Where

$$x(t) = (x_1(t), x_2(t))^T, \top \text{ is the transpose of the vector}$$

$$A_1 = \begin{bmatrix} e^{-a/q} & 2e^{-b/q} \\ e^{-2a/q} & 3e^{-2b/q} \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} e^{-a/q} f_1(a) & 2e^{-b/q} f_1(b) \\ e^{-2a/q} f_2(a) & 3e^{-2b/q} f_2(2b) \end{bmatrix}$$

$$f_1(a) := \int_{-a}^0 e^{-s/q} ds, f_1(2a) := \int_{-2a}^0 e^{-s/q} ds.$$

Therefore

$$L(\dot{x}(t)) = -x_0 + 1/qx_c(q)$$

$$L(Ax(t) + Bu(t)) = A_1x_c(q) + A_2x_c(q) + Bu(q)$$

Therefore

$$(20) \quad x_c(q) = (A_1 + A_2 + 1/qE)^{-1}[x_o + Bu(q)]$$

$$H^{-1}(q) = (A_1 + A_2 + 1/qE)$$

$$= \begin{bmatrix} e^{-a/q}(1 + f_1(a)) + \frac{1}{q} & 2e^{-a/q}(1 + f_1(b)) \\ e^{-a/q}(1 + f_2(b)) & 3e^{-a/q}(1 + f_2(2b)) + 1/q \end{bmatrix}$$

$$H(q) = \frac{1}{z(q)} \begin{bmatrix} 3e^{-a/q}(1 + f_2(2b)) + 1/q & -2e^{-b/q}(1 + f_1(b)) \\ -e^{-2a/q}(1 + f_2(a)) & e^{-a/q}(1 + f_1(a)) + 1/q \end{bmatrix}$$

Where

$$z(q) = 3e^{-2b/q}(1 + f_2(2b) + 1/q)(e^{-a/q}(1 + f_1(a)) + 1/q) - 2e^{(-b+2a)/q}(1 + f_1(b))(1 + f_2(a)).$$

$H(q)$ being the transfer function associated with the (ICDS)

$$B = \begin{bmatrix} e^{-a/q} \int_{-a}^{\infty} e^{-\tau} u_1(\tau) d\tau \\ e^{-b/q} \int_{-3a}^{\infty} e^{-\tau} u_2(\tau) d\tau \end{bmatrix} \text{ Therefore}$$

$$(21) \quad B_n'x(t) = x(q) = L_cx(t) + L_Ix(t)$$

$$= H(q)[x_o + Bu(q)]$$

$$+ \sum_{t_0 < t_k < t} I(x(t_k)).$$

We return to the model by applying B- stability concept to investigate the stability of (ICDS).

Theorem 2

Let there exist constants $M > 0$ and $N > 0$ and $r_0 > 0$ such that

$$M = \text{Max}[2 + |f_1(a)| + |f_1(2a)| + 1/q, 3 + |f_2(b)| + |f_1(a)| + 1/q],$$

$$N = \text{Max}\left[\int_{-a}^{\infty} e^{\tau} u_1(\tau) d\tau, \int_{-3a}^{\infty} e^{-3\tau} u_2(\tau) d\tau\right]$$

$$\text{Max}_{\tau \in R^+} [|u_1(\tau)|, |u_2(\tau)|] \leq r_0$$

Then the (ICGS) is B-Stable in the interval $(\frac{1}{r} - 3/4, 0)$ for any arbitrary $r > 0$, which is the apriori bound for $H(q)$, then the system is stable in that interval.

Proof

Given $\epsilon > 0$ and let $\varphi(\epsilon, r) = \frac{2\pi}{r} \left(\frac{\epsilon - \sum n_k \epsilon^{t_k/q}}{\int_{\epsilon} \epsilon^{t_k/q} dq} \right) - Nr_0$ where $|H(q)| < r$

Then

$$|x(q)| \leq |H(q)|[|x_0| + |B||u(q)|]$$

But

$$z(q) \geq -3 - \frac{4}{q} + 2e^{-(2a+b)/q} - 1/q^2$$

Hence

$$\frac{1}{|z(q)|} \leq \frac{1}{-3-4/q+2e^{1/q(2a+b)}-1/q^2} \leq \frac{1}{-3-4/q-1/q^2}$$

Thus, it is easy to observe that if $0 \leq \frac{1}{r} - 3/4$ then $H(q)$ is apriori bounded by r

Since

$$\begin{aligned} |H(q)| &\leq \frac{1}{|z(q)|} \max \left[|3e^{-2b} + \frac{1}{q}| + |e^{2a/q}| + |3e^{-a/q} + \frac{1}{q}| \right] \\ &\leq \frac{1}{|z(q)|} \max \left[4 + \frac{1}{q}, 3 + \frac{1}{q} \right] \leq \frac{4 + \frac{1}{q}}{3 + \frac{2}{q} + \frac{1}{q^2}} \end{aligned}$$

$H(q)$ is apriori bounded by a kind of Pade's (continuous fraction) approximation to e^θ .

Therefore, $|H(q)| < r$ then for $q_0 = \frac{2r}{4-3r}$, (IDCS) is B-Asymptotically stable by Theorem 1 and if T is chosen to be $T = \frac{\sqrt{4r^2 - (4-r)(1-r)}}{4-3r}$ and $|q - t_k| > \eta$, it is easy to show that the stability is also in Lyapunov's sense.

Since

$$\begin{aligned} |x(t)| &\leq \frac{1}{2\pi} \int_C |x(q)| e^{tq} dq \leq \frac{1}{2\pi} \int_C r(\delta_0 + M + Nr_0) e^{tq} dq \\ &+ \sum_{t_0 < t_k \leq t} r_k e^{t_k/q}. \end{aligned}$$

Thus given $\epsilon > 0$ and $\delta_0 > 0$, for $|x_0| < \delta$ it follows that $|x(t)| < \epsilon$, $t \geq t_0 + T$ which follows that the (IDCS) is asymptotically stable in the Lyapunov's sense.

5.1 Impulsive Drug Compatible Model

Consider the Impulsive drug compartment (IDC) model of an impulsive delay control system for the distribution of drugs which is a simple two compartment of drug administration model [11]. The impulsive analogue of the Kruger-Thiemer's model in [5] is special type of it. That is, when

$h_i = 0$ for $i = 1, 2, 3, 4$, $I = 0$ and $u_i = 0$ for $i = 1, 2$.

The evolution equation governing this model is given by the following control system

$$\begin{aligned} \dot{x}_1(t) &= k_1 x_1(t - h_1) + k_2 x_2(t - 2h_2) + u_1(t - h_3), t \neq t_k, k = 0, 1, 2, \dots \\ \dot{x}_2(t) &= k_3 x_1(t - h_1) + k_4 x_2(t - 2h_2) + u_2(t - h_3), t \neq t_k, k = 0, 1, 2, \dots \\ \Delta x_1(t = t_k) &= I_1(x_1(t_k)) \\ \Delta x_2(t = t_k) &= I_2(x_2(t_k)) \end{aligned}$$

For strictly increasing sequence of times $\{t_k\}$ such that $0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = \infty$

$x_i(t)$ and $x_2(t)$ denotes the amount of drugs' intake at time t into gastrointestinal tract and the apparent volume (e.g. blood, plasma, muscle, tissue, cell etc.) respectively. Assume that the drug intake are lagged by constants $h_i, i = 1, 2, 3$ where $t - h_i$ are taken to be effective times for the drugs to act $k_i, i = 1, 2, 3, 4$ are some relevant rate constants. u_i measures the controls that regulates the metabolic activity of the drugs and the absorption or inhibiting of the drugs. The regulators of the metabolic activity of the drugs may be assumed to be lagged also like $x_i(t), i = 1, 2$. In practice, it is not always easy to say whether the drug intake lagged by a constant or by a variable. Through experiment this may be determined. We note that the effect of $x_i(t)$ are impulsive because of various metabolic activities that take place in the body. The presence of enzymes and some other hormone interference with the drug and immune suppressing drugs in the transplanting surgery tends to show some impulsive behavior ([15]). For sake of simplicity, we will not consider the impulsive aspect of $u_i(t), i = 1, 2$. This may offer another interesting problem in its own right which may also offer feasible solution to some challenging problems in medicine in recent times.

Remark1

The IDKT model in section 4.1 is a special case (IDC) model where $k_1 = 1, h_1 = a, k_2 = 2, b = 2h_2, b_1 = 1,$

$$h_3 = 1, 2a = h_1, k_4 = 3, 2a = h_3, b_2 = 1, h_3 = -3a.$$

Applying the B-transform to (E_2) , we have

(23)

$$\begin{aligned} x_o(q) &= (A_3 + A_4 + 1/qE)^{-1}(x_o + Bu(q)) = \\ &H(q)[x_o + Bu(q)] \end{aligned}$$

where

$$A_3 = \begin{pmatrix} k_1 e^{-h_1/q} & k_2 e^{-2h_2/q} \\ k_3 e^{-h_1/q} & k_4 e^{-3h_2/q} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} k_1 e^{-h_1/q} f_3(h_1) & k_2 e^{-2h_2/q} f_3(h_2) \\ k_3 e^{-h_1/q} f_4(h_1) & k_4 e^{-3h_2/q} f_4(h_2) \end{pmatrix}$$

and

$$f_3(h_1) = \int_{-h_1}^0 e^{-s/q} ds, f_3(h_2) = \int_{-2h_2}^0 e^{-s/q} ds$$

$$B = \begin{pmatrix} e^{-h_3/q} \int_{-h_2}^{\infty} e^{-t} u_1(t) dt \\ e^{-4h_2/q} \int_{-4h_2}^{\infty} e^{-t} u_2(t) dt \end{pmatrix}$$

where

$$H(q) = \frac{1}{z^*(q)} \begin{pmatrix} k_4 e^{-3h_1/q} (1 + f_4(h_2)) + 1/q & -k_2 e^{2h_2/q} (1 + f_3(h_2)) \\ -k_3 e^{2h_1/q} (1 + f_4(h_1)) & -k_1 e^{-h_2/q} (1 + f_3(h_1)) + 1/q \end{pmatrix}$$

and

$$z^*(q) = k_1 k_4 e^{(-h_1+3h_2)/q} (1 + f_3(h_1)) (1 + f_4(h_2)) + \frac{k_1}{q} e^{-h_1/q} (1 + f_3(h_1))$$

$$+ \frac{k_4}{q} e^{-3h_2/q} (1 + f_4(h_2)) - k_2 k_3 e^{-2(h_1+h_2)/q} (1 + f_3(h_2)) (1 + f_4(h_1))$$

$$+ \frac{1}{q^2}$$

Thus, we can show that

$$z^*(q) \geq -k_1 k_2 (1 + f_3(h_1)) (1 + f_4(h_2)) - \frac{k_1}{q} (1 + f_3(h_1))$$

$$- \frac{k_4}{q} (1 + f_4(h_2)) + k_2 k_3 (1 + f_3(h_2)) (1 + f_4(h_1))$$

$$- \frac{1}{q^2}$$

Hence

$$\frac{1}{z^*(q)} \leq \frac{1}{-k_1 k_2 (1 + f_3(h_1)) - \frac{k_1}{q} (1 + f_3(h_1)) - \frac{k_4}{q} (1 + f_4(h_2)) + k_2 k_3 (1 + f_3(h_2)) (1 + f_4(h_1)) - 1/q^2}$$

$$\leq \frac{1}{(k_2 k_3 - k_1 k_2) (1 + f_3(h_1)) (1 + f_4(h_2)) - 1/q^2}$$

it follows that

$$H(q) = \frac{1}{|z^*(q)|} [\max((k_3 + k_4) (1 + f_4(h_2)) + 1/q, (k_1 - k_2) (1 + f_3(h_1)) + 1/q^2)]$$

$$\leq G(k_1, k_2, k_3, k_4, q).$$

For $f_4(h_2) \geq f_3(h_1)$, $k_3 + k_4 \geq k_1 - k_2$,

$$G(k_1, k_2, k_3, k_4, q) = \frac{(k_3 + k_4) (1 + f_4(h_2)) + 1/q}{(k_2 k_3 - k_1 k_2) (1 + f_3(h_1)) (1 + f_4(h_2)) - 1/q^2}$$

And for $f_4(h_2) \leq f_3(h_1)$, $k_3 + k_4 < k_1 - k_2$,

$$G(k_1, k_2, k_3, k_4, q) = \frac{(k_1 - k_2) (1 + f_3(h_1)) + 1/q}{(k_2 k_3 - k_1 k_2) (1 + f_3(h_1)) (1 + f_4(h_2)) - 1/q^2}$$

From all the derivations so far made, we found that

$$x(q) = (A_3 + A_4 + 1/qE)^{-1}(x_0 + Bu(q)) + \sum_{t_0 < t_k < t} e^{-t_k/q} f(x(t_k))$$

If we pack G such that $G(k_1, k_2, k_3, k_4, q) \leq r$ for all k_1, k_2, k_3 , and k_4 and

$$\phi(q, r) = (\frac{2\pi q}{r \int_C e^{t/q} dq} - |B|r_0 - \sum_{t_0 < t_k < t} r_k e^{-t_k/q}) > \delta_0$$

For $|x_0| < \delta$ for every $\epsilon > 0$ then the system is B-stable in the sub- interval $J_1 \subseteq R^+$.

5.2 Hydrocephalus Model

One painful physiological problem found in children is the inflammation of hydrocephalus valve. The hydrocephalus valve controls the cerebrospinal fluid that drains into the bloodstream (see[7]).When the valve becomes inflamed or works ineffectively there is rapid increase in the amount of fluid flowing into the head thereby causing discomfort and swollen in the head. We will not go into detail discussion about the physiological process that leads to this disease and the detail formulation of the mathematical model for the disease.

The mathematical equation for the construction of an artificial valve (hydrocephalus) regulating the flow of the cerebrospinal fluid is similar to a vibrating mechanism. Many researchers have demonstrated that vibrating mechanism are impulsive in nature (see for example [4],[15]) just to mention but few.

The equation governing the problem is

$$(24) \quad m\ddot{x} = -k(x + x_o + a) - C\dot{x} + (P - P_0)A$$

$$x(t_k^+) = \beta_k x(t_k)$$

For strictly increasing sequence of times $\{t_k\}$ such that

$$0 < t_0 < t_1 < t_2 < \dots < t_k, \lim_{k \rightarrow \infty} t_k = \infty$$

Where m is the mass of the bearing (valve) and the linear damping function is the term $C\dot{x}$.

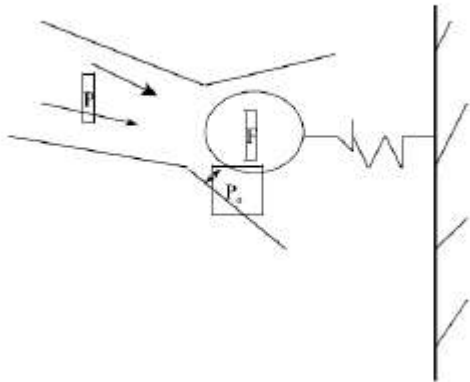
The hydraulic control variable for the problem is given by $u(t) = p(t)A$ (Fig. 1: A schematic diagram for hydrocephalus valve)

In the problem we have the following nomenclature :

x	is the length of the tract for the expansion of valve
P_o	is the ambient pressure
P	is the cerebrospinal pressure

F is the bearing (valve)

Fig 1:A Scientific diagram for hydrocephalus valve



The above equation is similar to the following canonical form of equation:

$$\begin{aligned}\dot{x} &= x_1 \\ \dot{x} &= \dot{x}_1 = \frac{-k}{m}(x + x_o + a) - cx_1 + (P - P_o)A \\ x(t_k^+) &= \beta_k x(t_k)\end{aligned}$$

Hence, in matrix form, we have

$$\begin{aligned}\dot{x} &= Ax + f, t \neq t_k, k = 0, 1, 2 \\ x(t_k^+) &= \beta_k x(t_k) \\ \text{where} \\ A &= \begin{pmatrix} 1 & 0 \\ \frac{-k}{m} & -c \end{pmatrix}, f = \begin{pmatrix} 0 \\ \frac{-k}{m}x_0 - \frac{k}{m}a + (P - P_o)A \end{pmatrix}.\end{aligned}$$

Applying the B-transform, we found that

$$\begin{aligned}H(q) &= \frac{1}{W(q)} \begin{pmatrix} -c + 1/q & 0 \\ \frac{k}{m} & 1 + 1/q \end{pmatrix}, w(q) = (1 + 1/q)(-c + 1/q) \\ \text{Thus} \\ x(q) &= H(q)(x_o + f(q)) + \sum_{t_0 < t_k < t} e^{-t_k/q} x(t_k) \\ f(q) &:= \begin{pmatrix} 0 \\ (\frac{-k}{m}x_0 - \frac{k}{m}a - P_o A)q + AP(q) \end{pmatrix}\end{aligned}$$

It is not difficult to show that if $|H(q)| < r$ then given a positive ϵ and $\varphi(q, r)$ can be found such that

$$\begin{aligned}\varphi(q, r) &= \frac{1}{(1 + \frac{k}{m})} \left(\frac{q}{r \int_{t_0}^t q e^{t/q} dq} + \frac{r|A|}{2\pi} \int_c^p p(q) e^{t/q} dq - \frac{k}{m}a - P_o|A| \right) \\ &+ \sum_{t_0 < t_k < t} r_k e^{t_k/q}\end{aligned}$$

We found after some estimation that $|x(t)| < \epsilon$ for $\varphi(t, r) > \delta_o$ which implies that the system is B-stable.

In practical term, stability about the equilibrium $x = 0$ is that of the hydrocephalus value to regulate the draining cerebrospinal fluid into the head properly. The control function $u(t)$ performs this function and the B-stability guarantees that such hydrocephalus valve can be constructed whether through surgery or artificially.

5.3 Epidemiological Model

The epidemiological model involving division of total population into n distinct groups $x_1, \dots, x_n$ each of which is homogenous within itself.

This is given by the equation

$$(25) \quad \begin{aligned} \frac{dx_i}{dt} &= -r_i x_i + (C_i - x_i) \sum_{j=1}^n a_{ij} x_j + f_i(u), t \neq t_k, k = 0, 1, 2, \dots \\ x_i(0) &= x_i^0, \Delta x_1 = \beta_k x_i(t_k) \\ C_i &= \sum_{k=1}^i x_k, x_i \text{ is the number of infectious persons in the group.} \end{aligned}$$

r_i is the recovery rate of the member x_i

a_{ij} is the rate at which the uninfected member acquires the disease.

$f_i(u)$ is a control function lagged by a constant h .

In [1] we reported that population is impulsive under rapid changes, while [11] it was emphasized that epidemiological models are impulsive if the susceptible are removed from environment. For these reasons, the above model is considered to be impulsive. For sake of analysis, we assume that the impulse is governed by Δx_i .

Let us return to the model, from the problem

$$F(x_i, u_i) = r_i x_i + (C_i - x_i) \sum_{j=1}^n a_{ij} x_j + f_i(u)$$

We can cast the population model into this form

$$(26) \quad \begin{aligned} \dot{y} &= Ay + Bu, t \neq t_k, k = 0, 1, 2, \dots \\ \Delta y &= \beta_k y(t_k) \end{aligned}$$

Where

y is the linearized form of the non-linear model for the population. It, therefore, follows that

$$(27) \quad A = (a_{ij})$$

$$\begin{aligned}
 & a_{ij} = -\gamma + C_i a_i + \frac{\delta f_i(0)}{\delta x_i}, i = 1, 2, \dots, n, j = 1, 2, \dots, n \\
 & = a_{ij} = (C_i - x_i) \sum_{j=1}^n a_{ij} + \frac{\delta f_i(0)}{\delta x_j}, i = 1, 2, \dots, n, j = 1, 2, \dots, n \\
 B &= \begin{pmatrix} \frac{\delta f_1(0)}{\delta u_1} \\ \dots \\ \frac{\delta f_n(0)}{\delta u_n} \end{pmatrix}
 \end{aligned}$$

Therefore

$H(q)$ is the inverse of the following matrix

$$\begin{aligned}
 & -r_i + c_i a_i + \frac{\delta f_i(0)}{\delta x_i}, i = 1, 2, \dots \\
 a_{ij}^* &= (C_i - x_i) \sum_{j=1}^n a_{ij} + \frac{\delta f_i(0)}{\delta x_j} + 1/q, j = 1, 2, \dots, n
 \end{aligned}$$

The equilibrium population can be obtain from the solution of the following system of equations:

$$-r_i x_i + (C_i - x_i) \sum_{j=1}^n a_{ij} x_j + f_i(u) = 0 \text{ for } \beta_k = 0, k = 0, 1, 2, \dots$$

If x_ξ is the equilibrium point then $\sup_{u \in U, \forall \epsilon > 0} |f_i(u) - x_\xi| < \epsilon$. If the inverse a_{ij}^* exists and it is apriori bounded and the norm of B is very small, then the zero solution $x = x(t) = 0$ of the system is B-stable.

Since the system is B-stable, we are guarantee that the population can be controlled in practical term, this will happen if r_i is increased and C_i is reduced and the control function $f_i(u)$ is maintained so that the population is kept within the vicinity of the equilibrium population. Otherwise, the trajectory of infectiousness will grow without bound and becomes instable.

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Model simulation for bioavailability and biodegradation: A mathematical approach to the bioremediation of polycyclic aromatic hydrocarbon contaminated site

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Abstract

A mathematical model for a one-dimensional convective-dispersive solute transport in both axial and radial directions of flow in a soil matrix is presented. The interplay of equilibrium sorption and first order degradation were incorporated into the formulation of the model. The model took into consideration the overall effects of the solid and bulk liquid phase mass transfer resistances which includes; intraparticle, interparticle and interphase mass transport.

The functional parameters; dispersion coefficient, pore-water velocity, first order degradation rate constant and the retardation factor were independently estimated to reduce the dimensionality of the search process.

The solution to the model equations was achieved by the use of the backward finite difference scheme. Results obtained showed that naphthalene was more selectively degraded than pyrene and anthracene with a residual concentration of naphthalene 1.12E-5mg/l, 1.48mg/l, pyrene 3.11E-4mg/l, 1.58mg/l and anthracene 7.67E-4mg/l, 1.61mg/l in the axial and radial directions respectively. The modeling results showed the progressive bioaccumulation of these compounds from the soil particle surface inwards to the fissures and cavities of the soil particles. This renders them not readily bioavailable and thus inaccessible to microbial degradation.

Keywords: Pore-water velocity, Dispersion coefficient, Retardation factor, Degradation rate constant, temporal moments, curve-fitting.
Mathematics Subject Classification 2000:92D40 &92C40

1.0 Introduction

Interest in the biodegradation mechanisms and environmental fate of polycyclic aromatic hydrocarbons (PAHs) is prompted by their ubiquitous distribution and their potentially deleterious effects on human health. Severe contamination is often located on former gasworks, while diffuse contamination is located in urban areas. Bioremediation is a fast developing technology, which has been described as the optimization of biodegradation (1). The biodegradation of PAHs by microorganisms is the subject of very many excellent reviews (2, 3, 4, and 5).

Research reports have shown that natural degradation occurs in both soil and groundwater depending on the molecular size, i.e., the number of aromatic rings, molecule topology or pattern of ring linkage and to a large extent on soil properties (6). The rate and extent of contaminant removal was also found to decrease with contaminant-soil contact time. The observation was particularly demonstrated for hydrophobic organic contaminants such as polycyclic aromatic hydrocarbons (7).

The use of models and simulations to assess the availability of contaminant solutes and the behavior of these solutes over relatively long spatial and temporal scales has become the trend in the studies of hydrocarbon degradation. The reason is not far from the fact that experimental studies over sufficiently long distances and/or time periods are cumbersome and usually very expensive. However, knowledge of the physical process is required to provide the basis for validating the mathematical expression describing the model.

Generally, the overwhelming reported literatures and articles have specified clean-up techniques for PAH contaminated sites using bacterial cultures, pure strains or strains in association (5, 8, 9, 10, 11, and 12). The few available reports on the modeling of PAH degradation (13, 14) have however, been unable to perform optimally relative to availability, mobility, toxicity and concentration of polycyclic aromatic hydrocarbons in the soil system ranging from surface soils to deep aquifers.

The development of the mathematical model carried out in this research work is based on the fundamental principle of conservation of mass or material balance. It takes into consideration the gas-liquid interface film and the biofilm between liquid and solid interface. It also accounts for the interparticle, intraparticle and interphase mass transport. The effects of axial dispersion in the fluid phase and transport resistances are not neglected. It responds to the pressing research needs for bioremediation with the underlying task of determining the factors, which govern bioavailability.

In this study, the solutions to the model equations are presented. The method of temporal moments (MOM) described by 15) and a CXTFIT version 2.0 have been applied for analyzing concentration breakthrough curves (BTCs) for the contaminants used in this transport study. The pore- water velocities and dispersion coefficients for the non reactive solute, sodium hexametaphosphate (tracer) and the retardation factor and degradation rate constant of the contaminant solutes will be estimated from the experimental column BTC data.

2.0 Summary of Experimental Investigation

Evaluation of substrate bioavailability and biodegradation in contaminated aqueous-solids systems was monitored in a soil microcosm reactor. A schematic illustration is shown in Figure 1. The contaminant hydrocarbons used for the study include naphthalene, anthracene and pyrene. Unimpacted surface and subsurface soils were excavated and placed inside the microcosm. The soil was spiked with a mixture of the contaminant hydrocarbons dispersed in water, which contained a surfactant (sodium hexametaphosphate) and nutrients. Standard solvent extraction using n-hexane and dichloromethane (HPLC grade) and gas chromatography methods were used to determine the aqueous phase concentration of the contaminants in the soil microcosm reactor.

1.

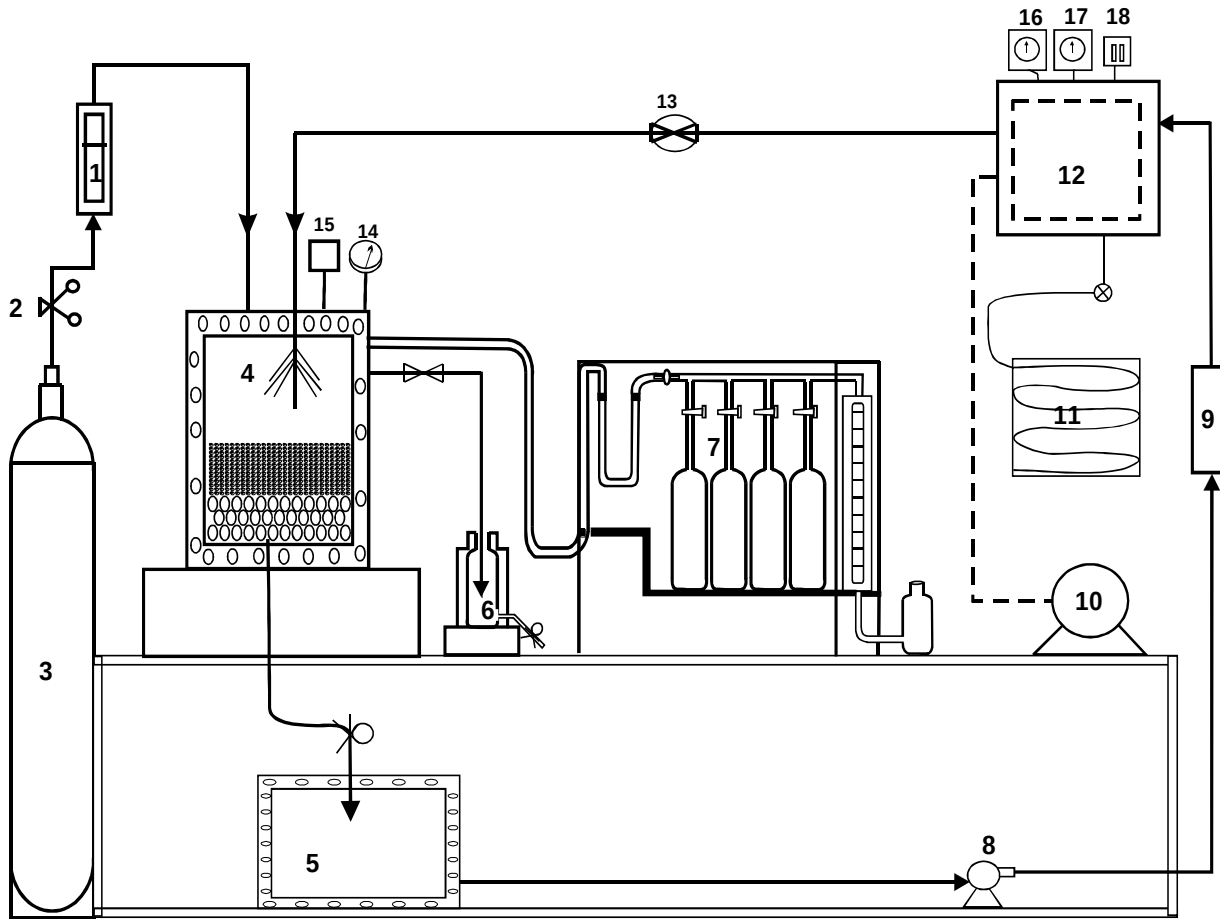


Fig 1: Schematic representation of experimental setup

Key: (1) rotameter, (2) regulator, (3) oxygen bottle, (4) microcosm reactor, (5) leachate holding tank, (6) oxygen absorption bottle, (7) orsat gas analyzer, (8) pump, (9) filter, (10) compressor, (11) condenser, (12) refrigerated nutrient tank, (13) solenoid valve, (14) pressure gauge, (15) digital multimeter, (16) delay timer, (17) programmable timer and (18) electrical switch.

3.0 Model Development

3.1 Parameters needed for simulation

The constant transport parameters D , V , needed for the simulation, of the biodegradation process, were estimated prior to the resolution of the unsteady state model. The D and V for the non reactive solute (surfactant) were obtained using the MOM (16) described by 15 and CXTFIT software, version 2.0, as described by 17. The Program assumes one-dimensional flow and estimates parameters by fitting the parameters to observed or experimental data. In this study, the program was used to solve direct problems where it predicted solute distribution against time.

The solutions of the degradation rate constant (λ) and retardation factor (R) were obtained using the expressions of 15.

$$R = \frac{(\mu_1 - 0.5t_o)\sqrt{V^2 + 4D\lambda}}{x} \quad (1)$$

$$\lambda = \frac{V^2}{4D} \left[\left(1 - \frac{2D}{xV} \ln \frac{M_o}{C_o t_o} \right)^2 - 1 \right] \quad (2)$$

The nth temporal moment of a concentration distribution at a location x was defined by Kucera, 1965 and Valocchi, 1985 and described by 15:

$$M_n = \int_0^{\infty} t^n C(x, t) dt \quad (3)$$

And the nth normalized moment of the distribution was defined as

$$\mu_n = \frac{M_n}{M_o} = \frac{\int_0^{\infty} t^n C(x, t) dt}{\int_0^{\infty} C(x, t) dt} \quad (4)$$

Equations 3 and 4 may be used to obtain experimental temporal moments from concentration breakthrough curves.

The pore water velocity (V) and dispersion coefficient (D) are estimated in order to calculate R and λ from equations (1) and (2)

$$V = \frac{x}{\mu_1 - 0.5t_0} \quad (5)$$

and

$$D = \frac{V^3}{2x} \left(\mu_2 - \mu_1^2 - \frac{t_0^2}{12} \right) \quad (6)$$

3.3 Unsteady State Models

The model accounts for contaminant transport through a homogenous, porous media in a one-dimensional uniform flow by considering convection, dispersion, linear equilibrium sorption and first order degradation. The governing transport equations were described by the following second order partial differential equations (18).

a. Macroporous System

$$\frac{\partial C_i}{\partial t} = D \frac{\partial^2 C}{\partial z^2} - \frac{V \partial C_i}{\varepsilon_b \partial z} - \left(\frac{1 - \varepsilon_b}{\varepsilon_b} \right) \left(\frac{3}{R} \right) k_f C_i \quad (7)$$

The initial and boundary conditions are:

i. $C_i = C_{oi}$ at $t \leq 0$, $z_T \geq z \geq 0$

ii. Inlet condition ($z = 0$, $t \leq 0$):

$$C_{oi} + D \frac{\partial C_i}{\partial z} = C_i$$

iii. Outlet condition ($z = z_T$, $t > 0$):

$$\left. \frac{\partial C_i}{\partial z} \right|_{z=z_T} = 0$$

b. Microporous System:

$$\frac{1}{\alpha} \frac{\partial C_{si}}{\partial t} = \frac{\partial^2 C_{si}}{\partial r^2} + \frac{2 \partial C_{si}}{r \partial r} - \frac{k C_{si}}{\alpha} + \frac{R + k C_{psi}}{\alpha} \quad (8)$$

Where $\alpha = \varepsilon_p D_{psi}$.

The initial and boundary conditions are:

$$i \quad C_{Si}(r, t) = C_{Si}(r) \text{ at } t \leq 0, \quad 0 \leq r \leq R$$

$$ii \quad \left. \frac{\partial C_{Si}}{\partial r} \right|_{r=R} = 0 \quad 0 \leq r \leq R, \quad t \geq 0$$

4.0 Method of Solution

The numerical method adopted for the solution of the second-order partial differential equations given in equations (7) and (8) is the Backward Finite Difference Scheme also called Fully Implicit Method. This method is preferred to the central difference method because its truncation error is of the second order, and it often involves implicit method of solution, which is stable and valid for any chosen interval. Both term on the R.H.S and L.H.S of the equations have the unit of concentration. The scheme involves the discretization of both depth (z) and time (t) simultaneously into mesh or grid points with constant intervals. The z-t plane is subdivided into equal time steps $\Delta \tau = k$ and depth step $\Delta z = h$. The domain of the function is overlaid by a grid whose mesh size is of h units, in the z direction and k units in the τ direction. The value of the function (z, τ) at the i j th grid point is denoted as $f_{i,j} \equiv f(ih, jk)$. The representative mesh point p is $z = ih$ and $\tau = jk$ where $i, j = 0, 1, 2, 3, \dots, N$

Thus

$$C_p = C(ih, jk) = C_{ij}$$

The backward finite difference scheme is valid and converges for all values of k/h^2 i.e. $k/h^2 \geq 0$. This is known as the stability criterion. To allow for more flexibility of the result to the equations, the following dimensionless variables are defined:

$$Z = \frac{z}{z_T} \quad (9)$$

Z = dimensionless depth

$$\tau = \frac{t}{T} \quad (10)$$

τ = dimensionless time

$$\bar{C} = \frac{C_i}{C_{oi}}$$

$\bar{C}$ = dimensionless concentration

$$\alpha = D, \quad \beta = \frac{V}{\varepsilon_b}, \quad \gamma = \left(\frac{1 - \varepsilon_b}{\varepsilon_b} \right) \frac{3}{R} k_f \quad (11)$$

$$\eta = \frac{k}{\alpha}, \quad \psi = \frac{1}{\alpha}, \quad \phi = \frac{R + kC_p}{\alpha} \quad (12)$$

Subject to

$$\bar{C} = \bar{C}_{O1} \quad 1 > z > 0, \quad \tau = 0 \quad (13)$$

$$\bar{C}(z, 0) = 1, \quad \tau = 0 \quad (14)$$

$$\bar{C} = \bar{C}_{O1} \quad 1 > r > 0, \quad \tau = 0 \quad (15)$$

$$\bar{C}(r,0)=1, \quad \tau=0 \quad (16)$$

Therefore non dimensionalizing equations (7) and (8), yields:

$$\frac{\partial \bar{C}}{\partial \tau} = \alpha \frac{\partial^2 \bar{C}}{\partial z^2} - \beta \frac{\partial \bar{C}}{\partial z} - \gamma(\bar{C}_i) = 1 \quad (17)$$

$$\psi \frac{\partial \bar{C}}{\partial \tau} = \frac{\partial^2 \bar{C}}{\partial r^2} + \frac{\partial \bar{C}}{\partial r} - \eta \bar{C} + \varphi = 1 \quad (18)$$

Discretizing equations (7),

$$C = C_{i,j} \quad (19)$$

$$\frac{\partial C}{\partial z} = \frac{C_{i,j+1} - C_{i,j}}{h} \quad (20)$$

$$\frac{\partial^2 C}{\partial z^2} = \frac{C_{i,j-1} - 2C_{i,j} + C_{i,j+1}}{h^2} \quad (21)$$

$$\frac{\partial C}{\partial t} = \frac{C_{i,j+1} - C_{i,j}}{k} \quad (22)$$

Similarly, discretizing equation (8),

$$C = C_{i,j} \quad (23)$$

$$\frac{\partial C}{\partial r} = \frac{C_{i+1,j} - C_{i,j}}{h} \quad (24)$$

$$\frac{\partial^2 C}{\partial r^2} = \frac{C_{i-1,j} - 2C_{i,j} + C_{i+1,j}}{h^2} \quad (25)$$

$$\frac{\partial C}{\partial t} = \frac{C_{i+1,j} - C_{i,j}}{k} \quad (26)$$

$$h=0.001, k=7$$

Substituting known values from equations (19) through (26) into equations (7) and (8), the iterative expressions for the macro and micro porous systems become

$$C_{i-1,j+1} - 2C_{i,j+1} + C_{i+1,j+1} = -2.49 \times 10^{-3} C_{i,j} + 1.49 \times 10^{-3} C_{i+1,j} \quad (27)$$

$$C_{i-1,j+1} - 3.4C_{i,j+1} + C_{i+1,j+1} = -1.18C_{i,j} - 0.2C_{i+1,j} - 0.74 \quad (28)$$

Substituting the initial and boundary values into equations (27) and (28), nine simultaneous equations result which are resolved using the matrix method. In this method, the sets of equations were put in the form

$$AX = R$$

A = coefficient matrix

R = R.H.S constant matrix

X = solution matrix = $A^{-1}R$

A^{-1} is the inverse matrix of A.

4.0 RESULTS

Table 1.0 shows some of the properties of the investigated PAHs (19). These properties were relevant in the study of the behaviour of the tested PAHs.

5.1 PAHs breakthrough curves

The variation of the relative concentration with time obtained for the experimental PAHs and non-reactive solute (tracer) using the MOM and non-linear least square curve-fitting program CXTFIT are illustrated with Figures 2 to 5.

The breakthrough profiles were characterized by a initial low level of normalized concentration C/C_o , in a water flow rate of 8.06cm³/min. Naphthalene was zero throughout the 40hour period, tracer was zero for 1hour, pyrene zero for 3hours and anthracene zero for 5hours. This was followed by a sharp but rather steep ascent to the exhaustion point as illustrated by the smooth and sigmoid shape and a subsequent decline in C/C_o values. Anthracene was found to elute only after the breakthrough of the tracer and other PAHs (naphthalene and pyrene).

Tables 2 and 3 below show and compares the values of the pore-water velocity V, dispersion coefficient D, calculated for the tracer (non-reactive solute) and the first order degradation rate constant λ and the retardation factor R for the reactive PAHs using the MOM and CXTFIT.

Table 2 shows that the D estimated from the MOM was higher than the CXTFIT estimate while the reverse is the case for V. This is probably attributable to the fact that MOM estimated value of D is a function of the difference between the second moment and the first moment squared. In Table 3, the calculated degradation and transport parameters for the contaminants PAHs using the two analytical methods are compared. Results indicate that naphthalene had the highest degradation rate and lowest retardation factor using both approaches. Anthracene had the least values.

a. Modelling Results

The mathematical equations for the unsteady state model contain a number of parameters, k_f and D_p , which must be determined independently to reduce the dimensionality of the search process. The film mass transfer coefficient (k_f) was determined from the experimental data on adsorption/desorption, using the relationship of Parvatiyar (1992), described by 13:

$$k_f = 0.32 \frac{D_{pi}^{2/3}}{D_T} V^{1/3} \text{Re}_{dp}^{3/4} \left[\frac{D_s^{1/2} d_p^{-3/4}}{D_T^{1/2} H_L^{1/4}} \right] \quad (29)$$

The pore diffusivities D_{pi} may also be estimated from literature (20):

$$D_p = \frac{E_p}{T} \left[\frac{3}{4\bar{r}} \left(\frac{\pi m}{2R\theta_A} \right)^{1/2} + \frac{1}{D_f} \right]^{-1} \quad (30)$$

While the diffusivity of the fluid phase D_f may be obtained from the correlation of Wilke Chang, reported by (21):

$$D_f = 7.4 \times 10^{-8} \frac{\sqrt{\psi_B M_B T}}{\mu V_A^{0.6}} \quad (31)$$

The summary of these parameters and their estimated values are shown in Table 4. The film mass transfer coefficient k_f represents the resistance to mass transfer between the fluid phase and solid phase. The values affirm that the resistance to transfer for the PAHs was

significantly higher than the diffusivities. The pore diffusivities suggest the occurrence of a slow diffusive mechanism for the contaminant PAHs. The results of the modeling equations describing the degradation with linear equilibrium sorption for the macroporous and microporous systems for the experimental PAHs are shown in Tables 5.1 to 5.6.

The results show the surface concentration with time for the PAHs in both axial and radial directions of flow. From the simulated results, it is apparent that the accumulation of the PAHs is more prominent in microporous system. The degree of accumulation is in the order naphthalene < pyrene < anthracene in the axial direction and anthracene > pyrene > naphthalene in the radial.

The non-steady state model developed for macroporous and microporous systems which had a number of variables whose numerical values were independently estimated was validated to assess the variation of the simulated composition of naphthalene, anthracene and pyrene with exposure time within the soil matrix. A comparison of the experimental and simulated results for both systems is shown in Figures 6 and 7.

6.0 Discussion

The comparative analysis of the concentration breakthrough curves in the contaminant transport studies for non-reactive solute (tracer) and the experimental PAHs using the MOM and CXTFIT, presented in the profiles in Figures 2 to 5, showed that they both fit the experimental breakthrough curve; although the MOM had better fit. This was because with MOM, no assumptions about the initial conditions for the experiment were required. The profiles showed that the non-reactive solute attained its maximum concentration in 5 hours, pyrene in 17 hours and anthracene in 30 hours. The time as at exhaustion i.e., at the peak of C/C_o profiles, indicate the breakthrough time i.e., the time each PAH would elute.

The observed result suggest that naphthalene would elute first before pyrene and anthracene. This behaviour can again be attributed to the differences in their properties such as aqueous solubility, molecular weight and diffusivity in water. The drop in the C/C_o peaks may be as a result of the weakening of the hydrogen-carbon bonds with time. This weakening, which results from the solubility of the contaminants, promotes diffusive mechanism, facilitates the transport of the compound and thus enhances its availability in solution. The dissolution of the tracer is of utmost benefit for the reaction because it promotes the degradation process by enhancing the solubility and bioavailability of the recalcitrant PAHs. However, naphthalene was observed to solubilize completely almost at the start of the experiment.

This suggests that its solubility and transport was not facilitated by the presence of the surfactant. This behavior can be attributed to its physical properties when compared to the other PAHs in the soil matrix (see Table 1), as aqueous solubility for naphthalene was highest at 0.93 (20). The marked variation in time between pyrene and anthracene in spite of the fact that pyrene has a higher molecular weight may be related to the differences in their structural configuration, the very low solubility of anthracene and the close diffusivity exhibited between anthracene and pyrene. Anthracene has a linear structure in which the benzene rings are tightly fused together while pyrene has a tetrahedral structure in which the rings are loosely fused. This creates porosity within the interstitial spaces between the rings making it susceptible for microbial attack. The results therefore implicates pattern of ring linkage and molecule topology as factors which could be considered as very important in the study of the kinetics of PAH degradation.

The two methods were further used to estimate the transport parameters; pore-water velocity and dispersion coefficient using the non-reactive solute; the retardation factor and the degradation rate constant of the contaminant solutes. In the result shown in Table 2 and 3, the lower order of magnitude (one order) observed for the coefficient of deviation of V when compared with those for D, R and λ , was attributed to the fact that only the first normalized moment is required to estimate V for a non-reactive solute. Conversely, estimation of D, R and λ required both the first and second moments, hence the marked deviation for those parameters. This observation was found to be consistent with the reports of (22) and (15) where it was noted that the higher the orders of the moments, the less stable the calculation.

The higher values of D obtained using MOM when compared with the least squares curve-fitting showed that the first moment calculation is positively influenced by calculation in the BTC tail. This is predicated on the fact that the MOM estimated values is a function of the difference between the second moment and the first moment squared. R is a dimensionless parameter which describes the bioavailability (the accessibility of a chemical for assimilation and possible toxicity) of one PAH relative to another and it increases with increasing solute hydrophobicity.

The results in Table 3 showed that naphthalene had lowest retardation factor with a corresponding higher degradation rate constant. The good agreement as shown by the coefficient of deviation ε , therefore suggests that the MOM could provide an additional and useful means of parameter estimation for transport involving equilibrium sorption and first order degradation.

The transport parameters V and D obtained from the temporal moment solutions and the independently estimated diffusion parameters k_f , D_p and D_f were used for the resolution of the dimensionality of the non steady state model. A reasonable agreement was observed between the experimental data on the concentration-time behaviour and the model predictions for the PAHs depicted in Tables 5.1 to 5.6. The lower diffusivities obtained in the soil when compared to diffusivities in water further suggest that diffusion in the pores of the soil may be retarded by surface adsorption/desorption effects on soil organic carbon. The mass transfer coefficient was significantly high for the PAHs compared to the diffusivities and this can be directly attributed to their aqueous solubility. Thus low solubility will account for low transfer/diffusion rate.

A critical appraisal of the values from the axial and radial directions of flow revealed that there was a consistent decrease in the concentration of the PAHs with time in the axial direction. In contrast, with increasing contact time, more PAHs were found within the soil particle pores than on the soil particle surface in the radial direction. This observation was found to be consistent with the findings of (13), (23), (24) and (4), where it was reported that more microbial activities take place on the soil particle surface than within the pores. This therefore suggested that with prolonged contact time, PAHs become occluded within the micropores (fissures and cavities) of the soil particles, making them not bioavailable and thus inaccessible to microbial degradation. The simulated concentration as given in Table 5.1 through 5.6 showed that the decay rate was faster in the macroporous system than in the microporous system. The concentration of naphthalene was lower in both systems followed by pyrene and anthracene in that order.

These results affirmed the fact that the microbial utilization of pyrene and anthracene for metabolic activities were greatly limited by their resistance to mass transfer between the fluid phase and the soil matrix occasioned. by their low aqueous solubility. Given the initial and boundary conditions, the results showed that for macro and microporous systems with C , z , r as dependent variables, the residual concentrations were in the order naphthalene < pyrene < anthracene. This result affirms that the microbial utilization of anthracene for their metabolic activities is greatly limited by the resistance to mass transfer within the soil/sediment matrix. Finally, the result confirmed that naphthalene was more selectively mineralized in the microcosm reactor than pyrene and anthracene.

Conclusion

Results of the detailed non-steady state model which accounts for the intraparticle, interparticle and interphase mass transport developed for both macro and microporous systems showed that long-term exposure of PAHs was characterized by a progressive accumulation of the PAHs from the soil particle surface towards the pores of the soil particle. This suggested PAH occlusion in the micropores of the soil particle. Anthracene was found to elute after the breakthrough of naphthalene and pyrene respectively. Unlike most other models in microbial degradation study, the developed non-steady state model adequately predicted the concentration profiles of PAHs within the soil matrix.

The parameter values and the simulated BTCs obtained from applying the temporal moment solutions and the curve-fitting were found to be very similar and showed good agreement as indicated by the coefficient of deviation ε . The temporal moment solutions were thus satisfactorily verified in this study.

i. Nomenclature

C_{oi}	Initial concentration
C_{si}	Concentration of contaminant in the solid phase (mg/l)
	Porosity of soil
D	Dispersion Coefficient (m^2/day)
K_f	Mass transfer coefficient for external film diffusion (m/day)
r	distance from the centre of the particle (cm)

R	Radius of soil particle (cm)
R	Retardation factor
t	time (days)
V	Pore water velocity (m/day)
Z	Axial distance (cm)
λ	first order degradation rate (per min)
ε	Coefficient of deviation

Dimensionless parameters

τ	time
z	space variable
Z_T	length of soil column
α, β, γ	constants defined by equation 11
ψ, η, ϕ	constants defined by equation 12

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12. Table 1: Some properties of investigated PAHs

Properties	Naphthalene	Anthracene	Pyrene
Molecular formula	C ₁₀ H ₈	C ₁₄ H ₁₀	C ₁₆ H ₁₀
Molecular weight (g/mol)	128	178	202
Density (g/cm ³)	1.14	1.099	1.271
Melting point (°C)	80.5	217.5	145-148
Boiling point (°C)	218	340	404
Aqueous solubility (g/m ³)	0.93	0.07	0.14

*Adapted from 19, 20 and 6.

TABLE 2 COMPARISON OF THE PORE-WATER VELOCITY AND DISPERSION COEFFICIENT OBTAINED FROM METHODS OF TEMPORAL MOMENTS AND CXTFIT CURVE FITTING PROGRAM.

Experimental data of the study (sandy soil microcosm reactor)

Tracer	V (m/day)	D (m ² /day)
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	MOM	CXTFIT	ε	MOM	CXTFIT	E
Sodium hexameta phosphate	2.15	2.24	-0.04	2.79	2.47	0.11

13. Table 3 Comparison of the Degradation and Transport parameters for the Contaminant PAHs

PAHs	R			λ (per day)		
	MOM	CXTFIT	ε	MOM	CXTFIT	E
Naphthalene	25.77	20.23	0.21	3.54	4.22	-0.19
Anthracene	41.62	28.43	0.32	1.21	2.05	-0.69
Pyrene	35.66	25.89	0.27	2.25	3.26	-0.45

Where $\varepsilon = \frac{MOM_{value} - CXTFIT_{value}}{MOM_{value}}$

Table 4
 Estimated values of the parameters for the solution of the modelling equations

Parameters	Naphthalene	Anthracene	Pyrene
k_f (m/day)	1.924E-3	1.72E-3	1.69E-3
D_p (m ² /day)	8.61E-6	8.59E-6	8.57E-6
D_f (m ² /day)	7.28E-5	6.10E-5	6.08E-5

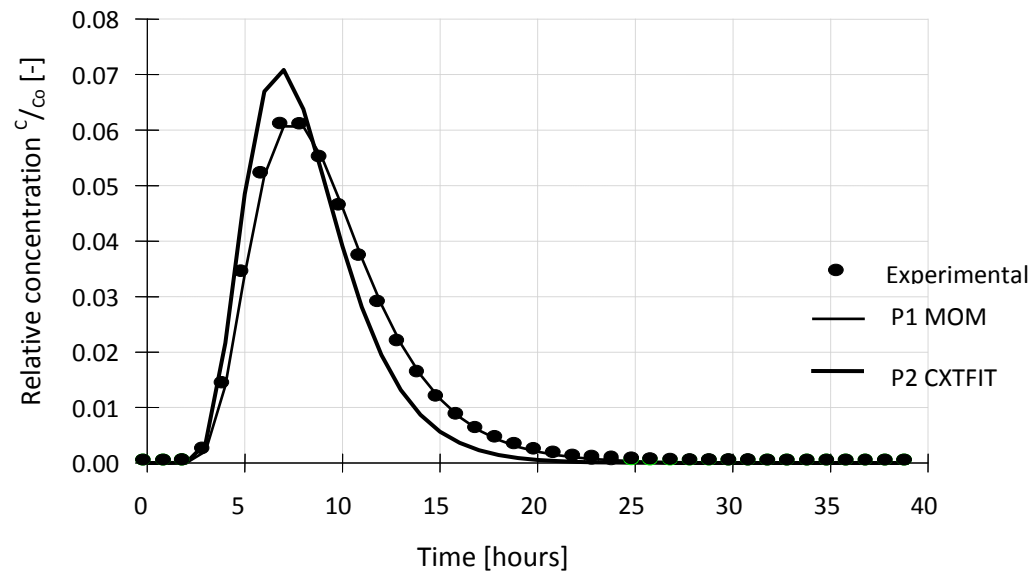


Fig 2: Comparison of MOM AND CXTFIT simulated breakthrough curves for Tracer using

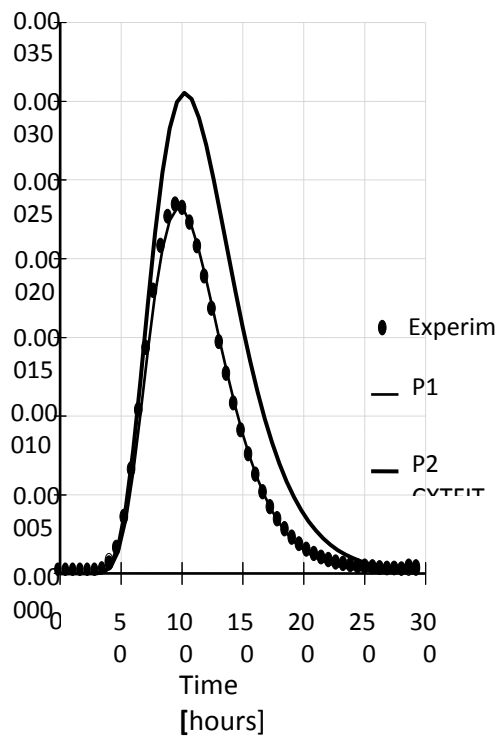


Fig.3: Comparison of MOM and CXTFIT simulated breakthrough curve for Naphthalene using experimental data.

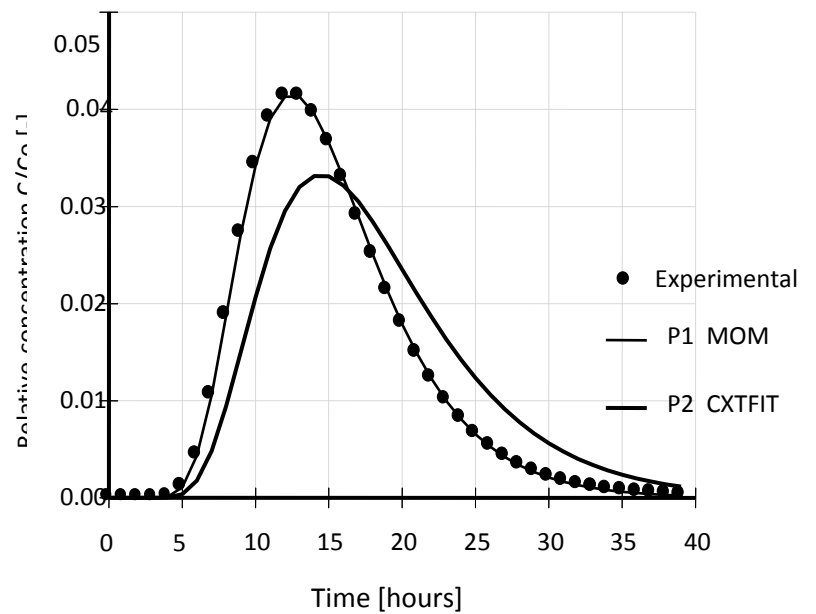


Fig.4: Comparison of MOM and CXTFIT simulated breakthrough curve for Anthracene using experimental data

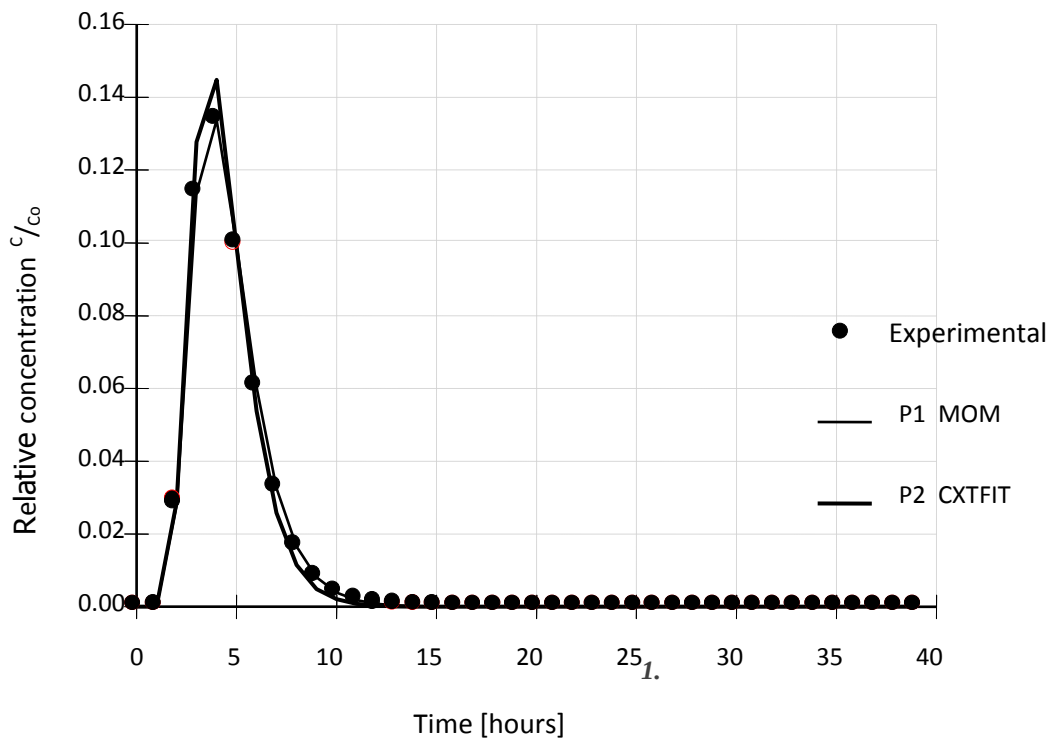


Fig.5: Comparison of MOM and CXTFIT breakthrough curves for Pyrene using experimental data.

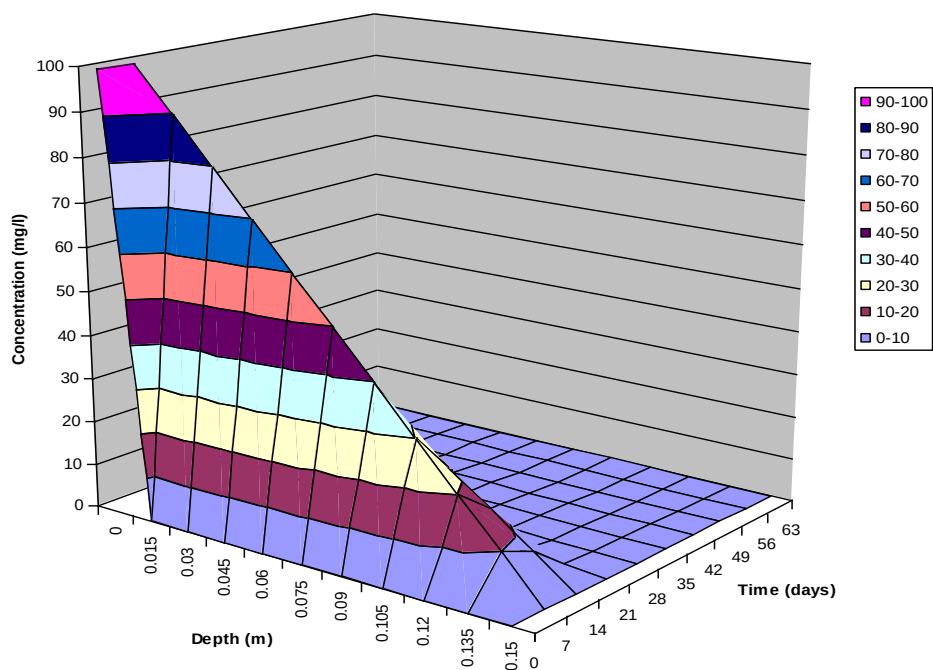


Fig 6: Variation of Concentration of Naphthalene with time and depth of soil particle

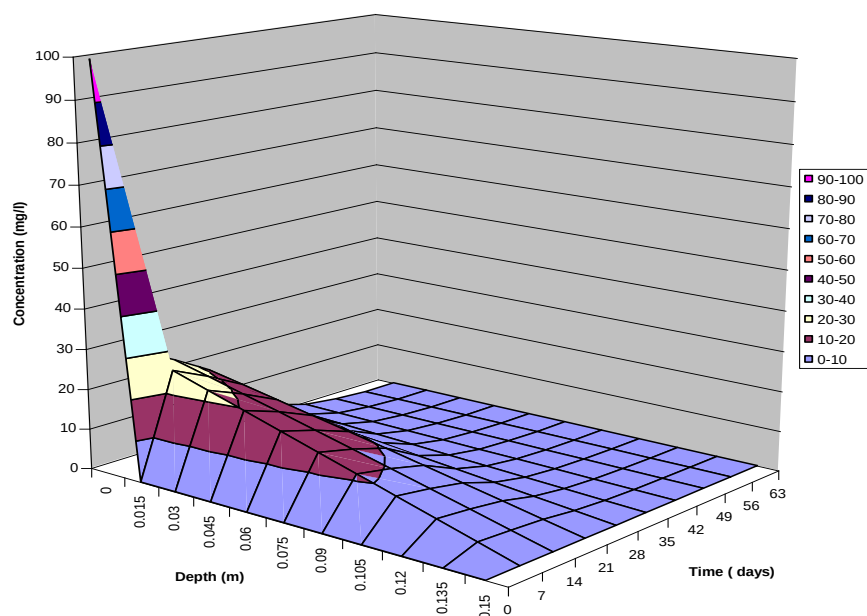


Fig 7 Variation of Concentration of Anthracene with time and depth

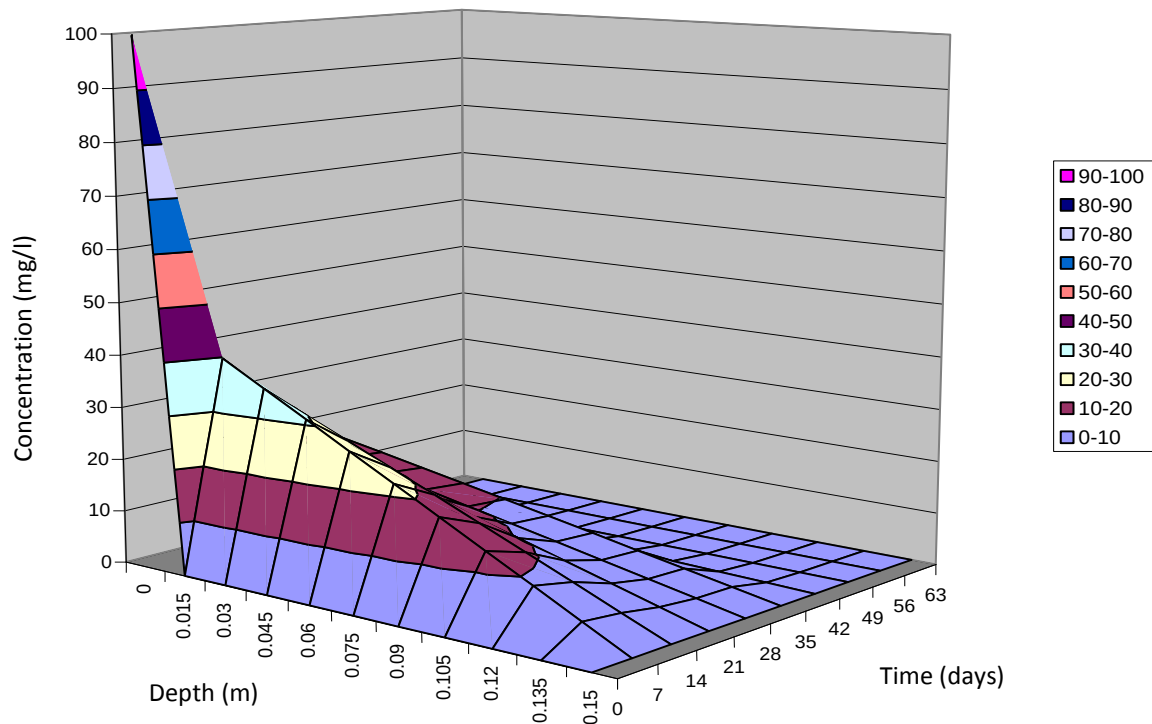


Fig 8: Variation of Concentration of Pyrene with time and depth

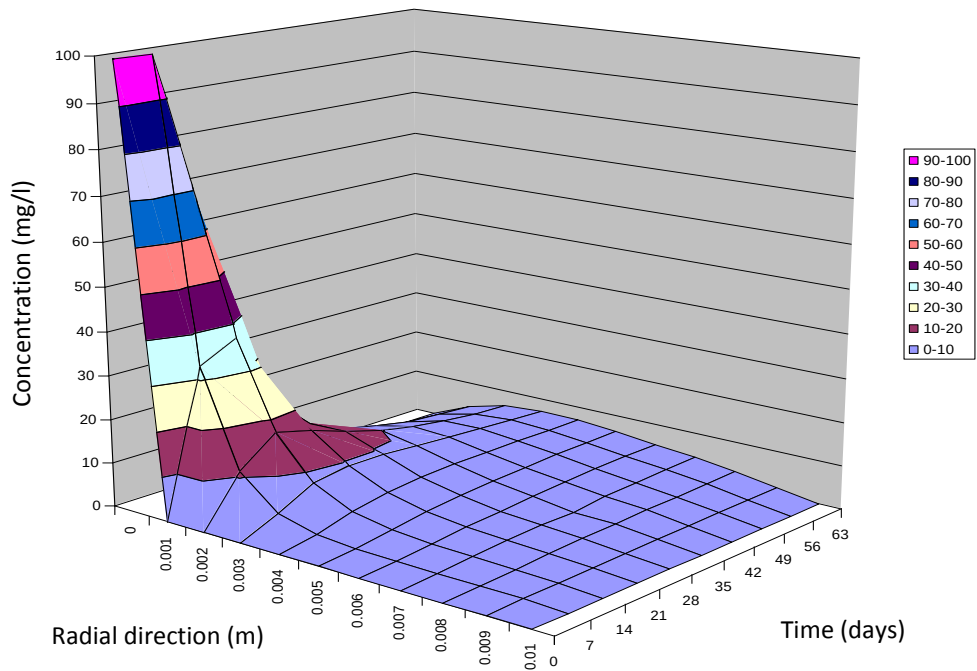


Fig 9: Variation of Concentration of Naphthalene with time and radial direction

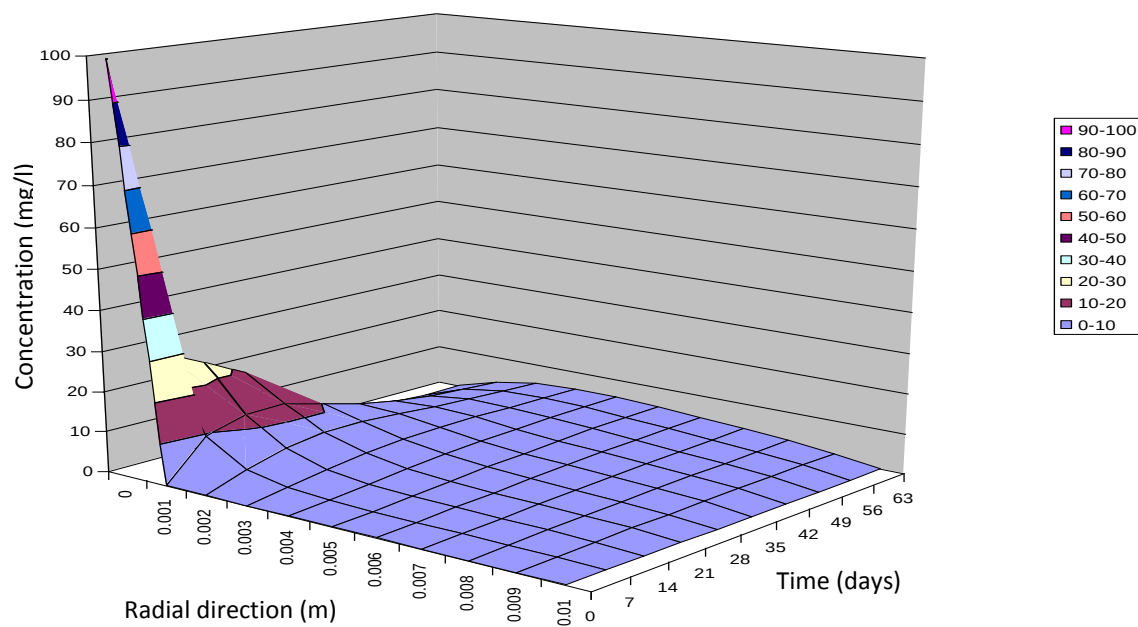


Fig 10: Variation of Concentration of Anthracene with time and radial direction

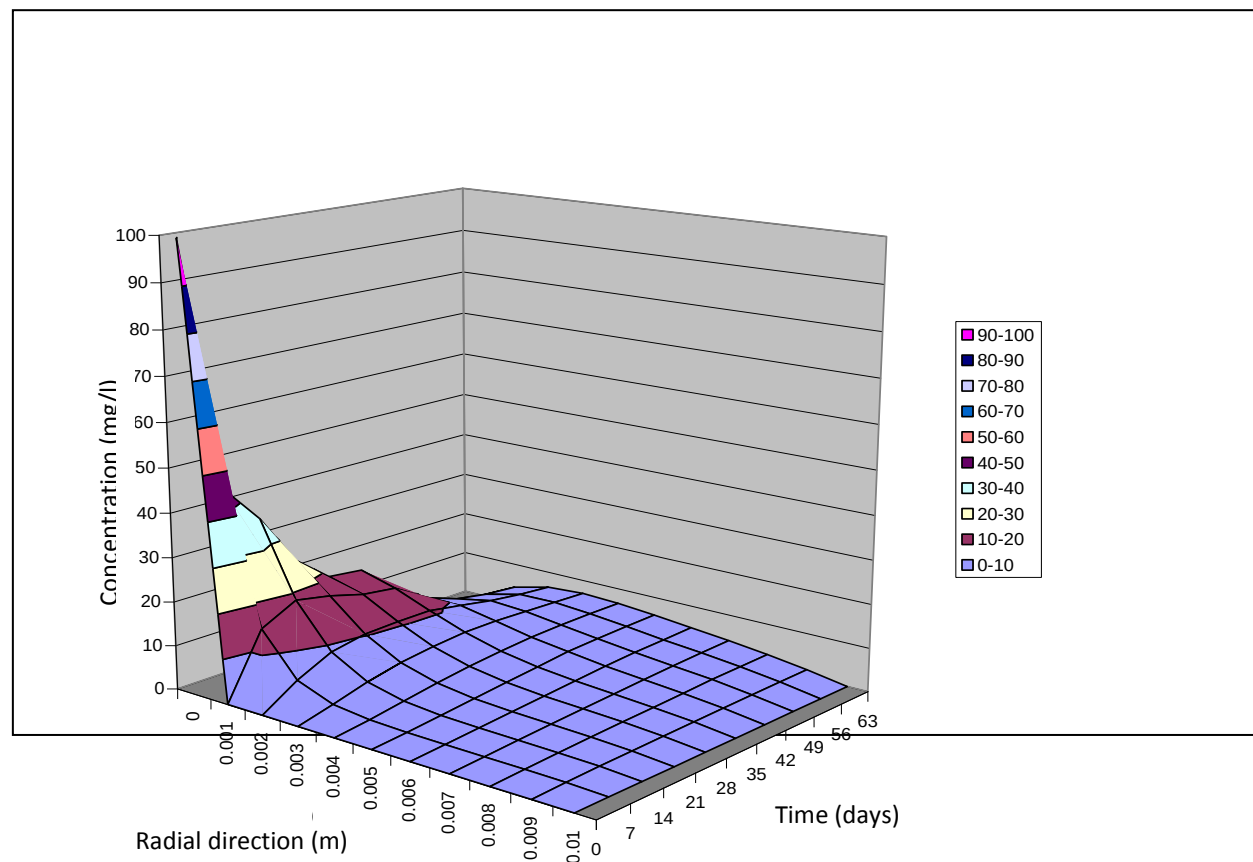


Fig 11: Variation of Concentration of pyrene with time and radial direction

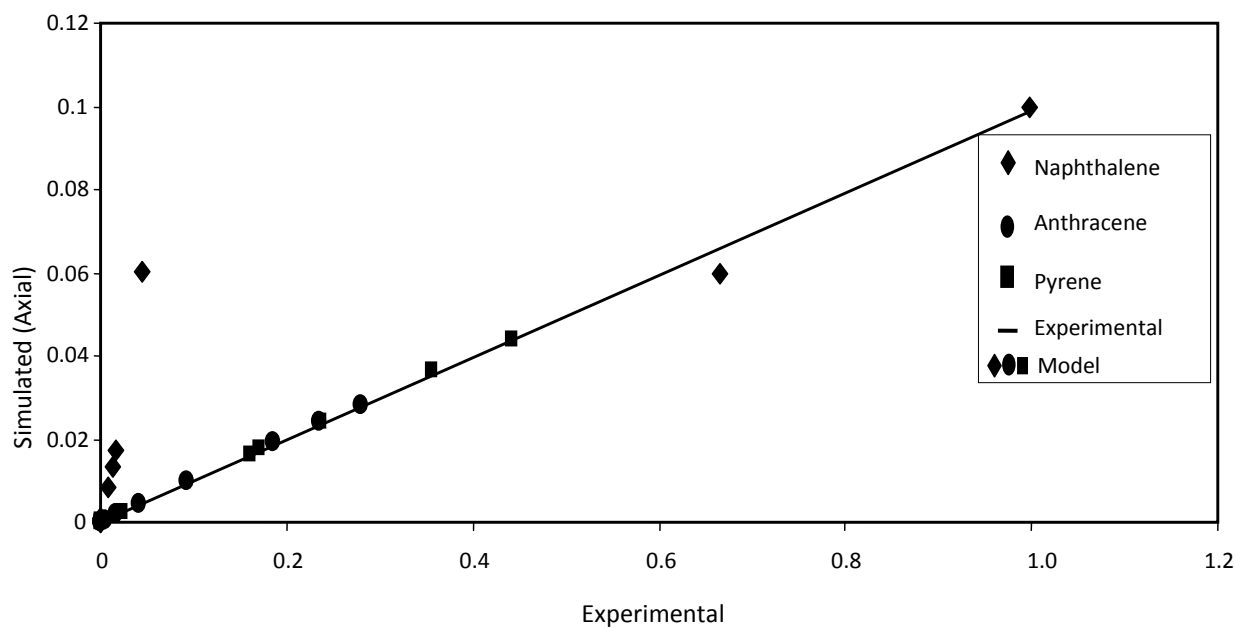
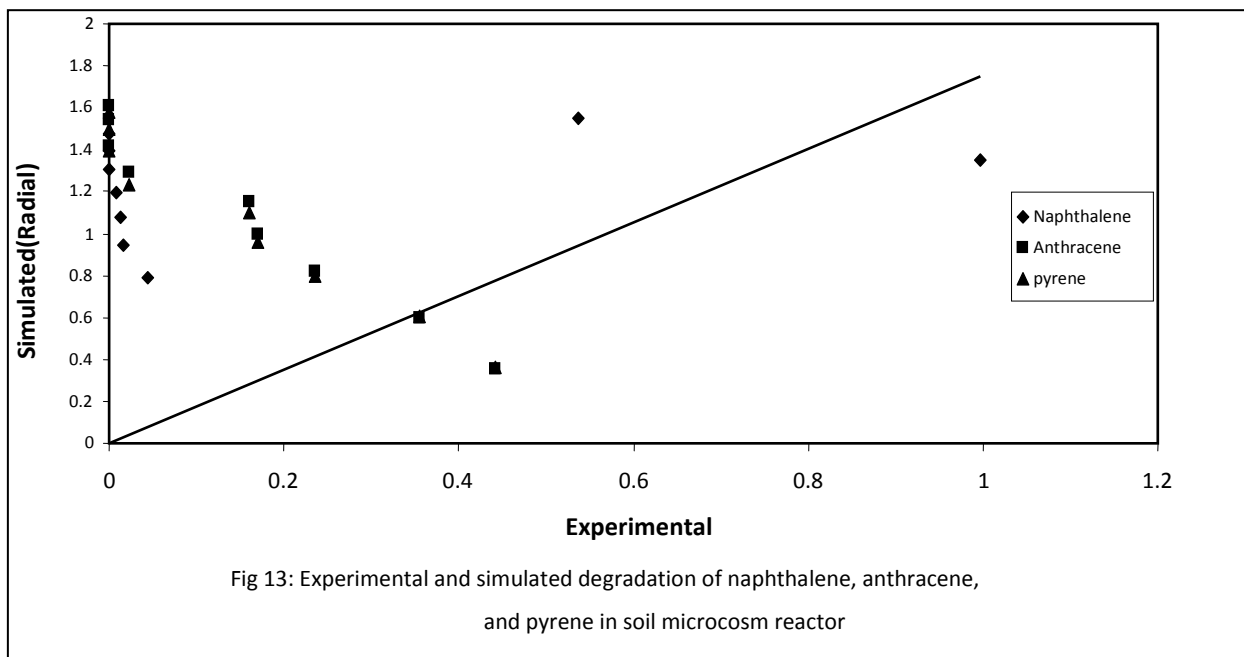


Fig 12: Experimental and simulated degradation of naphthalene, anthracene and pyrene in a soil microcosm reactor



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EMPERICAL MATHEMATICAL MODEL FOR PALM KERNEL OIL EXPRESSION USING SCREW PRESS

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ABSTRACT

Some of the operational parameters that influence the efficiency of screw press are pressure, feeding rate and speed of rotation. Application of mathematical model to determine the degree of influence of these parameters on performance of screw press is essential in its design and production. Empirical mathematical constructs were developed through statistical analysis based on experimental data. Dependent variables were oil yield, free fatty acid content and colour while pressure, feeding rate were the independent variables. Three levels each of independent variables used are Pressure (10, 20, 30 MPa), feeding rate (50, 100, 150 kg/h) and rotational speed (50, 80, 110 rpm). The data were analyzed statistically using regression. Among the studied parameters, only pressure significantly influenced the dependent variables at $p < 0.05$ but no relationships exist between all the parameters and the free fatty acid showing that machine parameters have no influence on the free fatty acid. The use of mathematical model to describe the relationship between process variables and product characteristics encourage experimental design whereby maximum amount of information can be obtained from minimum number of experiments. The mathematical relationships obtained are useful in engineering designs.

Key words: Modeling, Operational parameters, Screw press, Oil yield, Free Fatty Acid, Colour
Mathematics Subject Classification 2000:92C10 & 92C40

INTRODUCTION

Oil content for vegetable and oil-bearing materials varies between 3 and 70 % of the total weight of the seed, nut, kernel or fruit (Langstrat, 1976; Bachman, 2004). The rate of vegetable oil consumption is increasing compared to animal fat due to its health implication. The industry is challenged by demands for high quality products at reduced prices. Oil crops are vital part of the world's food supply as the world trade in oilseeds and oilseed products was estimated to be \$76 billion; equivalent to 13% of the total agricultural trade and the third most valuable component in total world agricultural trade, next to meat and cereal (FAO, 2005). This report further revealed that palm kernel oil accounted for 4.7% of the total value. Oil is obtained from oilseed by either solvent extraction or mechanical expression or the combination of the two processes. Mechanical oil expression is the process of removing oil from oil bearing materials by application of pressure using oil presses or expeller. Organic solvent such as hexane or dilute miscella is used to remove oil content from kernel flakes in solvent extraction method. Veloso *et al.* (2005) work on vegetable oil extraction revealed that direct solvent extraction is suitable for oil seeds containing less than 20 % oil, while mechanical expression is appropriate for high oil content seeds exceeding 30 %.

The oil palm tree (*Elaeis guineensis* Jacq.) is acclaimed to be the richest vegetable oil plant (Kheiri, 1985) with products such as palm oil, palm kernel oil, palm wine, fatty alcohol, broom and wood plank derivable from it. Palm kernel oil is used locally for cooking, lighting and production of soap, cosmetic and margarine. Good quality edible oil is fresh, pure, free from odour, and any sign of rancidity. The acceptability of a product at world edible oil market depends on its ability to satisfy basic standard tests for fats and oil.

Theoretically models are developed based on simulations of applicable laws and theories, while empirical ones depend on experimental data. The mathematical construct relates the process variables, which are categorized into dependent and independent variables. Inputs are independently manipulated while the outputs are dependent upon the inputs. Manipulating the variables within the boundary condition gives optimisation.

A number of studies have developed empirical mathematical models to predict effects of process parameters on the yield and quality of many oilseeds. These include duration of pressing on cake residual oil content in sunflower (singh et al., 1984) the use of response surface method to show the conditions for maximum oil expression from peanut using the screw press (sivakumaran et al., 1985), process optimisation and modelling of oil expression from groundnut and shearnut kernel (olajide, 2000), development of a model equation to predict effect of moisture content, roasting duration and temperature on sesame seed oil yield (Akinoso and Igbeka, 2006), prediction of the effects of temperature, pressure and pressing time with a moisture content on oil yield of soybean (Khan and Hanna, 1984; Tunde-Akintunde et al., 2001, 2002). Others are studies on groundnut (Hamzat and Clarke, 1993), rapeseed (Sukumaran and Singh, 1987).

theoretical predictions have also been done for oilseeds and these include the work on oil expression characteristics of rape seed by Raji and Favier (2004) who applied a simulation technique based on the discrete element method (dem) to numerically model the bulk compression of beds of three oilseeds: canola palm kernel and soybean. The model was able to predict quite closely the bed strains at the oil point to those observed in experiment for each seed type. Mrema and mculty (1985) developed a mathematical model for mechanical oil expression from oil seeds. The model revealed that the rate of oil expression was dependent on the flow of oil across the cell wall. Singh and singh (1991) developed mathematical model for oil expression from a thin bed of rapeseeds under uniaxial compression. It correlates oil extraction with pressure, coefficients of consolidation and permeability, the time of compression, area of cross section through which oil flows occurs and the density of the oil. The model described rape seed oil extraction's behaviour having close agreement with experimental values. They concluded that the models are useful tools in the study of mechanical seed oil expression and other agricultural particulate compression processes as well as provision of data necessary in the design of appropriate machinery.

Modelling expression from oil seed in mechanical expression will involve biological process modelling which will involve heavy theoretical formulations and complex relationships. This will take into consideration oil seed structure (s), engineering properties (e [strength modulus], r [rupture], μ [friction]), applied pressure, (p or σ) and heat (q), flow process (v, viscosity) and failure criterion. The functional relationship will be of the form in equation 1 which will end up being partial differential equation

$$Oy = f(s, e, r, \mu, \xi, v, p, q) \quad 1.$$

solvent extraction on the other hand will involve some of these parameters, chemical reactions in addition to diffusivity index being a mass movement and osmotic process hence

$$Oy = f(s, \mu, \xi, v, \theta, q) \quad 2$$

This study is an empirical study which combines experimentation with model development for prediction but not the theoretical modeling. It is a preliminary study for the full theoretical study.

The general objective is however to investigate the machine parameters that affect the efficiency of screw press using experimental data. The specific objectives of the research are (a) To determine the effect of operational parameters on efficiency of screw press. The parameters to be considered are applied pressure, feeding rate and speed of rotation. (b) To develop model equations that relates screw press efficiency to parameters under study. Moisture content, roasting duration and roasting temperature have been reported to significantly influence yield and quality of palm kernel oil. However, effect of operational parameters which include pressure, feeding rate and

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rotational speed cannot be neglected. Application of mathematical model to determine the degree of influence of the above mentioned factors on efficiency of oil expeller is essential in its design and production. The use of mathematics to describe the relationship between process variable and product characteristics encourage experimental design whereby maximum amount of information can be obtained from minimum number of experiment.

METHODOLOGY

A 3 x 3 factorial experimental design (Table1) was used to study the effect of machine parameters Compressive stress, feeding rate and worm shaft speed on oil yield and quality characteristics (free fatty acid and colour).

Table 1. The experimental design

Parameters	Levels		
	1	2	3
Compressive stress MPa	10	20	30
Feeding rate kg/h	50	100	150
Speed rpm	50	80	110

A tenera variety of palm kernel was procured from Nigeria Institute for Oil Palm Research NIFOR Benin City, Nigeria. Using ASABE (2008) standard for oil seed, moisture content of the palm kernel was determined, having cleaned the seed. A Tite 002 helical thread oil expeller manufactured by Tiny Tech Plant, India with rated capacity of 180 kg/h, powered by a 30 kW electric motor with interchangeable speed was used for oil expression. Expressed oil was collected and clarified by allowing it to stand for 96hr, then bottled and labeled as samples. Each sample weight was converted to percentage of the initial weight of raw material that is 5 kg. This procedure was repeated 3 times for each treatment, mean values were recorded as oil yield. Ratio of oil yield to total oil content as determined by soxhlet oil extraction method was recorded as expressed efficiency.

The samples of oil obtained from the experiments were analysed for Free Fatty Acid (FFA) and colour change. The FFA was analysed using the AOAC (1994) standard method for estimation of oil in oilseeds. This involved mixing 5mg of each oil sample with 50ml of hot neutral alcohol and phenolphthalein and then titrating with 0.5N NaOH until a pink colour was obtained. The colour changes observed in the samples after expression were obtained by Computer Image Analysis using Paint Shop Pro (a graphics application) to analyse the colour components of the images of the PKO samples taken with a digital camera. The histogram distribution of the Red-Green-Blue (RGB) components of each image was obtained. Experimental Wavelength Method involving the use of chemical method to obtain the wavelength corresponding to the absorptivity of each of the three principal colour components in each of the samples. These when converted to pixels (picture element) were compared with the results of the mean pixels obtained from the computer image analysis method.

The results of the oil expression were analysed statistically by regression to obtain the relationship between the independent variables; compressive stress, speed of operation of the oil expeller and feed rate and the dependent variable; oil yield and Free Fatty acid and colour. Mathematical modeling relating the expeller performance with all the parameters gave functional relationships for predicting and selecting appropriate parameter and variables combinations for the best oil.

RESULTS AND DISCUSSION

Oil Yield: The result of the regression analysis of the results is as presented in Table 2. The result presented is for a full quadratic analysis. The result shows that predictors: (Constant), Speed (S, rpm), Feeding rate (F, kg/h),

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Compressive stress (P, MPa) correlated positively and directly proportional to the oil yield (OY, % mean) with a positive and a value close to unity for the correlation coefficient. This is also evident in the R^2 value (0.952) which shows that the model fits well and that there is a relationship between the parameters. However from the coefficient table it was discovered that only the compressive stress significantly affect the oil yield at $p < 0.05$ from the value of the t-stats. This is also evident in the regression functional relationship which is given as Equation 3:

The functional relationships obtained between the combinations of two of the predictors with the oil yield are as presented in Equations 4 and 5 for compressive stress and speed, compressive stress and feeding rate respectively. The relationship with the third possible combination speed and feeding rate showed a very poor fit with an R^2 of 0.06 hence the equation is not presented.

$$OY = -6.94 + 2.661P - 0.076F + 0.348S - 0.049P^2 + 0.00067PF + 0.0031PS - 0.00031F^2 + 0.0014FS - 0.0033S^2 \quad (R^2 = 0.952) \quad 3$$

Where

OY-Oil yield (%)

P-Compressive stress (Mpa)

F-Feeding rate (kg/h)

S- Speed of rotation (rpm)

Table 2. Coefficient of the predictors for oil yield

	Coefficient	P value	t Stat	VIF
Constant	-6.494	0.540	-0.625	
P	2.661	3.039E-05	5.623	65.67
F	-0.07556	0.436	-0.798	65.67
S	0.348	0.08956	1.800	98.33
P ²	-0.04944	0.000138	-4.888	49.00
PF	0.000667	0.647	0.466	13.00
PS	0.00306	0.217	1.282	17.67
F ²	-0.000311	0.452	-0.769	49.00
FS	0.00139	0.00969	2.913	17.67
S ²	-0.00327	0.00973	-2.911	86.33

$$OY = -17.68 + 2.728P + 0.486S - 0.049P^2 + 0.003PS - 0.003S^2 \quad (R^2 = 0.922) \quad 4$$

$$OY = -1.593 + 2.906P + 0.035F - 0.049P^2 + 0.006PF - 0.0003F^2 \quad (R^2 = 0.895) \quad 5$$

See Appendix for the coefficients and statistical factors.

These predictions as presented in Equations 3 to 5 has shown that the oil yield is highly dependent on the compressive stress i.e. the pressure applied during expression. The speed and feeding rate are found to have no considerable effect on the oil yield at constant pressure. In order to obtain the optimized combinations of the predictors for the oil yield, the predicted functional relationships were used to obtain the surface response curves for the models. The optimized values were obtained from the plots in Figures 1 – 3 as Oil yield of 45.2 % at Pressure of 30MPa, Speed of 90 rpm and Feeding rate of 90 kg/hr

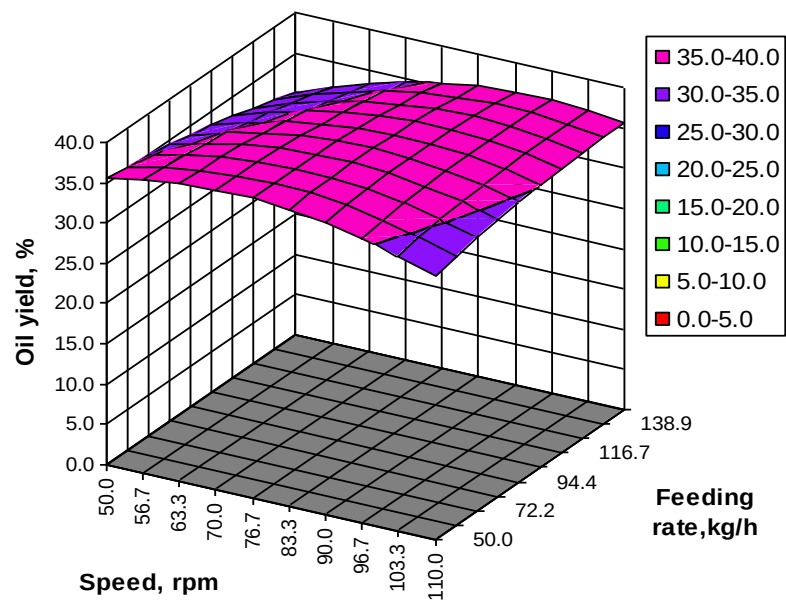


Fig.1. Plot of oil yield against speed and feeding rate at constant compressive stress

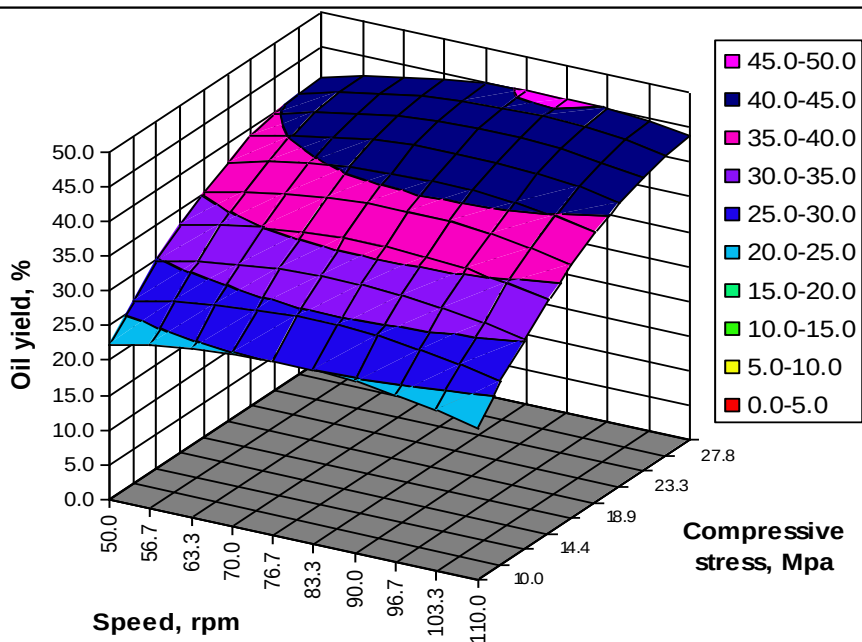
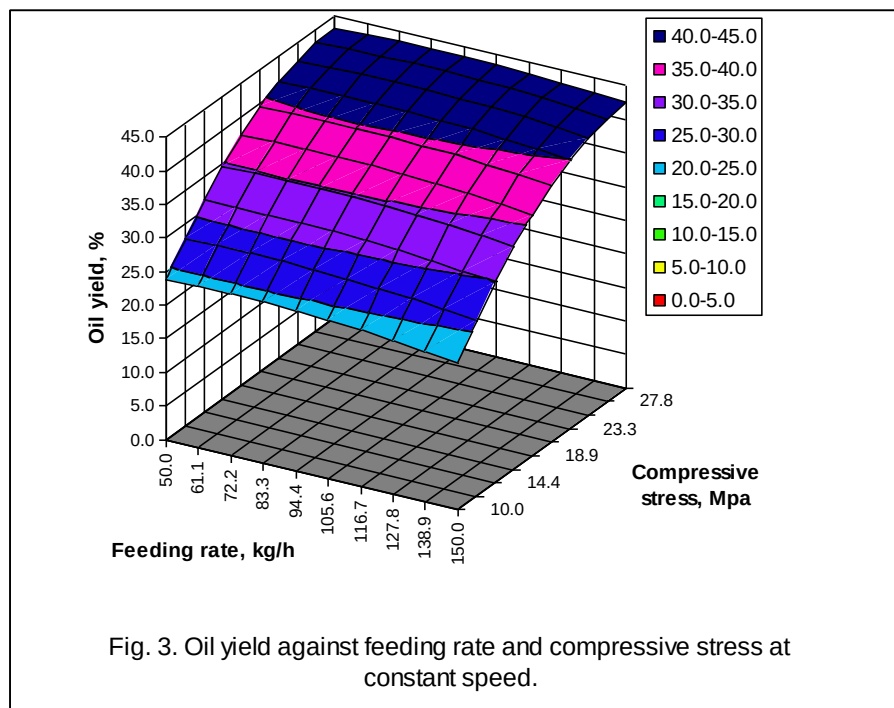


Fig.2. Oil yield against speed and compressive stress at constant feeding rate.



Free Fatty Acid (FFA). The free fatty acid content of the expressed oil did not have a good fit with the predictors: (Constant), Speed (S, rpm), Feeding rate (FR, kg/h), Compressive stress (P, MPa) although a relationship exists. This is evident in the very low value of R^2 (0.322). However from the coefficient table it was discovered that none of the predictors have significant effect on the fatty acid content ($p < 0.05$). This is also evident in the regression functional relationship with very low coefficients for all the predictors or the independent variables which is given as

$$\text{FFA} = 5.759 - 0.165P - 0.025F - 0.034S + 0.0032P^2 + 0.00037PF + 0.00019PS + 5.70 \times 10^{-5}F^2 + 8.96 \times 10^{-5}FS + 8.99 \times 10^{-5}S^2 \quad (R^2 = 0.322)$$

Colour. In a similar trend, the coefficient of determination R^2 obtained from the regression analysis for the colour is too low which shows that there is no good relationship between the colour change and the predictors (Constant), Speed (S, rpm), Feeding rate (F, kg/h), Compressive stress (P, MPa). This indicates that the FFA content and colour change are not heavily dependent on the combined effect of these machine parameters. The revelation discouraged further analysis.

CONCLUSIONS

It can be concluded from the foregoing that the predictions through statistical mathematical modeling of oil expression for palm kernel oil has revealed that oil yield and its properties are affected mostly by pressure. The machine parameters however have no significant effect on the quality parameters such as FFA and colour. There is however the need to obtain comprehensive functional relationships for the screw presses efficiency with more machine and oil quality parameters as well as kernel properties. This will be useful in machine design, process planning and theoretical studies.

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APPENDIX

Statistical coefficients for PF, PS, and FS

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$$\text{Oil_yield, \%}(1) = b_0 + b_1 * \text{Compressive_stress, Mpa} + b_2 * \text{Feeding_rate, kg/h} + b_3 * \text{Compressive_stress, Mpa} * \text{Compressive_stress, Mpa} + b_4 * \text{Compressive_stress, Mpa} * \text{Feeding_rate, kg/h} + b_5 * \text{Feeding_rate, kg/h} * \text{Feeding_rate, kg/h}$$

($R^2 = 0.895$)

		P value	Std Error	-95%	95%	t Stat	VIF
b0	-1.593	0.839	7.739	-17.69	14.50	-0.206	
b1	2.906	5.38467E-05	0.576	1.708	4.103	5.045	55.00
b2	0.03556	0.761	0.115	-0.204	0.275	0.309	55.00
b3	-0.04944	0.00141	0.01345	-0.07741	-0.02147	-3.676	49.00
b4	0.000667	0.729	0.00190	-0.00329	0.00462	0.350	13.00
b5	-0.000311	0.569	0.000538	-0.00143	0.000808	-0.578	49.00

$$\text{Oil_yield, \%}(1) = b_0 + b_1 * \text{Compressive_stress, Mpa} + b_2 * \text{Speed, rpm} + b_3 * \text{Compressive_stress, Mpa} * \text{Compressive_stress, Mpa} + b_4 * \text{Compressive_stress, Mpa} * \text{Speed, rpm} + b_5 * \text{Speed, rpm} * \text{Speed, rpm}$$

($R^2 = 0.922$)

		P value	Std Error	-95%	95%	t Stat	VIF
b0	-17.68	0.08428	9.756	-37.97	2.609	-1.812	
b1	2.728	3.13905E-05	0.517	1.652	3.803	5.275	59.67
b2	0.486	0.03396	0.214	0.04051	0.932	2.269	92.33
b3	-0.04944	0.000345	0.01159	-0.07356	-0.02533	-4.264	49.00
b4	0.00306	0.276	0.00273	-0.00263	0.00874	1.118	17.67
b5	-0.00327	0.01908	0.00129	-0.00595	-0.000592	-2.539	86.33

$$\text{Oil_yield, \%}(1) = b_0 + b_1 * \text{Feeding_rate, kg/h} + b_2 * \text{Speed, rpm} + b_3 * \text{Feeding_rate, kg/h} * \text{Feeding_rate, kg/h} + b_4 * \text{Feeding_rate, kg/h} * \text{Speed, rpm} + b_5 * \text{Speed, rpm} * \text{Speed, rpm}$$

$R^2 = 0.06$

		P value	Std Error	-95%	95%	t Stat	VIF
b0	23.65	0.493	33.88	-46.79	94.10	0.698	
b1	-0.06222	0.864	0.359	-0.809	0.685	-0.173	59.67
b2	0.409	0.589	0.745	-1.140	1.957	0.549	92.33
b3	-0.000311	0.849	0.00161	-0.00366	0.00304	-0.193	49.00
b4	0.00139	0.472	0.00190	-0.00256	0.00534	0.732	17.67
b5	-0.00327	0.473	0.00447	-0.01257	0.00603	-0.731	86.33

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Some Malaria Models Treating both Sensitive and Resistant Strains in Single and Multigroup Populations

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Abstract

This paper presents a model for malaria infection in which all infecteds are treated initially with a certain drug. Under this setting we assume that the class of (partial) immunes can only arise from the class of resistant infecteds. Human population is assumed fixed in all considerations while that of vectors vary. All the models presented are ordinary differential equation models. Our focus in analysing the models is on establishing a unique positive endemic state for each of them. It is shown that (under suitable conditions) the equilibrium points are (locally) asymptotically stable. The biological significance of these points also emerges as a by-product.

Keywords

Model,malaria,drugs,population and blood

Mathematics Subject Classification 2000:92C60 & 92C45

1 Introduction

Malaria is a mosquito-borne infection caused by protozoa of the genus plasmodium. Four species of the parasite, namely: *P. falciparum*, *P. vivax*, *P. ovale*, and *P. malariae* infect humans. Malaria remains the most important of the tropical diseases, being widespread throughout the tropics, but also occurring in many temperate regions.

The parasites are transmitted by the bite of infected female mosquitoes of the genus *Anopheles*. Mosquitoes become infected by feeding on the blood of infected people, and the parasites then undergo another phase of reproduction in the infected mosquito. Clinical symptoms such as fever, pains, and sweats may develop a few days after an infected mosquito bite.

In many parts of Africa, where malaria has long been highly endemic, people are infected so frequently that they develop a degree of acquired immunity, and may become asymptomatic carriers of the infection [8]. Treatment and control have become more difficult in recent years with the spread of drug resistant strains of malaria parasites [6, 8, 17]. Drugs such as chloroquine, nivaquine, quinine, and fansidar are used for treatment. More recent and more powerful drugs include mefloquine, and halofantrine.

It is estimated that 267 million people are presently infected, with 107 million clinical cases annually; the number of countries affected is put at 103 [17].

The biology of the four species of plasmodium is generally similar and consists of two discrete phases-sexual and asexual . The parasite migrates to the liver where it remains latent for several days while replicating. The latent period is followed by penetration of red blood cells and asexual replication within them. Asexual parasites in the blood, after surviving some developmental period, give rise to sexual stages called gametocytes. Gametocytes can remain in the blood for more than two years [8].

The emergence of drug-resistant strains of malaria parasite has become a significant health problem. Recent pronouncements by the World Health Organisation indicate the availability of strains that are resistant to virtually all known drugs.

Among the four species of plasmodium, *P. falciparum* causes the most serious illness and it is the most widespread in the tropics. This paper therefore focuses on the dynamics of *P. falciparum* malaria, although the analysis is similar for all forms of malaria infections.

An early fundamental model in the art of mathematical modelling of malaria, due to Ross-Macdonald describes the basic features of the interaction between infected humans (y) and infected mosquito vectors (q). The model is defined as follows:

$$\begin{aligned}\dot{y} &= \alpha q(1 - y) - ry, \\ \dot{q} &= \beta y(1 - q) - \mu q,\end{aligned}$$

where $\alpha = b\beta\frac{M}{N}$, β , r , μ are some constants defined as follows (see for e.g. [4]):

N is the size of the human population;

M is the size of the female mosquito population;

$\frac{M}{N}$ is the number of female mosquitoes per human host;

β is the rate of biting on man by a single mosquito (number of bites per unit time);

b is the proportion of infected bites on man that produce an infection;

r is the per capita rate of recovery for humans ($\frac{1}{r}$ is the average duration of infection in the human host);

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μ is the per capita mortality rate for mosquitoes ($\frac{1}{\mu}$ is the average lifetime of a mosquito).

In this simple model, the total population of both humans and vectors is assumed fixed, so that the dynamical variables (y and q) are the proportion infected in each population. The first equation describes changes in the proportion of humans infected. New infections are acquired at a rate that depends on the following factors:

- (i) the number of mosquito bites per person per unit time ($\beta \frac{M}{N}$)
- (ii) the probability that the biting mosquito is infected (q)
- (iii) the probability that a bitten human is uninfected ($1 - y$)
- (iv) the probability that an uninfected person thus bitten will actually become infected (b).

Infections are lost by infected people returning to the uninfected class, at a characteristic recovery rate r . Similarly, the second equation describes changes in the proportion of mosquitoes infected. Population changes are determined by the following factors:

- (i) the number of bites per mosquito per unit time (β)
- (ii) the probability that the biting mosquito is uninfected ($1 - q$)
- (iii) the probability that the bitten human is infected (y).

The loss term (μq) arises from the death of infected mosquitoes. The loss terms for the infected humans and infected vectors both involve death and recovery. For human hosts the recovery rate is typically faster than the death rate, whereas for vectors the opposite is the case. The origin is a local asymptotic equilibrium for this model if $\mu r > \alpha \beta$. Thus infection dies out if the product of the death rates (μ and r) for the two populations is large in the sense that $\mu r > \alpha \beta$. Thus the above formulation is a sensible approximation.

However, this model is highly simplified. The model simply assumes that an infected individual either recovers to join the susceptible group $N(1 - y)$ or dies. It fails to distinguish between the various infected categories of human and vector hosts. Thus it cannot describe accurately the recent trend in malaria infection.

This basic model has been studied, modified and generalized in different directions by various authors (see for e.g., [4, 5, 6, 11]). Aron and May [4] extended this model by introducing another population group z – the latent infected humans (infected but not yet infectious). They conjectured

that if the incubation interval in the mosquito has duration τ the second equation in the basic model could be replaced by the two dynamical systems:

$$\begin{aligned}\dot{z} &= ay(1 - q - z) - a\hat{y}(1 - \hat{q} - \hat{z})e^{-\mu\tau} - \mu z, \\ \dot{q} &= a\hat{y}(1 - \hat{q} - \hat{z})e^{-\mu\tau} - \mu q,\end{aligned}$$

where the circumflex denotes evaluation at time τ in the past: $\hat{y} = y(t - \tau)$; etc. Here, the two categories of mosquito (uninfected and infected-and-infectious) are now replaced by three categories: a proportion, $1 - q - z$, that are uninfected; a proportion q that are infected and infectious; and a third, new proportion z that are latent (infected but not yet infectious).

Bailey [5] considered two interacting populations—human hosts and mosquito vectors with each group consisting of three subgroups, viz susceptibles, infectives, and isolated (recovered and immune). These are designated by x , y , and z respectively for the human populations and x' , y' , and z' respectively for the vector populations. It follows that the number of new infections occurring in the human population in time interval δt is $\beta xy'\delta t$, where β is the infection rate. Since the converse arrangement is required to hold for vectors, namely that susceptible vectors are infected by human infectives, the number of new infections occurring in susceptible vectors in time interval δt is therefore given by $\beta' x'y\delta t$. If in addition the overall removal rates for the two populations are assumed to be γ and γ' , respectively, then the numbers of removals occurring in time δt are $\gamma y\delta t$ and $\gamma' y'\delta t$ for humans and vectors, respectively. The system of differential equations for the dynamic process involved is given as

$$\begin{aligned}\dot{x} &= -\beta xy', & \dot{x}' &= -\beta' x'y + \gamma' y', \\ \dot{y} &= \beta xy' - \gamma y, & \dot{y}' &= \beta' x'y - \gamma' y', \\ \dot{z} &= \gamma y,\end{aligned}$$

where β , β' , γ , γ' are some constants. Due to its relatively short life-cycle, isolation by immunity is negligible in the mosquito vectors, and hence the only isolation process is by death which is assumed to occur equally in all groups. Hence $z' \equiv 0$ (see [6], pp. 61–68).

In this paper we continue our study of drug resistant strains of malaria parasites which was first considered in [2]. We consider a nonlinear model treating both sensitive and resistant strains, first in a single population and secondly in a multigroup population. We assume in each case that all infected individuals receive initial treatment. This consists in the administration of chloroquine or other 4-aminoquinoline derivatives. Each of our models has only two equilibrium points - a trivial and a nontrivial equilibria. Our observation is that resistant parasites constitute the major

health problems since sensitive parasites are destroyed by drug action. It then follows that under this setting, sensitive parasites grow by an influx, metamorphosis, and /or by evolution. However our model does not recognise infected vector growth by influx. Our models reveal that either both infections die out simultaneously or an endemic state is reached for both of them.

Throughout this part, we shall adopt the following notations where applicable.

- N denotes the total human population;
- x denotes susceptibles (the number of people that are uninfected);
- y denotes infectives (the number of people that are severely infected);
- v denotes infected vectors (the population of vectors that can transmit the disease).

Where y and Y occur simultaneously, we designate y as the number of individuals infected with parasites that are sensitive to drugs and Y as the number of individuals that are infected with the resistant parasites with or without the sensitive parasites.

2 Single population

We assume that all infected individuals receive certain popular drugs such as chloroquine or other 4-aminoquinoline derivatives as their initial and only treatment. We consider infections with both drug sensitive and resistant strains of the parasite. Let r and R be the cure rates for the sensitive and resistant infecteds respectively; q and Q the populations of mosquitoes carrying sensitive and resistant parasites respectively. Note that it is possible for some mosquitoes of the class Q to be carrying both strains of the parasite. If a human susceptible is infected with both strains of the parasite, we simply call it resistant infection since the sensitive infection is absorbed in the resistant. Let u be the probability that a susceptible coming into infective contact with the mosquito class Q is infected with a resistant strain, given that the susceptible is infected and $1 - u$ the complementary probability, that is the probability that the susceptible is infected with a sensitive strain of the parasite. Since infecteds are treated initially, sensitive infecteds return directly to the susceptible group while resistant infecteds gradually acquire immunity and pass onto the immune class. Thus, sensitive infecteds cannot (possibly) produce *gametocytes* (matured strains) that are responsible for the transmission of the disease. Hence, the class of (partial) immunes can only arise from the class of resistant infecteds. Thus, we have the following mosquito-sensitive-resistant model:

$$\dot{y} = cxq + c(1 - u)xQ - cuyQ - ry, \quad (1)$$

$$\dot{Y} = cuXQ + cuyQ - RY, \quad (2)$$

$$\dot{z} = RY - \gamma z, \quad (3)$$

$$\dot{q} = \alpha z - \eta q, \quad (4)$$

$$\dot{Q} = \beta z - \zeta Q. \quad (5)$$

$$\dot{x} = -cx(q + Q) + ry + \gamma z, \quad (6)$$

where $x + y + Y + z = N$, γ is the rate of loss of immunity, c is the unit contact rate between susceptibles and infected mosquitoes with which new cases occur, δ is the death or removal rate of sensitive/ resistant type mosquitoes. Recovery from resistant infection with per capita rate R is simply a transition to the immune class. The contribution $cuyQ$ is a case of superinfection. Such phenomenon has been known to occur in reality (see for e.g. [4, 6]). Here, α and β are the rates at which new infections are acquired by the sensitive and resistant vectors respectively; η and ζ are the death or removal rates of sensitive and resistant vectors respectively.

The principal evolutionary pressure favourable to the propagation of any of the strains is the geographical distribution of the region under consideration.

Eliminating x from (1) and (2) (using $x + y + Y + z = N$) we get:

$$\begin{aligned} \dot{y} = & -ry + cNq + cN(1-u)Q - cyq - cYq - czq - cyQ - c(1-u)YQ \\ & - c(1-u)zQ, \end{aligned} \quad (7)$$

$$\dot{Y} = -RY + cuNQ - cu(Y+z)Q, \quad (8)$$

Equations (3), (4) (5), (7) and (8) have two equilibrium points, out of which the origin is one. The other equilibrium point is positive provided

$$N > \max \left\{ \frac{\gamma\zeta}{cu\beta}, \frac{[\alpha\zeta(\gamma + R) + \eta(1-u)\beta\gamma + \eta(1-u)\beta R]z_0}{\alpha\zeta R + \eta R(1-u)\beta} \right\}, \quad (9)$$

where,

$$z_0 = \frac{R(cu\beta N - \gamma\zeta)}{cu\beta(\gamma + R)}, \quad Y_0 = \frac{\gamma}{R}z_0 = \frac{\gamma(cu\beta N - \gamma\zeta)}{cu\beta(\gamma + R)}. \quad (10)$$

Let $P_0 = (y_0, Y_0, z_0, q_0, Q_0)$ be the nontrivial equilibrium point of the system (3), (7), (8), (4) and (5). Then the Jacobian matrix of the system at P_0 is

$$J(P_0) = \begin{bmatrix} -(r + cq_0 + cQ_0) & -c(q_0 + (1-u)Q_0) & -c(q_0 + (1-u)Q_0) & j_{14} & j_{15} \\ 0 & -(R + cuQ_0) & -cuQ_0 & 0 & -j_{25} \\ 0 & R & -\gamma & 0 & 0 \\ 0 & 0 & \alpha & -\eta & 0 \\ 0 & 0 & \beta & 0 & -\zeta \end{bmatrix}, \quad (11)$$

where,

$$j_{14} = c(N - y_0 - Y_0 - z_0), \quad j_{15} = c[(1 - u)(N - Y_0 - z_0) - y_0], \quad j_{25} = cuN - cu(Y_0 + z_0)$$

and P_0 is given by

$$P_0 = (y_0, Y_0, z_0, q_0, Q_0) = (y_0, Y_0, \frac{R}{\gamma}Y_0, \frac{\alpha R}{\eta\gamma}Y_0, \frac{\beta R}{\zeta\gamma}Y_0), \quad (12)$$

with Y_0 as in equation (10).

The eigenvalues of the matrix $J(P_0)$ are

$$\lambda_1 = -(r + cq_0 + cQ_0) < 0, \quad \lambda_2 = -\eta < 0,$$

and the three eigenvalues of the matrix

$$J_1 = \begin{bmatrix} -(R + cuQ_0) & -cuQ_0 & cuN - cu(Y_0 + z_0) \\ R & -\gamma & 0 \\ 0 & \beta & -\zeta \end{bmatrix}. \quad (13)$$

The characteristic polynomial of J_1 is

$$\lambda^3 + (R + cuQ_0 + \gamma + \zeta)\lambda^2 + ([R + cuQ_0]\gamma + [R + cuQ_0]\zeta + \gamma\zeta + cuQ_0R)\lambda + (R + cuQ_0)\gamma\zeta + cuQ_0R\zeta + cu\beta R(Y_0 + z_0 - N).$$

The Routh–Hurwitz condition for all eigenvalues of J_1 to have negative real parts is

$$(R + cuQ_0 + \gamma + \zeta)(R\gamma + cuQ_0\gamma + R\zeta + cuQ_0\zeta + \gamma\zeta + cuQ_0R) > cuQ_0R\zeta + (R + cuQ_0)\gamma\zeta + cu\beta R(Y_0 + z_0) - \beta RcuN > 0,$$

which simplifies to

$$N < \frac{Q_0\zeta}{\beta} + \frac{\gamma\zeta Q_0}{\beta R} + \frac{\gamma\zeta}{cu\beta} + Y_0 + z_0. \quad (14)$$

Since $\frac{Q_0\zeta}{\beta} + \frac{\gamma\zeta Q_0}{\beta R} + \frac{\gamma\zeta}{cu\beta} = N$, (14) further simplifies to

$$0 < Y_0 + z_0. \quad (15)$$

Thus the nontrivial equilibrium P_0 is asymptotically stable if and only if condition (15) holds.

By considering the linearized system near the origin it is easy to see that the origin is an asymptotic equilibrium point if and only if

$$N < \frac{\gamma\zeta}{cu\beta}. \quad (16)$$

It follows that the origin is an unstable equilibrium point if and only if

$N > \frac{\gamma\zeta}{cu\beta}$ or $cu\beta N > \gamma\zeta$. But if this is the case, equation (15) is satisfied since both Y_0 and z_0 automatically assume positive values. Thus,

the origin is a locally asymptotically stable equilibrium point of the system (3), (7), (8), (4) and (5) provided that $N < \frac{\gamma\zeta}{cu\beta}$ when it is the only equilibrium point. If $N > \frac{\gamma\zeta}{cu\beta}$ then the origin is unstable and there is a unique endemic equilibrium $(y_0, Y_0, z_0, q_0, Q_0)$ (where $(y_0, Y_0, z_0, q_0, Q_0)$ is given by equations (10) and (12)) which is locally stable.

Hence the issue of the existence and stability of the unique nontrivial endemic state is not a global phenomenon; it is dependent on parameter variations. If there is no resistant infection ($u=0$) the only equilibrium point is the trivial one and it is asymptotically stable for all parameter values. We can strengthen the ongoing discussion in a theorem as follows, and thus obtain a complete analytical and biological characterization of the equilibrium points of the system (3), (7), (8), (4) and (5) as follows:

THEOREM 1

The system (3), (7), (8), (4) and (5) has the following stability analysis.

1. If $0 < N < \frac{\gamma\zeta}{cu\beta}$
then the origin is locally asymptotically stable, the point P_0 is unstable and biologically infeasible.
2. If $N > \frac{\gamma\zeta}{cu\beta} > 0$
then the origin is unstable, P_0 is locally stable and biologically feasible.
3. If $N > \frac{\gamma\zeta}{cu\beta} = 0$
then the origin is unstable, P_0 is locally stable and feasible.

3 Multigroup population

In this section we consider a case of a k-interacting human population groups: $N_i, i = 1, 2, \dots, k$, with $N = \sum_{i=1}^k N_i$. We assume that infecteds in each group receive initial treatment. Thus as in the 2-strain model, sensitive infecteds cannot produce gametocytes since

the parasites are destroyed by drugs early enough before they get to the exoerythrocytic stage. The degree of resistivity of parasites to a particular antimalaria drug is a function of geographical distribution. Denote by y_i , Y_i , and z_i respectively the numbers of sensitive, resistant and (partial) immunes associated with group N_i ; also denote by q_i and Q_i respectively sensitive and resistant vectors associated with group N_i . The emergence and propagation of resistant parasites in each group depend on certain evolutionary pressure. For simplicity we shall assume that the growth of each of the vector groups depends on the population of (partial) immunes in that group. Vectors from group j infect susceptibles of group i at a rate of c_{ij} . Let u_j be the probability that a susceptible in group i is infected by a resistant strain from group j ; the complementary probability, $1 - u_j$, is the probability that a susceptible is infected with sensitive strain only. Let r_i and R_i respectively denote the cure rates for the sensitive and resistant infecteds. Note that resistant cure is just a transition to the corresponding immune class. The model for this k-population group is then described by the following system of differential equations:

$$\dot{y}_i = \sum_{j=1}^k [c_{ij}(N_i - y_i - Y_i - z_i)q_j + c_{ij}(1 - u_j)(N_i - y_i - Y_i - z_i)Q_j - c_{ij}u_j y_i Q_j] - r_i y_i, \quad (17)$$

$$\dot{Y}_i = \sum_{j=1}^k u_j c_{ij} Q_j (N_i - Y_i - z_i) - R_i Y_i, \quad (18)$$

$$\dot{z}_i = R_i Y_i - \gamma_i z_i, \quad (19)$$

$$\dot{q}_i = \alpha_i z_i - \eta_i q_i, \quad (20)$$

$$\dot{Q}_i = \beta_i z_i - \zeta_i Q_i, \quad (21)$$

$c_{ij} \geq 0$, $0 < u_i \leq 1$, $r_i > 0$, $R_i > 0$, $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$, $\eta_i > 0$, $\zeta_i > 0$ and $0 \leq y_i + Y_i + z_i \leq N_i$.

Let $\mathbf{X} = (y_1, \dots, y_k, Y_1, \dots, Y_k, z_1, \dots, z_k, q_1, \dots, q_k, Q_1, \dots, Q_k)^T$. Then $\mathbf{X}$ is a 5k-dimensional vector whose first, second, third, fourth, and fifth k components are respectively those of y_i ($i=1, \dots, k$), Y_i ($i=1, \dots, k$), z_i ($i=1, \dots, k$), q_i ($i=1, \dots, k$), and Q_i ($i=1, \dots, k$). The analysis of the system (17)–(21) is more complex than that of the corresponding 2-strain system. We begin with the following definitions/preliminaries.

For vectors a and b , where $a = (a_1, \dots, a_k)^T$, $b = (b_1, \dots, b_k)^T$, let $a \otimes b$ be the vector whose components are the products of the corresponding components of its arguments, i.e.

$$a \otimes b = (a_1 b_1, \dots, a_k b_k)^T.$$

It is easy to see that the operation $\otimes$ is symmetric, bilinear, associative and commutative. If $\text{diag}(d)$ for a vector d means the diagonal matrix (of appropriate dimension) whose i th

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diagonal entry is d_i and $\mathbf{1}$ means the vector with all components equal to 1, then

$$a \otimes b = \text{diag}(a)b \quad \text{and} \quad a.b = (a \otimes b).\mathbf{1}$$

(where $.$ means dot product). If B is a matrix, $a \otimes B$ is a matrix with $(a \otimes B)_{ij} = a_i B_{ij}$.

Proposition PS [16] *Consider the system*

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_2)\end{aligned}$$

with $f(.,.)$ and $g(.)$ continuous throughout a compact subset E of $\mathcal{R}^2$. Define $P(E)$ as the projection of E onto the x_2 axis. Assume

(a) E is positively invariant for the system,
and

(b) $x_2 = \hat{x}_2$ is an equilibrium point that is globally asymptotically stable on a subset A of $P(E)$.

Then every trajectory starting in $A' = \{(x_1, x_2) : x_2 \in A\}$ tends asymptotically to a point of the form $(\hat{x}_1, \hat{x}_2)$, where $\hat{x}_1$ is an equilibrium of $\dot{x}_1 = f(x_1, \hat{x}_2)$.

Proof (see [16])

The author [?] extended Proposition PS to $\mathcal{R}^4$ and thus obtained a unique globally asymptotic equilibrium point for a system in $\mathcal{R}^4$.

The set of feasible values for the system (17)—(21) is

$$\Omega = \{\mathbf{X} = (y, Y, z, q, Q)^T \in \mathcal{R}^{5k} : y_i \geq 0, Y_i \geq 0, z_i \geq 0, q_i \geq 0, Q_i \geq 0, y_i + Y_i + z_i \leq N_i, \forall i\}$$

THEOREM 2

The set Ω is positively invariant for the system (17)—(21).

Proof. By Nagumo's theorem [14], if the outward-pointing normal $\mathbf{n}$ at each point on the boundary $\partial\Omega$ of a set Ω when dot multiplied with the vector field $\dot{X}$ at that point yields a nonpositive result, then the set is positively invariant for the differential equation.

The boundary of Ω corresponding to the condition $y_i \geq 0$ or $-y_i \leq 0$ gives rise to the outward-pointing 5k-dimensional normal vector

$$\mathbf{n} = (0, \dots, -1, \dots, 0)^T, \quad -1 \text{ in the } i\text{th position.}$$

We are to show that

$$\mathbf{n} \cdot \dot{X} \leq 0$$

when $y_i = 0$. From equation (17), when $y_i = 0$, $-\dot{y}_i$ is

$$-(N_i - Y_i - z_i) \sum_{j=1}^k c_{ij} [q_j + (1 - u_j)Q_j] \leq 0.$$

The boundary of Ω corresponding to $-Y_i \leq 0$ gives rise to the 5k-dimensional normal vector

$$\mathbf{n} = (0, \dots, -1, \dots, 0)^T, \quad -1 \text{ in the } (k + i)\text{th position.}$$

From equation (18), when $Y_i = 0$,

$$\mathbf{n} \cdot \dot{X} = -Y_i = - \sum_{j \neq i} u_j c_{ij} Q_j (N_i - z_i) \leq 0.$$

Similar results hold for the boundaries of Ω corresponding to

$$-z_i \leq 0, \quad -q_i \leq 0, \quad -Q_i \leq 0.$$

The boundary of Ω corresponding to $y_i + Y_i + z_i \leq N_i$ gives rise to the 5k-dimensional normal,

$$\mathbf{n} = (0, \dots, 1, \dots, 1, \dots, 1, \dots, 0)^T, \quad 1's \text{ in the } i\text{th}, (k + i)\text{th and } (2k + i)\text{th positions.}$$

At boundary, $y_i + Y_i + z_i = N_i$, so that $N_i - Y_i - z_i = y_i$; hence, from equations (17), (18) and (19)

$$\begin{aligned} \mathbf{n} \cdot \dot{X} &= \dot{y}_i + \dot{Y}_i + \dot{z}_i \\ &= - \sum (c_{ij} u_j y_i Q_j - r_i y_i + \sum (c_{ij} u_j y_i Q_j) - R_i Y_i + R_i Y_i - \gamma_i z_i) \\ &= -(r_i y_i + \gamma_i z_i) \leq 0. \end{aligned}$$

Hence Ω is positively invariant for the system (17)–(21).

Let $y = (y_1, \dots, y_k)^T$, $Y = (Y_1, \dots, Y_k)^T$, $z = (z_1, \dots, z_k)^T$, $q = (q_1, \dots, q_k)^T$, $Q = (Q_1, \dots, Q_k)^T$. With these definitions, the vector form of system (17)–(21) is

$$\begin{aligned} \dot{y} &= -r \otimes y + (N \otimes C)q + (N \otimes C)((1 - u) \otimes Q) - ((y + Y + z) \otimes C)q \\ &\quad - ((y + Y + z) \otimes C)((1 - u) \otimes Q) - (y \otimes C)(u \otimes Q), \end{aligned} \quad (22)$$

$$\dot{Y} = -R \otimes Y + (N \otimes C)(u \otimes Q) - ((Y + z) \otimes C)(u \otimes Q), \quad (23)$$

$$\dot{z} = R \otimes Y - \gamma \otimes z, \quad (24)$$

$$\dot{q} = \alpha \otimes z - \eta \otimes q, \quad (25)$$

$$\dot{Q} = \beta \otimes z - \zeta \otimes Q. \quad (26)$$

This 5k system satisfies

$$\begin{pmatrix} \dot{y} \\ \dot{Y} \\ \dot{z} \\ \dot{q} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} A_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} y \\ s \end{pmatrix} - T(y, s),$$

where $s = (Y, z, q, Q)^T$. The origin is locally asymptotically stable if $s(A_{11}) < 0$ and $s(B_{22}) < 0$

Consider the system

$$(1.7) \quad \dot{y}_i = \left(\sum_{j=1}^n w_{ij} y_j (N_i - y_i) \right) - g_i y_i,$$

where w_{ij} , N_i , g_i are constants and $w_{ij} > 0$, $N_i > 0$, $g_i > 0$, $y_i \geq 0$. Let y be the vector whose components are y_i , $i=1, \dots, n$; A the matrix of linear terms, and $Q(y)$ the vector of quadratic terms in (1.7). Then the vector form of the system is

$$(1.8) \quad \dot{y} = Ay - T(y).$$

Lajmanovich and Yorke (see [10]) proved that solutions to system (1.8) are globally asymptotically stable with limiting value determined by the stability modulus $s(A)$ of the matrix A . This is defined as the maximum real part of the eigenvalues of A , i.e.,

$$s(A) = \max\{\operatorname{Re} \lambda : \lambda \text{ an eigenvalue of } A\}.$$

Precisely, the following theorem was proved by Lajmanovich and Yorke.

THEOREM LY *The solutions to system (1.8) approach the origin if $s(A) \leq 0$ and approach a unique positive equilibrium $\hat{y}$ if $s(A) > 0$, provided $y_i(0) > 0$ for some i . Furthermore in this case $0 < \hat{y}_i(0) < N_i$ for each $i=1, 2, \dots, n$.*

Now, back to system (17)–(21). I do not know a complete analytical solution to this problem. However, with much restrictions as below, we obtain some reasonable results.

At equilibrium, $\dot{z}_i = 0$, $\dot{q}_i = 0$, $\dot{Q}_i = 0$. This implies that

$$z_i = \frac{R_i}{\gamma_i} Y_i = \rho_i Y_i \quad (\rho_i = \frac{R_i}{\gamma_i}), \quad (27)$$

$$q_i = \frac{\alpha_i}{\eta_i} z_i = \mu_i Y_i \quad (\mu_i = \frac{\alpha_i R_i}{\eta_i \gamma_i}), \quad (28)$$

$$Q_i = \frac{\beta_i}{\zeta_i} z_i = \omega_i Y_i \quad (\omega_i = \frac{\beta_i R_i}{\zeta_i \gamma_i}). \quad (29)$$

At this point, equation (18) becomes

$$\dot{Y}_i = \sum_{j=1}^k [c_{ij} u_j \omega_j Y_j (N_i - Y_i - \rho_i Y_i)] - R_i Y_i. \quad (30)$$

System (30) is amenable to the LY system with matrix of linear terms given by

$$A_Y = \text{diag}(N) C \text{diag}(u \otimes \omega) - \text{diag}(R), \quad (31)$$

and the vector of quadratic terms given by

$$T_Y = ((\rho + 1) \otimes Y \otimes C)(u \otimes \omega \otimes Y). \quad (32)$$

Thus Y tends asymptotically to 0 if $s(A_Y) \leq 0$ and to a fixed nonnegative vector Y^* (say) if $s(A_Y) > 0$.

Hence:

Q tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\omega \otimes Y^*$ if $s(A_Y) > 0$;

z tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\rho \otimes Y^*$ if $s(A_Y) > 0$;

q tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $\mu \otimes Y^*$ if $s(A_Y) > 0$.

By substituting these values (in view of Proposition PS extended to $\mathcal{R}^{5k}$) into equation (22) we also get

y tends asymptotically to 0 if $s(A_Y) \leq 0$ and to $y^* \neq 0$ if $s(A_Y) > 0$.

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All these prove the following theorem:

THEOREM 3

The solutions to system (17)—(21) approach the origin if $s(A_Y) \leq 0$ and approach a unique positive equilibrium $(y^*, Y^*, z^*, q^*, Q^*)^T$ provided $Y(0) \neq 0$; where $z^* = \rho \otimes Y^*$, $q^* = \mu \otimes Y^*$, $Q^* = \omega \otimes Y^*$ and y^* is obtained by setting the right hand side of equation (22) to zero and substituting for the known vectors.

It follows that both infections die out if $s(A_Y) \leq 0$ and that an endemic state is reached for both infections if $s(A_Y) > 0$. This result together with that of theorem 1 shows that both models have similar results. These results should be a guide to health officials. Since major health efforts should be geared towards eliminating the parasites (and hence the infection), conditions leading to the origin being an asymptotic equilibrium point should be put in place. Finally we remark that a complete analytical solution of the system (17)—(21) is desirable.

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A MATHEMATICAL MODEL TO ASSESS THE IMPACT OF COUNSELLING AND ANTIRETROVIRAL THERAPY ON THE SPREAD OF HIV/AIDS

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NIGERIA

ABSTRACT

A Mathematical model of HIV transmission dynamics is proposed to assess the impact of counseling and antiretroviral therapy (ART) on the spread of HIV/AIDS. From the analysis of the model equations, threshold conditions are obtained, in terms of the given model parameters, for the existence and stability of the disease-free and endemic equilibrium states of the model, as well as the proportion of infected people to receive ART. Analytical and numerical results obtained indicate that ART and counseling could be effective methods in the control and eradication of HIV/AIDS, especially when there is a sufficient reduction in the average number of sexual contact partners for the infected individuals.

Key-words: HIV, ART, Mathematical Model, Threshold conditions, Numerical results, Control programme.
Mathematics Subject Classification 2000:92C60 & 92D30

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1.0 INTRODUCTION

A major method, apart from the use of the condom, in the control of HIV/AIDS, is Antiretroviral Therapy (ART). By this approach, HIV positives are detected and placed on antiretroviral drugs. Generally, there are public awareness campaigns which are intended to educate the general public on the spread of HIV and how to control it. Members of the public are encouraged to go for tests in order to determine their HIV/AIDS status so as to benefit from ART. ART does not cure HIV infection, it only boosts the immune system of infected people against

secondary infections, thereby prolonging their life-span. HIV positives are also detected through random screening and contact tracing.

Here, we propose a mathematical model to assess the impact of counseling and ART on the spread of HIV/AIDS. The population is partitioned into three compartments of susceptible $S(t)$, infected $I(t)$, and removed $R(t)$. A susceptible is an individual that is yet to be infected, but is open to infection as he or she interacts with members of the I-class. An infected individual is one who has contracted HIV and is at some stage of infection. A removed individual is one that is confirmed to be HIV positive, counseled, and is receiving ART.

It is assumed that the recruitment into the S-class is only through birth, at a rate b , and it is proportional to the total population $N(t) = S(t) + I(t) + R(t)$ at time t . Death is explicit in the model and it occurs in all classes at a constant rate μ . However, there is an additional death rate α_0 in the I and R-classes due to infection. There is a maximum period of time, T after infection, which a member in class I must leave the class through death. The death rate in the R-class is therefore given by $\alpha = \alpha_0 e^{-kT}$, where k is the efficacy of the antiretroviral drug. The higher the value of k the smaller the value of α and vice versa. Clearly $\alpha \leq \alpha_0$, and $\alpha = \alpha_0$ when $k = 0$ (i.e. no ART).

The recruitment into the R-class from the I-class depends on the effectiveness of public campaign or counseling, and this is done at a rate σ . σ can also be referred to as the treatment rate. We ignore vertical transmission and age structure in the formulation. An age-structured formulation of a similar model has been proposed by Akinwande (2006), although not quite a SIR model as ours. Mathematical models to investigate the effect of treatment and vaccination on the spread of HIV/AIDS can be found, for example, in Kaosimore and Lungu (2004), Yang and Ferreira (1999), Hsu-Schmitz (2000), Swanson *et al* (1994). Models for the control of HIV using the condom can be found, for example, in Hsieh and Velasco-Hernandez (1995). Hsieh (1996), Mastro and Limpakarnjanarat (1995), Kimbir and Aboiyar (2003), Kimbir *et al* (2006).

2.0 FORMULATION OF THE MODEL EQUATIONS

The following diagram will be found useful in formulating the model equations.

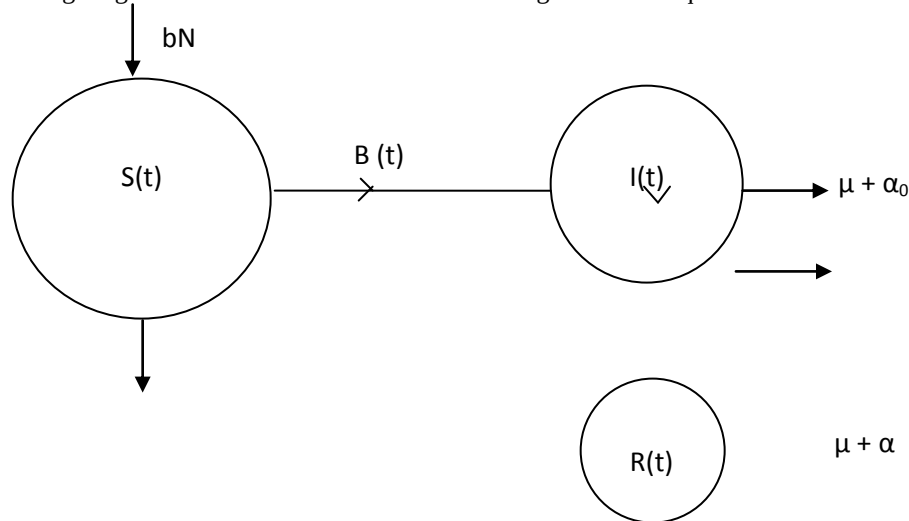


Figure 1. A Flow diagram of the transmission of HIV considering counseling and ART.

From the assumptions in section 2 and the above diagram (figure 1) the following model equations are derived.

$$\frac{dS}{dt} = bN - B(t)S - \mu S, \quad \dots \quad (2.1)$$

$$\frac{dI}{dt} = B(t)S - (\mu + \alpha_0 + \sigma)I, \quad \dots \quad (2.2)$$

$$\frac{dR}{dt} = \sigma I - (\mu + \alpha)R, \quad \dots \quad (2.3)$$

where

$$N(t) = S(t) + I(t) + R(t), \quad \dots \quad (2.4)$$

and

$$\alpha = \alpha_0 e^{-kT}, \quad \dots \quad (2.5)$$

The incidence rate $B(t)$ at time t is given as in Hsieh (1996), namely

$$B(t) = \frac{c\beta I + c'\beta'R}{N}, \quad \dots \quad (2.6)$$

where β is the probability of transmission by an individual in class I and β' is the probability of transmission by an individual in class R; c and c' are, respectively, the average number of sexual partners per unit time for individuals in class I and R. b is the reproduction rate of the population, α_0 , α , and σ are as defined in section one. $c\beta$ and $c'\beta'$ are therefore the net transmission rates for the classes I and R, respectively. As a result of counseling, it is assumed that $c'\beta' < c\beta$. For ease of reference we redefine the model parameters in the following table.

Table 1

$S(t)$ =	Number of susceptible at time t
$I(t)$ =	Number of infected at time t
$R(t)$ =	Number of infected receiving ART at time t
b =	Population birth rate
μ =	Population death rate
α_0 =	Population death rate of infected not receiving ART
α =	Population death rate of infected receiving ART
T =	Maximum lifespan after infection
k =	Efficacy of ART per unit time
c =	Average number of sexual partners of members of class I
c' =	Average number of sexual partners of members of class R
β =	Probability of transmission by members of class I
β' =	Probability of transmission by members of class R
σ =	Proportion of infected receiving ART per unit time

Adding equations (2.1) – (2.3), we have

$$\frac{dN}{dt} = (b - \mu)N - \alpha_0 I - \alpha R \quad \dots \quad (2.7)$$

Let $s = \frac{S}{N}$, $i = \frac{I}{N}$, and $r = \frac{R}{N}$, then we have $s = 1 - i - r$ and the governing equations of the model, in proportions i and r , are given below.

$$\frac{di}{dt} = (c\beta i + c'\beta' r)(1 - i - r) - (\alpha_0 + b + \sigma)i + \alpha_0 i^2 + \alpha r i \quad \dots \quad (2.8)$$

$$\frac{dr}{dt} = \sigma i - (\alpha + b)r + \alpha_0 i r + \alpha r^2 \quad \dots \quad (2.9)$$

The equations in proportions have biological meaning as they define prevalence of infection.

3.0 EXISTENCE AND STABILITY OF EQUILIBRIUM STATES

It is easy to see that $(0, 0)$ is an equilibrium state of the model (2.8) and (2.9).

The Jacobian matrix J_0 associated with the equilibrium state $(0, 0)$ is given by

$$J_0 = \begin{pmatrix} c\beta - (\alpha_0 + b + \sigma) & c'\beta' \\ \sigma & -(\alpha + b) \end{pmatrix}$$

Let $R = \frac{c\beta}{\alpha_0 + b + \sigma}$ and define $D = \{(i, r) : i > 0, r > 0, i + r = 1\}$, then we have the following result.

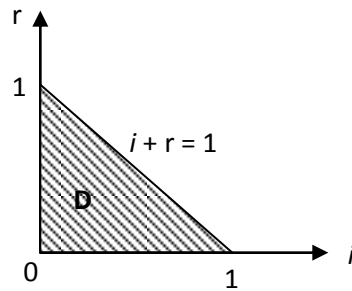


Figure 2: The region D in $\mathbb{R}^2$

Theorem 1

Given $\alpha, \alpha_0, b, \sigma, c\beta, c'\beta' > 0$. If $\alpha + b > \alpha_0$ and $0 < R < 1$, then there exists a disease-free equilibrium state (DFE), $(0, 0)$, which is locally and asymptotically stable (LAS), otherwise there exists an endemic state (i^*, r^*) which is LAS in $D - \{(0, 0)\}$.

Proof

We see from the hypotheses of the theorem that

$$\text{Tr}J_0 = (\alpha_0 + b + \sigma)(R - 1) - (\alpha + b) < 0, \text{ and}$$

$$\det J_0 > -(\alpha_0 + b + \sigma)^2(R - 1) > 0.$$

Therefore, the disease-free state is locally and asymptotically stable. We know that the DFE is unstable if the condition $R < 1$ does not hold, that is if $R \geq 1$. In this case we only need to show that the region D is invariant, containing no periodic solutions of the system (2.8) and (2.9), so that all solutions tend to the endemic equilibrium state (i^*, r^*) .

First, we shall show that D is an invariant region. We do this by showing, as in Beltrami (1989), that the inner product of the vector field defined by equations (2.8) and (2.9) with the inward normal to D is non-negative.

Let $f_1(i, r)$ and $f_2(i, r)$ denote, respectively, the rhss of equations (2.8) and (2.9). Going back to figure 2, we see that the inward normal to the i -axis is $(0, 1)$, therefore $(0, 1) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_2 = (\sigma i - (\alpha + b))r + \alpha_0 r i + \alpha r^2 > 0$ (since $r = 0$ on this axis). Next the inward normal to the r axis is $(1, 0)$, so that $(1, 0) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1 = c' \beta r (1 - r) > 0$ (since $i = 0$ on this axis and $r < 1$). Finally, on the line $i + r = 1$, we have $\frac{1}{\sqrt{2}}(-1, -1) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} > 0$, using similar arguments. Thus we have proved that D is invariant. It remains to prove, using the Bendixon-Dulac criterion, as in Hsieh (1996), that there are no periodic solutions of the system (2.8) and (2.9). Let $g = \frac{1}{ir}$, then we have $\frac{\partial}{\partial i}(g f_1) + \frac{\partial}{\partial r}(g f_2) < -\left(\frac{\sigma}{r^2} + \frac{\alpha}{i}\right) < 0$. Therefore there are no periodic solutions of the system in D . Hence the proof.

4.0 DISCUSSION AND CONCLUSION

In this paper we formulated and studied a mathematical model for the transmission of HIV/AIDS considering counseling and antiretroviral therapy (ART). The model parameters are shown in table 1. The model equations are derived with the help of a flow diagram in figure 1.

The main result of the study is found in theorem 1, where threshold conditions are given for the stability of the disease-free and the endemic equilibrium states of the model. Whereas the condition $\alpha + b > \alpha_0$ holds vacuously, the number $R = \frac{c\beta}{\alpha_0 + b + \sigma}$ may not always be less than 1. From the rhs of the expression for R , we see that increasing the value of σ (i.e. increasing the treatment rate) reduces the value of R below 1. Similarly, reducing the value of c or β may achieve the same purpose. Thus, for an effective ART programme, it may be necessary to also reduce the transmission probability and the average number of sexual partners of the infected individuals. These can be done through counseling and education. From the expression $R < 1$, we see that the minimum proportion σ of infected individuals to receive ART is $c\beta - (\alpha_0 + b)$.

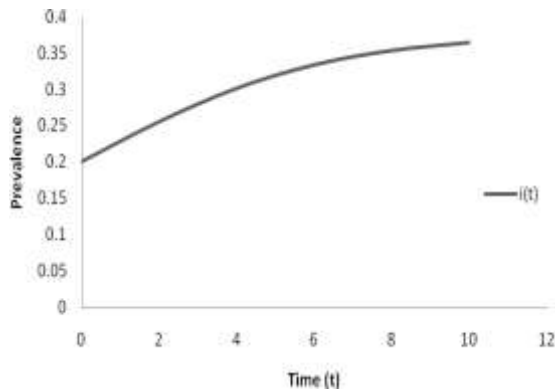


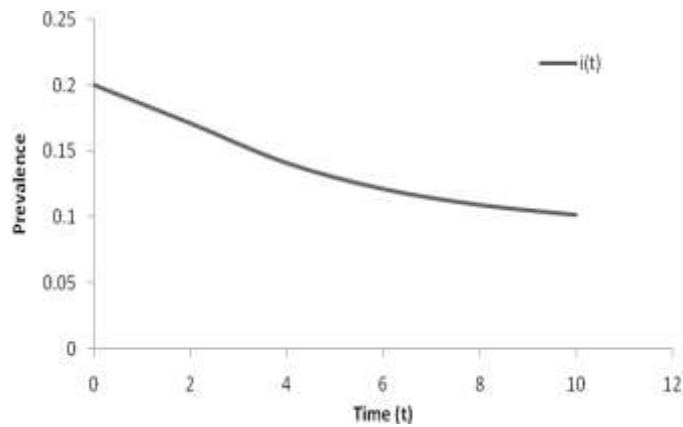
Figure 3. Prevalence of Infection without any intervention. ($\sigma = 0$).

Other parameter values are: $b = 0.5$, $\alpha_0 = 0.2$, $T = 10$, $k = 0$, $c = 1$, $c' = 0$, $\beta = 1$, $\beta' = 0$.

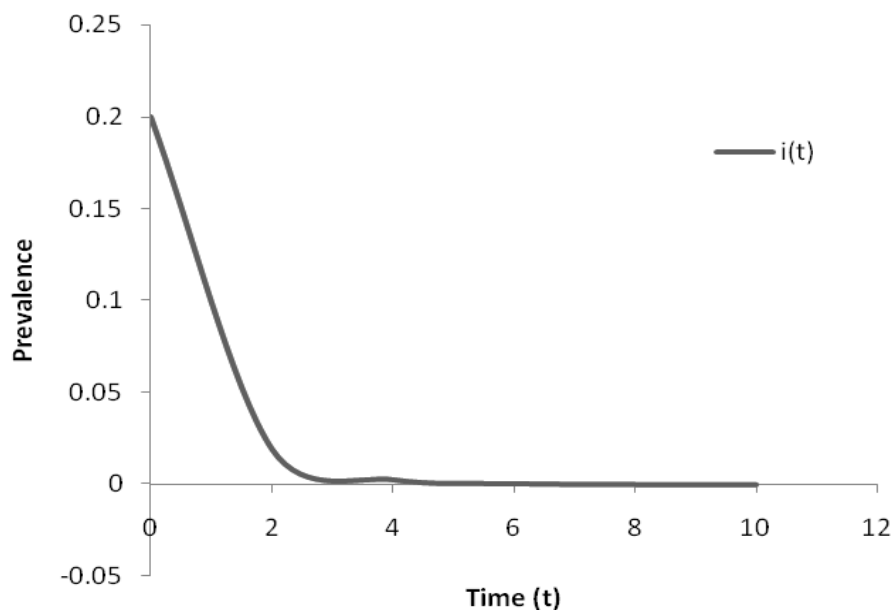
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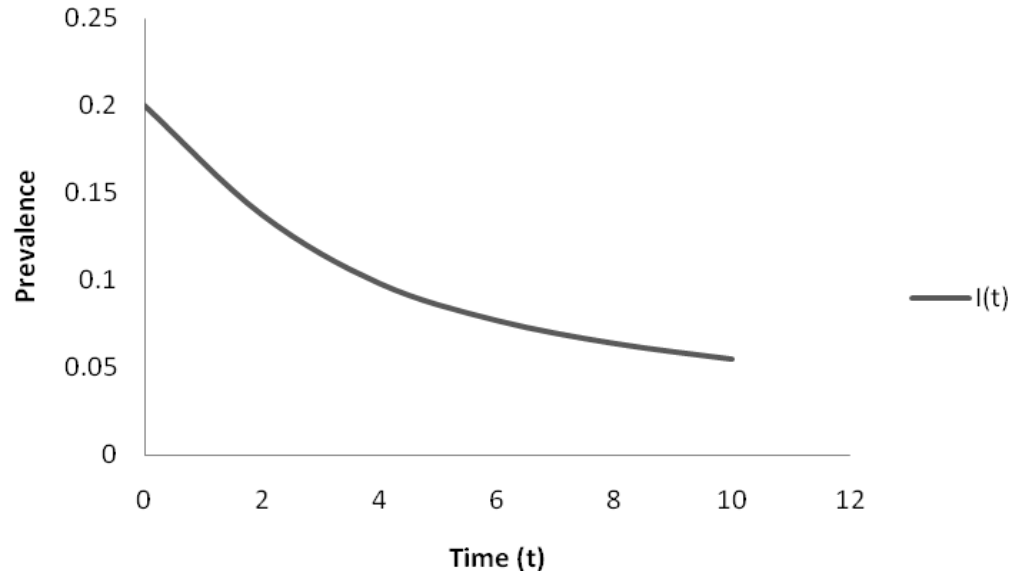


Figure 6: Prevalence of infection with a single sexual partner ($c = 1$). Other parameter values are: $b = 0.5$, $\sigma = 0.8$, $\alpha_0 = 0.2$, $T = 10$, $k = 0$, $c' = 0$, $\beta = 1$, $\beta' = 0$.

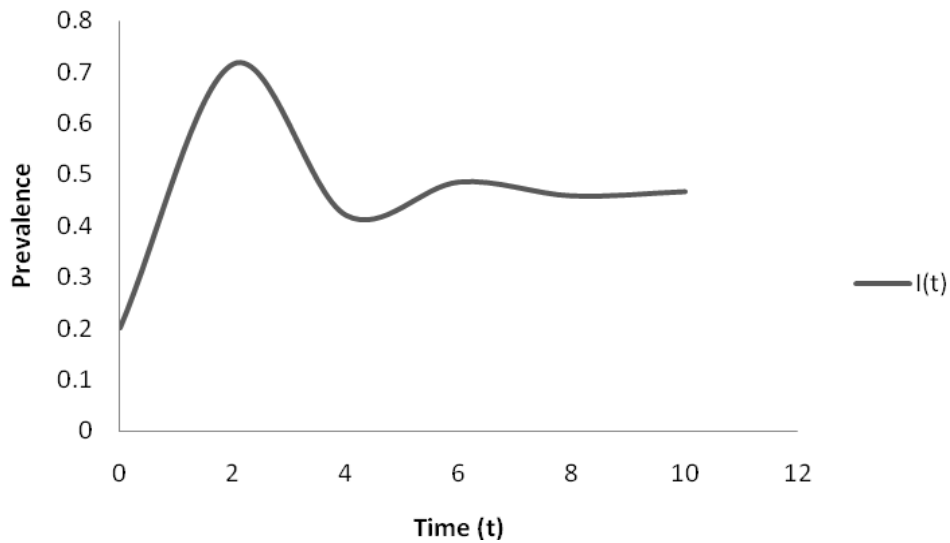


Figure 7: Prevalence of infection with three sexual partners ($c = 3$). Other parameter values are: $b = 0.5$, $\sigma = 0.8$, $\alpha_0 = 0.2$, $T = 10$, $k = 0$, $c' = 0$, $\beta = 1$, $\beta' = 0$.

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Numerical examples, using hypothetical data, satisfying the inequality $c\beta > (\alpha_0 + b)$ give the following results. Figure 3 shows an increasing prevalence in the absence of ART (i.e. $\sigma = 0$). Figure 4 shows the prevalence of infection when the treatment rate is low (i.e. $\sigma = 0.2$), while figure 5 shows prevalence of infection when the treatment rate is high (i.e. $\sigma = 0.8$).

Figure 6 and 7 illustrate the importance of multiple sexual partners in the transmission of the disease. From figure 6, we see that it is possible to control the infection in the situation where everybody engages a single sexual partner, while in figure 7 it is seen that the prevalence of infection remains high for three sexual partners. This is so, even when there is high transmission rate ($\beta = 1$) and low treatment rate ($\sigma = 0.3$). Hence, this study confirms that counseling and ART could be useful methods for the control and eradication of HIV/AIDS.

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A note on generalised model for estimating life expectancy of populations

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ABSTRACT

In this paper, we discussed some models for estimating life expectancy under all severities of life situations as an application of operations research to a development problem. In human mortality context, these models are important in several ways, for example, in the setting of assumptions regarding mortality improvement in population projections. We have shown that the impact of frailty on life expectancy of moving from the homogeneous to non homogeneous situation (and vice-versa) depends on the analytical form of the survival function , the way that unobserved frailty is modeled and life expectancy at birth. Life expectancy, normally regarded as a problem in demography has also been shown to depend crucially on questions of statistical specification.

Numerical examples which illustrate assessment of the impact of frailty on life expectancy are also presented which could also be made the basis for projection of mortality in developing countries. Finally, the use of the satisfactory life expectancy concept developed here can be also seen as part of the efforts to find population requirements aimed at increasing existing levels of safety at reduced cost especially in developing countries. It's application can also be extended to social and medical sciences (e.g. times spent in unemployment or before changing jobs, residential mobility, survival after medical intervention).

Keywords: life expectancy; homogeneous and non-homogeneous population; mortality rate, mean survival age, frailty, Gompertz model.

Mathematics Subject Classification 2000:90B50,90B70 & 62Q10

Introduction

In the studies of histories of events and their times of occurrence (whether the event is dying, migrating, changing job, etc) , different survival models have been used to examine the relationship between the probability of an event and either the age or duration since the previous event. In many populations, some of the units are more likely (whether by reason of biology or behavior) to undergo the event (dying or making a transition) than others. For example, most of the previous standard analytical models assume that all individuals at a given age or duration face the same probability of change and so largely ignore this heterogeneity (Manton et al ¹) .

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In particular , the conventional life-tables are generated from population distribution and age specific mortality rates. The assumption used is that the population is homogeneous in which all lives of the same age face equal mortality risks. It is also well known that mortality differentials exist even among people of the same age and environmental risks because some people are more likely to die than others due to individual frailty (differences in longevity due to biological and behavioural reasons). Whittaker² has shown how the rate of change in life expectancy can be used as the basis of calculating acceptable risk. Wajiga and Adekola³ gave the effect of additional mortality risks on life expectancy but do not consider individual risks. Congdon⁴ uses microlevel and aggregate level methods to compare the influence of known and unknown risk factors on mortality and stated that life expectancy based on homogeneous assumption will be overstated for all ages due to the unobserved risk factors .

However and more recently for mortality applications, an increasing number of applied studies have departed from the assumption of homogeneity and allow for differences in individual frailty which remain after allowing for duration or the age gradient and measurable attribute of individuals or their context. For example , Wajiga and Adekola⁵ using a mathematical and empirical life expectancy model for a non-homogeneous population showed that frailty can significantly affect age- specific life expectancies outside the immediate neighbourhood of the mean of survival age . Adekola⁵ presented a simple life expectancy model for a non homogeneous population to demonstrate that life expectancy is function of life expectancy at birth and age of the population which compares favourably with a more complex model in Wajiga and Adekola⁵ but it does not take into consideration the life expectancy at the maximum survival age. Using the Australian (males) 1961 data , it was also shown that frailty can significantly affect age specific life-expectancies outside the immediate neighbourhood of the mean survival age, and that life expectancies are overstated for ages less than the mean, and also overstated for higher ages.(For further definition of homogeneous and non-homogeneous, see Congdon⁴).

These applications can also be extended to social and medical sciences (e.g. times spent in unemployment or before changing jobs, residential mobility, and survival after medical intervention). However, in this paper , we have focused our presentation on the analysis of human mortality because of its immense use and in particular, is the long tradition of actuarial graduation of aggregated rates and investigation of the most suitable forms of age dependence in the hazard function (Forfar et al⁶). The hazard rate is the probability of death (or change) in an infinitesimal interval $(t, t+dt)$ given that a unit has survived until age or duration t : its analogue in aggregate analysis is the number of new cases of a diseases or the number of deaths divided by the “ population time” (persons years of risk before diseases or death) over which they occur. In the models consider here , the hazard is strongly related to age (i.e. there is clear age gradient) though in some applications the rate may be independent of t .

At birth, an individual has a certain life expectancy. Life expectancy is the average length of life lived in a population. It is a function of several risk factors , such as war , environmental pollution, disease, feeding and hygiene habits, random events such as lightning and tornadoes, and numerous forms of accident. A population exposed to an increased mortality risk will have a reduced life expectancy. Life expectancies can be obtained using a numerical integration over the range of age variable (with the upper limit set by a maximum conceivable age) and sometimes, analytically by integrating the survival function over relevant ages. This approach may sometimes have a number of obstacles and drawbacks including the rather tedious numerical procedure and the inability to arrive at a reasonable solution for certain details. The conventional life-table assumes that population is homogeneous in which all lives of the same age face equal mortality risks. However, some people are more likely to die than others due to individual frailty (differences in longevity due to biological and behavioural reasons). Whittaker² has shown how the rate of change in life expectancy can be used as the basis of calculating acceptable risk. Wajiga and Adekola³ gave the effect of additional mortality risks on life expectancy, but do not consider individual risks. Congdon³ stated that life expectancy based on homogeneous assumption will be overstated for all ages due to the unobserved risk factors.

A simple life expectancy model

A simple mathematical life expectancy model which is presented here for completeness depends on the life expectancy at birth and the age of the population which also compares favourably with a more complex empirical model from Wajiga and Adekola⁵. In essence, it will be more convenient to work with the reparametrized form because of its simplicity and appeal. Here, it is now assumed that the life expectancy curve may be reasonably represented by an equation of the form.

$$L(A) = L(0)(1-\beta A)^{\theta} \quad A \geq 0, \theta \geq 0$$

where $L(A)$ and $L(0)$ are non-negative constants and denote the life expectancy at Age, A and birth respectively, $\theta \geq 0$ and $\beta \geq 0$ represents as a measure of frailty variance and rate of change in life expectancy respectively (For more details see Adekola⁷). The model assumed that the measure of frailty parameter does not change with time which is not too realistic since it may well be that frailty parameter changes with time as environment and socioeconomic situation changes. There are several possible choices for distribution of frailty and one being a parametric distribution, however, for ease of illustration here we assume one of the simplest and most commonly used parametric model a Gompertz model that $\theta = \eta \exp(\eta t)$, where η is a constant parameter and t is the unit time of exposure (see Congdon⁴ for more details). Using the Australian (males) data, it was also observed that the life expectancies for the homogenous and non-homogeneous models is significantly different except around the mean survival age. In addition, there is intersection at the mean survival age and the maximum age of the population for the homogeneous and non-homogeneous models. These are approximately equal to mean (30) and (105) years lived for this Australian data. It means at the point of intersection, the most frail would have all died by the mean survival age and all the members of cohort cannot live beyond the maximum age of the of the population. Moreover, in comparing the life expectancies of homogeneous and non-homogeneous models, we also observe that those who survive to the mean survival age tend to live longer.

A generalized life expectancy model

Using Australian male 1961 data, Wajiga and Adekola⁵, showed empirically that life expectancies computed for homogeneous population models are overstated for ages less than the mean survival age and understated for higher ages. It was also observed that the life expectancies curves for homogeneous and non-homogeneous populations intersect at the mean survival age (when the most frail would have died off) and the maximum age which all members of the cohort can live. One therefore has separate curves for the two regions corresponding to moderately high ages and low ages. The problem is to produce a simple continuous life expectancy curve which blends these two curves together in a satisfactory way.

In this section we now present a generalized mathematical and analytical model which is useful in estimating an acceptable life expectancy under all severities of life situation. In human mortality context this is important in several ways, for example, in the setting of assumptions regarding mortality improvement in population projections.

For the purposes of the present discussion, it will be more convenient to work with the reparametrized form.

$$L = L_- + \frac{L_0 - L_-}{(1 + \beta A)^{\theta}} \quad (1)$$

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Here, L_0 = Life expectancy at birth and L_- = Life expectancy at age $-$, where scale parameter, $\beta = \beta_-$ and shape parameter $\theta = \theta_-$

The mathematical problem arising from the discussion of the previous section amounts to choosing a function, preferably of the standard form which

- (i) satisfies life expectancy requirements (i.e. life expectancy decreases as Age, A increases and when $A=0$ or $-$, then L_0 or L_- results). Moreover, as age becomes very large, L_- approximately becomes negligible.
- (ii) satisfactorily blends into non-homogeneous life-expectancy curve as A becomes very large and
- (iii) blends into (i.e. touches) the homogeneous life expectancy curves at some suitable point (i.e. at mean survival age, when the most frail in the population would have all died off) and at the maximum age lived in the population (when the last members of the cohort would have died off). It is assumed, of course, that each of these three requirements corresponds to the same probability of occurrence.

Write $L_N(A)$, $L_s(A)$ for the life expectancy of non-homogeneous population and satisfactory life expectancy population functions respectively and let L'_0 be the life expectancy for homogeneous population at birth. Mathematically, requirements (i-iii) may be written.

- (i) $L_s(0)=L'_0$
- (ii) $L_s(-)=L'_-$
- (iii) At some point A' one has

$$L_s(A') = L_N(A'), \left(\frac{dL_s}{dA}\right)A = A' = \left(\frac{dL_N}{dA}\right)A = A'$$

We investigate solutions of the form (i) which satisfies (i)-(iii). Since parameters L'_0 and L'_- of $L_s(A)$ are set by requirements (i) and (ii) respectively, there are two free parameters θ' and β' of $L_s(A)$. We shall call the pair (θ', β') feasible if for some A' the function

$$L_s(A) = L_- + \frac{L_0 - L_-}{(1 + \beta_- A)^{\theta'}} \quad (2)$$

and its gradient at $A=A'$ match $L_s(A')$ and $\left(\frac{dL_H}{dA}\right)A = A'$ respectively.

Numerical example 1

In order to illustrate the computations involved, we describe a simple numerical example using Australian (male) 1961 data. Suppose that the life expectancy curve parameters are given by $\theta' = 1.475$, $L'_0 = 67.92$, $L'_- = 2.29$ and $\beta' = 0.009$ which are measure of frailty, life expectancy at birth and maximum survival age and rate of change in life expectancy respectively and finally giving.

$$L_s(A) = 2.29 + \frac{65.63}{(1 - 0.009A)^{1.475}} \quad (3)$$

Using spreadsheet calculation for Ages 0 to 95, one may obtain their approximate values as illustrated in Table I. Note that age, A in Table I represents the interval $[A, A+5)$ and accuracy improves with the reduction in the age interval.

The life expectancy values for ages $[0, 30)$ and $(30, 95]$ years compares favourably with that in Wajiga and Adekola⁵ for homogeneous and non-homogeneous population models respectively.

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Table I Satisfactory life-expectancies using $L_s(A)$ model

Age, A	0	5	10	15	20	25	30	35
$L_s(A)$	67.92	63.61	59.4	55.28	51.27	47.35	43.55	39.85
Age, A	40	45	50	55	60	65	70	75
$L_s(A)$	36.27	32.81	29.46	26.25	23.17	20.23	17.43	14.8
Age, A	80	85	90	95				
$L_s(A)$	12.33	10.04	7.96	6.09				

In this section , we have suggested here a convenient and simple mathematical model for a satisfactory life-expectancy curve which blend into a life-expectancy curve for non-homogeneous at moderately higher ages, (2) blends into life-expectancy curve of homogeneous population at lower ages, (3) satisfies life expectancy requirements and (4) is of a standard parametric form. There is an infinite class of such curves. However, an analytical proof to generalise this statement is desirable. The method vastly compares favourably with other techniques previously used for the construction of life expectancy curves for population. In human mortality context, this model is important in several ways, for example, in the setting a assumptions regarding mortality improvement in population projections. The use of the satisfactory life expectancy concept developed here can be also seen as part of the efforts to find population requirements aimed at maintaining existing levels of safety at reduced cost especially in developing countries.

Generalized life expectancy models with time dependent frailty

Here, as a follow up to Adekola⁷ and Adekola⁸ , we presented a wider generalised mathematical and analytical model for estimating an acceptable life expectancy under all severities of life situation. However, the models in earlier studies assumed that the measure of frailty parameter does not change with time which is not too realistic since it may well be that frailty parameter changes with time as environment and socioeconomic situation changes. In particular, Adekola⁸ is now extended to cover a wider class of situation and time dependent frailty distribution . It is however assumed here that the life expectancy curve may be reasonably represented by an equation of the form

$$L_A^1 = L_\omega + \{L_0 - L_\omega\} \exp [f(t)\gamma(A)] \quad 0 \leq A \leq \omega, -\infty \leq \gamma(A) \leq \infty \quad (1)$$

where $\gamma(A)$ is a constant and $f(t)$ is the frailty distribution reflecting the measure of rate of change in life expectancy at age , A and frailty respectively, L_0 and L_ω are non-negative constants that denotes life expectancy at

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birth and age ω (maximum survival age of the population) respectively and L_A^t represents life expectancy at age A during exposure to some unobserved risk at a unit time t. This form can also be easily adopted from Adekola⁷ where $f(t) = -\theta$ and $\gamma(A) = \ln(1+\beta A)$. It is assumed, of course that any life expectancy model will correspond to the form of (1) and when $f(t) < 0$, $\gamma(A) > 0$ and $f(t)\gamma(A)$ becomes significantly large then the life expectancy at age, A approximately becomes negligible and also as $f(t)$ or $\gamma(A)$ tends to zero, life expectancy at birth result for all ages. Moreover, there are generally many feasible and intermediate solutions to equation (1).

Nevertheless, sometimes it is more convenient to exclude life expectancy over 95 years of age since recording the death or age of old people is often unreliable (Heligman and Pollard⁹). Assuming L_ω becomes significantly negligible and also, from equation (1) and when $f(t) > 0$, $\gamma(A) < 0$ and $f(t)\gamma(A)$ becomes approximately negligible then the life expectancy at birth results. In particular, Adekola⁷ results when $f(t) = \theta$ and $\gamma(A) = \ln(1-\beta A)$, (where β may denote the rate of change in life expectancy) and therefore, much details are omitted here for brevity. Moreover, when $f(t) = 1$ and $\gamma(A) = \ln(1-\beta A)$, a linear life expectancy model result. There are several possible choices for distribution of frailty and one being parametric distributions, however, for ease of illustration here we can also assume one of the simplest and most commonly used parametric model, a Gompertz model where $f(t) = \phi \exp(\phi t)$ and ϕ is a constant parameter (see Congdon⁴ for more details). The numerical example illustrated in the next section shows that the modeling of life expectancy and movement from the homogeneous and to non-homogeneous model (and vice-versa) depends on the analytical form of the frailty distribution, life expectancy at birth, rate of change in life expectancy and age of the population.

Numerical example 2

In order to illustrate the computations involved, we describe a simple numerical example using Australian (male) 1961 data. Suppose that the life expectancy curve parameters are given by $L_0 = 67.92$, $L_\omega = 105$ and $\beta = 0.85$ which is life expectancy at birth and rate of change in life expectancy respectively. Moreover, assuming $\phi = 0.9999$ and let $L_t(A)$, $L_\omega(A)$ and $L_\delta(A)$ denote life expectancy at age, A when $t = 10^{-4}$, 0.26 and 0.40 respectively. Using spreadsheet calculation for Ages 0 to 95, one may obtain their approximate values as illustrated in Table I. Note that age, A in Table I represents the interval $[A, A+5)$ and accuracy improves with the reduction in the age interval.

The life expectancy values $L_t(A)$ for ages $[0, 75)$ years compares favourably with that in Wajiga and Adekola⁵ for homogeneous model and however, for some values of t at old ages we are unable to produce satisfactory life expectancy curve $L_t(A)$. However, the life expectancy values $L_\delta(A)$ for ages $[0, 95]$ and $L_\omega(A)$ for ages $(30, 95]$ compares favourably with that in Adekola⁸ (i.e. a single life expectancy curve that blends the homogeneous and non homogeneous curve together in a satisfactory way) and Wajiga and Adekola⁵ (non-homogeneous life expectancy model) respectively. Thus the impact on life expectancy of moving from the homogeneous to non-homogeneous model depends on the way that unobserved frailty are modeled. This is on the ceteris paribus assumption that the life expectancy at birth and rate of change in life expectancy is unchanged.

A close correspondence can be seen between these models and most of the discrepancies are significantly negligible for lower ages, however, for the moderately high and older ages $L_\omega(A)$ also compares favourably with Adekola⁷ and generally, as frailty increases the life expectancy decreases. At older ages, there is also universal tendency for a reduced life expectancy once allowance is made for frailty however, the overall statements on the direction of change in life expectancy from a homogeneous (or from non-homogeneous) cannot be made. The life expectancies also increased when a more generalized parametric gamma frailty is assumed while other related forms of frailty distribution are also subject to the same tendency.

Table I Life-expectancies using an $L(A)$ model based on time dependent frailty

Age, A	0	5	10	15	20	25	30	35
$L_{\tau}(A)$	67.92	63.67	59.42	55.17	50.92	46.67	42.42	38.17
$L_{\omega}(A)$	67.92	63.70	59.54	55.45	51.43	47.48	43.60	39.80
$L_{\delta}(A)$	67.92	63.71	59.60	55.57	51.64	47.81	44.07	40.44
Age, A	40	45	50	55	60	65	70	75
$L_{\tau}(A)$	33.92	29.67	25.42	21.17	16.92	12.67	8.42	4.17
$L_{\omega}(A)$	36.08	32.45	28.91	25.46	22.12	18.89	15.78	12.81
$L_{\delta}(A)$	36.91	33.49	30.18	26.98	23.91	20.96	18.14	15.46
Age, A	80	85	90	95				
$L_{\tau}(A)$	–	–	–	–				
$L_{\omega}(A)$	9.99	7.34	4.89	2.70				
$L_{\delta}(A)$	12.93	10.55	8.33	6.33				

Conclusion

We have shown that life expectancy for population is sensitive to the form of frailty distribution. The life expectancy curve for a homogeneous population blend into a life-expectancy curve for non-homogeneous population at moderately higher ages as the frailty parameter increases. This is on the ceteris paribus assumption that the life expectancy at birth and rate of change in life expectancy is unchanged . The change in life expectancy for low ages are relatively small and at old age , the overall statements on the direction of change in life expectancy from a non-homogeneous cannot be made and there is also universal tendency for a reduced life expectancy once allowance is made for frailty. There is also consistent increase (or decrease) in life expectancy under different form of frailty and such an improvement (or deterioration) could be made the basis for projection of mortality. The analysis could obviously be extended and compared in various ways by using other parametric forms for frailty. Life expectancy, normally regarded as a problem in demography has been shown to depend crucially on questions of statistical specification.

Using the Australian (males) 1961 data , it was also shown that frailty can significantly affect age specific life-expectancies outside the immediate neighbourhood of the mean survival age, and that life expectancies are overstated for ages less than the mean, and also overstated for higher ages. It was also observed that life expectancy curves for both homogenous and non-homogeneous populations intersect at the mean survival age (where most frail would have all died off) and the maximum age lived in the population (when the last members of cohort would have died off) . However, an analytical proof to generalize this statement is desirable. Moreover, in

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comparing the life expectancies of homogeneous and non-homogeneous models , we also observed that those who survive to the mean survival age tend to live longer.

We have also presented a generalized life expectancy model which covers a wider class of situations. It was also shown that life expectancy and the change from homogeneous to non homogeneous situation (and vice-versa) depends on the analytic form of the frailty distribution, life expectancy at birth and function of age of the population. A simple numerical example which illustrate an assessment of the impact of frailty on life expectancy is also presented which could also be made the basis for projection of mortality. There is also consistent increase (or decrease) in life expectancy under different analytic form of frailty and such an improvement (or deterioration) could be made the basis for projection of mortality. The analysis could obviously be extended and compared in various ways by using other parametric forms for frailty. The assessment of life expectancy trends is essentially important for mortality projections and moreover, the assumption that life expectancy will improve overtime is one of the more certain assumptions about future demography but the extent of improvement remain a major question.

Finally, there is considerable scope for applying such methodology to other spatio-temporal data , using different specifications for frailty and other forms of hazard. For example in reliability engineering fields, where generally the parameters of the life distribution are usually estimated from fatigue test data with due allowance for variation. Fatigue is one of the most common causes of in –service failure of components and structures. For future work and application , Sweeting¹⁰ can be extended to cover wider class of models in structural engineering fields. A generalized S-N model would definitely be useful in estimating an acceptable safe service life under all severities of service loading.

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Volatility Modelling For Stock Prices

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ABSTRACT

In this study, we compared the optimum order of the linear GARCH and EGARCH models in modeling the volatility of the daily closing prices of First Bank Nigeria Plc. traded in the Nigeria Stock Exchange from January 2001 to July 2008. The returns series of the daily closing prices showed leptokurtic, significantly skewed and deviated from normality. Akaike and Bayesian information criteria (AIC and BIC respectively) were used to obtain the optimum order in each case and the results showed that linear GARCH (1,2) model would provide a good fit in analyzing the volatility process. In order to account for the fat tails, the chosen model was compared based on the Normal distribution, Student-t distribution and Generalized Exponential distribution (GED). The results showed that the kurtosis and skewness displayed would be minimized if GED is used for the parameter estimation. The standardized residuals from this model were checked for serial correlation using Ljung-Box test. Lagrange Multiplier (LM) test was also used to test for the presence of ARCH effects and the selected model satisfied all these diagnostic checks assumptions.

Keywords: *GARCH Models, Distribution, Diagnostic Checks.*

Mathematics Subject Classification 2000:91B28,62M10 & 62M20

1. INTRODUCTION

Volatility is a key variable that plays a central role in many areas of finance and accurate measures of volatility are crucial for the implementation and evaluation of asset and derivative pricing models in stock exchange markets. The issue of volatility modeling has been in the scope of researchers for a considerable time and many researchers have provided evidence concerning this issues of stock market return volatility modeling using the class of autoregressive conditional heteroscedasticity (ARCH)/ General ARCH (GARCH) models but the existing literatures provide little guidance on how to select optimal p and q values in GARCH (p, q) model.

McMillan et al (2000) analyzed the performance of a variety of volatility model, including GARCH, Threshold ARCH (TARCH), Exponential GARCH (EGARCH) and component-GARCH (Engle and Lee, 1993) models to forecast the volatility of the daily, weekly and monthly UK FTA and Financial Times Stock Exchange (FTSE) 100 stock indices. They found that GARCH model provided the most consistent forecasting performance of all frequencies. Ng and McAleer (2004) used simple GARCH (1, 1) and EGARCH (1, 1) models for testing, estimation and forecasting the volatility of daily returns in S and P 500 composite index and the Nikkei 225 index. Their empirical results indicated that the forecasting performance of both models depends on the data set used. The EGARCH (1, 1) model seems to perform

better with Standard and Poor's (S and P) 500 data, whereas the GARCH (1, 1) model is better in some cases with Nikkei 225. West and Cho (1995) in their research performed to evaluate the predictive ability of alternative models of exchange rate volatility, and develop an asymptotic procedure to test for the equality of the forecast error statistics. Using weekly data they compared the performance of historical, autoregressive, GARCH and Gaussian Kernel models in one, 12 and 24-week-ahead forecasts. GARCH forecasts were found more accurate for longer horizons.

Pagan and Schwert (1990) used the approach of Fair and Schiller (1989, 1990) in which the 'actual' series is regressed on two competing forecasts to test the in and out of sample performance of GARCH, EGARCH, Markov-switching Gaussian kernel and Fourier models for conditional stock volatility. They ran the above regression both in levels and logs, motivated by a symmetric and an asymmetric loss function respectively, test the regression coefficients and compare their coefficient of determination (R^2). Their evidence suggests that EGARCH and GARCH models to be less biased in out of sample prediction. Day and Lewis (1992) compared the information content of implied volatilities from weekly prices of call option on the S and P 100 index to GARCH and EGARCH models. Their out-of-sample evidence suggests that implied volatility and the GARCH and EGARCH forecasts are on average unbiased but results regarding their relative information content are not as clear.

Cao and Tsay (1993) used a Threshold Autoregressive (TAR) model in describing monthly volatility series. They compared the TAR model with ARMA, GARCH and EGARCH models using Mean Square Error (MSE) and Average Absolute Deviation (AAD) as the criteria, out-of-sample forecasts are compared and the result shows that TAR models consistently out perform ARMA models in multi-step ahead forecasts for S and P and value-weighted composite portfolio excess returns. Their result also pointed out that EGARCH model is the best in long-horizon volatility forecasts for equal-weighted composite portfolios. From the forecasting point of view, Hansen and Lunde (2005) conducted a comparison of more than 300 ARCH-type models, on DM-S and IBM stock, and find no evidence that a GARCH (1,1) is outperformed by more sophisticated model.

Some suggested criteria to determine the suitable p and q include Pagan and Sabu (1992) suggested a mis-specified volatility equation that can result in inconsistent maximum likelihood estimates of the conditional variance parameters. Further, West and Cho (1995) showed how appropriate GARCH model selection can be used to enhance the accuracy of the exchange rate volatility modeling. It becomes obvious that there is a wide range of different approaches to volatility modeling but on the basis of unbiasedness, the standard GARCH and EGARCH models tend to be less biased than their competitors.

Therefore, this study is concerned with finding an optimal order of the GARCH models for modeling the daily closing stock prices of the First Bank Nigeria Plc. traded in the Nigeria Stock Exchange by comparing the optimum order of the linear GARCH model with the optimum order of the non-linear EGARCH model using the Akaike and Bayesian information criteria together with their likelihood values.

2. COMPUTING VOLATILITY

In financial time series analysis, the return rate of stock market prices is used because it has some statistical properties such as stationarity. Hence, the continuously compounded returns are obtained by taking the logarithmic difference of the daily prices, that is:

$$y_t = \ln \left(\frac{P_t}{P_{t-1}} \right)$$

Where y_t is the daily returns that is the generating function, P_t and P_{t-1} are the closing prices on t and t-1 days respectively. It could be seen that equation (1) follows a random walk model known as geometric random walk model.

THUS, THE DAILY VOLATILITY IS DEFINED AS THE STANDARD DEVIATION OF CONTINUOUSLY COMPOUNDED DAILY RETURN AS FOLLOWS:

$$\sigma_t = \sqrt{\frac{\sum_{t=1}^n (y_t - \bar{y})^2}{n-1}}$$

Where the mean daily return stands for

$$\bar{y} = \left(\frac{1}{n}\right) \sum_{t=1}^n y_t$$

3.1 Generalized ARCH- GARCH

This section describes a generalization of the ordinary Autoregressive Conditional Heteroscedastic, ARCH model. The model structure was introduced by Bollerslev (1986). The generalization is similar to the extension of an Autoregressive, AR (p), to an Autoregressive Moving Average, ARMA (p,q). Formally the process can be written as:

$$y_t = \sigma_t Z_t$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^p a_i y_{t-i}^2 + \sum_{j=1}^q b_j \sigma_{t-j}^2$$

where $p, q \in \mathbb{Z}^+$ and

$$\left\{ \begin{array}{l} a_0 > 0 \\ a_i \geq 0, i = 1, 2, \dots, p \\ b_j \geq 0, j = 1, 2, \dots, q \end{array} \right\}$$

Thus the additional feature is that the process now includes lagged σ_{t-j}^2 . For p=0, the process is an ARCH (p). For p=q=0 (an extension allowing p=0 if q=0), y_t is white noise (Bollerslev, 1986).

3.2 Exponential GARCH- EGARCH

Black (1976) pointed out that bad news tends to drive down the stock price, thus increasing the leverage (i.e., the debt-equity ratio) of the stock and causing the stock to be more volatile. The presence of skewness and leverage effects in financial data has motivated the introduction of EGARCH model by Nelson (1991).

A simple EGARCH (1,1) model is

$$y_t = \sigma_t Z_t$$

with conditional variance

$$\sigma_t^2 = \exp \left(a_0 + a_1 \frac{|y_{t-1}| + \alpha_1 y_{t-1}}{\sigma_{t-1}} + b_1 \ln(\sigma_{t-1}^2) \right)$$

where $Z_t = \frac{y_t}{\sigma_t}$ is the normalized residual series.

The presence of leverage effect is due to the fact that $\alpha_1 < 0$. This implies that the impact is asymmetric if $\alpha_1 \neq 0$.

EGARCH (1,1) process can be extended to EGARCH (p,q) process such that

$$y_t = \sigma_t Z_t$$

$$\sigma_t^2 = \exp \left(a_0 + \sum_{i=1}^p a_i \frac{|y_{t-i}| + \alpha_i y_{t-i}}{\sigma_{t-i}} + \sum_{j=1}^q b_j \ln(\sigma_{t-j}^2) \right)$$

3.3 Distribution

Bollerslev (1987) showed that financial time series are typically characterized by high kurtosis. In order to model and minimize the fat tails displayed by the residuals of the conditional heteroscedasticity models, Student- t distribution and Generalized Error distribution are considered in addition to the Normal distribution.

If a random variable y_t has a student-t distribution with h degrees of freedom, the probability density function (pdf) of y_t is given by:

$$f_y(y_t; h) = \frac{\Gamma[(h+1)/2]}{\sqrt{h\pi} \Gamma(h/2) \left(1 + \frac{y_t^2}{h}\right)^{(h+1)/2}}$$

where $\Gamma(\cdot)$ is the gamma function, with $y = \frac{z}{\sqrt{\frac{u}{h}}}$;

where z is a standard normal and u is a χ^2 distribution with h degrees of freedom. Under the Generalized Error distribution (GED), y_t follows the pdf

$$f(y_t) = \frac{v \exp \left[-(\frac{1}{2}) \left(\frac{y_t}{\lambda} \right)^v \right]}{\lambda 2^{\frac{(v+1)}{v}} \Gamma(1/v)}$$

$$\text{with } \lambda = \left[\frac{2^{-2/v} \Gamma(1/v)}{\Gamma(3/v)} \right]^{1/2}$$

and v is a positive parameter governing the thickness of the tail behaviour of the distribution.

3.4 Diagnostic Checking of Serial Correlation

Testing for serial correlation is a fundamental problem in time series analysis. To determine whether a time series is independent, the autocorrelation function (ACF) of the series is examined. If the ACF is significantly different from zero, this implies that there is dependence between observations. Therefore, ACF is a powerful complementary tool for testing independence (Janacek and Swift, 1993; Fernando et al, 2000). The residual autocorrelation should be obtained to determine whether the residuals are white noise. In this study, the Ljung-Box Q statistic is used as alternative approaches for the diagnostic checking of residuals for serial correlation and is given as (Ljung and Box, 1978):

$$Q_k = n(n+2) \sum_{i=1}^k \frac{r_i}{(n-i)}$$

where n is the number of samples, k is the number of lags and r_i is the i^{th} autocorrelation. If Q_k is large then the probability that the process has uncorrelated data decreases. Q_k is asymptotically approximate as χ_k^2 .

3.5 Diagnostic Checking of Normality

There are several statistical tests used for the diagnostic checking of normality. In this study, Jarque Bera (JB) test was used for the diagnostic checking of residuals for normality and it is denoted by:

$$JB = \left(\frac{N}{6} \right) (S^2 + (K-3)^2 / 4)$$

Where S=Skewness, K=Kurtosis.

It follows a χ^2 distribution with 2 d.f. If the JB Statistic is greater than critical value of χ^2 , then we reject the null hypothesis of normality.

3.6 Testing For ARCH Effects

Lagrange Multiplier test for ARCH disturbances proposed by Engle (1982) is adopted. The methodology is as follows:

- (a) Use the Ordinary Least Square (OLS) to estimate the most appropriate AR (p) (or regression) model:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \dots + \alpha_p y_{t-p} + e_t$$

- (b) Obtain the squares of the fitted error $\hat{e}_t^2$. Regress these on a constant and the p lagged values, that is

$$\hat{e}_t^2 = a_0 + a_1 \hat{e}_{t-1}^2 + \dots + a_p \hat{e}_{t-p}^2$$

If there are no ARCH or GARCH effects, the estimated values of a_1 through a_p should be zero. Hence this regression will have little explanatory power so that the coefficient of determination R^2 , will be quite low. With a sample of T residuals, under the null hypothesis of no ARCH errors, the test statistic TR^2 converges to a χ_p^2 .

If TR^2 is sufficiently large, rejection of the null hypothesis that a_1 through a_p are jointly equal to zero is equivalent to rejecting the null hypothesis of no ARCH error. On the other hand, if TR^2 is sufficiently low, it is possible to conclude that there are no ARCH effects.

3.7 Model Order Selection

Since Garch model can be treated as ARMA model for squared residuals, traditional model selection criteria such as AIC and BIC can also be used for selecting models. When a model involving q independently adjusted parameters is fitted to data, the AIC is defined by:

$$AIC(q) = -2\ln(\text{Maximized likelihood}) + 2q$$

Plotting AIC (q) against q the graph will, in general, show a definite minimum value, and the appropriate order of the model is determined by that value of q at which AIC (q) attains its minimum value. The Bayesian information criterion (BIC) which is a modification of AIC is given as:

$$BIC(q) = -2\ln(\text{Maximized likelihood}) + q\ln(n)$$

where n is the number of observation.

4. RESULTS

Figure 1 shows the time series plot of the returns of First Bank Nigeria Plc. The empirical phenomenon of time –varying volatility and volatility clustering (i.e. periods of large changes are followed by periods of large changes while periods of small changes are followed by small changes) can be easily observed. From the plot, the returns series is stationary. However, to confirm this conjecture, Figure 2 shows the ACF plot of First Bank returns series. There is little correlation at lags 1 and 3 but dies out with time. Also, the squared returns exhibit autocorrelation at lag 3 and dies out with time. This indicates that the returns series is stationary.

From Table1, we see that the returns series is leptokurtic in the sense that the kurtosis exceeds positive three with positive non-zero skewness. The positive skewness shows that the upper tail of the distribution is thicker than the lower tail implying that the market increases occur more than the market declines. So the results showed that the returns have significant skewness, excess kurtosis and hence the assumption of normal distribution is not satisfied.

Different p and q values for the standard GARCH and EGARCH models were tested using the AIC and BIC techniques. So from Table 2, if the various BIC values are compared, we established that GARCH (1,2) is the one with the minimum BIC which is also one with the minimum AIC and highest likelihood value. Therefore, GARCH (1,2) is chosen as the optimum order of the linear GARCH model. Similarly for EGARCH model, the optimum order is EGARCH (1,0) as shown in Table 3. Furthermore, the optimal orders of the linear GARCH and non-linear EGARCH models were compared and the results showed that standard GARCH (1,2) would best describe the characteristics of the stock prices.

To evaluate which conditional distribution better describes the observed characteristics of daily stock price returns of the First Bank Nigeria Plc., three different distributions were compared for the identified GARCH (1,2) model using BIC and their likelihood values as shown in Table 5. According to the table, generalized exponential distribution (GED) has the smallest AIC and BIC values with highest likelihood value. This implies that the excess kurtosis and skewness displayed by the residuals of GARCH (1,2) is reduced with the use of GED. Therefore, the parameters of the GARCH (1,2) model are going to be estimated using the identified distribution.

However, Table 6 shows the values of the parameters for the identified model. Consequently, the following GARCH (1,2) process is fitted to the data in order to model the volatility:

$$y_t = \sigma_t Z_t$$

$$\sigma_t^2 = 0.0002553 + 0.8291y_{t-1}^2 + 0.0009142\sigma_{t-1}^2 + 0.1463\sigma_{t-2}^2$$

When we check for persistency of our chosen model, we have that the sum

$$0.8291 + 0.0009142 + 0.1463 = 0.9763142 < 1$$

This showed that the parameters satisfy the second order stationary (weakly stationary) conditions with high degree of persistence in the conditional variance.

Various test statistics carried out to assess the performance of the GARCH (1,2) model are shown in Tables 7 and 8. From Table 7, all the parameters including the constant value except GARCH (1) parameter, are significantly different from zero at 5% level of significance. Also, from Table 8, the test for serial correlation structure showed that no autocorrelation left in the standardized residuals and the squared standardized residuals since we fail to reject both the Null hypotheses in the Ljung-Box test. The results from the Engle's ARCH test at all lags indicated that we have successfully removed the conditional heteroscedasticity that existed when the test was performed on the pure return series. Furthermore, this was elaborated in figure 3- the ACF of the standardized residuals and the ACF of the squared standardized residuals, which showed that there is no autocorrelation left in both cases.

5. CONCLUSION

This study is concerned with obtaining the optimum model order of the linear GARCH and non-linear EGARCH models in modeling the First Bank returns series. The results from the descriptive statistics of the daily return series supported the claim that most of the financial series are leptokurtic. By comparing the orders of the models using AIC and BIC techniques as well as their likelihood values, GARCH (1,2) model was identified to be the most appropriate for the time-varying volatility of the data.

In order to account for the fat tails, the chosen model was compared based on the Normal, Student-t and Generalized Exponential distributions and the parameters of GARCH (1,2) model were obtained using GED. Serial correlation structure of the residuals of the identified model was examined using the Ljung-Box statistic. Also for ARCH test, LM was used. The results from all these tests showed that the selected model fulfilled all the diagnostic checks assumptions.

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Figure 1: The Daily Returns of First Bank Nigeria Plc.

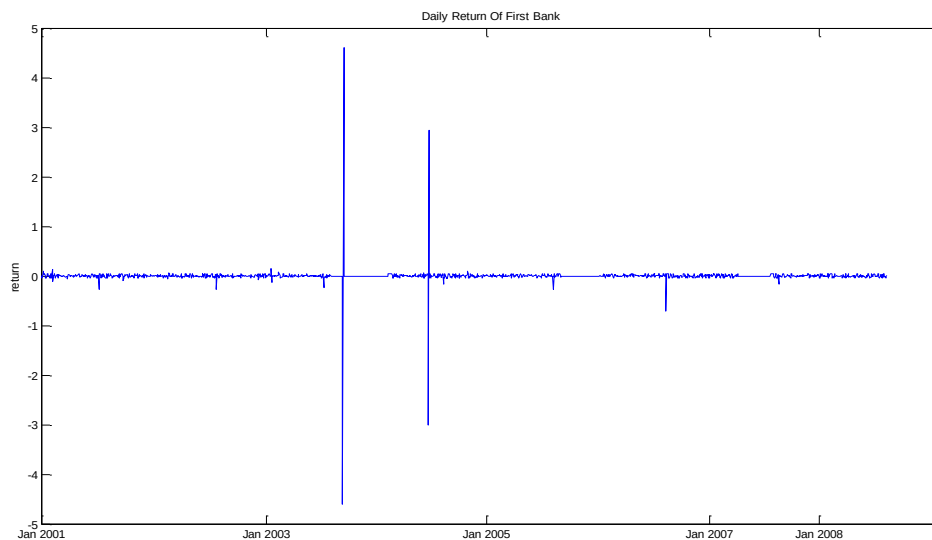


Figure 2: ACF Plot of First Bank Returns and Square Returns Series

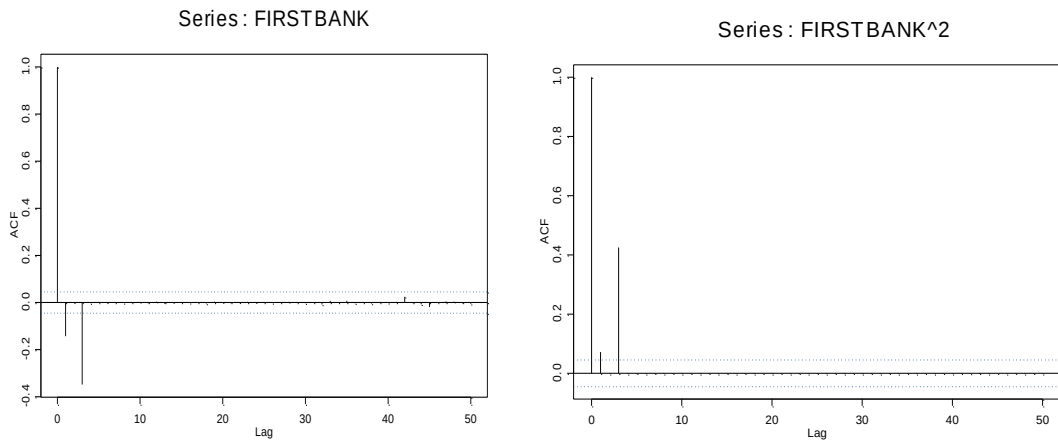


Figure 3: ACF of The Standardized Residuals and Squared Standardized Residuals

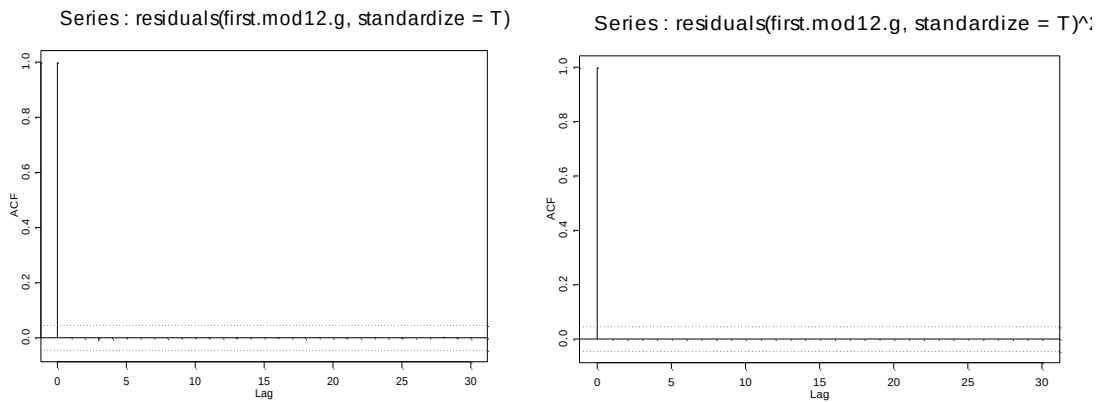


Table 1: The Descriptive Statistics for the Returns Series

Mean	Min.	Max.	Median	Std.Dev.	Skewness	Kurtosis
-0.0002735642	-4.60517	4.60517	0.0000	0.1817525	0.229075	512.5148

Table 2: The Various Fitted GARCH (p,q) Model

Order	AIC	BIC	Likelihood
0,1	-995.169	-978.557	500.6
0,2	-663.8	-641.6	335.9
1,0	-1319.41	-1302.798	662.7
1,1	-2189.236	-2167.086	1099
1,2	-2216.163	-2188.476	1113
2,0	8877	8900	-4435
2,1	-2202.176	-2174.489	1106
2,2	-2085.656	-2052.432	1049

Table 3: The Various Fitted EGARCH (p,q) Model

Order	AIC	BIC	Likelihood
0,1	90921	90937	-45457
0,2	163152580	163152602	-81576286
1,0	-454.2	-432.1	231.1

1,1	9494	9522	-4742
1,2	3.754e+013	3.754e+013	-1.877e+013
2,0	710.1	743.3	-349.1
2,1	1547.6	1586.4	-766.8
2,2	3.754e+013	3.754e+013	-1.877e+013

Table 4: Comparison of Optimum Order Of Linear and Non-Linear GARCH Models

Order	AIC	BIC	Likelihood
GARCH (1,2)	-2216.163	-2188.476	1113
EGARCH (1,0)	-454.2	-432.1	231.1

Table 5: Conditional Distribution for The Identified GARCH (1,2) Model

Distribution	AIC	BIC	Likelihood
Normal	-2216	-2188	1113
Student-t	-9247	-9214	4629
Generalized Exponential	-9287	-9253	4649

Table 6: Estimation of GARCH (1,2) with Conditional GED Distribution with Estimated Parameter, $V=0.7434626$, and Standard Error 0.006131138.

Parameters	Value
Constant	2.553e-004
ARCH (1)	0.8291
GARCH (1)	0.0009142
GARCH (2)	0.1463

Table 7: Test Results for The Parameters of The Chosen Model

Parameter	Value	Std. Error	t-value	Null Hypothesis	P-value
Constant	2.553e-004	0.00001744	14.6404817	Zero	0.0000
ARCH(1)	0.8291	0.08250219	10.0493057	Zero	0.0000
GARCH(1)	0.0009142	0.00183175	0.4990830	Zero	0.3089
GARCH(2)	0.1463	0.02085932	7.0152251	Zero	1.598e-012

Table 8: Test Results for Testing Various Hypotheses for The Chosen Model

Test	Test Statistic	Null Hypothesis	P-value
Jarque-Bera	139840117	Normally Distributed	0.0000
Ljung-Box(Std. Residuals)	0.6138	No Autocorrelation	1.0000
Ljung-Box (Squared Std. Residuals.)	0.03133	No Autocorrelation	1.0000

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LM (Lag 1)	0.0010	No ARCH	0.9743
LM (up to Lag 2)	0.0021	No ARCH	0.9990
LM (up to Lag 3)	0.0028	No ARCH	1.0000
LM (up to Lag 30)	0.03168	No ARCH	1.0000



Preventive Repair Policy and Overhaul Policy of Repairable System

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Abstract

Preventive maintenance is one of the most popular maintenance policies with wide useful application in engineering for keeping repairable system working, if one assumes that a system can be made as good as new by preventive maintenance through periodic overhaul. In this paper we shall present a model that obtains an optimal number of overhauls and optimal time of overhauling which minimizes the cost of maintenance.

Keywords: Preventive repair, overhaul, preventive repair policy and replacement policy

Mathematics Subject Classification 2000:62P30& 62N05.

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Introduction

Many systems become more large-scale and more complicated and influence our society greatly, such as airplanes, computer networks. Reliability/maintainability theory plays a very important role in maintaining such complicated large scale system. In early research of system reliability, it was usually assumed that the systems could be repaired to be as good as new. However due to the obvious differences between the assumption and reality , in the last four decades many new models that cannot be repaired to be as good as new have been presented by many researchers. In the literature there is a considerable report on the reliability of failing system. It has been observed that the reliability of a system can be increased substantially by setting either a preventive or scheduled maintenance polices in place whereby units which are about to enter a wear-out life or are partially worn out, or aged ,or are due for minor or major overhaul, are replaced with new units at predetermined periods of operation ;Manortey(2006). Kececioglu, (1995) observed that when these policies are implemented effectively, they have the advantage of reducing the average failure rate of the equipment, reduce the cost and inconveniences associated with failures and increase the equipment availability and productivity and if it is a production equipment will decrease the unit cost of production. It is observed that in the literature some of the researchers focus on the determination of optimal preventive maintenance cycles ;Barlow and Hunter(1960), Brown and Proschan(1983) , Stadje and Zuckerman(1990) The basic objective of the analysis is to find out time and conditions when it is optimal to shot down and carry out a major overhaul of the system. Such action is assumed to restore the system to an “as-new” state such that the ensuing force of mortality has been

refreshed In view of the above observation it is pertinent to note that preventive maintenance consists of regularly scheduled inspection, adjustments, cleaning, and repair of components and entire system. Some time preventive maintenance is called time driven or calendar based maintenance. Traditional preventive maintenance is keyed to failure rates and time between failures. It assumes that these variables can be determined statistically, and that one can therefore replace a part that is “due for failure” shortly before it fails. The availability of statistical failure information tends to lead to fixed schedules for overhaul of a system or the replacement of parts of the system subject to wear. Preventive maintenance is based on the assumption that the overhaul of a system by disassembly and replacement of parts restores the system to a like-new condition with no harmful side effects. In addition, this renewal task is based on the perception that new components installed are less likely to fail than old components of the same design.

Preventive maintenance is usually performed without regard to equipment condition or age of use. In this work attention is paid to repairable system. Here we mean a failed system whose probability of restoration to a satisfactory operating condition in a specified interval of active time is greater than zero. We believe that this measure is probably more valuable to the administration of the repair facility since it helps quantify workload for the repairmen. Furthermore, in this paper attention is paid to systems that will be available after being maintained. Here availability is defined as the probability that the system is operating satisfactorily at any point in time. Time here is only operating time and down time excluding idle time. Sometimes intrinsic availability term is really considered. This is defined as the probability that a system is operating in a satisfactory manner at any point in time when used under stated conditions.

Definition and Assumption

T	Scheduled interval between overhauls
N	Scheduled number of overhauls until the system is replaced at times NT
$V_n(T)$	Virtual age of the system at n th overhaul
$h_n(t)$	Hazard rate in the period of n th overhaul, i.e. during $(n-1)T, nT$
C_1	Cost of minimal repair at failure
C_2	Cost of scheduled overhaul
C_3	Cost of replacement
$C(N,T)$	expected cost rate of the system
$X_{(n,k)}$	System lifetime after $(k - 1)$ th minimal repairs in the period of n th overhaul
$Y_{(n,k)}$	Time at k th system failure
$F_{(n,k)}$	Distribution function of $Y_{(k,n)}$
R	Maintenance time

Problem formulation

Since an overhaul may affect only a limited number of components, the overhaul makes a system ‘better’ than old but not as good as new. Recently, Liu et al. (1995), Nakagawa (1986) and Pham and Weng (1996) emphasize the importance of overhaul.

The model describing the effect of overhaul is fundamental for establishing an appropriate PM policy. Liu et al. (1995) and Nakagawa (1979) proposed a ‘virtual age model’ and a ‘reduction model’ respectively. The virtual age model assumes that each overhaul decreases the hazard rate of a system by a fixed factor, and the reduction model assumes that the hazard rate after overhaul increases more quickly than that before overhaul. The virtual age model is relatively easy to analyze. A weakness of this model is the assumption that an overhaul decreases hazard rate of a system but never changes the hazard rate function. The reduction model overcomes this; each overhaul resets the hazard rate to zero (i.e. each overhaul makes the system as good as new).

In reality however, each overhaul may not be able to eliminate all the impending failures. As a result, unlike replacement, the overhaul cannot make the system as good as new. In other words, the overhaul can only rejuvenate the system and bring the condition of the system to a level somewhere between as good as new and just prior to the overhaul. Since the impending failures does not affect future reliability of the system, the hazard rate function may become higher after each overhaul is performed on the system. Therefore, we propose a model which not only decreases the hazard rate of the system to a certain value,

but also changes the hazard rate function after overhaul. We shall consider the following PM policy: an overhaul is made at periodic times T^* and the system is replaced by a new system at the N^{th} overhaul.

The expected rates under the proposed model are obtained and numerical studies will be performed in the cases of negligible maintenance times.

The following notations will be used in this paper:

The model is constructed by using the virtual age function and increasing the slope of the hazard rate function after overhaul. Figure 1 depicts hazard rate functions before and after the first overhaul. The figure shows that the overhaul decreases the hazard rate but not to zero, and the slope of the hazard rate function becomes larger, i.e. the hazard rate right after overhaul is $h_1(\theta T)$. and the slope is the same as $h_2(t)$. Therefore, the hazard rate right after overhaul can be described as $h_2[v_1(T)]$, where $v_1(T)$, called virtual age which satisfies $h_2[v_1(T)] = h_1(\theta T)$.

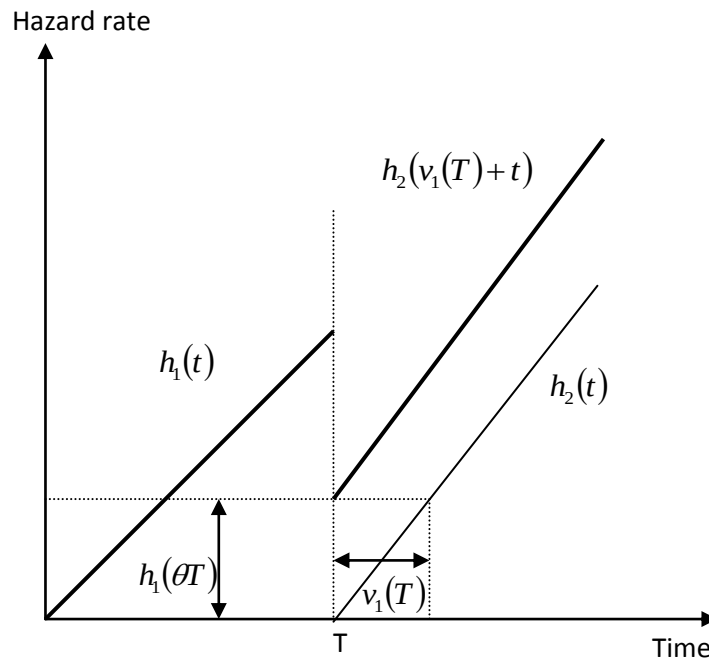


Figure 1 Hazard rate of the model

We now derive the virtual age of the system after the n th overhaul. Let t_n be the time of the n th overhaul, where $t_0 = 0$; that is $t_n = nT$ ($n = 1, L, N$). Let $v_n(T)$ be the virtual age right after the n th overhaul. The virtual age function of the system is a function of two variables, $V(v, T)$, that specifies the functional relationship between v and T . If the system has hazard rate function $h_n(t)$ in the period of n th

overhaul, then the hazard rate is $h_n \{V[v_{n-1}(T), T]\}$ right after t_n . Since the hazard rate function changes to $h_{n-1}(t)$ after nth overhaul, $v_n(T)$ is obtained by formula.

$$h_{n-1}[v_n(T)] = h_n \{V[v_{n-1}(T), T]\} \quad (1)$$

Kijima and Nakagawa (1992) measured the effect of the overhaul on the virtual age by a multiplier $0 \leq \theta \leq 1$, and used virtual age function $V(v, X) = v + \theta X$, using this virtual age function, virtual age at nth overhaul becomes.

$$v_n(T) = h_{n-1}^{-1} [h_n (v_{n-1}(t) + \theta T)] \quad (2)$$

Where $v_0(T) = 0$.

PREVENTIVE MAINTENANCE POLICIES

PREVENTIVE MAINTENANCE POLICY – Negligible Maintenance Time

Suppose that the system undergoes minimal repair at failure and maintenance time is zero. Then, downtime occurs according to a non stationary Poisson process with failure rate $h_n(t)$ in the period of nth overhaul and the expected repair number in $[v_{n-1}(T), v_{n-1}(T) + T]$ is $\int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t) dt$ as shown in Blischke and Murthy (1994). Therefore, the expected cost in a renewal cycle is:

$$c_1 \sum_{n=1}^N \int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t) dt + (N)c_2 + c_3$$

Where c_2 is the cost of the overhaul and c_3 is the cost of replacement with $c_3 > c_2$, and the expected length of a renewal cycle is NT . Thus the expected cost rate over infinite time horizon is given by

$$C(N, T) = \frac{c_1 \sum_{n=1}^N \int_{v_{n-1}(T)}^{v_{n-1}(T)+T} h_n(t) dt + (N)c_2 + c_3}{NT}$$

The optimal number of overhauls and time of overhauling - (N, T) that minimizes the expected cost rate can be obtained by using a procedure similar to that in Liu et al (1995) and Nakagawa (1986). The intention is to seek both the optimal number of overhauls N^* and the optimal time T^* which minimizes $C(N, T) < C(N-1, T)$, which implies that

$$C(N, T) \geq \frac{c_3 - c_2}{c_1} \text{ and } C(N-1, T) < \frac{c_3 - c_2}{c_1} \quad (4)$$

$$C(N, T) = \begin{cases} N \int_{V_n(T)}^{V_n(T)+T} h_{n+1}(t) dt - \sum_{n=1}^N \int_{V_{n-1}(T)}^{V_n(T)+T} h_n(t) dt, & \text{if } N=1, 2, 3, \dots, I \\ 0 & \text{if } N=0 \end{cases}$$

From the assumption that $h_{n+1}(t) > h_n(t)$ for any $t > 0$, we obtain:

$$C(N, T) \geq \int_{V_N(T)}^{V_N(T)+T} h_{N+1}(t) dt - \int_0^T h_1(t) dt$$

Thus $C(N, T)$ is increasing in N and, if $\lim_{N \rightarrow \infty} h_N(t) \rightarrow \infty$, then it tends to ∞ as $N \rightarrow \infty$

Therefore, there exists a finite and unique N^* which satisfies (4) for any $T > 0$. Next, differentiating $C(N, T)$ with respect to T and setting it equal to 0, we obtain:

$$\sum_{k=1}^N \left[\{h_n(V_{n-1}(T) + T)(V_{n-1}(T) + 1) - h_n(V_{n-1}(T))V_{n-1}(T)\} T - \int_{V_{n-1}(T)}^{V_{n-1}(T)+T} h_n(t) dt \right] = \frac{(N-1)C_2 + C_3}{C_1}$$

6

If $h_n(t)$ is differentiable and strictly increasing to ∞ , then the left-hand side of (6) strictly increases to ∞ .

Thus, there exists a finite and unique T^* which satisfies (4) for any integer N .

(N^*, T^*) can be obtained using the following procedure:

- i. Let $N_1 = 1$ and compute $T = T_1$ satisfying $\frac{\partial C(N_1, T)}{\partial T} = 0$.
- ii. Find $N = N_2$ satisfying the inequalities $C(N+1, T_1) \geq C(N, T_1)$ and $C(N, T_1) < C(N-1, T_1)$
- iii. Compute $T = T_2$ satisfying $\frac{\partial C(N_2, T)}{\partial T} = 0$.
- iv. If $N_j = N_{j+1}$ ($j=1, 2, 3, L$), set $(N^*, T^*) = (N_j, T_j)$ and stop, otherwise, go to (ii).

For the Weibull hazard rate, we obtain $C(N, T)$ and compute the optimal T^* , if the hazard rate in the n th overhaul period is $h_n(t) = \alpha_n \beta t^{\beta-1}$ and virtual age function satisfies equation (2), then:

$$V_n(t) = \sum_{k=1}^n \left(\frac{\alpha_k}{\alpha_{n+1}} \right)^{\frac{1}{\beta-1}} \theta T \quad 7$$

This can be shown by induction on n . For $n=1$, (7) holds since

$$\alpha_2 \beta (V_1(T))^{\beta-1} = \alpha_1 \beta (\theta T)^{\beta-1}, \text{ and } V_1(T) = \left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{\beta-1}} \theta T. \text{ Suppose that equation 7 holds for}$$

$(n-1), n \geq 1$. With $h_n(t) = \alpha_n \beta t^{\beta-1}$, equation (2) becomes:

$$\alpha_{n+1}\beta(V_n(T))^{\beta-1} = \alpha\beta(V_{n-1}(T) + \theta T)^{\beta-1}$$

And,

$$V_n(T) = \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} \{V_{n-1}(T) + \theta T\} = \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} \left[\sum_{k=1}^{n-1} \left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}} + 1 \right] \theta T$$

Therefore, equation 7 holds for every $n \geq 1$.

$$V_n(T) = \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} \{V_{n-1}(T) + \theta T\} = \left[\sum_{k=1}^{n-1} \left(\frac{\alpha_k}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} + \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} \right] \theta T = \sum_{k=1}^n \left(\frac{\alpha_k}{\alpha_{n+1}}\right)^{\frac{1}{\beta-1}} \theta T$$

From equations 3 and 7,

$$C(N, T) = \frac{C_1 R(N, T) + (N-1)C_2 + C_3}{NT}$$

8

Where

$$R(N, T) = T^\beta \sum_{n=1}^N \alpha_n \left[\left\{ \sum_{k=1}^{n-1} \left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}} \theta + 1 \right\}^\beta - \left\{ \sum_{k=1}^{n-1} \left(\frac{\alpha_k}{\alpha_n}\right)^{\frac{1}{\beta-1}} \theta \right\}^\beta \right]$$

NUMERICAL INVESTIGATION

Suppose that the time to failure of the system follows Weibull distribution with $\beta = 2, 3, 4$

and $\frac{1}{\alpha_n} = 100x(0.9^\beta)^{n-1}$, ($n = 1, 2, \dots, L$). That is, the mean time to failure in the n th period of overhaul

becomes 10 percent shorter for every overhaul.

Table 1 A: Optimal maintenance policy for negligible maintenance time

		$\beta = 2$		$\beta = 3$		$\beta = 4$	
Θ	$\frac{c_3}{c_2}$	N	T	N	T	N	T
		$C(N^*, T^*)$		$C(N^*, T^*)$		$C(N^*, T^*)$	
0.1	3	1	17.32	1	5.31	1	3.16
	10	0.3464		0.8469		1.2649	
	20	3	19.06	3	5.21	3	2.91
	50	0.5970		1.5367		2.4460	
	100	3	24.29	4	5.20	4	5.20
		0.7135		2.0929		3.4525	
		4	28.95	5	5.61	5	5.61
		1.0190		3.3168		5.7833	
		5	32.68	6	5.86	6	5.86
		1.3711		4.9057		8.9668	
0.2	3	1	17.32	1	5.31	1	3.16
	10	0.3464		0.8469		1.2649	
	20	2	22.31	2	5.88	2	3.24
	50	0.5852		1.6557		2.6723	
	100	3	22.68	3	5.59	3	2.97
		0.7641		2.3242		3.8930	

		4	26.44	4	5.82	4	2.91
		1.1156		3.7988		6.7509	
		5	29.33	5	5.86	5	2.80
		1.5273		5.7089		10.6584	
0.3	3	1	17.32	1	5.31	1	3.16
	10	0.3464		0.8469		1.2649	
	20	2	21.42	2	5.60	2	3.07
	50	0.6070		1.7402		2.8222	
	100	2	28.49	3	5.15	3	2.70
		0.8074		2.5255		4.2776	
		3	31.35	3	6.65	3	3.27
		1.1910		4.2119		7.6052	
		4	33.29	4	6.42	4	3.00
		1.6372		6.3697		12.0924	

Table 1A gives the value of N^* , T^* , and the corresponding expected cost rate $C(N^*, T^*)$ when $\theta = 0.1, 0.2, 0.3$; $\frac{C_2}{C_1} = 3$ and $\frac{C_3}{C_2} = 3, 10, 20, 50, 100$ and shows that :

- The replacement time and the number of overhauls become larger as replacement cost gets larger.
- For a fixed θ , as β gets larger (and hence the systems hazard rate becomes larger), T^* becomes smaller.
- For a fixed β , as θ gets larger (and hence overhaul is less effective), the replacement time and the number of overhauls become smaller.

To investigate further the effect of overhaul, additional numerical analysis is performed. Let Δ be the percentage of cost savings of the optimal maintenance policy with overhauls over cost of optimal maintenance policy with only minimal repairs and replacement (which means that there will be no overhauls for the system);

$\Delta = 100 \times \frac{c(1, T_1^*) - C(N^*, T^*)}{C(1, T_1^*)}$. Table 2 and figure 2 gives the optimal N^* , T^* , and the effect of

overhaul for $\theta = 0.2$ and $\frac{C_3}{C_1} = 10$, and shows that as the system's hazard rate gets larger, overhaul becomes more effective.

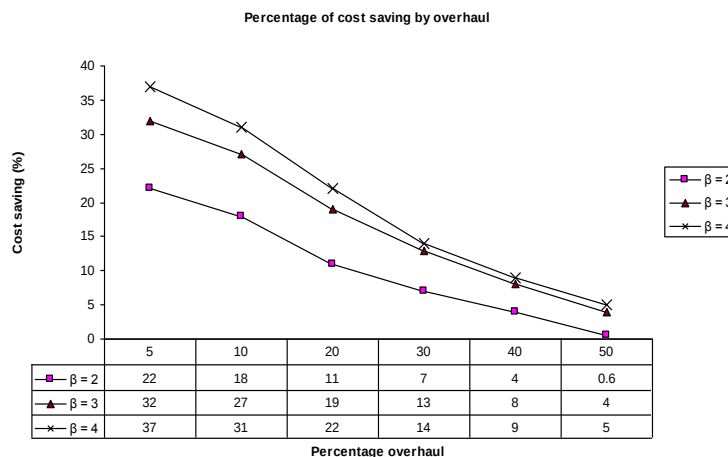


Fig. 2: Percentage of cost saving by overhaul

Table 2: Optimal maintenance policy and cost saving for selected values of overhaul cost

$c_2/c_3 \times 100 (\%)$	N* T* C(N*, T*) $\Delta(\%)$	N* T* C(N*, T*) $\Delta(\%)$	N* T* C(N*, T*) $\Delta(\%)$
5	4	4	4
10	11.67	3.38	1.94
20	0.4925	1.2771	1.9804
30	22	32	37
40	3	4	4
50	15.41	3.52	2.00
	0.5191	1.3858	2.1711
	18	27	31
	3	3	3
	16.65	4.56	2.54
	0.5607	1.5383	2.4471
	11	19	22
	2	2	2
	22.21	5.88	3.24
	0.5852	1.6577	2.6723
	7	13	14
	2	2	2
	23.05	6.03	3.30
	0.6073	1.7416	2.8250
	4	8	9
	2	2	2
	23.86	6.17	3.36
	0.6286	1.8236	3.1205
	0.6	4	5

The estimate of the repair cost for a complex system is laborious, so that the effects of incorrect estimate of repair cost should be investigated. The percentage error of $C(N^*, T^*)$ caused by incorrect estimate of repair cost is defined as:

$$PE = \frac{C(N', T') - C(N^*, T^*)}{C(N^*, T^*)} \times 100(\%) \quad (9)$$

Where $C(N^*, T^*)$ is the minimal cost obtained with the correct repair cost and $C(N', T')$ is the cost obtained with incorrect estimate of repair cost.

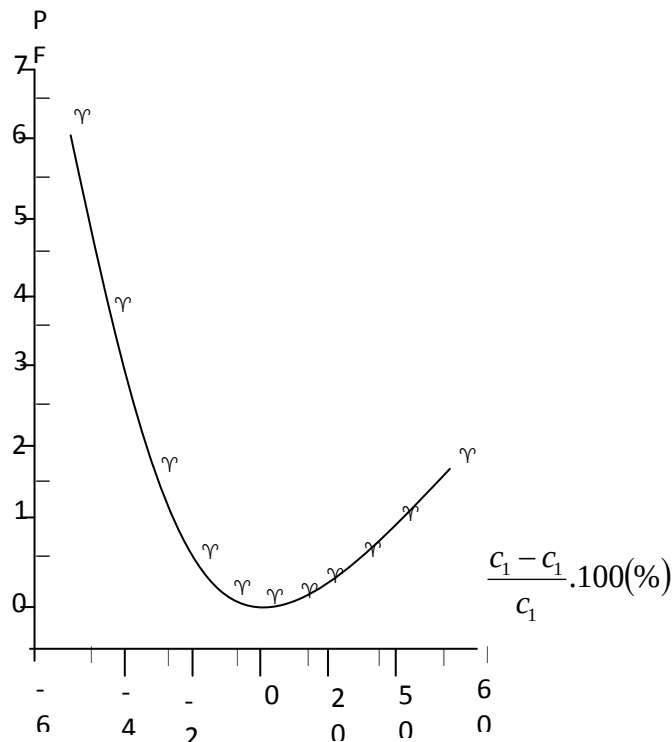


Figure 3: Percentage errors by wrong repair cost

Figure 3 gives PE versus $\frac{C_i - C_1}{C_1} \times 100(\%)$ for $\beta = 2$, $\theta = 0.2$, $\frac{C_2}{C_1} = 3$, and $\frac{C_3}{C_1} = 10$, and shows that underestimating repair cost causes a larger PE increase than overestimating repair cost.

Conclusion

Preventive maintenance has long been recognized as extremely important in the reduction of maintenance costs and improvement of equipment reliability. In practice it takes many forms. Two major factors that should control the extent of a preventive program are first, the cost of the program compared with the carefully measured reduction in total repair costs and improved equipment performance; second, the percent utilization of the equipment maintained. In this paper we have presented an improved model for describing a system subject to minimal repair and overhaul. We have also established optimal maintenance policies in the case of negligible maintenance time. The numerical studies indicate that overhaul becomes more effective as the system's hazard rate gets larger. Finally the study shows that the effect of under estimating repair cost has a more drastic effect than over estimating it.

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A MATHEMATICAL MODEL TO DETERMINE THE GROWTH OF INVESTMENTS USING SHARE PRICES

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Abstract

The current global crisis has called for attention from all quarters. Thus, we developed a mathematical model using share prices to determine the rate of growth or decline of investments that are listed on a stock exchange. The model formulation was based on Hamilton-Jacobi-Bellman equation with a constant discount rate. Analyses of the formulation were carried out and certain interesting results were obtained. An expose on the market trends was presented to enlighten and guide policy makers and investors.

Key words: Investment growth model, share price, discount rate and present value.

Mathematics Subject Classification 2000:91B28 & 91B70

1. Introduction

It is well known now that most of the leading economies such as those of USA, Japan, Germany, etc are recording a down-turn. This is as a result of the melt-down at the micro-level. In fact, some of the big investment companies in these countries have already accepted running at a deficit. This paper attempts to study the behaviour of such investments through the main economic index (share price) used in stock exchange markets. The effort is to study the performance of the investments from the view point of mathematical modelling. This will help determine the effect of a number of economic indices on their performance. A control mechanism could then be considered towards improving their aggregate performance, which in turn will lead to improvement of the entire economy at the macro-level.

The paper has extended the horizon of that by Ugbebor et al (2001) in the sense that the market growth rate has been considered from a wider perspective. The object of interest, the share price, has been expressed in time and the corresponding model equations considered accordingly. These have made the study in this paper more realistic and capable of revealing useful information on the current economic down-turn.

After this introductory part, the rest of the paper is organised as follows: In Section 2, the model formulation and its solutions are presented. This is followed by the analysis of the market trends in Section 3. Finally, the conclusions are given in Section 4.

2. Problem Formulation and Solution

According to Ugbebor et al (2001) and the references therein, the behaviour of an investment under a fixed capital, satisfies the model equation

$$\frac{1}{2}\beta^2 p^2(t) \frac{d^2 w(p(t))}{dp^2(t)} + \alpha p(t) \frac{dw(p(t))}{dp(t)} - rw(p(t)) = -f(p(t)), \quad (2.1)$$

for any time, t ; where the investment output is $w(p(t))$, the constant discount rate is r and the production function is $f(p(t))$, which could be called stimulating function, inducement function or motivating function, the parameters, α and β are, respectively, the average rate of change of share prices and average standard deviation of the changes in the share prices.

In this study we shall neglect the family of exogenous factors $B(t)$, (see Ugbebor et al, 2001) that influences changes in the share prices.

Thus, we shall have

$$p(t) = p_0 e^{\left(\alpha - \frac{1}{2}\beta^2\right)t} \quad (2.2)$$

where p_0 , is the initial share price. In line with Cobb-Douglas function under constant return to scale, we shall adopt the following:

$$f(p(t)) = p(t), \quad (2.3)$$

Then the solution of (2.1) becomes

$$w(p(t)) = C_1(p(t))^{\lambda_1} + C_2(p(t))^{\lambda_2} + \frac{p(t)}{r - \alpha}. \quad (2.4)$$

where

$$\lambda_1 = \frac{1}{\beta^2} \left[-\left(\alpha - \frac{1}{2}\beta^2\right) + \sqrt{\left(\alpha - \frac{1}{2}\beta^2\right)^2 + 2\beta^2 r} \right] \quad (2.5)$$

Which implies that $\lambda_1 > 0$, or

$$\lambda_2 = \frac{1}{\beta^2} \left[-\left(\alpha - \frac{1}{2}\beta^2\right) - \sqrt{\left(\alpha - \frac{1}{2}\beta^2\right)^2 + 2\beta^2 r} \right] \quad (2.6)$$

and hence,

$$\lambda_2 < 0.$$

Note (see (2.2)) that whenever $\alpha < \frac{1}{2}\beta^2$ then the share price $p(t)$, is declining, showing a down-turn of the investment whereas $p(t)$ is appreciating whenever $\alpha > \frac{1}{2}\beta^2$, thereby indicating a growth of the investment.

The constants of integration C_1 and C_2 in (2.4) could be obtained if the following conditions (natural) are imposed.

That:

$$w(0) = 0 \quad (2.7)$$

and

$$\frac{dw(\hat{p})}{dp(t)} = 0 \quad (2.8)$$

With these two conditions, a finite difference method could be used to solve the problem in situations where theoretical methods are not handy.

where $\hat{p}$ is the equilibrium price.

Applying (2.8) to (2.4) gives

$$w(p(t)) = \frac{p(t)}{r - \alpha} + C_1(p(t))^{\lambda_1} \quad (2.9)$$

and applying (2.8) to (2.4) gives

$$\frac{\hat{p}}{r - \alpha} + C_1 k \lambda_1 (\hat{p})^{\lambda_1} = 0 \quad (2.10)$$

$$\therefore C_1 = -\frac{\hat{p}}{r - \alpha} \cdot \frac{1}{\lambda_1 (\hat{p})^{\lambda_1}} \quad (2.11)$$

Remark: Equations (2.9) and (2.10) are so obtained because C_2 is allowed by Assumption (v) to take value zero in order to avoid dealing with indeterminate cases. According to Chiang and Wainwright (2005, p.530), this makes the equilibrium price an unstable one since the only root left is a positive one. Also, since the particular solution as shown in (2.4) is linear in $p(t)$, it means that the equilibrium price is dynamic. Hence, we conclude that the equilibrium price in the study is a dynamically unstable one.

Applying (2.11) to (2.9) gives

$$w(p(t)) = \frac{p(t)}{r - \alpha} - \frac{\hat{p}^{1-\lambda_1} \cdot p(t)^{\lambda_1}}{\lambda_1 (r - \alpha)} = \frac{1}{r - \alpha} \left[p(t) - \frac{\hat{p}^{1-\lambda_1} \cdot p(t)^{\lambda_1}}{\lambda_1} \right] \quad (2.12)$$

Where $w(p(t))$ is the investment output per unit of the quantity of trade.

Since the share price is the rate of total value traded (output) to the quantity (volume) of trade, we shall have

$$w(p(t)) \propto p(t) \quad \text{or} \quad w(p(t)) = Qp(t),$$

where Q is the quantity of product traded. At the equilibrium price, $\hat{p}$, we let

$$w(\hat{p}) \equiv \bar{p} \quad (2.13)$$

where $\bar{p}$ is the average share price for a given period of time.

Putting (2.13) into (2.12) gives

$$\hat{p} = \frac{\lambda_1(r - \alpha)\bar{p}}{\lambda_1 - 1} \quad (2.14)$$

And (2.14) into (2.12) gives

$$w(p(t)) = \frac{p(t)}{r - \alpha} - \frac{1}{\lambda_1} \left[\frac{\lambda_1}{\lambda_1 - 1} (r - \alpha) \bar{p} \right]^{(1-\lambda_1)} \frac{p(t)^{\lambda_1}}{(r - \alpha)} \quad (2.15)$$

where $(r - \alpha)$ and $(\lambda_1 - 1)$ have been found empirically (See Ugbebor et al, 2001) to always have the same sign.

Also, the investment output at the equilibrium price is

$$w(\hat{p}) = \hat{w} = \frac{\hat{p}}{r - \alpha} - \frac{\hat{p}^{(1-\lambda_1)}}{\lambda_1} \cdot \frac{\hat{p}^{\lambda_1}}{(r - \alpha)} = \frac{\hat{p}(\lambda_1 - 1)}{\lambda_1(r - \alpha)} \quad (2.16)$$

Next, we shall express the investment output in terms of time. This is achieved by putting (2.2) into (2.12) to get

$$w(t) = \frac{p_0 e^{(\alpha - \frac{1}{2}\beta^2)t}}{r - \alpha} - \frac{\hat{p}^{1-\lambda_1}}{\lambda_1(r - \alpha)} \cdot p_0^{\lambda_1} e^{(\alpha - \frac{1}{2}\beta^2)\lambda_1 t} \quad (2.17)$$

Equation (2.17) enables us to tract the output with respect to time.

From (2.12) the market growth rate with respect to $p(t)$ is given as

$$\frac{dw(p(t))}{dp(t)} = \frac{1}{r - \alpha} - \frac{\hat{p}^{1-\lambda_1}}{(r - \alpha)} \cdot p(t)^{\lambda_1-1} = \frac{1}{r - \alpha} \left[1 - \left(\frac{\hat{p}}{p(t)} \right)^{1-\lambda_1} \right] \quad (2.18)$$

Similarly, the market growth rate with respect to time, t is obtained from (2.17) as

$$\begin{aligned} \frac{dw(t)}{dt} &= \frac{p_0(\alpha - \frac{1}{2}\beta^2)}{r - \alpha} \cdot e^{(\alpha - \frac{1}{2}\beta^2)t} - \frac{\hat{p}^{1-\lambda_1} \cdot p_0^{\lambda_1}}{(r - \alpha)} \cdot (\alpha - \frac{1}{2}\beta^2)\lambda_1 e^{(\alpha - \frac{1}{2}\beta^2)\lambda_1 t} \\ &= \frac{(\alpha - \frac{1}{2}\beta^2)}{(r - \alpha)} \left[p_0 e^{(\alpha - \frac{1}{2}\beta^2)t} - \hat{p}^{1-\lambda_1} \cdot p_0^{\lambda_1} \cdot e^{(\alpha - \frac{1}{2}\beta^2)\lambda_1 t} \right] \end{aligned} \quad (2.19)$$

From (2.18) and (2.19) the economic indices that affect the growth of an investment are α , β^2 , r and $p(t)$ (i.e $\hat{p}$ or $\bar{p}$).

Note: The second term inside the square bracket in (2.19) is greater than the first term. Therefore,

$$\dot{w}(t) < 0 \text{ if } \frac{(\alpha - \frac{1}{2}\beta^2)}{r - \alpha} > 0 \text{ and } \dot{w}(t) > 0 \text{ if } \frac{(\alpha - \frac{1}{2}\beta^2)}{r - \alpha} < 0. \text{ In particular, } \dot{w}(t) < 0 \text{ when } r < \frac{1}{2}\beta^2 \text{ and } \dot{w}(t) > 0 \text{ when } r \geq \frac{1}{2}\beta^2.$$

Assumptions

The model equations derived in Section 2 is based on the following assumptions:

- i) The model equations are concerned with the production sector of the economy. In the light of that, the production function in the formulations is assumed to have constant return to scale.
- ii) The equilibrium position is taken to be one which does not depend on time. That is, the position is time-homogeneous of degree zero. The analyses of the market growth are carried out at this position.
- iii) The analyses of the performance of shares are based on the assumption that the parameters α and β are such that $\alpha < \frac{1}{2}\beta^2$. This is to reflect the downward trend of share prices.
- iv) The family of exogenous factors are neglected. This is to limit the study to a non-stochastic one (as the downward trend is already envisaged).
- v) The equilibrium price in the study is a dynamically unstable one. Thus, it properly captures the volatility of the share prices.

3. Market Trends

Here, we shall discuss investment growth or decline rate with respect to the discount rate and also the prices (equilibrium and average). The analyses will be done from two perspectives, namely,

- The case when the growth or decline rate is known and the other economic indices (discount rate and prices) are to be determined and
- The reverse case.

The equations that will be used for the analyses are (2.14), (2.18) and (2.19).

Case I: Growth/Decline rate, $w'(p(t))$, in (2.18) is Known

One of the following conditions may occur:

- (a) When there is no growth or decline then $w'(p(t)) = 0$. This happens when $\hat{p} = \bar{p}$ or $r = \alpha$.
- (b) When there is a decline, then $w'(p(t)) < 0$ and $r > \alpha$.
- (c) The growth in investment implies that $w'(p(t)) > 0$ and $r < \alpha$.

These scenarios can be simulated whereby the relationship, say, between equilibrium price and discount rate could be established.

Case II: Growth/Decline rate, $w'(p(t))$, is not known in (2.18)

In order to show some relationships between the economic indices, we would like to use the following figures obtained by Ugbebor et al (2001):

Average price, $\bar{p} = 4.86$ naira,

Rate of share price changes over a given trading period, $\alpha = 0.143$,

The variance of share price changes over a trading period, $\beta^2 = 0.975$.

For the set of values of $r = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 1.0$, we find the corresponding values of λ_1 , $\hat{p}$ and $w'(p(t))$. The diagrams in Figures 1- 2 shown below give the various relationships of interest.

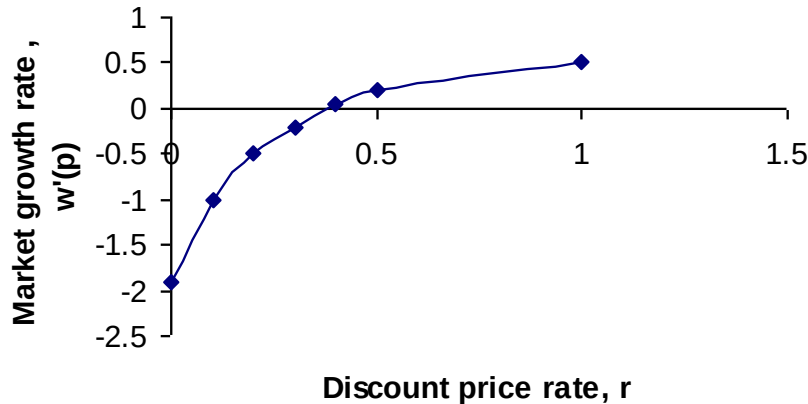


Figure 1: Change in Market growth with Discount rate.

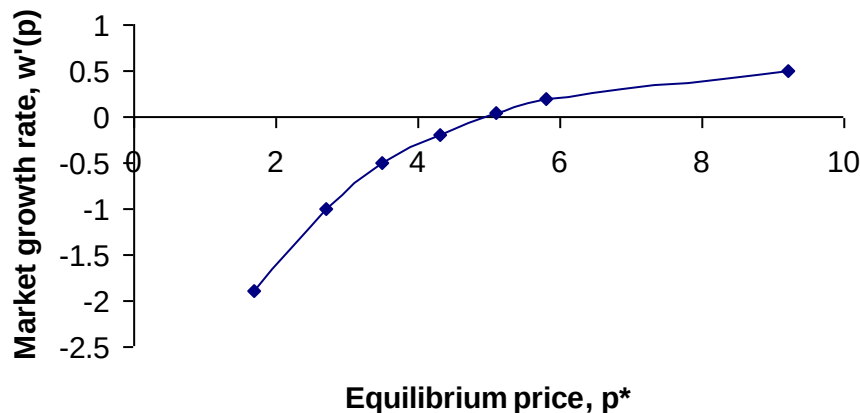


Figure 2 : Change in Market growth with equilibrium price
Under the time dependent case, the following observations could be made:

- A choice of p_0 only indicates the point of $\dot{w}(t)$ at $t = 0$.
- Once the initial share price is at or below the equilibrium point, then the company is in the disinvestment condition. We shall, therefore, be interested in cases when $p_0 > \hat{p}$.
- The parameters r, α and β are, however, seen to influence the trend also.
- The growth /decline rate is found under a long run condition. Hence we have,

$$\dot{w}_\infty = \lim_{t \rightarrow 0} \dot{w}(t) = \frac{\alpha - \frac{1}{2}\beta^2}{r - \alpha} \left[p_0 - p_0^{\lambda_1} \hat{p}^{1-\lambda_1} \right] \quad \dots\dots\dots(3.1)$$

because $\alpha < \frac{1}{2}\beta^2$, (see Assumption (iii)).

NOTE: In (2.19), when $\left(\alpha - \frac{1}{2}\beta^2\right) < 0, \exists$ a decline of the market. When $\left(\alpha - \frac{1}{2}\beta^2\right) = 0, \exists$ no growth. When $\left(\alpha - \frac{1}{2}\beta^2\right) > 0$, the market trend is not discernable.

At a glance, we observe in (2.18) and (2.19) that, $r - \alpha$, suppresses the growth rate. Therefore, r should be as close as possible to α since r can be more easily controlled by entrepreneurs.

4. Conclusion

There are three economics parameters (α, β^2 and r) and one independent variables, $p(t)$ referred to as economics indices that are built into the model equation (2.1) with natural boundary conditions as given in (2.7) and (2.8). Practically, it has been found that the average rate of change of share prices, α and the average variance of the share prices changes β^2 , are difficult to control by government and entrepreneur, for the simple reason that they are subject to market forces (which may be influence by political and/or social situations). On the other hand, the discount rate, r is the one that could be easily controlled by government/entrepreneur (by taking certain actions). It is against this background that the graphs, (see Fig. 1 - 2) in Section 3, are drawn.

The paper has shown that there is growth in an investment company when there is increase in discount rate. For the particular company with $\alpha = 0.143$ and $\beta^2 = 0.975$, Fig. 1 shows that the rate of growth is positive as from $r = 0.38$ upward. For such a company, the entrepreneurs are expected to take actions that will ensure that the discount rate is not less than 0.38. Here, it should be noted that the negative rate of growth is referred to as a decline.

It has been shown that the higher the equilibrium price the more the growth in the investment. It has also been found that an investment undergoes growth only when the periodic share prices, $p(t)$ is lower than the equilibrium price $\hat{p}$. The entrepreneurs could use this fact to know the state (healthy or otherwise) of their companies. Further, the periodic share prices in the exchange market do not indicate growth or

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otherwise of the company. For the company with the above figures, Fig.2 shows that the growth is positive when $\hat{p} = 4.90$ naira and above .Therefore, it is a bad signal whenever $p(t)$ is higher than 4.90 naira.

Equation (3.1) shows that the growth/Decline rate is asymptotically stable. Further analysis could be carried out from this point. The method of solution similar to that by Sjoberg and Glad (2008) is being considered for the problem stated here.

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On Steady Flow and Heat transfer in a pipe with temperature dependent viscosity and convective cooling

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Abstract

This paper investigates the effect of convective cooling on a temperature dependent viscosity liquid flowing steadily through a cylindrical pipe. The system is assumed to exchange heat with the ambient following Newton's cooling law and the fluid viscosity model varies as an inverse linear function of temperature. Analytical expressions for fluid velocity and temperature are derived which essentially expedite to obtain expressions for thermal satiability criterion. Our results reveal that both the fluid velocity and temperature decrease with an increase in convective cooling and increase with a decrease in fluid viscosity.

Keywords: Pipe flow; Variable viscosity; Lubrication Approximation, Convective cooling.

Mathematics Subject Classification 2000:35Q35,35K55 & 80A23

1. Introduction

There have been considerable interests in investigating the effect of convective cooling on the variable viscosity pipe flow because of its implications in the engineering and biological systems such as fluid transport in petrochemical industries, food processing, coating and polymer processing, bio-fluid mechanics [1, 11]. In industrial and physiological flow processes, fluid can be subjected to extreme conditions such as high temperature, pressure and shear rate. External heating such as the ambient temperature and high shear rates can lead to a high temperature being generated with the fluid. This may have a significant effect on the fluid properties. Fluid used in industries such as polymer fluids as well as physiological fluid like blood have a viscosity that varies rapidly with temperature and may give rise to strong feedback effects, which can lead to significant changes in the flow structure of the fluid [2-4, 14]. Due to the strong coupling effect between the Navier-Stokes and energy equations, viscous heating also plays an important role in fluid with strong temperature dependence. Costa et al. [2] applied the

temperature dependent viscosity model to study magma flows. Elbashbeshy et al. [3] investigated the effect of temperature dependent viscosity on heat transfer over a moving surface. In their investigation the fluid viscosity model varies as an inverse linear function of temperature. Makinde [5, 6] studied the flow of liquid film with variable viscosity along an inclined heated plate. The effects of temperature dependent fluid viscosity on heat transfer and thermal stability of reactive flow in a cylindrical pipe with isothermal wall was reported in Makinde [7].

The main objective of this study is to investigate the effect of convective cooling on steady flow of a variable viscosity fluid through a cylindrical pipe with convective. The plan of this paper is as follows; in sections 2 and 3 we describe the theoretical analysis of the problem with respect to the fluid velocity and temperature fields. Section 4 describes the thermal stability criterion for the flow system. The results are presented graphically and discussed quantitatively in section 5.

2. Mathematical Model

The configuration of the problem studied in this paper is depicted in Fig.1. The flow is considered to be steady in the $\bar{z}$ -direction through a cylindrical pipe of radius a and length L under the action of a constant pressure gradient, viscous dissipation, convective cooling at the pipe surface. It is assumed that the pipe is long enough to neglect both the entrance and exit effects. The fluid is incompressible and the temperature dependent viscosity ($\bar{\mu}$) can be expressed as [3]

$$\bar{\mu} = \frac{\mu_0}{1 + m(\bar{T} - T_a)}, \quad (1)$$

where μ_0 is the fluid dynamic viscosity at the ambient temperature T_a .

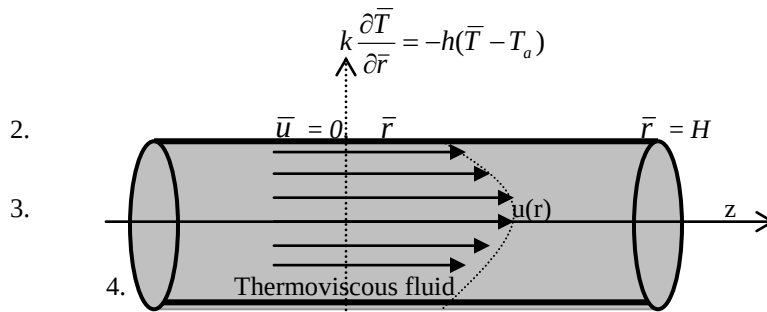


Fig.1: Schematic diagram of the problem

Under these conditions the continuity, momentum and energy equations governing the problem in dimensionless form may be written as [1, 7, 13]

$$\frac{\partial(ru)}{\partial z} + \frac{\partial(rv)}{\partial r} = 0, \quad (2)$$

$$\varepsilon Re \left(u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} \right) = -\frac{\partial p}{\partial z} + 2\varepsilon^2 \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left[r \mu \left(\frac{\partial u}{\partial r} + \varepsilon^2 \frac{\partial v}{\partial z} \right) \right], \quad (3)$$

$$\varepsilon^3 Re \left(u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} \right) = -\frac{\partial p}{\partial r} + \frac{2\varepsilon^2}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial v}{\partial r} \right) + \varepsilon^2 \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial u}{\partial r} + \varepsilon^2 \frac{\partial v}{\partial z} \right) \right] - 2\mu \varepsilon^2 \frac{v}{r^2}, \quad (4)$$

$$\varepsilon Re Pr \left(u \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial r} \right) = \varepsilon^2 \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \mu \Phi, \quad (5)$$

where

$$\Phi = Br \left[2\varepsilon^2 \left(\frac{\partial u}{\partial z} \right)^2 + 2\varepsilon^2 \left(\frac{\partial v}{\partial r} \right)^2 + \left(\frac{\partial u}{\partial r} \right)^2 + 2\varepsilon^2 \left(\frac{v}{r} \right)^2 + 2\varepsilon^2 \frac{\partial v}{\partial z} \frac{\partial u}{\partial r} + \varepsilon^4 \left(\frac{\partial v}{\partial z} \right)^2 \right]. \quad (6)$$

We have employed the following non-dimensional quantities in Eqs. (2)-(6):

$$r = \frac{\bar{r}}{\varepsilon L}, z = \frac{\bar{z}}{L}, u = \frac{\bar{u}}{U}, v = \frac{\bar{v}}{\varepsilon U}, \varepsilon = \frac{a}{L}, \mu = \frac{\bar{\mu}}{\mu_0}, T = \frac{\bar{T} - T_a}{T_a}, P = \frac{a^2 \bar{P}}{\mu_0 UL}, \quad (7)$$

$$\alpha = mT_a, Br = \frac{\mu_0 U^2}{kT_a}, Pr = \frac{\mu_0 c_p}{k}, Re = \frac{\rho U a}{\mu_0}, Bi = \frac{ah}{k}.$$

where ρ is the fluid density, k is the thermal conductivity, $\bar{T}$ is the fluid temperature, U is the velocity scale, α is the viscosity variation parameter, wall temperature, $\bar{u}$ is the axial velocity, $\bar{v}$ is the normal velocity, c_p is the specific heat at constant pressure, $\bar{P}$ is the pressure, Pr is the Prandtl number, Br is the Brinkman number, Bi is the Biot number, h is the transfer coefficient, Re is the Reynolds number, $\bar{x}$ and $\bar{y}$ are distances measured in streamwise and normal direction respectively. Since the pipe is narrow and the aspect ratio $0 < \varepsilon \ll 1$, the lubrication approximation based on an asymptotic simplification of the governing equations (2) –(6) is invoked and we obtain,

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial u}{\partial r} \right) + O(\varepsilon), \quad (8)$$

$$0 = \frac{\partial p}{\partial r} + O(\varepsilon^2), \quad (9)$$

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \mu Br \left(\frac{\partial u}{\partial r} \right)^2 + O(\varepsilon), \quad (10)$$

where $\mu = 1/(1 + \alpha T)$. The dimensionless corresponding boundary conditions at the pipe wall is are the usual no slip condition for the fluid velocity together with the exchange of heat with the ambient following Newton's cooling law:

$$u = 0, \quad \frac{dT}{dr} = -BiT \quad \text{at} \quad r = 1, \quad (11)$$

and the regularity of the solution along the pipe centreline i.e.,

$$\frac{du}{dr} = \frac{dT}{dr} = 0 \quad \text{at} \quad r = 0. \quad (12)$$

3. Solution Method

Eqs. (8)–(10) subject to the boundary conditions can be easily combined to give

$$\frac{du}{dr} = -\frac{rG}{2}(1 + \alpha T), \quad \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{r^3 G^2 Br}{4}(1 + \alpha T) = 0, \quad (13)$$

where $G = -\partial P / \partial z$ is the constant axial pressure gradient. Eq. (13) with the corresponding boundary conditions is solved exactly and we have the solutions for fluid velocity and temperature profiles as;

$$\begin{aligned} u(r) = & \frac{1}{4} \left(G Bi \left(-2 r^2 \text{BesselJ} \left(0, \frac{1}{4} G \sqrt{\alpha Br} r^2 \right) \right. \right. \\ & - r^2 \pi \text{StruveH} \left(0, \frac{1}{4} G \sqrt{\alpha Br} r^2 \right) \text{BesselJ} \left(1, \right. \\ & \left. \frac{1}{4} G \sqrt{\alpha Br} r^2 \right) + r^2 \pi \text{StruveH} \left(1, \right. \\ & \left. \frac{1}{4} G \sqrt{\alpha Br} r^2 \right) \text{BesselJ} \left(0, \frac{1}{4} G \sqrt{\alpha Br} r^2 \right) + 2 \text{BesselJ} \left(0, \right. \\ & \left. \frac{1}{4} G \sqrt{\alpha Br} \right) + \pi \text{StruveH} \left(0, \frac{1}{4} G \sqrt{\alpha Br} \right) \text{BesselJ} \left(1, \right. \\ & \left. \frac{1}{4} G \sqrt{\alpha Br} \right) - \pi \text{StruveH} \left(1, \frac{1}{4} G \sqrt{\alpha Br} \right) \text{BesselJ} \left(0, \right. \\ & \left. \frac{1}{4} G \sqrt{\alpha Br} \right) \left. \right) \Bigg/ \left(-\sqrt{\alpha Br} G \text{BesselJ} \left(1, \frac{1}{4} G \sqrt{\alpha Br} \right) \right. \\ & \left. + 2 Bi \text{BesselJ} \left(0, \frac{1}{4} G \sqrt{\alpha Br} \right) \right) \end{aligned} \quad (14)$$

$$T(r) = \left(2 \text{BesselJ}\left(0, \frac{1}{4} G \sqrt{Br \alpha} r^2\right) Bi \right) / \left(\left(-\sqrt{Br \alpha} G \text{BesselJ}\left(1, \frac{1}{4} G \sqrt{Br \alpha}\right) + 2 Bi \text{BesselJ}\left(0, \frac{1}{4} G \sqrt{Br \alpha}\right) \right) \alpha \right) - \frac{1}{\alpha} \quad (15)$$

4. Thermal Stability Criterion

For the temperatures in the flow field to remain finite at any given value of $\alpha > 0$, the denominator of Eq. (15) should not vanish [10, 12]. The imposition of this restriction leads to the following thermal stability criterion

$$\frac{\text{BesselJ}\left(0, \frac{1}{4} G \sqrt{Br \alpha}\right)}{\text{BesselJ}\left(1, \frac{1}{4} G \sqrt{Br \alpha}\right)} \neq \frac{1}{2} \frac{G \sqrt{Br \alpha}}{Bi} \quad (16)$$

Equation (16) indicates that the thermal stability of the flow system depends not only on the convective cooling parameter but also on the viscous heating parameter and its pressure gradient as well as the parameter characterizing the fluid viscosity variation.

5. Results and Discussion

For the numerical validation of our results we have chosen physically meaningful values of the parameters entering into the problem. It is important to note that a positive increase in the parameter value of α indicates a decrease in the fluid viscosity while the convective cooling in the flow system is enhanced by increasing the Biot number (Bi). In Figs. 2-5, the axial velocity distributions are reported for increasing values of α , Bi , G and Br . Generally a parabolic velocity profile is observed with maximum value along the pipe centerline and minimum at the wall. The velocity increases with increasing values of α , Br and G but decreases with increasing values of Bi . Thus, a decrease in the fluid viscosity coupled with an increase in the viscous heating will enhance the flow velocity, although similar effect is observed by increasing the flow pressure gradient. However, it is noteworthy that an increase in convective cooling slows down the flow process.

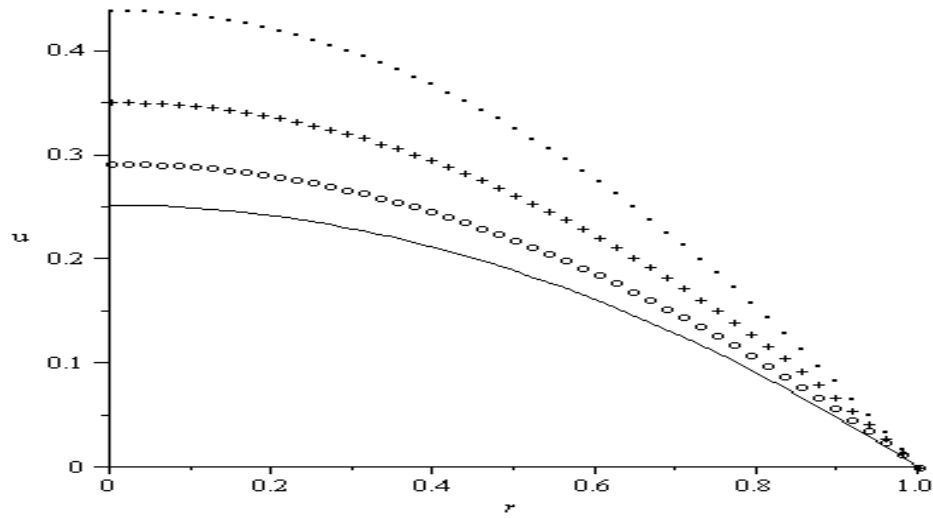


Fig. 2: Velocity profile: $G = 1$; $Br = 1$; $Bi = 1$; _____ $\alpha = 0.1$; ooooo $\alpha = 2$; +++++ $\alpha = 4$; $\alpha = 6$.

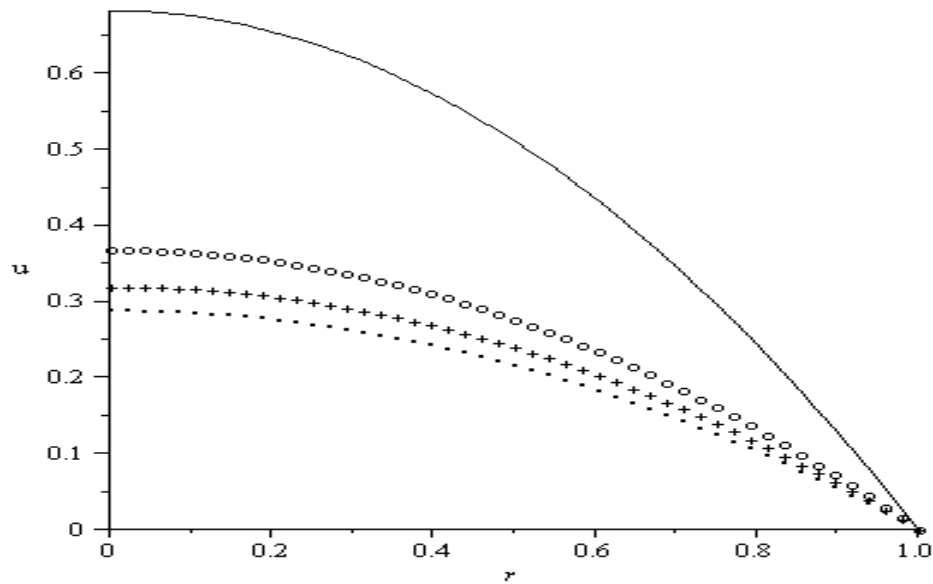


Fig.3: Velocity profile: $G=1$; $\alpha = 1$; $Br = 1$; _____ $Bi = 0.1$; ooooo $Bi = 0.2$; +++++ $Bi = 0.4$; $Bi = 0.5$

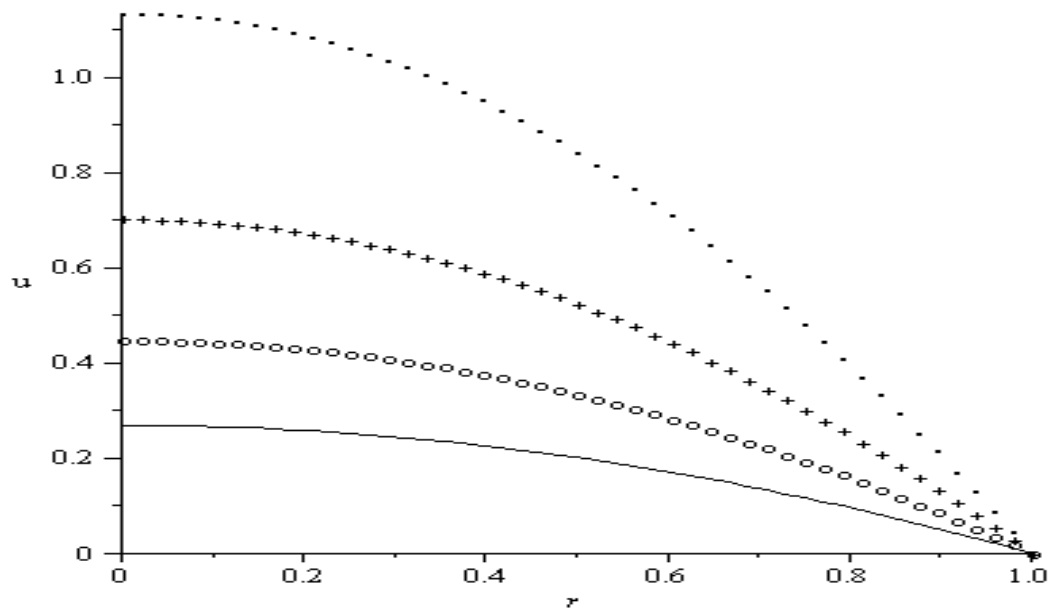


Fig.4: Velocity profile: $Br = 1$; $\alpha = 1$; $Bi = 1$; _____ $G = 1$; ooooo $G = 1.5$; ++++ $G = 2$; $G = 2.5$

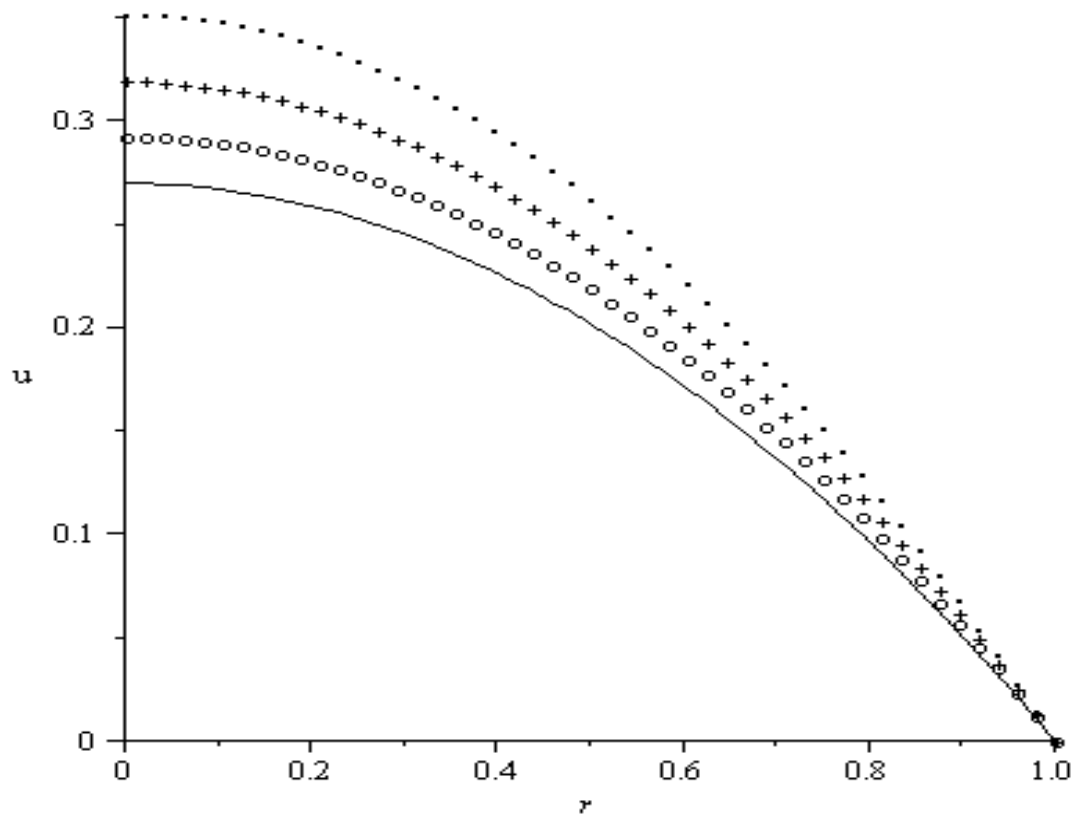


Fig.5: Velocity profile: $G = 1$; $\alpha = 1$; $Bi = 1$; _____ $Br = 1$; ooooo $Br = 2$; ++++ $Br = 3$; $Br = 4$

Typical variations of the fluid temperature profiles in the normal direction are shown in figs. 6-8. Generally, the fluid temperature is attained its peak value along the pipe centreline and decreases gradually towards the wall due to convective heat exchange with the ambient at the wall. However, the fluid temperature increases with increasing values of α , br and decreases with increasing values of biot number bi .

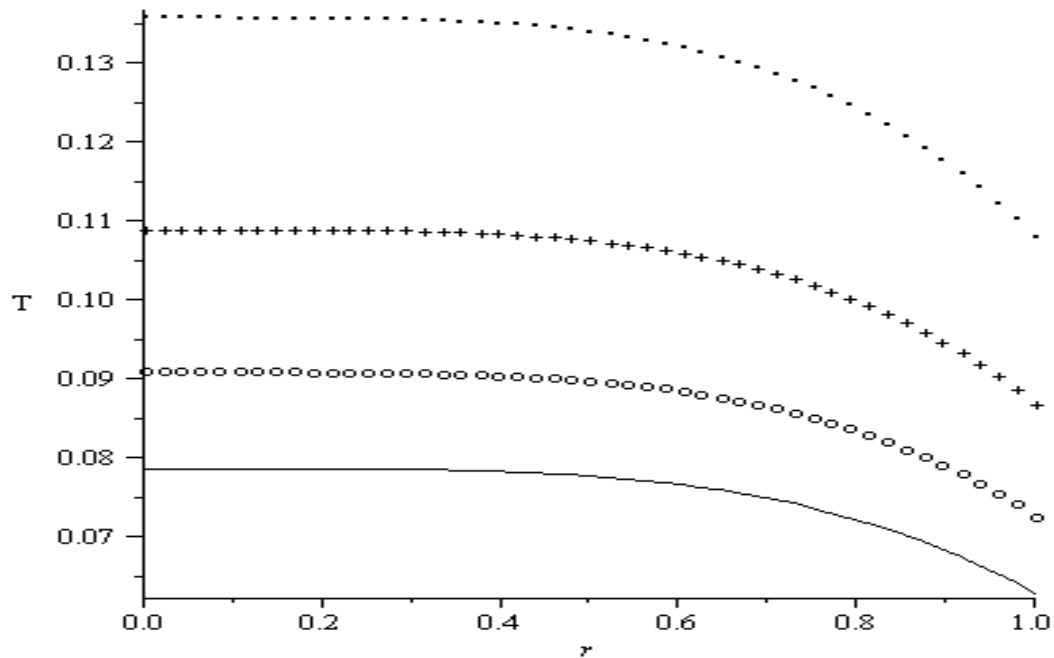


Fig.6: Temperature profile: $G = 1$; $Br = 1$; $Bi = 1$; _____ $\alpha = 0.1$; ooooo $\alpha = 2$; +++++ $\alpha = 4$; $\alpha = 6$.

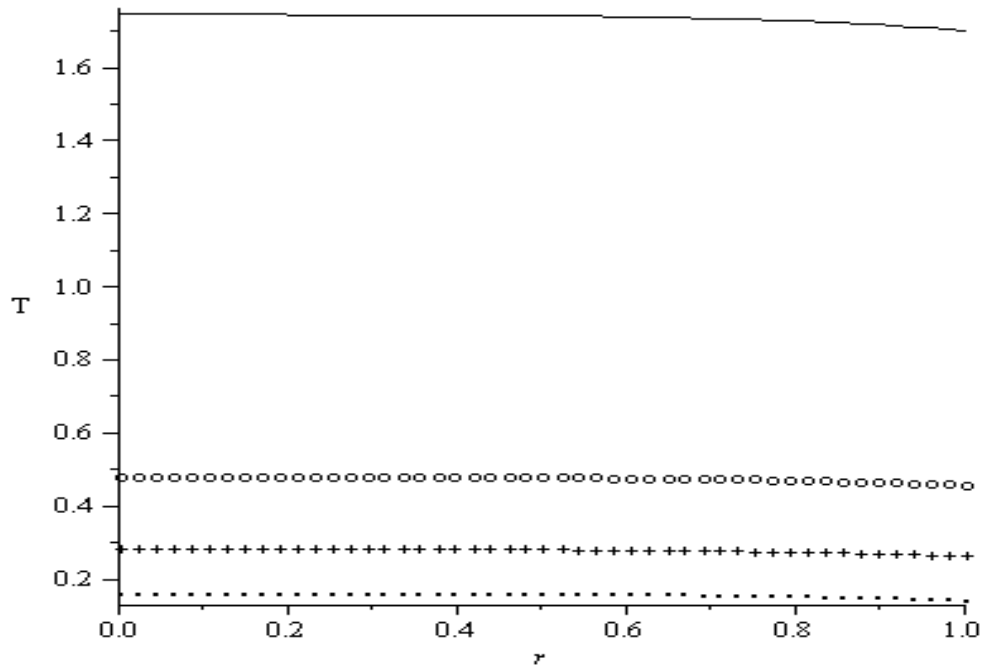


Fig.7: Temperature profile: $G=1$; $\alpha = 1$; $Br = 1$; _____ $Bi = 0.1$; ooooo $Bi = 0.2$; ++++ $Bi = 0.3$; $Bi = 0.5$

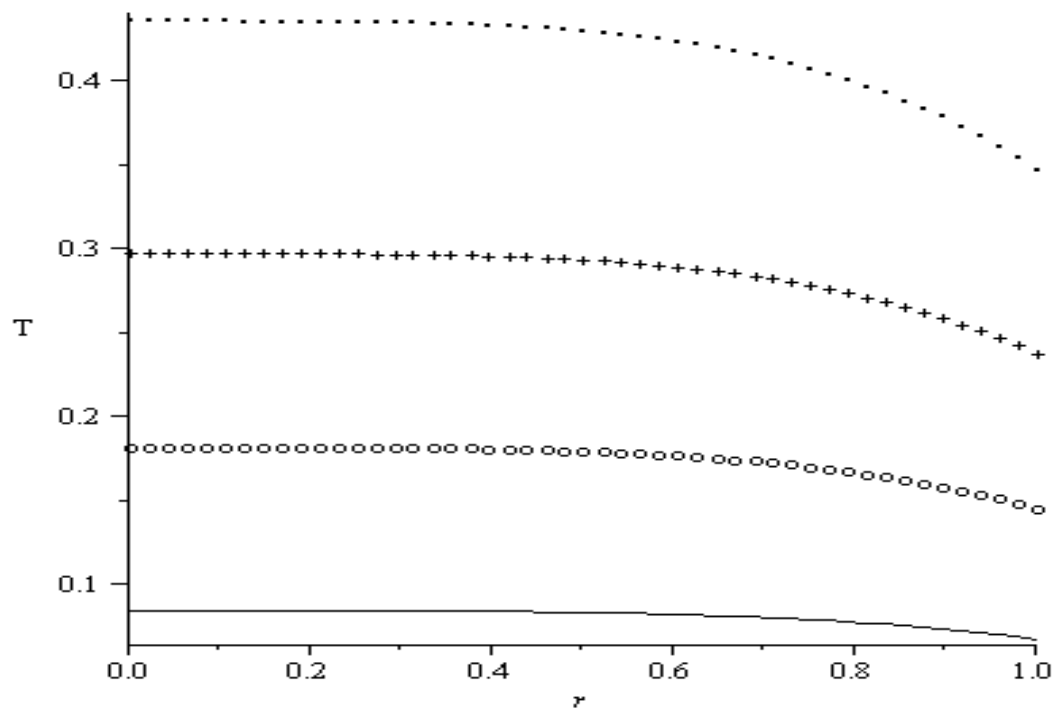


Fig.8: Temperature profile: $G=1$; $\alpha = 1$; $Bi = 1$; _____ $Br = 1$; ooooo $Br = 2$; +++++ $Br = 3$; $Br = 4$

6. Conclusions

In this paper, the effect of convective cooling on temperature dependent viscosity fluid flowing steadily in a cylindrical pipe is investigated. The velocity and temperature profiles are obtained and used to evaluate the thermal stability criterion. Both the fluid velocity and temperature increase with increasing values of α , G , Br and decreases with increasing value of Bi . The results will no doubt be of biological and engineering interest.

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EFFECT OF NOISE ON BLOOD PRESSURE

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ABSTRACT

The effect of noise on blood system is studied mathematically. Mathematical models are built that can be used to study the flow of blood, and it is known that blood flow at the Aorta and Vena Cava, as well as at the arteries and veins is affected by the heartbeat. However, at the venule, arterioles and capillaries, the walls are in elastic, hence not affected by the heartbeat and flow is regarded as non-pulsatile. We observed computed results for pressure variation of the different arteries and veins with respect to their radii, and venule, and capillaries are constant. The effect of constriction at venule and capillaries is not established. A pressure level varies directly with time.

Keyword words

Noise, blood system, transmission.

Mathematics Subject Classification 2000:92C35 &92C17

1. INTRODUCTION

Sound level greater than 85 DB triggers off danger signal in the brain, such that necessary glands have to prepare the body, to take up the impending challenge. This goes to show that the rise in the blood pressure and also heartbeat is in response to the noise, see Mbah et al (2004).

Much work has been done on the flow of blood through the artery, among which are those of Metea et al (2006), Schonfdder et al (1978), Butryn et al (1995), Hauck et al (2004), Takano et al (2006), Berg et al (1997) etc. We shall, for convenience here, consider blood as homogeneous and Newtonian. These assumptions are made possible by the size of the diameter of the artery, as compared to a much smaller diameter of the red blood cells, contained in the blood plasma. It might be necessary for us to state here the a typical blood plasma consists of water, the red blood cells, the white blood cell, the platelets and other constituents.

However, the cell that plays vital role in pressure build-up is the red blood cell, because of its demand by the cells in the body. We all know that cells in the body require oxygen for their normal activities, and this is provided by the blood through the red blood cells, which combine with oxygen to form the oxygenated blood at the lungs, which are eventually returned to the heart for circulation. When the blood is pumped out from the heart, it flows down from the aorta to the lesser sized arteries, until it gets to the capillary bed. It is here at the capillary bed that the blood pressure of the body is determined. This is because the velocity of flow at the capillary bed (usually referred to as creeping flow) is very small, (about 0.025cm/s) as compared to the velocity at the aorta and larger arteries (about 2.5cm/s). Because of this, therefore, we consider the flow of blood in the arteries with particular emphasis on the capillary bed.

Since the artery is considered as cylindrical and shown as:

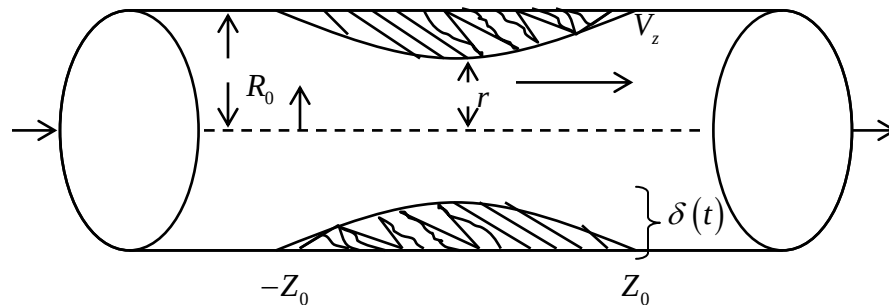


Fig. 1.0:A diagram of an artery

Where R_0 is the radius of the artery at the unstricted region, and r is the radius at the clumped/constricted region, V_z is the axial velocity of flow while Vr is the radial velocity of flow of the blood.

We know that the artery is Hookean elastic and, therefore, constricts. The constriction is usually as a result of the rhythmic nature of the heartbeat. Thus, this means that the flow of blood through the artery is affected by the rhythmic nature of the heartbeat. This is why the blood flow through the artery is considered as a pulsatile flow. The radius of the artery due to this rhythmic constriction can be determined. See (9).

$$r = R_0 \left[1 - \frac{\delta(t)}{2R_0} \left(1 + \cos \pi \frac{z}{z_0} \right) \right] \dots\dots\dots 1.0$$

where $\delta(t) = \delta_m \left(1 - e^{-t/s}\right)$, δ_m = maximum height of constriction attainable, r = rate of increase

in constriction, z_0 = radius of the constriction, and z the point in constricted region that is of interest.

Since the artery is considered cylindrical and blood flows through it, we can study the flow pattern in the artery by using the Navier-Stokes equation for the study in the cylindrical tubes. Thus we have the equations:

$$\rho \partial V_r / \partial t = -\partial p / \partial r + \mu \left(\partial^2 V_r / \partial r^2 + 1/r \partial V_r / \partial r - V_r / r^2 + \partial^2 V_r / \partial z^2 \right) \quad 1.2$$

$$\rho \partial V_z / \partial t = -\partial p / \partial z + \mu \left(\partial^2 V_z / \partial z^2 + 1/r \partial V_z / \partial r + \partial^2 V_z / \partial r^2 \right) \quad 1.3$$

with the continuity equation

$$\partial V_r / \partial r + \partial V_z / \partial z + V_r / r = 0 \quad 1.4$$

where ρ = density of the blood, μ = blood viscosity which we shall assume to be greater than the inertia force.

Many studies carried out on blood flow, have been done using these equations, though with modifications as the case may be. Equation (1.2) describes the radial flow of blood in the artery, equation (1.3) describes the axial flow, while equation (1.4) as stated already, is the equation of continuity. Because of the cylindrical nature of the artery, one expects that there should be a component equation for the angular flow. In theory, this exists, but is of no practical importance, as it has been shown to have no significant contribution, to flow pattern when neglected.

We shall not, in particular, be interested in the flow of blood at this region of the artery. We shall rather be interested in the flow at the capillary bed. We have to state here, that as we go down the artery to the capillary region (arterioles) loses their elasticity and become rigid. Also at this region, the diameter of the artery becomes extremely small (about $8\mu m$), such that the flow of blood past this region becomes very slow. Hauck et al (2004), had considered the flow of blood through a constricted capillary, and had shown that when the diameter of the capillary is less than $8\mu m$, the red blood cells will not pass through

such capillary, thereby causing possible blockage of such capillary. In addition, there could be complications at the region beyond such points in the human system. Therefore, at the capillary bed, because of its distance from the heart, the flow will be steady and no longer pulsatile; the wall of the capillary is ridged and inelastic. Also, for the red blood cells to pass, they need to deform. It is this time taken in the deformation of the red blood cells that eventually lower the velocity of flow. This reduced velocity eventually builds up pressure in the artery.

According to Takano et al (2006), Metea (2006), at such level of blood flow, the size of the red blood cells affect the flow. Thus, the fluid (which is the entire blood plasma) will be considered Newtonian and incompressible with suspended deformable bodies (red blood cells), whose shape at any instant depend on the flow field around it. Because of the loss of elasticity of the artery here, we modify the Navier-strokes equation, in which case now, the viscous term dominates over the inertia term, so that our model equations from (1.3) becomes:

$$\partial p / \partial z = \mu \left(\partial^2 u / \partial r^2 + 1/r \partial u / \partial r \right) + K N_0 (v - u) \quad 1.5$$

$$m \partial v / \partial t = K (u - v) \quad 1.6$$

where u and v are the velocities of flow of the plasma and the red blood cells respectively, r is the radius of the capillary, K is the strokes resistance coefficient, which for spherical objects of radius 'a' is given as $K = 6\pi\mu a$; μ is the viscosity of the blood; N_0, m is the number density, and mass of the red blood cells respectively, while P is the pressure in the flow.

Equations (1.5) and (1.6) were solved for a case, when the constriction cause by the red blood cells is fully developed. Where the above equations are non-dimensionalized, by using the following dimensionless variables:

$$r = \bar{r} / R_0, z = \bar{z} / z_0, u = v\mu / V_0^2, p = PR_0 / V_0^2, t = \bar{t}V_0 / R_0 \text{ and } \delta = \bar{\delta} / R_0$$

where V_0 is characteristic velocity of the plasma. In the blood flow capillary, the red blood flow past the capillary is shown in the diagram below

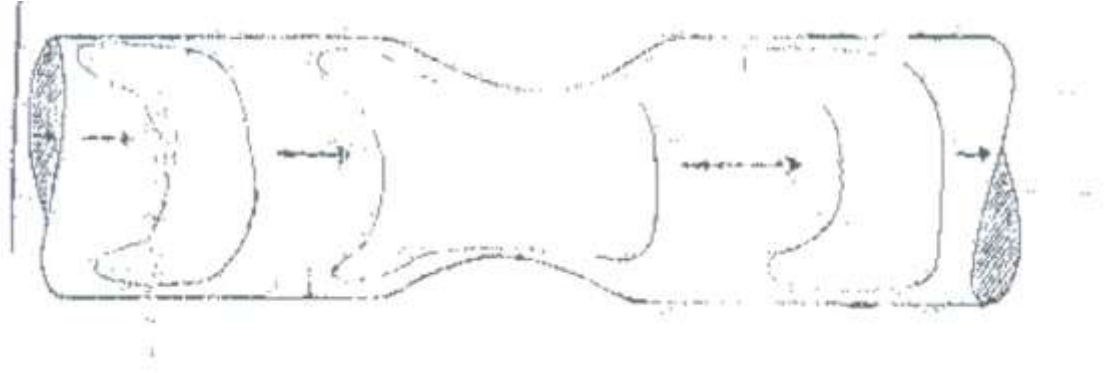


Fig. 1.1: The red blood flow past the capillary

Consequently even at non-constricted region, the flow is dragged because of the relative equal diameter of the red blood cells and the capillary. Hence, using the dimensionless variable, equations (1.5) and (1.6) change to:

$$\partial p / \partial z = \partial^2 u / \partial r^2 + 1/r \partial u / \partial r + \alpha(V - u) \quad (1.7)$$

$$\delta v / \delta t = (U - V) / \eta \quad (1.8)$$

where $\alpha = R_0^2 K N_0 / \mu, \eta = V_0 M / K R_0 R_0 \approx 0(Z_0)$.

If we expect that the red blood cells at the constricted region will finally deform to the shape of a cylinder, then the Stroke's coefficient $K = 3A\mu / (r+1)$, where $A = 2\pi r(r+1)$ for a cylinder (i.e. the surface area of a cylinder). Note that, no matter the deformation level of the red blood cells, the surface area remains the same.

Using the initial condition for the flow as:

$$u - v = \partial v / \partial t = 0 \text{ for } t < 0 \text{ and } \partial v / \partial t = 0 \text{ at } t > 0$$

We rearrange (1.7) as:

$$\partial p / \partial z = \partial^2 u / \partial r^2 + 1/r \partial u / \partial r - (\alpha u + b^2 u / r^2) + b^2 u / r^2 + \alpha v_z \quad 1.9$$

we choose

$$\partial^2 u / \partial r^2 + 1/r \partial u / \partial r - (\alpha u + b^2 u / r^2) = 0$$

1.10

which is Bessel's differential equation of order b, and its solutions is Bessel function of the form.

$$u(r) = AI_b(r/\alpha) + BK_b(r\sqrt{\alpha})$$

1.11

Since the velocity in the artery (i.e. axial flow) is finite, then B is zero.

Hence:

$$u(r) = AI_b(r\sqrt{\alpha}) = A \sum \left(\frac{1}{2} r \sqrt{\alpha} \right)^{b+2s} / s! \Gamma(b+s+1) \quad 1.12$$

which satisfy the equation (1.10)

The equation (1.7) and (1.8) transform to

$$\partial p / \partial z = b^2 u / r^2 + \alpha v_z \quad 1.13$$

$$\partial v / \partial t + v_z / \eta = u / \eta \quad 1.14$$

solving equation (1.14) yields:

$$\begin{aligned} v_z &= u / \eta \int_0^t e^{-1/\eta(t-s)} \delta s + C(r) e^{-t/\eta} \\ &= u / \eta \cdot 1/1/\eta e^{-1/\eta(t-s)} \Big|_0^t = -u \left(1 - e^{-1/\eta t} \right) + C(r) e^{-1/\eta} \end{aligned}$$

Hence:

$$v_z = -u \left(1 - e^{-t/\eta} \right) + C(r) e^{-t/\eta} \quad 1.15$$

If we assume that $V_z(r, 0) = 0$, then $C(r) = 0$

Hence equation (1.15) becomes:

$$V_z = -u(1 - e^{-t/\eta}) = -A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} (1 - e^{-t/\eta}) \quad 1.16$$

Substituting equations (1.12) and (1.15) in (1.13) yields:

$$\begin{aligned} \frac{\partial p}{\partial z} &= b^2 / r^2 A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} (1 - e^{-t/\eta}) - A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} \alpha (1 - e^{-t/\eta}) \\ &= A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} \left\{ b^2 / r^2 - \alpha (1 - e^{-t/\eta}) \right\} \\ &= A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} \left\{ \frac{b^2 r^{b+2s}}{r^2} - r^{b+2s} [\alpha (1 - e^{-t/\eta})] \right\} \\ &= \beta [b^2 r^{b+2(s-1)} - \lambda r^{b+2s}] \end{aligned} \quad 1.17$$

$$\text{where } \beta = A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} \text{ and } \lambda = \alpha (1 - e^{-t/\eta})$$

Since we are only interested in the constricted region, the order of the Bessel function must be $b > 2$ at $t = 0$; the velocity temporarily goes to zero, so as to enable the red blood cell assume a new velocity.

Here $b = 2 + \varepsilon$, where $\varepsilon = \frac{1}{2}$

We expand as follows:

$$\frac{\partial p}{\partial z} = b^2 / r^2 A \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2} r \sqrt{\alpha}\right)^{b+2s}}{s! \Gamma(b+s+1)} [b^2 r^{b+2(s-1)} - \lambda r^{b+2s}]$$

where $s = 0$ and $b = 2 \frac{1}{2}$

$$\frac{\partial p}{\partial z} = -\frac{A\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{5}{2}}}{\Gamma\left(3\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{1}{2}} - \lambda r^{\frac{5}{2}}\right]$$

where $s = 1$ and $b = 2\frac{1}{2}$

$$\frac{\partial p}{\partial z} = -\frac{A\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{9}{2}}}{\Gamma\left(4\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{5}{2}} - \lambda r^{\frac{9}{2}}\right]$$

where $s = 2$ and $b = 2\frac{1}{2}$

$$\frac{\partial p}{\partial z} = -\frac{A\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{13}{2}}}{2\Gamma\left(5\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{9}{2}} - \lambda r^{\frac{13}{2}}\right]$$

$$\begin{aligned} \therefore \frac{\partial p}{\partial z} &= -A\left[\frac{\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{1}{2}} - \lambda r^{\frac{5}{2}}\right] + \frac{\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{9}{2}}}{\Gamma\left(4\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{5}{2}} - \lambda r^{\frac{9}{2}}\right] + \frac{\left(\frac{1}{2}\sqrt{\alpha}\right)^{\frac{13}{2}}}{2\Gamma\left(5\frac{1}{2}\right)}\left[\left(\frac{5}{2}\right)^2 r^{\frac{9}{2}} - \lambda r^{\frac{13}{2}}\right]\right] \\ &= -A\left[0.4298\left\{6.25r^{\frac{1}{2}} - \lambda r^{\frac{5}{2}}\right\} + 0.0938r^{\frac{5}{2}} - \lambda r^{\frac{9}{2}} + 0.0159\left\{6.25r^{\frac{9}{2}} - \lambda r^{\frac{13}{2}}\right\} + \dots\right] \dots\dots\dots \\ &\dots\dots\dots 1.18 \end{aligned}$$

In order to analyze the effect of constriction, we obtain certain figures from models for blood flows by Kapur J. N. (1989) as follows: $R_0 = 0.0004cm$, $z = 0$, $z_0 = 0.0004$, Mass of red blood cell (m) = 8.57×10^9 per / cc, $K = 0.0001206$, calculated α and η from the figures gives 3.0552 and 17765.34 respectively. Hence, we draw the graphs as shown below, which is pressure variation axially at various radii at different times t .

ANALYSIS AND DISCUSSION

It is known that blood flow at the aorta and vena cava, as well as at the arteries and veins, is affected by the heartbeat. Thus, the flow of the blood at these regions are pulsatile. However, at the veinoules, arterioles and capillaries, the walls are in-elastic (i.e. Rigid walls), hence not affected by the heartbeat, and the flow is regarded as non-pulsatile.

We observed computed results for the pressure variation of the different arteries and veins, with regard to their radii. We observed that the pressure variation at the arterioles, venules and the capillaries are constant, respectively. This can be interpreted to mean that, at these points, the pressure variation has stabilized,

since the walls are rigid due to their in-elastic nature. Secondly, the effect of constriction at these regions is not established, and this goes to show that there may not be the presence of stenosis at these regions (Mbah, 1998). Figure 3, shows the negative of the blood pressure variation levels against time. we observe that negative blood pressure variation levels varies directly as time increases, which is in agreement with the theory works of Metea et al (2006) on analysis of Pulsatile blood flow (through Stenosed arteries and its application to cardio vascular diseases).

The same thing can be said of figures 1,2 and 4 figure 5 shows the comparison of blood pressure variation levels, at the aorta and vena cava. The graph shows two parallel horizontal lines. A good look at figures 4 and 5 show differences in pressure variation levels. The difference can be explained by the fact that the vein has less thick wall than the arteries, and such can easily show the effect of pressure variation on it. It is also to be noted that bursting of the neural system occurs mainly in the veins, and this is due to the reason put forward above.

$$p(t) = - \left[\begin{aligned} &0.4298 \left[6.25x^{0.5} - \alpha \left(1 - \exp\left(\frac{-t}{\eta}\right) \right) x^{\frac{5}{2}} \right] + 0.938 \left[6.25x^{\frac{5}{2}} - \alpha \left[\left(1 - \exp\left(\frac{-t}{\eta}\right) \right) x^{\frac{9}{2}} \right] \right] \dots \\ &+ 0.0159 \left[6.25x^{\frac{9}{2}} - \alpha \left[\left(1 - \exp\left(\frac{-t}{\eta}\right) \right) x^{\frac{13}{2}} \right] \right] \end{aligned} \right]$$

$$\alpha = 3.0552 \quad \eta = 17765.34$$

$x = 0.2$ Blood Pressure Level at the Arteries
Veins

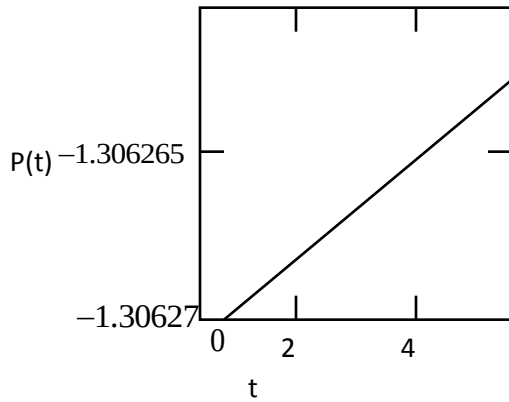


Fig. 1

$x = 0.25$ Blood Pressure Levels at the

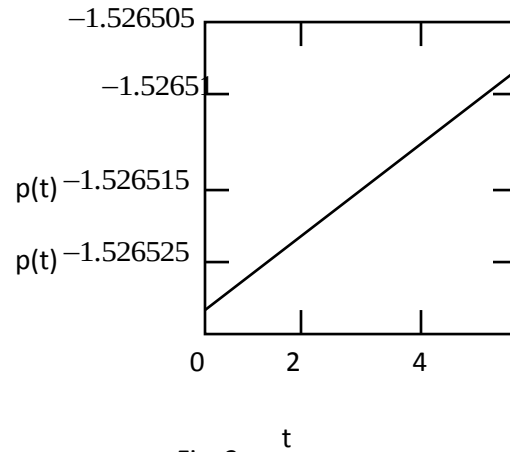


Fig. 2

$x = 1.25$ Blood Pressure Levels at the Aorta

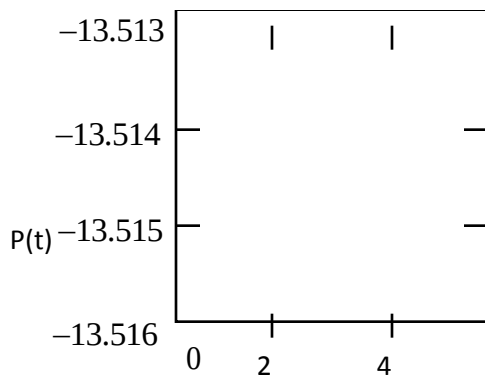


Fig. 3

$x = 1.5$ Blood Pressure Levels at the Vena-Cava

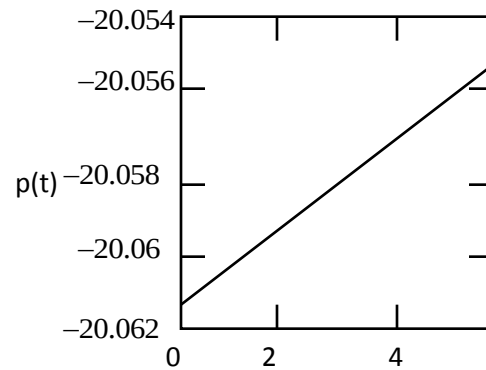


Fig. 4

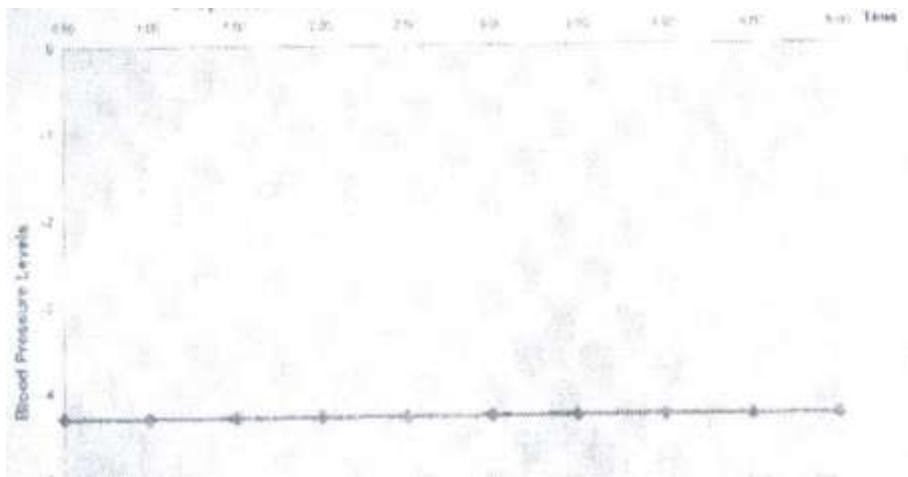


Fig. 5

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THE FLUID MECHANICS OF THE COCHLEA DUE TO NOISE

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ABSTRACT

The spaces above and below the basilar membrane are filled with a non-viscous, incompressible fluid which the velocity potential was calculated. The mass of the fluid in the cochlear canals also favours a topographical selection: a low frequency sets more fluid in motion than a high one, and a greater mass of fluid needs to be displaced to vibrate the basilar membrane at the apex than at the base of the cochlear .

Mathematics Subject Classification 2000:92C35,92C17 & 35Q30

INTRODUCTION

The cochlea is the part of the inner ear, which is a small fluid -filled chamber, and contains the biological structures that convert mechanical signals into neural signals. In addition to the signal conversion, it does process signals. Thus, a clear understanding of the mechanism requires that we understand the cochlea fully as it relates to audition. For more details see Barbel et al (2), Lesser & Berkley (7), Ranke, (10), Lamb (6).

In modeling the fluid motion in the cochlea the following assumptions are made:

- (1) The model is a two-dimensional model in an enclosed cavity containing a structure of spatially variable elastic properties.
- (2) The spiral cochlea is unwound.
- (3) The central duct in the cochlea which contains the organ of corti and which is enclosed by Reissner's membrane and basilar membrane will be represented as a single elastic partition.
- (4) The mechanical properties of each partition are represented by the assumption that each point acts as a damped harmonic oscillator point to point, coupling being only through the surrounding fluid. This assumption leads to representing the partition by a mechanical impedance $z(x_1, \omega)$, x_1 being the distance from the oval window along the partition.
- (5) The endolymph is considered incompressible for it has the same sound speed as water, which is likely; the wavelength of an acoustic signal at 500Hz (at high frequency for hearing) is about 30cm while the cochlea is only 35mm.

- (6) The fluid flow will be considered inviscid, though we shall regard this as a first step in an expansion procedure.
- (7) The endolymph is considered inviscid.

From all these points, we have that the flow pattern in a cochlea model excited by an oscillatory disturbance exhibits a steady streaming motion as well as motion typical of a fluid with a free surface. As the excitation is purely oscillatory, the steady motion must result from a non-linear interaction, see Pain (9). From Batchelor (1), we see that the governing parameter for the shearing effect is the Stroahal number,

$S = \omega L / u_{\infty}$ where L , typical length, and u_{∞} is the velocity amplitude of the driving oscillation.

The important fact from the above is that the streaming motion only affects the flow significantly after a number of acoustic periods. The assumption that basilar membrane motion is primarily controlled by the potential flow is used. A linearized theory is adequate on the true scale of an acoustic period, since for the range of frequencies of interest in auditory perception, physical measurement shows Bekesy (11) the maximum basilar membrane slope to be sufficiently small. Consistency of the numerical result (that is, those also showing small membrane slope), supported use of the linearized equations. These considerations lead us to postulate as a reasonable mathematical model or perhaps better analogue of the cochlea, potential cavity flow model presented below.

The model is an enclosed two-dimensional cavity and the basilar membrane appears in it as thin plate immersed in the fluid. The flexural deformation of the basilar membrane is derived from the theories of the elastic plate.

The linear short-time scale aspect of the cochlea behaviour is considered. Thus, we assume linearized two-dimensional potential flow in the configuration depicted in figure 1 below.

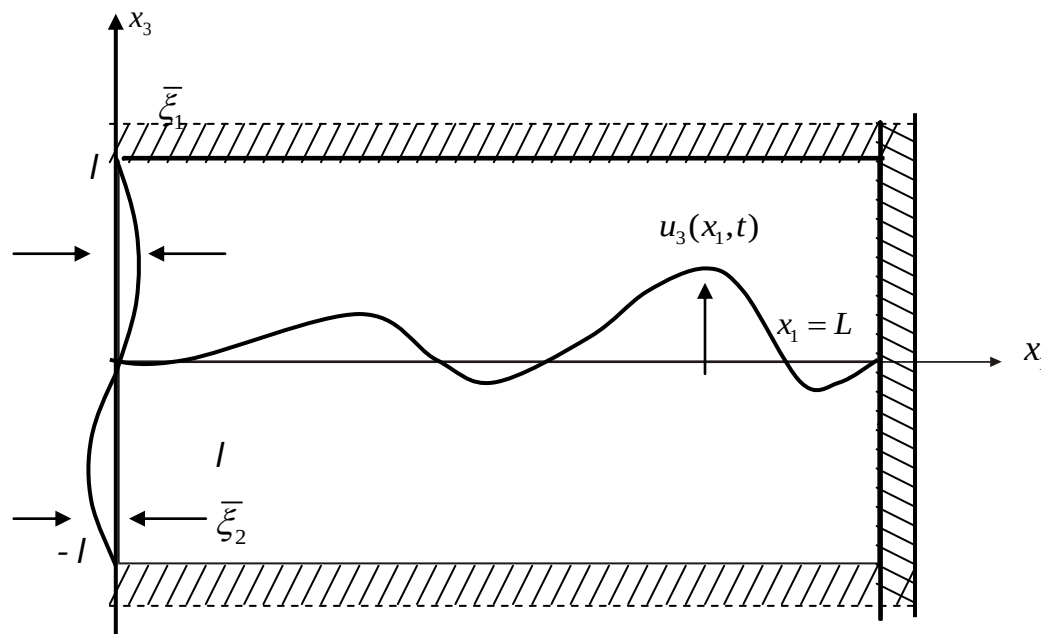


Fig 1 Potential flow model of the cochlea

THE MATHEMATICAL MODEL

We also begin by writing down the equations of motion of non-viscous, incompressible fluid. Thus, the equations characterizing fluid motion are:

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_3} = 0 \quad (1.1)$$

$$\rho \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + v \frac{\partial u}{\partial x_3} \right\} = - \frac{\partial p_1}{\partial x_1} \quad (1.2)$$

$$\rho \left\{ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + v \frac{\partial v}{\partial x_3} \right\} = - \frac{\partial p_2}{\partial x_3} \quad (1.3)$$

Where ρ is the fluid density and p_i , is the fluid pressure, $i = 0, 1, 2$. Since we are interested in the case of small amplitude motion, we neglect the product terms in equation (1.2) and (1.3) to obtain

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p_1}{\partial x_1} \quad (1.4)$$

$$\rho \frac{\partial v}{\partial t} = - \frac{\partial p_2}{\partial x_3} \quad (1.5)$$

The potential is $\bar{\phi}$ with $\nabla \bar{\phi} = (\bar{u}, \bar{v})$, where $\bar{u}$ and $\bar{v}$ are the x_1 and x_3 fluid velocity components.

Now, let

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \bar{\phi}}{\partial x_1} \right) = \frac{\partial}{\partial x_1} \left(\frac{\partial \bar{\phi}}{\partial t} \right) \quad (1.6)$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \bar{\phi}}{\partial x_3} \right) = \frac{\partial}{\partial x_3} \left(\frac{\partial \bar{\phi}}{\partial t} \right) \quad (1.7)$$

Equation (1.1) is satisfied identically, by introducing the potential function $\bar{\phi}(x_1, x_3)$ such that

$$u = \frac{\partial \bar{\phi}}{\partial x_1} \text{ and, } v = \frac{\partial \bar{\phi}}{\partial x_3} \quad (1.8)$$

Hence, substituting (1.8) into (1.1), gives

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_3} = \frac{\partial^2 \bar{\phi}}{\partial x_1^2} + \frac{\partial^2 \bar{\phi}}{\partial x_3^2} = \nabla_{(1)}^2 \bar{\phi} = 0 \quad (1.9)$$

It then follows that in the upper and lower chambers,

$$\nabla_{(1)}^2 \bar{\phi}_1 = \nabla_{(1)}^2 \bar{\phi}_2 = 0 \quad (1.10)$$

Where $\nabla_{(1)}^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$.

upon substitution of equations (1.6) and (1.7) into (1.4) and (1.5) respectively, we obtain:

$$\frac{\partial}{\partial x_1} \left(\frac{\partial \bar{\phi}}{\partial t} \right) + \frac{1}{\rho} \frac{\partial \bar{p}_1}{\partial x_1} = \frac{\partial}{\partial x_1} \left(\frac{\partial \bar{\phi}}{\partial t} + \frac{\bar{p}_1}{\rho} \right) = 0 \quad (1.11)$$

$$\frac{\partial}{\partial x_3} \left(\frac{\partial \bar{\phi}}{\partial t} \right) + \frac{1}{\rho} \frac{\partial \bar{p}_2}{\partial x_3} = \frac{\partial}{\partial x_3} \left(\frac{\partial \bar{\phi}}{\partial t} + \frac{\bar{p}_2}{\rho} \right) = 0 \quad (1.12)$$

In the lower and upper chambers equations (1.11) and (1.12) reduce to

$$\rho \frac{\partial \bar{\phi}_1}{\partial t} + \bar{p}_1 = 0 \quad (1.13)$$

$$\rho \frac{\partial \bar{\phi}_2}{\partial t} + \bar{p}_2 = 0 \quad (1.14)$$

As a model of the basilar membrane, on $x_3 = 0$, $0 < x_1 < L$

(See figure 1) the equation of flexural vibration of the basilar membrane is:

$$\alpha_3 \nabla^2 \bar{u}_3 - \rho \ddot{\bar{u}}_3 = \bar{\rho}_2(x_1, 0, t) - \bar{\rho}_1(x_1, 0, t) \quad (1.15)$$

Where $\nabla^2 = \frac{\partial^2}{\partial x_1^2}$ and $-F_3 = \bar{\rho}_2(x_1, 0, t) - \bar{\rho}_1(x_1, 0, t)$ is the load on

The Basilar membrane and the boundary condition on the Basilar membrane is given by:

$$\frac{\partial \bar{u}_3}{\partial t} = \frac{\partial \bar{\phi}_1}{\partial x_3}, \quad \frac{\partial \bar{u}_3}{\partial t} = \frac{\partial \bar{\phi}_2}{\partial x_3} \quad (1.16)$$

On $x_1 = 0$, $0 < x_3 < l$, the equation of motion of the oval window is given by:

$$m_0 \frac{\partial^2 \bar{\xi}_1}{\partial t^2} + r_0 \frac{\partial \bar{\xi}_1}{\partial t} + k_0 \bar{\xi}_1 = \bar{\rho}_0 - \bar{\rho}_1(0, x_3, t) \quad (1.17)$$

The velocity at the oval window and that of the fluid at the point of contact is given by:

$$\frac{\partial \bar{\xi}_1}{\partial t} = \frac{\partial \bar{\phi}_1}{\partial x_1}, \text{ on } x_1 = 0 \quad (1.18)$$

On $x_1 = 0$, $-l < x_3 < 0$, the equation of motion of the round window is given by:

$$m_0 \frac{\partial^2 \bar{\xi}_2}{\partial t^2} + r_0 \frac{\partial \bar{\xi}_2}{\partial t} + k_0 \bar{\xi}_2 = -\bar{p}_2(0, x_3, t) \quad (1.19)$$

The velocity at the round window and that of the fluid at the point of contact is given by:

$$\frac{\partial \bar{\xi}_2}{\partial t} = \frac{\partial \bar{\phi}_2}{\partial x_1}, \text{ on } x_1 = 0 \quad (1.20)$$

$\bar{\xi}_1$ and $\bar{\xi}_2$ are the displacements of the oval and round windows respectively. Equations (1.17 – 1.20) valid on $x_1 = 0$ are equations of motion of the oval and round windows with their boundary conditions respectively, which were adapted from the work of Lesser and Berkley, (7). The other boundary conditions are:

For $x_1 = L$, we take

$$\frac{\partial \bar{\phi}_1}{\partial x_1} = \frac{\partial \bar{\phi}_2}{\partial x_1} = 0 \quad (1.21)$$

For $x_3 = l$,

$$\frac{\partial \bar{\phi}_1}{\partial x_3} = 0 \quad (1.22)$$

For $x_3 = -l$,

$$\frac{\partial \bar{\phi}_2}{\partial x_3} = 0 \quad (1.23)$$

The constant values of the parameter for the oval and round windows are denoted by m_0 , mass per unit area, r_0 , damping in dyne sec/cm³ and k_0 , stiffness in dyne /cm³. We seek a solution such that the field variables will be proportional to $e^{i\omega t} = e^{st}$.

We write

$$\phi = \text{Re}(\bar{\phi} e^{st}), u_3 = \text{Re}(\bar{u}_3 e^{st}), p_i = \text{Re}(\bar{p}_i e^{st}), \xi = \text{Re}(\bar{\xi} e^{st}).$$

The equations of motion of fluid now become

$$\nabla_{(1)}^2 \phi_1 = 0, \quad \nabla_{(1)}^2 \phi_2 = 0 \quad (1.24)$$

$$p_1 + \rho s \phi_1 = 0, \quad p_2 + \rho s \phi_2 = 0 \quad (1.25)$$

$$\text{On } x_3 = 0, \quad 0 < x_1 < L$$

$$\alpha_3 \nabla_{(1)}^2 u_3 - \rho s^2 u_3 = p_2(x_1, 0) - p_1(x_1, 0) \quad (1.26)$$

The boundary conditions (1.16) become

$$s u_3 = \frac{\partial \phi_1}{\partial x_3}, \quad s u_3 = \frac{\partial \phi_2}{\partial x_3} \quad (1.27)$$

at the boundary, $x_3 = 0$

On $x_1 = 0, \quad 0 < x_3 < l$,

$$m_0 s^2 \xi_1 + r_0 s \xi_1 + k_0 \xi_1 = p_0 - p_1(0, x_3) \quad (1.28)$$

with the boundary condition

$$s \xi_1 = \frac{\partial \phi_1}{\partial x_1} \quad (1.29)$$

On $x_1 = 0, \quad -l < x_3 < 0$,

$$m_0 s^2 \xi_2 + r_0 s \xi_2 + k_0 \xi_2 = -p_2(0, x_3) \quad (1.30)$$

with boundary condition

$$s \xi_2 = \frac{\partial \phi_2}{\partial x_1} \quad (1.31)$$

$$\text{For } x_1 = L, \quad \frac{\partial \phi_1}{\partial x_1} = \frac{\partial \phi_2}{\partial x_1} = 0$$

$$\text{For } x_3 = l, \quad \frac{\partial \phi_1}{\partial x_3} = 0, \quad \text{for } x_3 = -l, \quad \frac{\partial \phi_2}{\partial x_3} = 0$$

The outer walls are rigid and the basilar membrane is an elastic plate where undisturbed position is the plane $x_3 = 0$, which is also a plane of symmetry. The spaces above and below the basilar membrane are filled with a non-viscous, incompressible fluid. The basilar membrane is light and taut at the basal end of the cochlea, it is thick and loose near its apex.

We assume the end $x_1 = 0$ is under stress and $x_1 = L$ is stress free and as such

$$\left. \frac{\partial \bar{u}_3}{\partial x_1} \right|_{x_1=L} = 0 \quad (1.32)$$

POTENTIAL FLOW SOLUTIONS

We seek solutions to the equation (1.24), viz.

$$\nabla_{(1)}^2 \bar{\phi} = \frac{\partial^2 \bar{\phi}}{\partial x_1^2} + \frac{\partial^2 \bar{\phi}}{\partial x_3^2} = 0 \quad (2.1)$$

Equation (2.1) is a second order linear partial differential equation with constant coefficients, and is of the type designated in theory of such equation as elliptic.

Separation leads to formal solutions

$$\bar{\phi}_1(x_1, x_3) = (k_1 \cos \lambda x_1 + k_2 \sin \lambda x_1)(A_1 \cosh \lambda x_3 + B_1 \sinh \lambda x_3) \quad (2.2)$$

$$\bar{\phi}_2(x_1, x_3) = (\bar{k}_1 \cos \lambda x_1 + \bar{k}_2 \sin \lambda x_1)(A_2 \cosh \lambda x_3 + B_2 \sinh \lambda x_3) \quad (2.3)$$

It is known that the solutions of equation (2.1) and all their derivatives with respect to component of x_1 , are finite and continuous at all points, except possibly at some point on the boundary of the field. Thus, the smoothness of the velocity distribution is ensured at all points of the fluid, except at those points of the boundary where a singularity of some kind – for example an abrupt change of the tangent plane to the boundary, as at a corner or edge – is prescribed as part of the boundary conditions, see Barbel et al, (2), Bell & Holmes (3), Gupta (4), Harold (5).

Applying the boundary condition (1.21) to (2.2), gives

$$\frac{\partial \bar{\phi}_1}{\partial x_1} = (-\lambda k_1 \sin \lambda x_1 + \lambda k_2 \cos \lambda x_1)(A_1 \cosh \lambda x_3 + B_1 \sinh \lambda x_3)$$

On $x_1 = L$, we obtain

$$-\lambda k_1 \sin \lambda L + \lambda k_2 \cos \lambda L = 0$$

For $\lambda \neq 0$, we thus obtain

$$k_2 = \frac{k_1 \sin \lambda L}{\cos \lambda L} \quad (2.4)$$

Also applying the boundary condition (1.22) to (2.2), gives

$$\frac{\partial \bar{\phi}_1}{\partial x_3} = (k_1 \cos \lambda x_1 + k_2 \sin \lambda x_1) (\lambda A'_1 \sinh \lambda x_3 + \lambda B'_1 \cosh \lambda x_3)$$

On $x_3 = l$,

$A'_1 \sinh \lambda l + B'_1 \cosh \lambda l = 0$, and gives

$$B'_1 = \frac{-A'_1 \sinh \lambda l}{\cosh \lambda l} \quad (2.5)$$

Substituting (2.4) and (2.5) into (2.2) gives

$$\bar{\phi}_1(x_1, x_3) = \frac{\beta \cos \lambda (L - x_1) \cosh \lambda (\ell - x_3)}{\cos \lambda L \cosh \lambda \ell} \quad (2.6)$$

Where $\beta = k_1 A'_1$.

Similarly, applying the boundary condition (1.21) to (2.3), gives

$$\frac{\partial \bar{\phi}_2}{\partial x_1} = (-\lambda \bar{k}_1 \sin \lambda x_1 + \lambda \bar{k}_2 \cos \lambda x_1) (A_2 \cosh \lambda x_3 + B_2 \sinh \lambda x_3)$$

On $x_1 = L$, we have $-\lambda \bar{k}_1 \sin \lambda L + \lambda \bar{k}_2 \cos \lambda L = 0$, which implies

$$\bar{k}_2 = \frac{\bar{k}_1 \sin \lambda L}{\cos \lambda L} \quad (2.7)$$

Applying the boundary condition (3.24) to (4.3), gives

$$\frac{\partial \bar{\phi}_2}{\partial x_3} = (\bar{k}_1 \cos \lambda x_1 + \bar{k}_2 \sin \lambda x_1) (\lambda A'_2 \sinh \lambda x_3 + \lambda B'_2 \cosh \lambda x_3)$$

On $x_3 = -l$, $-A'_2 \sinh \lambda l + B'_2 \cosh \lambda l = 0$,

$$\Rightarrow B'_2 = \frac{A'_2 \sinh \lambda l}{\cosh \lambda l} \quad (2.8)$$

Upon substitution of (2.7) and (2.8) in (2.3), we obtain

$$\bar{\phi}_2(x_1, x_3) = \frac{\gamma \{ \cos \lambda (L - x_1) \cosh \lambda (L + x_3) \}}{\cos \lambda L \cosh \lambda l} \quad (2.9)$$

where $\gamma = \bar{k}_1 A'_2$.

ANALYSIS

Figures 1 and 2 are the velocity potentials in upper and lower chambers of the cochlea. This represents the steady state case where the time (t) and the distance (x_1) vary simultaneously.

The pressure variation which is noise is thought to arise from the action of large number of variables. In this sense, it is usually understood that noise is multidimensional. The mathematical analysis of noise involves associating a random variable with high-dimensional physical process causing the noise. Nevertheless, it is remarkable that a wide range of electrogenic phenomena can be classified, ordered and explained by suitable application of the ionic theory for spike electrogenesis. This spiking nature of the graphs in figures 1 & 2, can be mistaken for chaos, which it is not, but rather as a result of noise in the system.

A sound wave vibrates the eardrum, rocks the ossicles and causes changes in pressure on the oval window, these changes are communicated directly to the fluid in the tunnel above the basilar membrane. A positive pressure in this tunnel, the scalar media, distorts the basilar membrane downwards increasing the pressure in the scalar tympani and bulges the round window outwards, conversely a negative pressure draws the basilar membrane upward, the attention of pressure in acoustic waves make the membrane vibrate.

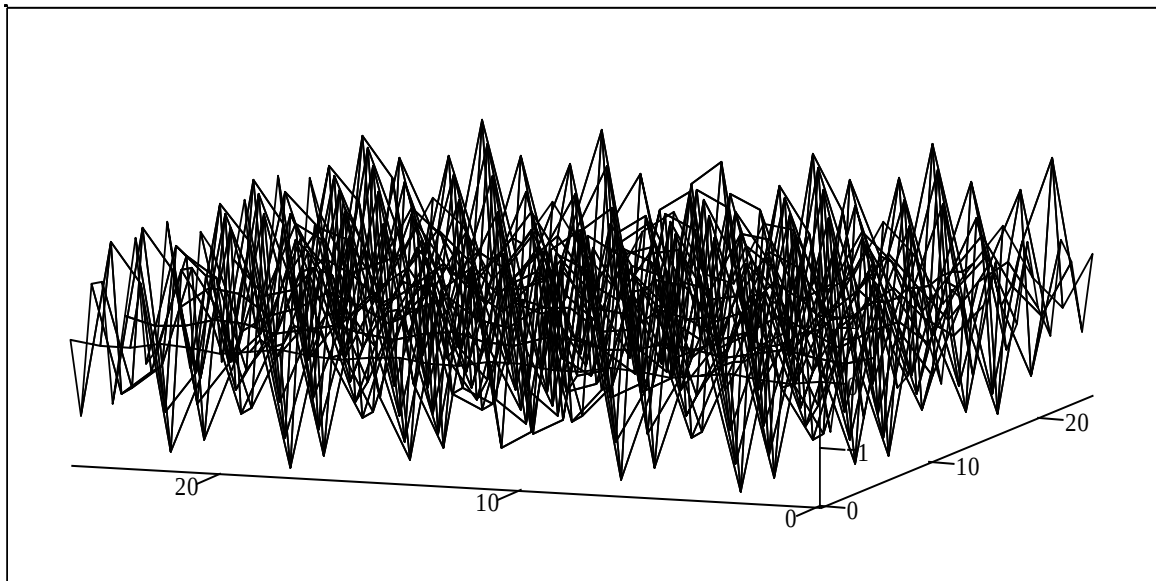
There is a large standing potential difference on the scalar media relative to the scalar vestibuli and scala tympani. This is known as the endocochlear potential, which is exquisitely sensitive to changes in oxygen supply, implying a dependent on metabolic rather than neuronal activity.

The microphonic active of the cochlea is undoubtedly a delicate mechanism for transforming mechanical force into electrical energy. The response recordable at the cochlea is, therefore, actual affect: the mechanical vibration causes the microphonic potential of Wever & Bray, while the action potential setup in the sensory pathways in the true auditory response.

The mathematical analysis of noise involves associating a random variable with the high-dimensional physical process causing the noise. One of the difficulties with modeling noise is that in general, we do not have access to the noise variable itself. Rather, we usually have access to a state variable of a system that is perturbed by one or more sources of noise.

The hallmark of non-linear behaviour in cochlea mechanics is caused by eddy in the cochlea, which is as a result of the combination of viscous and non-linear effects. The combination of noise and non-linearity can produce time series that are mistaken for chaos as seen figures 1 & 2.

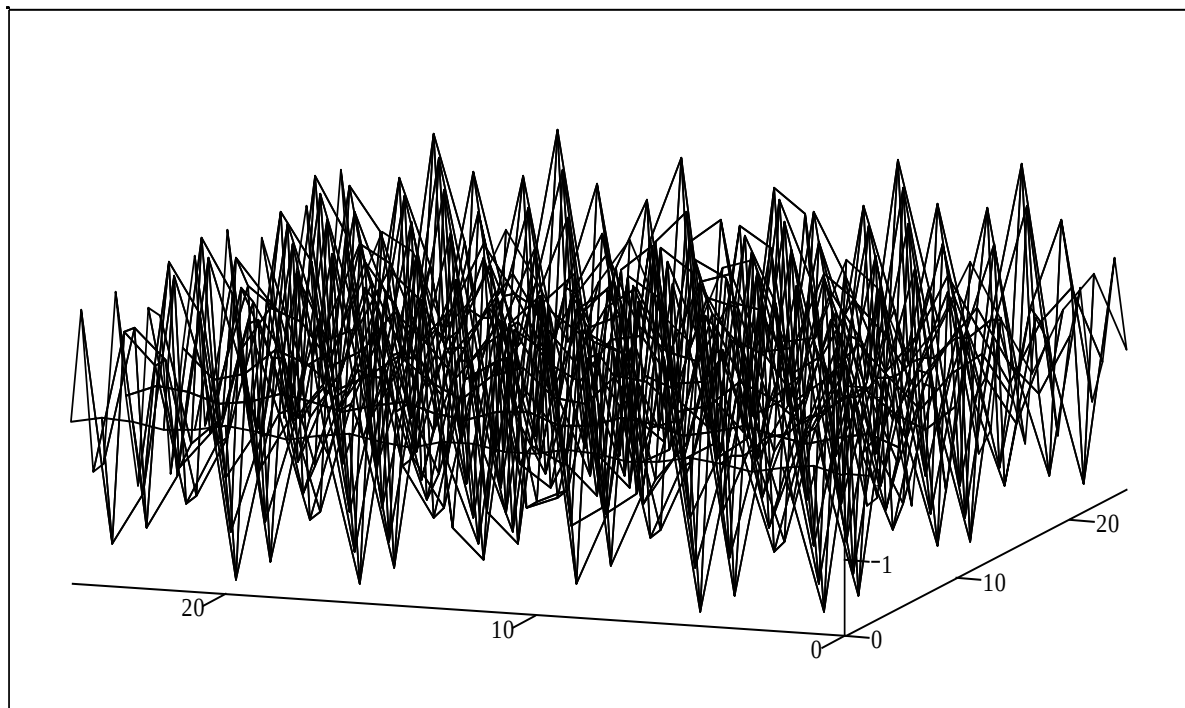
a. THE VELOCITY POTENTIAL AT UPPER DOMAIN



M

2. *Fig 1*

a. THE VELOCITY AT LOWER DOMAIN



M

Fig 2

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An Instability patterns in a Mathematical Model for Tumour Development

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ABSTRACT

The development of tumor has been explained in different ways by cancer researchers. Picturing the size of the tumored-mass as a function of reactions between normal and tumor cells requires understanding of spatio-temporal patterns as seen in morphogenesis. We demonstrate that by preventing the tumor cells from diffusing across a particular organ or imposing a movement function across the boundary of the cellular organ in question, gives rise to complications of various magnitudes. The tumor and normal cells interaction is represented with a system of two reaction-diffusion equations of the type of Turing with reaction terms similar to the Schenackenberg kinetics. The model is then analysed for pattern formation. The result confirms that imposing separate boundary conditions to each of the cell types can generate more complex tumor patterns than when subjected to scalar boundary conditions.

Keywords: Tumor, Schnackenberg, Reaction-diffusion, pattern formation.

Mathematics Subject Classification 2000:92C15 & 35K57

INTRODUCTION

The process of the development of the embryo after fertilisation requires a series of mitotic and meiotic cell divisions the outcome of which give rise to developmental pathways. For normal healthy cells, the study of the development of biological patterns and forms is known as morphogenesis. Neuplasia refers to the study of new growth of abnormal cells in a tissue.

Biologists and Mathematical Biologists have worked extensively in order to explain patterns formation using different approaches. While the biologists have used zoological, clinical and botanical experiments in their studies, the mathematical biologists and bioinformaticians have used mostly analytical methods. Some of the methods applied so far, make used of mathematical and computational modelling (See for example Ngwa, 1994; Dillon *et al.* 1994; Maini and Myerscough, 1996). Amongst the tools which have been widely used to explain pattern formation in biological systems, is the Reaction-Diffusion equations and the computational methods for solving these equations. The brain behind this model is the differentiation in time and dispersion in space of reacting cells during morphogenesis. As these cells differentiate (react) and disperse (diffuse) there is the creation of a metabolic pathway from which a pattern arise.

In developmental biology, it is well known that the fate of a cell in a developing organism depends both on its genomes and its position relative to other neighbouring cells. Hence the specificity of cell structures, functions and the arrangement of cells to form tissues and organs require mechanisms for the spatio-temporal control of cellular activity. Thus, a specific mechanism gives rise to a specific pattern. These patterns can be classified as Spatial (stationary in time) and Spatio-temporal (varying with time) patterns. Therefore, mathematically, spatial patterns come from time-independent solutions of reaction-diffusion (RD) systems and spatio-temporal patterns come from both time-dependent and space-dependent solutions of RD-systems.

Pattern formation in Turing's hypothesis (Turing 1952), is thought to be the result of the response of individual cells to an underlying spatial pattern of one or more chemicals (later called morphogens). Cells in the medium located at a point where the chemical concentration is of certain threshold simply divide. In this light, Wolpert (1971) explained that, cells at a position of high chemical or morphogens concentration will differentiate accordingly in a way that their relative positioning gives rise to appropriate developmental pathways. A cell, in a developing or regenerating system must therefore know its position relative to other cells. Hence the chemical pre-pattern or reaction diffusion models are based on the hypothesis that diffusing morphogens supply positional information that can be interpreted at later time by an appropriate cell (Wolpert 1971 a, b). In this light, the essence of pattern formation is that of detecting schemes, which generate positional information while the underlying problem is considered to be that of explaining the mechanism by which the spatial patterns can be generated and maintained in biological systems. One class of models that have been proposed as a mechanism for pattern formation in morphogenesis is the reaction-diffusion (also known as chemical pre-pattern) model.

The general form of a reaction-diffusion system is an equation of the form;

Where $\mathbf{u}$ is a vector whose components represent the various diffusing species (quantities), $\mathbf{D}$ is a matrix of

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla \cdot (\mathbf{D} \nabla \mathbf{u}) + \mathbf{F}(\mathbf{u}), \quad (1.1)$$

diffusion coefficients that is in general not a constant matrix and ∇ is the gradient operator in the appropriate space.

Turing's RD model, involve two or more chemicals that react and diffuse continuously throughout the system bringing about a heterogeneous distribution of chemical concentrations that serves as a blue print for cell differentiation in morphogenesis (Turing, 1952). In Turing's original analysis, no cells were distinguished a priori; all could serve as source or sink of the morphogens (Dillon *et.al.*, 1994). He however, considered only systems in which the same boundary conditions were imposed on all the species. We called such a system a periodic or closed system. In these types of models, patterns formation either does not involve the flow of information across the boundary of the domain (zero flux boundary condition) or the concentrations are fixed on the boundary. Ngwa (1994) reported that, patterns resulting from any of these boundary conditions are in most cases qualitatively similar. In addition, Maini and Myerscough (1996) explained that these patterns are qualitatively similar only to an extent.

Based on Turing's models, RD models have been proposed to account for spatial pattern formation in many biological systems. The segmentation pattern along the Anterio-posterior axis of an insect and the pattern observed in the skeletal element of a developing tetrapod limb are some examples of patterns explained using RD models (Murray, 1989).

Hence we have seen that Turing systems have limitations such as: the need for tight control of the parameters to obtain the onset of instability, the sensitivity of the resulting pattern to the overall scale and geometry and the existence of multiple stable solutions which make it difficult to study the problem of pattern selection. What is not very certain is whether the boundary conditions subjected to each of the reactant in a system can affect the resulting patterns.

Most related works in reaction-diffusion models for pattern formation focus on chemical kinetics or species interaction as agents of destabilisation or initiation of a pattern.

Ngwa (1994) and Dillon *et. al.* (1994) respectively considered the cases of mixed boundary conditions of the linearised problem and mixed boundary conditions on a restricted domain. In their respective analysis, they employed a finite dimensional approximation to solve the infinite system of algebraic equations that resulted from the system of partial differential equations. However, the relationship between the patterns observed in the two cases was not considered. The patterns observed in their simulation were interesting different from Turing patterns. Similarly, Shin-Ichino *et. al.*, considered the thermal self-ignition model proposed by Gelfand (1968) and Gavallas (1963) in the case where the two reactants involved have the same kinetics which are pre-supposed to be exponential. The observed patterns obtained by their model show slight deviation from Turing's pattern growing exponentially. Finally, Maini and Myerscough (1996) considered boundary driven instability in which they showed that Dirichlet boundary conditions can destabilise a steady state which is stable under Neumann boundary conditions giving rise to more interesting patterns. In other words, they considered destabilisation effects from one scalar boundary condition to another.

Our objective therefore will be that of studying the possibility of pattern formation in the case where one of the species is subjected to zero flux boundary condition and the other to fixed concentration on the boundary. Mathematically, we shall consider the full non-linear problem with mixed boundary conditions and study conditions necessary for pattern formation in such systems. We will compare our results with those of the same problem with scalar boundary conditions.

LINEAR ANALYSIS

In rd systems, the kinetic terms are very often non-linear (see for example; Schnackenberg, 1979; Thomas, 1957; Gierer and Meinhardt, 1972; Marek and Svobodova, 1975; and Murray, 1989). Thus before looking at the non-linear problem, it is good indeed to study the behaviour of the problem in the linear regime. Therefore, we shall carryout linear analysis for the problem with scalar boundary conditions and determine under what conditions the systems can generate biological patterns.

Generalized Turing system let $\Omega \subset \mathcal{R}^q$, $q \leq 3$, be a domain, with smooth boundary $\partial\Omega$ and outward normal $\mathbf{n}$. Turing's generalised model for pattern formation is a system of rd equations of the form:

$$\left. \begin{aligned} \frac{\partial u_i}{\partial t} &= D_i \nabla^2 u_i + f_i(u_1, u_2, \dots, u_n, \mathbf{p}), \text{ in } \Omega, t > 0 \\ \mathbf{n} \cdot \nabla u_i &= H_i(u_i^0 - u_i), \text{ on } \partial\Omega, t > 0, \\ u_i &= U_i^0(\mathbf{r}), \forall \mathbf{r} \in \bar{\Omega}, t = 0. \quad i = 1, 2, \dots \end{aligned} \right\} \quad (2.1)$$

Where, $i=1, 2, 3 \dots n$, u_i^0 is a fixed concentration, d_i is the diffusion coefficients of the i^{th} species, $u_i^0(r)$ is a given function of initial chemical concentration and h_i is a mass transfer coefficient. When $h_i = \infty$, we have a dirichlet-boundary condition on the i^{th} species meanwhile if $h_i = 0$, we have neumann-boundary conditions (Dillon *et al.*, 1994).

Based on the importance of dimensional analysis in mathematical modeling, we shall non-dimensionalise the system (2.1). To do this, we let L to be a measure of the size of the domain, ω^{-1} the time scale characteristic of the reactions, D_1 the smallest of the diffusion coefficients and U_1 a reference concentration

of the i^{th} species with diffusion coefficient D_i . Further, let P_j be an arbitrary scaling parameter for each of the parameters in p and define the following dimensional quantities

$$(2.2) \quad v_i = \frac{u_i}{U_i}, \quad \tau = \omega t, \quad \mathbf{x} = \frac{\mathbf{r}}{L}, \quad p_j^* = \frac{p_j}{P_j}, \quad d_i = \frac{D_i}{D_1}$$

Substituting (2.2) into (2.1), we have:

$$\begin{aligned} \frac{\partial v_i}{\partial t} &= d_i \nabla^2 v_i + \gamma f_i \quad \text{in } \Omega, \quad t > 0 \\ \mathbf{n} \cdot \nabla v_i &= Q_i (v_i^0 - v_i), \quad \text{on } \partial\Omega, \quad t > 0 \\ v_i &= V_i^0(\mathbf{x}), \quad \forall \mathbf{x} \in \bar{\Omega}, \quad t = 0, \quad i = 1, 2, \dots \end{aligned} \quad (2.3)$$

The above system represents a dimensionless system with the function F_i being a dimensionless function of chemical concentration. γ is a scaled parameter representing the activity constant of the reaction, diffusion coefficients of the reacting species, and domain size (length).

Clearly the temporal dynamics in a spatially uniform system are governed by the solution of the system of differential equations:

$$\frac{du_i}{dt} = f_i(\mathbf{u}; p), \quad i = 1, 2, \dots, n, \quad \mathbf{u} = (u_1, u_2, \dots, u_n), \quad (2.4)$$

where: f_i gives the net rate of production of the i^{th} chemical, f_i is usually a polynomial which is smooth and admit no singularities in the space of functions under consideration, $\mathbf{p}$ is a parameter vector that may include the kinetics constant and perhaps species that appear in the kinetics mechanism but do not have any significant changes in the time scale of interest (for example, catalysts). From a physical stand point, the problem (2.3) should be well-posed, that is, the solution should exist, be non-negative and bounded, for $t \in [0, \infty)$. Clearly, this condition will be satisfied whenever the functions f_i are such that for each u_j ,

$$f_i(u_1, u_2, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n, \mathbf{p}) \geq 0, \quad i \neq j. \quad (2.5)$$

That is, under this condition, the solution of (2.3) exists, non-negative and bounded. The initial condition gives rise to a unique solution if the functions f_i are locally lipschitz continuous in each u_j in the region of interest.

It has been shown by Conway *et al.*, (1978) that, under Neumann-boundary conditions, if; the domain Ω of a population is small enough, or the diffusion coefficient D is large enough, the long-term behaviour of the simple RD-system;

$$\frac{\partial u}{\partial t} = D\nabla^2 u + f(u, p) \quad (2.6)$$

is essentially the same as the corresponding solution of the associated kinetics system (2.4). Thus, Ashkenazi and Othmer (1978) stated and proved a theorem which revealed that, under most of the typical rate laws used, the condition (2.5), which guarantees invariance of domain space, also guarantees the non-negativity of the classical solution of the reaction diffusion system (2.3) for $t > 0$, provided that the initial data is non-negative. Furthermore, the solution exists and is unique for sufficiently small time and bounded in $L_1(\Omega)$, $t \in [0, \infty)$, under minimal smoothness conditions on the vector field defined by the kinetic terms. Under scalar boundary conditions, asymptotic stable solutions of the reaction-diffusion equations are stable solutions of the corresponding kinetics system (Fife, 1979). In the absence of reaction, under homogeneous Neumann boundary conditions, the reaction diffusion system reduces to the heat equation given by:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= D\nabla^2 u_i \text{ in } (\Omega) \\ \mathbf{n} \cdot \nabla u_i &= 0 \text{ in } \partial\Omega, \quad \text{where } u_i = u_i(x, t), \end{aligned} \quad (2.8)$$

The solution relaxes exponentially to the average concentration set by the initial condition. This is clear as the solutions of such equations are of the form;

$$u_i(x, t) = \sum_{n=0}^{\infty} a_{in} e^{-Dn^2 \pi^2 t} \cos(n\pi x) \quad (2.9)$$

where a_{in} can be determined by expanding the initial conditions in a Fourier series. Thus, one expects that a system will relax to a uniform state whenever the relaxation time for diffusion of each species is sufficiently short compared to that of chemical reaction (Dillon *et al.*, 1994).

As already seen above, diffusion could destabilise a steady state, which was originally stable in the absence of diffusion. Here we refer to this phenomenon as *Diffusion Driven Instability* (DDI). To precisely define this concept in mathematics lets linearise the system (2.3).

Assume that $\mathbf{u}^s = (u_1^s, u_2^s, \dots, u_n^s)$ is the time-independent solution of the non-dimensional system (2.3). To linearise (2.3), about $\mathbf{u}^s$, Let,

$$\mathbf{u}^* = \mathbf{u} - \mathbf{u}^s \text{ with } \|\mathbf{u}^*\| \ll 1. \quad (2.10)$$

Assume further that u^s is the uniform steady state corresponding to (2.3) such that for all i , $f_i = 0$ at u^s . Substituting for $\mathbf{u}$ into (2.3), and retaining only first order terms in the Taylor series expansion of the function f_i , we get the linearized system in u^* . For simplicity, we drop the asterisks to get;

$$\left. \begin{aligned} I \frac{\partial u}{\partial t} &= D \nabla^2 u + \gamma \mathbf{K} u \quad \text{in } \Omega, \quad t > 0 \\ \mathbf{n} \cdot \nabla u &= -Q_i u_i, \quad \text{on } \partial\Omega, \quad t > 0 \\ u &= u_i^0(\mathbf{x}), \quad \forall \mathbf{x} \in \bar{\Omega}, \quad t = 0, \end{aligned} \right\} \quad (2.11)$$

where I is the identity matrix and D is a diagonal matrix of the diffusivity ratios ordered in the form, $1, d_1, d_2, \dots, d_n$. The matrix $\mathbf{K}$ is the Jacobian matrix and is given by;

$$\mathbf{K} = \begin{pmatrix} k_{11} & . & . & . & k_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ k_{ni} & . & . & . & k_{nn} \end{pmatrix}, \quad \text{where } k_{ij} = \left. \frac{\partial f_i}{\partial u_j} \right|_{u=u^s}. \quad (2.12)$$

The equation (2.11) has solution of the form $u \propto \exp(\lambda t) \cdot \phi(r)$ which upon substitution to the system (2.11) gives

$$\begin{aligned} D \nabla^2 \phi + (\gamma \mathbf{K} - \lambda I) \phi &= 0 \quad \text{in } \Omega \\ \mathbf{n} \cdot \nabla \phi &= -Q \phi \quad \text{on } \partial\Omega. \end{aligned} \quad (2.13)$$

In general (2.13) is not a self-adjoint problem and the eigenfunctions do not have a simple form. In case of self-adjoint operators, if the eigenvalues form a denumerable sequence, then the orthogonality properties of the eigenfunctions will be preserved and the expansion theorem will hold as will be seen below. If the boundary conditions in (2.13) is a scalar condition, that is $Q = \rho I$, $\rho \in \mathcal{R}^+$, then the eigenfunctions of (2.13) may be written in the form $\phi_m = y_m \psi_m$, where ψ_m is the solution of the scalar eigenvalue problem;

$$\begin{aligned} \nabla^2 \psi_m + \alpha_m^2 \psi_m &= 0 \quad \text{in } \Omega \\ \mathbf{n} \cdot \nabla \psi_m &= -\rho \psi_m \quad \text{on } \partial\Omega, \end{aligned} \quad (2.14)$$

Where $\alpha_m \in \mathcal{R}$ and the eigenvectors y_m associated with the eigenfunctions ϕ_m satisfy the eigenvalue problem

$$(\gamma \mathbf{K} - \lambda I - \alpha_m^2 D) y_m = 0, \quad (2.15)$$

From where it follows that non-zero y_m exist if and only if

$$|\gamma \mathbf{K} - \lambda I - \alpha_m^2 D| = 0. \quad (2.16)$$

The solution of this gives n eigenvalues $\lambda_{i,m}$, $i=1,2,\dots,n$, with associated eigenvectors $y_{i,m}$. These eigenvalues form a denumerable sequence and can be ordered so that the corresponding eigenfunctions form a complete orthonormal set of functions. By the expansion theorem, the solution to the linearized problem is given in terms of the eigenvalues and eigenfunctions as

$$u_i = \sum_{m=0}^{\infty} \sum_{i=1}^n A_{i,m} y_{i,m} e^{\lambda_{i,m} t} \psi_m(r), \quad (2.17)$$

where the initial data determines the amplitudes $A_{i,m}$, $i=1,2,\dots,n$.

Therefore, under scalar boundary conditions, the eigenfunctions that span the null space can be determined completely. Hence the solution of the full non-linear problem can be characterized at the bifurcation point by specifying the amplitude spectrum relative to a basis comprising of these eigenfunctions.

The problem of stability in the sense of $L^2(\Omega)$ is reduced to the problem of getting the eigenvalues of the family of matrices $\{\gamma \mathbf{K} - \alpha_m^2 D\}$, $m=0,1,\dots$. The principle of linearized stability assures us that the stability properties of the solution of the full non-linear system can be characterized if those of the linear system are known. It should also be noted that in the case where the steady state varies with length, it is possible to characterize the solution by again using their amplitude spectrum relative to the basis of eigenfunctions, provided the eigenfunctions are complete (Dillon *et al*, 1994). Stability of the uniform steady state in the case where α_m is continuous is governed by the eigenvalues of the one parameter family of matrices $\{\gamma \mathbf{K} - \alpha^2 D\}$, $\alpha \in \mathcal{H}^+$, and we have the following as a condition for diffusion driven instability:

A zero Amplitude Diffusion Driven Instability of an asymptotically stable solution u^s of the linearized problem exists, if there exists α_+ and α_- with $0 < \alpha_- < \alpha_+ < \infty$ such that $\gamma \mathbf{K} - \alpha^2 D$ has at least one eigenvalue with positive real part, with $\alpha \in (\alpha_-, \alpha_+)$. The instability is stationary at u^s if there exists $\alpha^* \in (\alpha_-, \alpha^+)$ such that $\{\gamma \mathbf{K} - (\alpha^*)^2 D\}$ has a single real positive eigenvalue. If $\{\gamma \mathbf{K} - (\alpha^*)^2 D\}$ has complex eigenvalues with positive real part, then the instability is oscillatory at u^s . Generally, stationary instability leads to bifurcation of stationary solutions while oscillatory instability leads to bifurcation of periodic solutions (Dillon *et al*., 1994).

Two-Component Turing models

Since two-component reaction-diffusion system is the basis of this research, we highlight some of the known properties about such systems with scalar boundary conditions. For simplicity, we write the system as follows;

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \mathcal{H}(u, v, p) \\ \frac{\partial v}{\partial t} &= d \frac{\partial^2 v}{\partial x^2} + \mathcal{G}(u, v, p) \end{aligned} \right\} \text{in } \Omega \quad (2.18)$$

and

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= \rho(u^s - u), \quad \frac{\partial v}{\partial x} = \rho(v^s - v) \end{aligned} \right\} \text{on } \partial\Omega \quad (2.19)$$

From (2.3), we can easily see that (f, g) represents the reaction vector field (f_1, f_2) , while d_1 is 1 and $d_2 = d$. Again, γ and d are scaled parameters as in (2.3). In particular, this general form enables d and γ to have a wider biological interpretation than the dimensional parameter. Also, if the domains in parameter space where particular spatial pattern appears is considered, the result can be conveniently displayed in (γ, d) space.

Let (u^s, v^s) be the uniform steady state solutions of the system (2.18), (2.19). That is, u^s and v^s are such that $f(u^s, v^s) = 0$ and $g(u^s, v^s) = 0$. A steady state is uniform if it is steady both in time and space. From (2.1), the governing equation for the stability of (u^s, v^s) is;

$$|\gamma \mathbf{K} - \mathbf{D} \alpha^2 - \lambda| = 0, \quad (2.20)$$

where

$$\mathbf{K} = \begin{pmatrix} f_u(u^s, v^s) & f_v(u^s, v^s) \\ g_u(u^s, v^s) & g_v(u^s, v^s) \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solving we get,

$$\begin{vmatrix} \mathcal{H}_u - \alpha^2 - \lambda & \mathcal{H}_v \\ \mathcal{G}_u & \mathcal{G}_v - d\alpha^2 - \lambda \end{vmatrix} = 0 \quad (2.21)$$

$$\lambda^2 + C_1(\alpha^2)\lambda + C_2(\alpha^2) = 0,$$

$$C_1(\alpha^2) = (d+1)\alpha^2 - \gamma(f_u + g_v) \quad (2.22)$$

$$C_2(\alpha^2) = d\alpha^4 - \gamma(g_v + df_u)\alpha^2 + \gamma^2(f_u g_v - f_v g_u)$$

It is easily verified that when $\alpha = 0$, the solution (u^s, v^s) of (2.18) is (asymptotically) stable if and only if $\text{Trace}(K) < 0$ and $\det(K) > 0$. In other words,

$$f_u + g_v < 0 \text{ and } f_u g_v - f_v g_u > 0. \quad (2.23)$$

We shall assume these conditions to hold hereafter.

If $C_1(0) > 0$, then $C_1(\alpha^2) > 0$ for all α provided $d > 0$. If the eigenvalues λ_1, λ_2 are complex, then they have negative real parts since $C_1(\alpha^2) > 0$. This therefore shows that oscillatory diffusion-driven instability is impossible in this case. Also, if λ_1 and λ_2 are real, then oscillatory DDI is still impossible. Therefore, oscillatory DDI is impossible if diffusion ratio, d , is positive and $C_1(0) > 0$. What are therefore the conditions for the system to be driven linearly unstable? We make a sketch of the last equation of (2.22) as shown in figure 1 below.

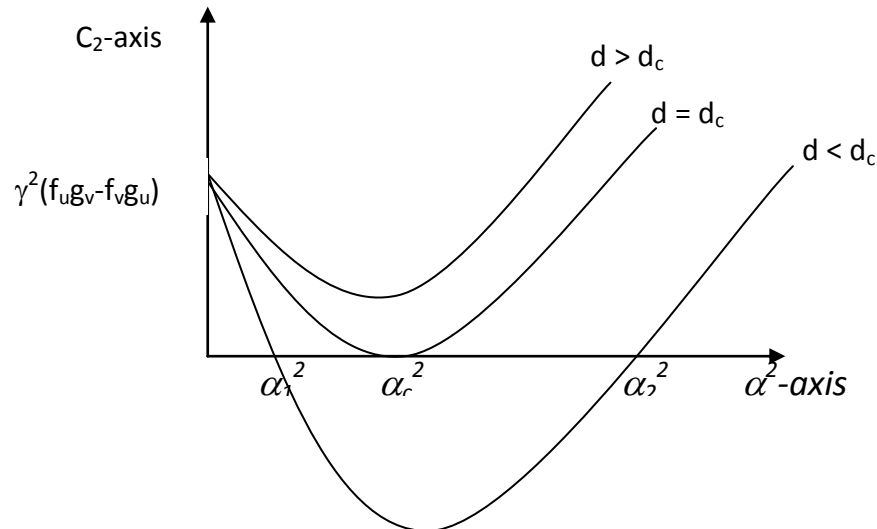


Fig.1: A Sketch of C_2 against α^2 for $d=d_1, d = d_c$ and $d = d_2$ with $d_1 < d_c < d_2$. As d increases, α_c^2 tends to $f_u/2$ that is, α_2^2 tend towards α_1^2 . As d decreases to zero, α_c^2 tends to ∞ .

From the dispersion relation (2.21) and (2.22) and the condition (2.23), we see that $C_2(\alpha^2) = 0$ if and only if, one of f_u and g_v is positive. Without loss of generality, we can assume that $f_u > 0$; then it is necessary that $g_v < 0$ and $f_v g_u < 0$ for (2.23) to be satisfied. Generally, for DDI, we require that, $\lambda_{1,2}$ should be such that, in the absence of diffusion, $\lambda_{1,2}(0) < 0$; long term solution should be bounded, $\lambda_{1,2}(\infty) < 0$; and for some $\alpha \in [0, \infty)$, $\lambda_{1,2}(\alpha^2) > 0$. The conditions for instability are therefore those for which $\lambda_{1,2}(\alpha^2) > 0$ whenever $\lambda_{1,2}(0) < 0$. Since $\lambda_{1,2}(\alpha^2) > 0$ if and only if $C_2(\alpha^2) < 0$, it implies that for instability, we must have $C_2(\alpha^2) < 0$. Again looking at C_2 , since $d > 0$, we easily verify that the graph C_2 is concave upward with a minimum point at

$$\alpha^2 = \frac{\gamma g_v + d f_u}{2d} = \alpha_c^2 \quad (2.24)$$

From (2.21) there is no positive roots for λ , hence the system becomes more and more stable in conformity with the condition $\lambda_{1,2}(\infty) < 0$. Also, from the first condition of (2.23), it follows that α_c^2 is positive if and only if $d \neq 1$. Therefore the interval (α_1^2, α_2^2) represents the range of dimensionless wave numbers within which $C_2(\alpha^2) < 0$. The values of α_1^2 and α_2^2 are obtained from the zeros of $C_2(\alpha^2)$ and are given by;

$$\alpha_{1,2}^2 = \frac{\gamma(g_v + d f_u) \pm \gamma \sqrt{(g_v + d f_u)^2 - 4d(f_u g_v - f_v g_u)}}{2d} \quad (2.25)$$

At $\alpha = \alpha_c$, we have,

$$C_2(\alpha_c^2) = \frac{\gamma^2}{4d} ((f_u)^2 d^2 + (4g_u f_v - 2g_v f_u)d + (g_v)^2).$$

The zeros of $C_2(\alpha_c^2)$ are;

$$d_{\pm} = \frac{-(4g_u f_v - 2g_v f_u) \pm 4\sqrt{g_u f_v (g_u f_v - g_v f_u)}}{2(f_u)^2} \quad (2.26)$$

$d_{\pm}$ is real if f_u and g_v have opposite signs. Thus, the interval for unstable wave numbers can be characterized in terms of the diffusion ratio d as in (2.24). Therefore, a minimum allowable d exists for which a zero-amplitude DDI occurs at α_c^2 . Such a d is called the critical diffusion coefficient and is denoted by d_c and defined by

$$2d_c\alpha_c = \gamma g_v + d_c f_u. \quad (2.27)$$

In a one-dimensional domain, with homogenous Neumann boundary conditions, on the nondimensional interval $[0,1]$, α is discrete and we represent it by $\alpha_n = n\pi$, $n = 1, 2$, and $\phi_n = \cos(n\pi x)$ are eigenfunctions corresponding to the eigenvalues. Under Dirichlet boundary conditions, fixed at the steady state, $\phi_n = \sin(n\pi x)$, $n \neq 0$. In either case, Ngwa, (1994) showed that the wave number is indexed by the integer n closest to

$$\frac{\alpha_c}{\pi} = \frac{1}{\pi} \sqrt{\frac{\gamma(g_v + df_u)^2}{2d_c}}, \quad (2.28)$$

$$(g_v + df_u)^2 - 4d(f_u g_v - f_v g_u) > 0. \quad (2.29)$$

where n is clearly determined by the size of γ . If the parameters in the chemical system are such that there is a positive eigenvalue α^2 of the linearized system in the range (α_1^2, α_2^2) , then the uniform steady state (u^s, v^s) will become linearly unstable to the eigenmode ϕ_n whenever γ is of a certain size. A non-zero interval of unstable modes (α_1^2, α_2^2) will exist whenever $d > d_c$ if and only if, the discriminant of (2.25) is greater than zero that is,

$$(g_v + df_u)^2 - 4d(f_u g_v - f_v g_u).$$

For DDI, we must have the derivatives of the kinetic functions satisfying the conditions

$$\left. \begin{array}{l} f_u + g_v < 0, \quad f_u g_v - f_v g_u > 0, \quad g_v + df_u > 0 \\ (g_v + df_u)^2 - 4d(f_u g_v - f_v g_u) > 0 \end{array} \right\} \text{which we have deduced.}$$

The above conditions will be assumed as we carried out a non-linear analysis of the problem with mixed boundary conditions. The parameter domain for which the above conditions will be satisfied is called a Turing space and this defines the conditions under which DDI is possible. For given kinetics $f(u, v, p)$ and $g(u, v, p)$ we can determine the domain of instability.

NON-LINEAR ANALYSIS

Here, we seek solutions of the full non-linear problem where the interactions between the tumour cells and the normal cells are represented with the Schnackenberg kinetics (Schnackenberg, 1979). These kinetic equations have been applied to a variety of situation corresponding to interactions between cells, the rod bending phenomenon etc. The section, which is divided into three sub sections, starts with a perturbation analysis, then a Fourier series representation of the solution and a study of the leading order behaviour of the solution of the non-linear problem and a numerical result.

Perturbation Analysis of the Non-Linear Problem

Consider equation (2.18) with the following initial and boundary conditions;

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= 0 \\ v &= v^s \end{aligned} \right\}, t > 0, x \in (0,1) \quad (3.1)$$

$$u(x,0) = u^o(x), v(x,0) = v^o(x)$$

The schnackenberg kinetics applicable to tumour growth can be deduced from the following facts; Let F is the reaction rate of tumour cells and G the rate of the normal reaction. Let K_1 represent the total number of cells of tumour at the point of diagnoses and K_4 be the number of normal cells at time of diagnosis; we let K_3 be the rate of consumption of normal cells by the tumour cells; K_2 the rate of death or disappearance of tumour cells, then we represent the interaction between the normal cells and the tumour cells as,

$$F(u_1, u_2) = K_1 - K_2 u_1 + K_3 u_1^2 u_2 \quad \text{and} \quad G(u_1, u_2) = K_4 - K_3 u_1^2 u_2, \quad (3.1)'$$

Where $K_2 u_1$ gives the proportion of tumour cells removed from the total tumour population. $K_3 u_1^2$ represent the total population removed from the normal population in favour of the tumour population.

In non-dimensional form, this kinetics (3.1)' is given by:

$$f(u,v) = a - u + u^2 v \quad \text{and} \quad g(u,v) = b - u^2 v. \quad (3.2)$$

Let u^s and v^s be the uniform steady states of the system (2.18) and consider perturbations of u^s and v^s of the form $u = u^s + \nu$ and $v = v^s + \nu$, where $|\nu| \ll 1$, and $|\nu| \ll 1$

Expanding f and g in a Taylor series as a function of two variables, we have;

$$f(u,v) = f(u^i + v, v^i + v) = f(u^i, v^i) + \int_v u + \int_v v + \frac{1}{2} \left(\frac{\partial^2 f}{\partial v^2} v^2 + \frac{2\partial^2 f}{\partial v^2} v v + \frac{\partial^2 f}{\partial v^2} v^2 \right) + O(h^3),$$

$$g(u,v) = g(u^i + v, v^i + v) = g(u^i, v^i) + g_u v + g_v v + \frac{1}{2} \left(\frac{\partial^2 g}{\partial v^2} v^2 + \frac{2\partial^2 g}{\partial v^2} v v + \frac{\partial^2 g}{\partial v^2} v^2 \right) + O(h^3). \quad (3.3)$$

Now, substituting (3.3) into (2.18) using the boundary conditions (3.1) we have;

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \mathcal{F}_u v + \mathcal{F}_v v + \text{terms of order } uv \text{ and higher} \quad (3.4)$$

$$\frac{\partial v}{\partial t} = d \frac{\partial^2 v}{\partial x^2} + \mathcal{G}_u v + \mathcal{G}_v v + \text{terms of order } uv \text{ and higher}$$

Using Schnackenberg kinetics in place of f and g , equation (3.4) reduces to

$$\left. \begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \mathcal{F}_u v - \mathcal{F}_v v &= \gamma(v^2 v + v^2 v^s + 2v v u^s) \\ \frac{\partial v}{\partial t} - d \frac{\partial^2 v}{\partial x^2} - \mathcal{G}_u v - \mathcal{G}_v v &= -\gamma(v^2 v + v^2 v^s + 2v v u^s) \end{aligned} \right\} \quad (3.5)$$

The system (3.5) has linear terms on the left and non-linear terms on the right. It is therefore a non-linear system of the first order. We shall consider this system for pattern formation by subjecting it to scalar and mixed boundary conditions.

SYSTEMS (3.5) with Scalar Boundary Conditions

Consider the system (3.5) with the boundary conditions $u(0,t)=0=v(0,t)$ and $u(1,t)=0=v(1,t)$. Since u and v vanishes at $\{0, 1\}$ they can be expanded as a Fourier sine series in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x), \quad v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x), \quad (3.6)$$

1. Without loss of generality, we can introduce the following notations for the sake of simplification. Let;

$$\alpha_{n,j} \equiv 2\langle \sin(n\pi x), \sin(j\pi x) \rangle = \begin{cases} 0, & \text{if } n \neq j \\ 1, & \text{if } n = j \end{cases};$$

$$\alpha_{n,j} \equiv 2\langle \cos(n\pi x), \cos(j\pi x) \rangle = \begin{cases} 0, & \text{if } n \neq j \\ 1, & \text{if } n = j \end{cases} \quad \text{and} \quad (3.7)$$

$$\beta_{j,n} \equiv \langle \sin(n\pi x), \cos(j\pi x) \rangle = \begin{cases} 0, & \text{if } n + j \text{ is even} \\ \frac{2j}{(j^2 - n^2)\pi}, & \text{if } n + j \text{ is odd} \end{cases}$$

We now substitute (3.6) into (3.5), multiply the result by $\sin(k\pi x)$ and integrate it on $[0,1]$, using identities (3.7) get

$$\sum_{n=1}^{\infty} \left(\frac{da_n}{dt} + ((n\pi)^2 - \mathcal{H}_u) a_n \right) \alpha_{n,k} - \mathcal{H}_v \sum_{n=1}^{\infty} b_n \alpha_{n,k} =$$

$$\frac{\gamma}{4} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{njmk} a_n a_j b_m + \frac{\mathcal{W}^s}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \beta_{nj k} a_n a_j + \mathcal{U}^s \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nmk} a_n b_m, \quad (3.8)$$

where

$$\alpha_{njmk} \equiv \{ \alpha_{n+j, m+k} - \alpha_{n+j, m-k} - \alpha_{n-j, m+k} + \alpha_{n-j, m-k} \};$$

$$\beta_{nj k} \equiv \{ \beta_{n-j, k} - \beta_{n+j, k} \}; \beta_{nmk} \equiv \{ \beta_{n-m, k} - \beta_{n+m, k} \}$$

Using the orthogonality conditions of the trigonometric functions, (3.8) reduce to;

$$\frac{1}{2} \left(\frac{da_n}{dt} + ((n\pi)^2 - \mathcal{H}_u) a_n - \mathcal{H}_v b_n \right) = \frac{\gamma}{4} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{njmk} a_n a_j b_m$$

$$+ \frac{\mathcal{W}^s}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \beta_{nj k} a_n a_j + \gamma \mathcal{U}^s \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nmk} a_n b_m \quad (3.9)$$

Similarly, the second equation of (3.5) with the substitution of $v(x,t)$ as in (3.6), leads to a corresponding system to that of $u(x,t)$ which we jointly quote as follows:

$$\frac{1}{2} \left(\frac{da_k}{dt} + ((k\pi)^2 - \mathcal{H}_u) a_k - \mathcal{H}_v b_k \right) = \frac{\gamma}{4} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{njmk} a_n a_j b_m$$

$$+ \frac{\mathcal{W}^s}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \beta_{nj k} a_n a_j + \gamma \mathcal{U}^s \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nmk} a_n b_m, \quad (3.10a)$$

$$\begin{aligned} \frac{1}{2} \left(\frac{db_k}{dt} + (d(n\pi)^2 - \gamma g_v) b_k - \gamma g_u a_k \right) &= \frac{-\gamma}{4} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{njmk} a_n a_j b_m \\ - \frac{\gamma^s}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \beta_{nj k} a_n a_j - \gamma u^s \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nm k} a_n b_m. \end{aligned} \quad (3.10b)$$

If we consider the system (3.10) and neglect higher order terms of the form $a_k b_m$, $a_k a_j$, $a_k b_m a_j$, $i, j, k = 1, 2, \dots$, we obtain a linearised homogeneous system given by

$$\left\{ \begin{aligned} \left(\frac{da_k}{dt} + (k\pi)^2 - \gamma f_u \right) a_k - \gamma f_v b_k &= 0 \\ \left(\frac{db_k}{dt} + (d(k\pi)^2 - \gamma g_v) b_k - \gamma g_u a_k \right) &= 0 \end{aligned} \right. \quad (3.11)$$

This system is the linearised system in the amplitude functions $a_k(t)$ and $b_k(t)$. If we seek solutions of (3.11) in the form $e^{\lambda t}$, then $\lambda = \lambda(k\pi)$ will satisfy the equations

$$\lambda^2 + C_1(k^2\pi^2) + C_2(k^2\pi^2), \quad (3.12)$$

$$\begin{aligned} \text{where, } C_1(k^2\pi^2) &= \gamma(f_u + g_v) - (d+1)(k\pi)^2 \\ C_2(k^2\pi^2) &= \gamma^2(f_u g_v - f_v g_u) - \gamma(df_u + g_v)(k\pi)^2 + dk^2\pi^4. \end{aligned}$$

Observe that this is precisely the system studied earlier linear analysis with α replaced by its discrete value $k\pi$, hence the results there carry over. However, because of the simplicity of the linear system, we can obtain a closed form solution for the amplitude functions a_k and b_k , by writing the equations in the form,

$$\begin{pmatrix} \frac{da_k}{dt} \\ \frac{db_k}{dt} \end{pmatrix} = \begin{pmatrix} \gamma f_u - (k\pi)^2 & \gamma f_v \\ \gamma g_u & \gamma g_v - d(k\pi)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}. \quad (3.13)$$

a. Let,
$$\mathbf{A} = \begin{pmatrix} \mathcal{H}_u - (k\pi)^2 & \mathcal{H}_v \\ \mathcal{G}_u & \mathcal{G}_v - d(k\pi)^2 \end{pmatrix}, \quad (3.14)$$

b. and set $\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \mathbf{P} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where $\mathbf{P}$ is such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$; then it follows that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix},$$

where λ_1 and λ_2 are the solutions of (3.12) or the eigenvalues of $\mathbf{A}$ and c_1 and c_2 are arbitrary constants. By finding the eigenvectors corresponding to λ_1 and λ_2 we see that $\mathbf{P}$ is of the form;

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \frac{\mathcal{H}_u - (k\pi)^2 - \lambda_1}{\mathcal{H}_v} & \frac{\mathcal{H}_u - (k\pi)^2 - \lambda_2}{\mathcal{H}_v} \end{pmatrix}. \quad (3.15)$$

If we let $\mathbf{B}_1 = \frac{\mathcal{H}_u - (k\pi)^2 - \lambda_1}{\mathcal{H}_v}$ and $\mathbf{B}_2 = \frac{\mathcal{H}_u - (k\pi)^2 - \lambda_2}{\mathcal{H}_v}$,

then the solution of the system (3.13) is given by;

$$\left. \begin{aligned} a_k(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ b_k(t) &= c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \end{aligned} \right\}, \quad (3.16)$$

Where, the constants c_1, c_2 are determined by expanding the initial conditions in a Fourier series. The general form of the linear solutions is then given by

$$u(x, t) = \sum_{K=1}^{\infty} a_k(t) \sin(k\pi x); \quad v(x, t) = \sum_{k=1}^{\infty} b_k(t) \sin(k\pi x) \quad (3.17)$$

From our linear analysis, if the parameters lie in the Turing space, then for a given set, there exists a set of numbers $k_1, k_2, k_3, \dots, k_n \in (1, \infty)$ for which $\max\{\lambda_1(k\pi), \lambda_2(k\pi)\} > 0$. For those values of k , the linear solution (3.16) indicates that the eigensolution associated with λ will grow exponentially. We expect that in the non-linear regime, these exponentially growing solutions will be bounded by nonlinearities and the result is a new heterogeneous distribution of chemical concentrations. The steady state (u^s, v^s) has been driven unstable. The fact that this instability has arise is an indication of possible patterns.

Systems (3.5) with Mixed Boundary Conditions

In this section, we consider the system (3.5) subject to homogenous Dirichlet and Neumann conditions on the u and v equation respectively. It is clear that on the boundary $\{0,1\}$, functions with vanishing derivatives have cosine series expansion, while those functions which vanish on the boundary have Fourier sine series expansion. Thus, for these boundary conditions, we take $u(x,t)$ and $v(x,t)$ to be

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos(n\pi x), \quad v(x,t) = \sum_{m=1}^{\infty} b_m(t) \sin(m\pi x). \quad (3.18)$$

Substitution of (3.18) into (3.5) and proceeding as in the former case leads to the corresponding system to (3.10) in the case with mixed boundary conditions given by

$$\begin{aligned} \frac{1}{2} \left(\frac{da_k}{dt} + ((n\pi)^2 - \mathcal{H}_u) a_k \right) &= \mathcal{H}_u \sum_m b_m \beta_{m,k} + \\ \frac{\gamma}{4} \sum_{njm} a_n a_j b_m \beta_{njmk} &+ \frac{\gamma v^s}{2} \sum_{nj} a_n a_j \alpha_{nj} + \mathcal{W}^s \sum_{nm} a_n b_m \beta_{nmk}, \end{aligned} \quad (3.19a)$$

$$\begin{aligned} \sum_m \left(\frac{db_m}{dt} + (d(m\pi)^2 - \mathcal{H}_v) b_m \right) \beta_{mk} &- \frac{\alpha g_u a_k}{2} = \\ - \frac{\gamma}{4} \sum_{njm} a_n a_j b_m \beta_{njmk} &- \frac{\mathcal{W}^s}{2} \sum_{nj} a_n a_j \alpha_{nj} - \mathcal{W}^s \sum_{nm} a_n b_m \beta_{nmk}, \end{aligned} \quad (3.19b)$$

Where

$$\left. \begin{aligned} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} &\equiv \sum_{njm}; \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \equiv \sum_{nj}; \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \equiv \sum_{nm} \end{aligned} \right\}$$

Expanding (3.19) and rearranging leads to equations of the form

$$\begin{aligned} \frac{1}{2} \frac{da_k}{dt} &= \frac{\gamma}{4} \beta_{kkkk} a_k^2 b_k + \theta_1 a_k^2 + \theta_2 a_k b_k + (\mu_k + \theta_3) a_k + \theta_4 b_k + \theta_5 \\ \sum_m \frac{db_k}{dt} \beta_{m,k} &= - \frac{\gamma}{4} \beta_{kkkk} a_k^2 b_k - \omega_1 a_k^2 - \gamma \omega_2 a_k b_k - \gamma \omega_3 a_k - \gamma \omega_4 b_k + \gamma \omega_5 \end{aligned}$$

The coefficients of $a_k, b_k, a_k^2 b_k, a_k b_k, a_k^2$ namely, $\theta_1, \theta_2 \dots \theta_5$ and $\omega_1, \omega_2 \dots \omega_5$ are infinite sums whose internal complexity is left out.

Similarly as in the case with scalar boundary conditions, we consider the system (3.19) and eliminate higher order terms to obtain the infinite system of ordinary differential equations given by:

$$\begin{aligned} \frac{1}{2} \left(\frac{da_k}{dt} + ((n\pi)^2 - \mathcal{H}_u) a_k \right) &= \mathcal{H}_u \sum b_m \beta_{m,k} \\ \sum_m \left(\frac{db_m}{dt} + (d(m\pi)^2 - \mathcal{G}_v) b_m \right) \beta_{mk} &= \frac{\alpha \mathcal{G}_u a_k}{2} \end{aligned} \quad (3.21)$$

The above system can be written in the form;

$$\mathbf{H} \frac{d\mathbf{a}}{dt} = \mathbf{G}\mathbf{a} \Rightarrow \frac{d\mathbf{a}}{dt} = \mathbf{H}^{-1} \mathbf{G}\mathbf{a} \quad (3.22)$$

$$\text{Where } \mathbf{H} = \begin{bmatrix} \mathbf{A}_{i,j} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{i,j} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{D}_{i,j} & \mathcal{H}_v \mathbf{E}_{i,j} \\ \mathcal{G}_u \mathbf{A}_{i,j} & \mathbf{C}_{i,j} \end{bmatrix}$$

$$\mathbf{E}_{i,j} = \beta_{i,j}, \quad \mathbf{A}_{m,k} = \begin{cases} \frac{1}{2}, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad \mathbf{D}_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ \frac{\mathcal{H}_u - (i\pi)^2}{2}, & \text{if } i = j \end{cases}$$

$$\mathbf{C}_{i,j} = (\mathcal{G}_v - d((j-1)\pi)^2) \beta_{i,j-1} \quad \text{With } i, j=1, 2,$$

This is an infinite system of equations, which we shall attempt to solve by making a Finite Dimensional Approximation of the system that is, truncating the series after a finite number of terms. We shall start by considering the first order terms (one term approximation) in order to gain some insight into the behaviour of the solutions of the infinite system. To start with, let $k = 0$ and $m = 1$, then we have the system

$$\left. \begin{aligned} \frac{da_o}{dt} &= \mathcal{H}_u a_o + \frac{4}{\pi} \mathcal{H}_v b_1 \\ \frac{db_1}{dt} &= (\mathcal{G}_v - d\pi^2) b_1 + \frac{\mathcal{G}_u \pi a_o}{4} \end{aligned} \right\} \quad (3.23)$$

This is a two-dimensional linear system and like in the scalar case, the solutions are given by

$$a_o(t) = s_1 e^{\lambda_1 t} + s_2 e^{\lambda_2 t}; \quad b_1(t) = s_1 E_1 e^{\lambda_1 t} + s_2 E_2 e^{\lambda_2 t}; \quad (3.24)$$

Where;

λ_1, λ_2 are the eigenvalues of the associated matrix system to (3.23), and satisfying the dispersion relation given by

$$\lambda^2 + C_1 \lambda + C_2 = 0, \quad \text{with} \quad \left. \begin{aligned} C_1 &= d\pi^2 - \gamma(f_u + g_v), \\ C_2 &= \gamma^2(f_u g_v - f_v g_u) - \gamma d f_u \pi^2 \end{aligned} \right\} \quad (3.25)$$

s_1 and s_2 are arbitrary constants obtain by the expansion of the initial conditions in a Fourier series and E_1, E_2 , components of eigenvectors corresponding the eigenvalues.

Results

Consider the following substitution

$$a = 0.1, b = 0.9, \quad (3.26)$$

Then with the Schnackenberg kinetics, the uniform steady state is given by $(u^s, v^s) = (1.0, 0.9)$ while the partial derivatives at the uniform steady states are given by;

We obtain the following matrices for one, two and three term approximations of the scalar and mixed boundary value problems.

$$f_u = \frac{b-a}{a+b} = 0.8, f_v = (a+b)^2 = 1.0, g_u = \frac{-2b}{a+b} = -1.8, g_v = -(a+b)^2 = -1.0$$

One term mixed BC.
One term scalar BC.

$$\begin{pmatrix} \mathcal{H}_u - (k\pi)^2 & \mathcal{H} \\ \mathcal{G}_u & \mathcal{G}_v - d\pi^2 \end{pmatrix}$$

Two terms mixed BC.

$$\begin{pmatrix} \mathcal{H}_u & \frac{4\mathcal{H}_v}{\pi} \\ \frac{\mathcal{G}_u \pi}{4} & \mathcal{G}_v - d\pi^2 \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{H}_u & 0 & 2\mathcal{H}_v & 0 \\ 0 & \mathcal{H}_u - \pi^2 & 0 & 4\mathcal{H}_v \\ \mathcal{G}_u & 0 & \mathcal{G}_v - d\pi^2 & 0 \\ 0 & \frac{3\pi\mathcal{G}_u}{8} & 240 & \mathcal{G}_v - 4d\pi^2 \end{pmatrix}$$

Three-term Mixed BC.

$$\begin{pmatrix} \mathcal{H}_u & 0 & 0 & \frac{2\mathcal{H}_v}{\pi} & 0 & \frac{2\mathcal{H}_v}{3\pi} \\ 0 & 4\mathcal{H}_u - \pi^2 & 0 & 0 & \frac{4\mathcal{H}_v}{3\pi} & 0 \\ 0 & 0 & \mathcal{H}_u - 4\pi^2 & \frac{-2\mathcal{H}_v}{3\pi} & 0 & \frac{6\mathcal{H}_v}{5\pi} \\ \frac{\mathcal{H}_u}{2\pi} \frac{27}{64} & 0 & \frac{\mathcal{H}_u}{2\pi} \left(\frac{-15}{64} \right) & \frac{21(\mathcal{H}_v - d\pi^2)}{16\pi^2} & 0 & 0 \\ 0 & \frac{\mathcal{H}_u}{2\pi} \frac{3}{4} & 0 & 0 & \frac{\mathcal{H}_v - 4d\pi^2}{\pi^2} & 0 \\ \frac{15}{64} \left(\frac{\mathcal{H}_u}{2\pi} \right) & 0 & \frac{\mathcal{H}_u}{2\pi} \frac{45}{64} & \frac{-15(\mathcal{H}_v - d\pi^2)}{16\pi^2} & 0 & \frac{\mathcal{H}_v - 9d\pi^2}{\pi^2} \end{pmatrix}$$

Table 1: Computed eigenvalues for the system with mixed and scalar boundary conditions for one-, two- and three-term approximation.

Parameters	Number of terms of approximation	Mixed Boundary Conditions (Real parts of eigenvalues)	Scalar Boundary Conditions (Real λ)
	1	0	$k\pi^2$, $k=1,2,\dots$
	2	0,0,0,- π^2	
	3	0,0,0,0,- π^2 , - $4\pi^2$	
	1	-4,-4	-6.931, -6.931
	2	-2.0, -2.0, -6.934, -6.935	
	3	-1.01, 2.05, 2.05, 0.14, -0.26, -0.38	
	1	1.19, -12.06	-10.37, -10.37
	2	2.34, -6.46, -13.2, -43.9	
	3	-1.618, -5.19, 3.61, -9.57, -35.39	
	1	-14.95, -14.95	-19.9, -19.9
	2	9.91, -14.95, -14.95, -79.3	
	3	-1.62, -1.62, 42.1, 42.1, 64.12, 64.12	
	1	6.4, -107.1	-2.01, -108.6

2	-2.09, 7.01, -107.7, -404.6	
3	-1.65, 9.66, 12.77, -31.59, 40.796, -90.87	
1	10.42, -113.1	0.05, -112.6
2	5.27, 20.68, 114.03, -413.9	
3	5.36, -14.8, 14.96, 23.89, -41.26, 91.58	

Discussion

From the dispersion relation (3.25) and the conditions for DDI obtained in the linear analysis, we make the following deductions;

If $d = 0$, it is clear that $C_1 > 0$ and $C_2 > 0$, hence there is no positive root for λ . This fact leads us to the conclusion that there can be no growing solution. Thus, all solutions will decay exponentially, leading to stability of the zero solution. Since $d = 0$ implies either the diffusion coefficient of the other species is zero, or D_1 is very large, the one term approximation reveals that no pattern can be observed when $d = 0$, even if the boundary conditions are mixed. However, we cannot make any conclusion yet as for the general case.

For equal diffusion coefficients that is, $d=1$ and by condition (2.23), we easily see that $C_1 > 0$ and since f is positive, it is possible to have $C_2 < 0$. If this is the case, then there will exist at least one positive root for λ . The existence of a positive root for λ indicates an exponentially growing solution in time. Therefore, for equal diffusion coefficients, a reaction diffusion system subjected to mixed boundary conditions can generate pattern. This was not the case for systems with scalar boundary conditions studied earlier. Thus, mixed boundary conditions can render a system, which is stable under scalar boundary conditions, unstable. However, as stated for the case $d = 0$, this conclusion is not good enough, since one term is a very crude approximation.

Now, suppose $d > 1$, then we see that $C_2 < 0$ for extremely small values of γ . This follows from the expression of C_2 as a function of γ .

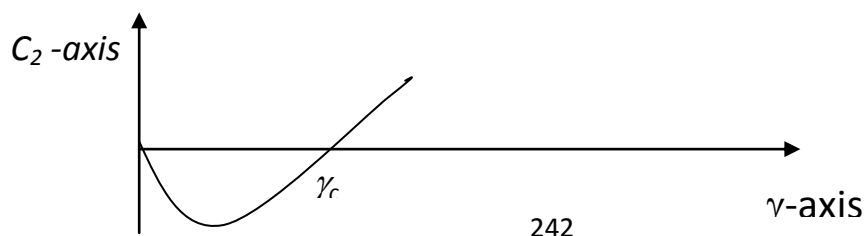


Fig.2 Sketch of C_2 as a function of γ showing the possibility of having unstable solutions for small values of γ .

From Fig.2, the graph of the C_2 is a parabola, concave upward and having 0 and γ_c as intercepts with the γ -axis (Fig.2). Above γ_c , C_2 becomes positive in conformity with the case of scalar boundary conditions. The fact, that $C_2 < 0$ for extremely small values for γ , leads to the observation of at least one growing solution for each value of γ between 0 and γ_c . Patterns can therefore be observed for extremely small values for γ , contrary to the case with scalar boundary conditions where patterns can only be observed for values of γ of a certain size as was seen in the linear analysis. Like in the case with scalar boundary conditions, we expect that the non-linear terms will bound the growth of these exponentially growing solutions for each of the conditions on d above. Since the one term approximation hardly gives any good results, to check however that the above trends persist for higher order terms, we consider an example in which many more terms are taken into consideration.

From the table 1, when $d = 0$ and $\gamma = 0$, a one term approximation of the system with mixed BC has all its eigenvalues zero so that nothing can be said about this case. Two- and three-term approximations of this system give one or two negative eigenvalues, respectively. This is an indication that for extremely small domains and zero diffusion, no pattern is possible. This is also true for the system with scalar BC.

Secondly, when $d = 0$ and $\gamma = 20$, the system with scalar BC and one-term and two-term approximation for the system with mixed BC gives all negative eigenvalues. But three-term approximation of the system with mixed BC shows that there are three positive eigenvalues, hence there exist three growing solutions. The emergence of three growing solutions indicates that the boundary conditions can, to some extent, destabilize a system even in the absence of diffusion.

When $d = 1$ and $\gamma = 5$, the system with scalar BC gives all negative eigenvalues, while for the system with mixed BC, one-term, two-term and three-term approximation gives one positive eigenvalue. Changing γ to 100, eliminate the positive eigenvalue for the one-term approximation, maintain it for the two-term while three more new positive eigenvalues emerge if three-term are considered in the approximation for mixed BC. The presence of one or four positive eigenvalues for the one-term, two-term or three-term approximation indicates the likelihood of pattern observation for the mixed boundary conditions. Thus, we see that when the BC is mixed, patterns can emerge even if the diffusion coefficients have the same magnitude.

For $d = 10$ and $\gamma = 10$, like in the previous cases, no positive eigenvalues is seen when the BCs are scalar but one-term and two-term approximation of the system with mixed BC gives one positive eigenvalue each, while three-term approximation gives three positive eigenvalues. Changing γ to 20 maintains the one

positive eigenvalue but add two and three more positive eigenvalues for the two-term and three-term approximation, respectively. This change also saw the emergence of one positive eigenvalue for the system with scalar BC. This shows that pattern formation is possible for the system with mixed BC for small γ but possible when the BC is scalar only when γ is of a certain size (i.e. for example, $\gamma=20$ in this example). By this example, we have shown, using finite dimensional analysis with one, two and three terms, that mixed boundary conditions can cause a system which is stable to be unstable.

For higher order terms, numerical simulation solves the problem better. When more terms are used in the finite dimensional approximation, we believe that the eigenvalues thus obtained will be better approximated. The convergence to the true eigenvalues, as the number of terms used in the series expansions (3.18) tend to infinity, is guaranteed by the expansion theorem (Currant and Hilbert, 1953). Thus as the number of terms used increases, the patterns will become more interesting as shown in a finite dimensional analysis by Ngwa (1994).

Numerical results

For the parameter sets presented above, we applied finite difference of the crack Nicason scheme to simulate the system with scalar and mixed boundary conditions. Some of the our results are presented below,

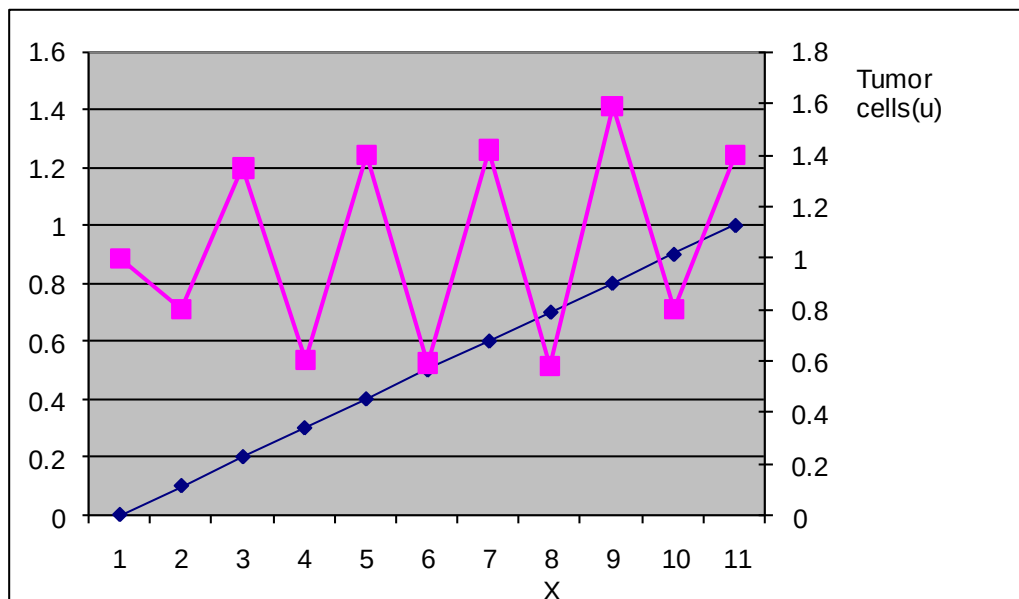
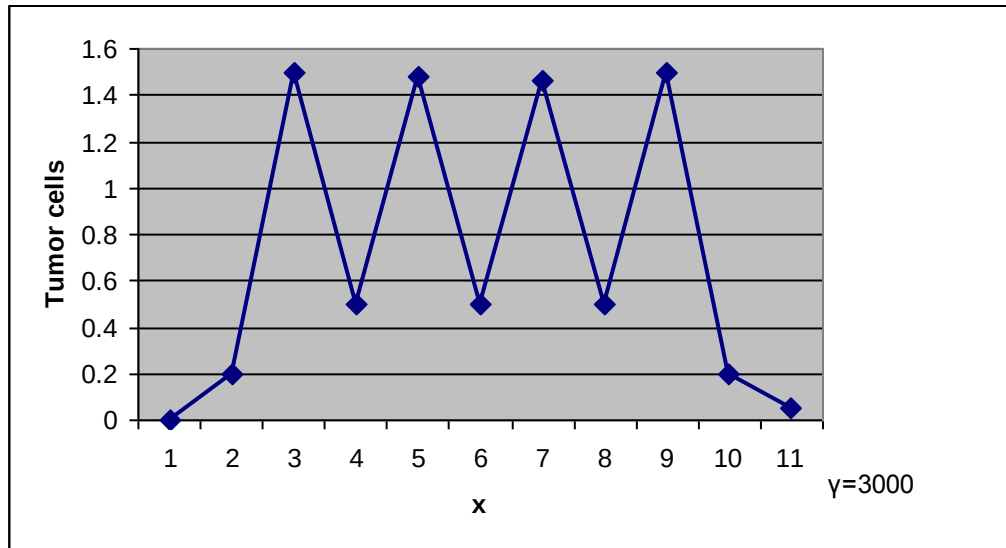


Figure A): The case with mixed boundary conditions for the tumour cells. The pattern looks more interesting than for scalar boundary conditions.



B) Scalar boundary conditions: The pattern looks more uniform than in A) where the boundary conditions are mixed.

The simulations when the scale parameter is 3000 for the tumour cells' population is presented in these figures A and b above. The simulations of the normal cells' population are similar with opposite polarity.

Finally, a comparison of these results indicates that given the same parameter regime within the Turing space for a reaction diffusion model for pattern formation, the boundary conditions have a significant effect on the observed pattern. In particular, for any given set of parameters, solutions selected when the boundary conditions are the same for each species may not be selected when the boundary conditions are different. This effect is most significant when one of the species is subjected to non-homogenous Dirichlet boundary conditions rather than when it is subjected to homogenous Dirichlet boundary condition.

Summary

It has been shown in this work that, the boundary condition plays an active role in driving a uniform steady state of a reaction diffusion system unstable. In particular, we showed that imposing mixed boundary conditions in Turing system greatly affects the system's behaviour. The gravity of the effect is more when the boundary conditions are non-homogeneous Dirichlet - Neumann than when it is homogeneous Dirichlet-Neumann. Similar trends were seen in Dillon *et al.* (1994), Miani and Myerscough (1996) and Ngwa (1994). The essential difference here lies in the complexity of the patterns and the fact that the scale

parameter γ plays a role in the pattern formation process. Since γ is a non-dimensional parameter which is a function of length, increasing the value of γ may be viewed as increasing the size of the domain. Therefore, we have shown that, not only for a small domain does the boundary condition gives rise to better patterns by also for a large domain.

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Mathematical Model of HIV/AIDS Pandemic with the Effect of Drug Application

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Abstract

In this paper we propose a mathematical model of the dynamics of HIV/AIDS pandemic and analyze the equilibrium states for stability. The total population of the community in view is partitioned into three distinct compartments of Susceptible, Removed and Infected classes, giving rise to a set of two ordinary differential equations and one partial differential equation. A parameter (k) is introduced to measure the effectiveness of anti-retroviral drugs application in slowing down the death of the members of infected class. It is observed that the zero equilibrium state will be stable if the birth rate is less than the death rate ($\beta < \mu$), while the non-zero equilibrium state, which is the state of population sustenance will be stable with the birth rate greater than the death rate ($\beta > \mu$) if k is high.

Keywords: *Stability, Equilibrium State, HIV/AIDS, Anti-retroviral Drugs*

Mathematics Subject Classification 2000:92D20 & 92C60

1.0 Introduction

Many aspect of human life can be transformed into a mathematical modeling. The solution to this transformation will then proffer a solution to that problem. The approach to this solution brings about special schemes, which may be analytical or numerical. Problems such as the existence of equilibrium states and their stability are of great interest in the mathematical models of population dynamics as pointed out by Akinwande [1]. In this work, we proposes a deterministic mathematical model of HIV/AIDS disease pandemic with the effects of drug application, which is a system of two ordinary and one partial differential equation. The population is partitioned into three compartments of the Susceptibles $S(t)$, Removed $R(t)$ ant the Infected $I(t)$.

The infected class $I(t)$ is structured by the infection age with the density function $\rho(t, \tau)$, where t is the time parameter and τ is the infection age. There is a maximum infection age T at which a member of the infected class $I(t)$ must leave the compartment via death; and so $0 \leq \tau \leq T$. This notwithstanding, a member of the class could still die by natural causes at a rate μ , the latter is also applicable to the Susceptibles class $S(t)$ and the Removed class $R(t)$.

Members of the class $S(t)$ move into the class $R(t)$ due to change in behaviour or/and as a result of effective public campaign at a rate γ .

The death rate via infection as in [2] is given by $\delta e^{-k(T-\tau)}$ where δ is the maximum death rate due to infection while k is a control parameter which could be associated with the measure of slowing down the death of the infected, such as the effectiveness of anti-retroviral drugs which give the victims longer life-span. A high value of k means high effectiveness of such measure and vice versa.

It is assumed that while the new births in $S(t)$ are born as Susceptibles, the offspring of $I(t)$ are divided between $S(t)$ and $I(t)$ in the proportions θ and $(1-\theta)$ respectively, i.e. a proportion $(1-\theta)$ of the offspring of $I(t)$ are born with the virus.

2.0 Model Equations

The model equations are given by (2.1) to (2.8) below; with the infected class structured into infection age using the pattern of Gurtin and MacCamy [6].

$$P(t) = S(t) + R(t) + I(t) \quad (2.1)$$

$$\frac{dS(t)}{dt} = \beta(S(t) + R(t)) - (\gamma + \mu)S(t) + \theta\beta I(t) - \alpha S(t)I(t) \quad (2.2)$$

$$\frac{dR(t)}{dt} = \gamma S(t) - \mu R(t) \quad (2.3)$$

$$\frac{\partial p(t, \tau)}{\partial t} + \frac{\partial p(t, \tau)}{\partial \tau} + \sigma(\tau)p(t, \tau) = 0 \quad (2.4)$$

$$\sigma(\tau) = \mu + \delta e^{-k(T-\tau)} \quad (2.5)$$

$$I(t) = \int_0^T p(t, \tau) d\tau \quad (2.6)$$

$$p(t, 0) = B(t) = \alpha S(t)I(t) + (1-\theta)\beta I(t); p(0, \tau) = \phi(\tau) \quad (2.7)$$

$$S(0) = S_0; R(0) = R_0; I(0) = I_0 \quad (2.8)$$

The parameters are defined as follows:

β = natural birth rate for the population $P(t)$.

μ = natural death-rate for the population $P(t)$.

α = rate of contracting the HIV virus, via interaction of $S(t)$ and $I(t)$.

$\sigma(\tau)$ = the gross death rate of the infected class $I(t)$.

δ = maximum death rate from infection for the class $I(t)$.

k = measure of the effectiveness of application of anti-retroviral drugs at slowing down the death of infected members in $I(t)$.

γ = rate of removal of the Susceptible $S(t)$ into the Removed class $R(t)$; due to public campaign; i.e. the measure of the effectiveness of the public campaign against infection.

θ = the proportion of the offspring of the infected class $I(t)$ which are virus-free at birth; with $0 \leq \theta \leq 1$.

t = time; τ = infection age.

T = maximum infection age; it is assumed that when $\tau = T$ the infected member dies of the disease.

3 Equilibrium States of the Model

At the equilibrium states let

$$S(0) = x; R(0) = y; I(0) = z \quad (3.1)$$

So that from (2.1) and (2.9) after much algebraic simplifications, we have the equilibrium states as:

(a) the zero equilibrium states given by $(x, y, z) = (0, 0, 0)$, and

(b) The non-zero equilibrium states (x, y, z) given by :

$$x = \frac{1 - (1 - \theta)\beta\bar{\pi}}{\alpha\bar{\pi}} \quad (3.2)$$

$$y = \frac{\gamma[1 - (1 - \theta)\beta\bar{\pi}]}{\alpha\mu\bar{\pi}} \quad (3.3)$$

$$z = \frac{(\beta - \mu)(\gamma + \mu)(1 - (1 - \theta)\beta\bar{\pi})}{\alpha\mu(1 - \beta\bar{\pi})} \quad (3.4)$$

4 The Characteristic Equation

We perturb the equilibrium state as follows: Let

$$S(t) = x + p(t); p(t) = \bar{p} e^{\lambda t} \quad (4.1)$$

$$R(t) = y + r(t); r(t) = \bar{r} e^{\lambda t} \quad (4.2)$$

$$I(t) = z + q(t); q(t) = \bar{q} e^{\lambda t} \quad (4.3)$$

Where $\bar{p}, \bar{q}, \bar{r}$ are constants.

From (4.1) - (4.3) and (2.1) and (2.9) after much algebraic simplifications, we have

$$(\beta - \gamma - \mu - \lambda - \alpha z) \bar{p} + \beta \bar{r} + (\theta \beta - \alpha x) \bar{q} = 0 \quad (4.4)$$

$$\gamma \bar{p} - (\lambda + \mu) \bar{r} = 0 \quad (4.5)$$

$$\alpha z b(\lambda) \bar{p} + \{(\alpha x + (1 - \theta) \beta) b(\lambda) - 1\} \bar{q} = 0 \quad (4.6)$$

With

$$b(\lambda) = \int_0^T \exp \left\{ - \int_0^\tau (\lambda + \sigma(s) ds) \right\} d\tau \quad (4.7)$$

The coefficients of $\bar{p}, \bar{r}$ and $\bar{q}$ in (4.4) - (4.7) give the Jacobian determinant for the system with the eigenvalue λ .

$$\begin{vmatrix} \beta - \gamma - \mu - \lambda - \alpha z & \beta & \theta \beta - \alpha x \\ \gamma & -(\lambda + \mu) & 0 \\ \alpha z b(\lambda) & 0 & [\alpha x + (1 - \theta) \beta] b(\lambda) - 1 \end{vmatrix} = 0 \quad (4.8)$$

and the characteristics equation for the model is therefore given by

$$-(\beta - \gamma - \mu - \lambda - \alpha z)(\lambda + \mu)\{(\alpha x + (1 - \theta) \beta) b(\lambda) - 1\} - \gamma \beta \{(\alpha x + (1 - \theta) \beta) b(\lambda) - 1\} + \alpha z (\lambda + \mu) (\theta \beta - \alpha x) b(\lambda) = 0$$

i.e.

$$\{(\lambda + \mu)(\beta - \gamma - \mu - \lambda - \alpha z) + \gamma \beta\} \{(\alpha x + (1 - \theta) \beta) b(\lambda) - 1\} - \alpha z (\lambda + \mu) (\theta \beta - \alpha x) b(\lambda) = 0 \quad (4.9)$$

5 Stability of the Equilibrium States

5.1 Stability of the Zero Equilibrium State

At the zero equilibrium state $(x, y, z) = (0, 0, 0)$. The characteristics equation (4.9) takes the form:

$$\{(\lambda + \mu)(\beta - \gamma - \lambda - \mu) + \beta \gamma\} \{(1 - \theta) \beta b(\lambda) - 1\} = 0 \quad (5.1)$$

i.e. either

$$(\lambda + \mu)(\beta - \gamma - \lambda - \mu) + \beta \gamma = 0 \quad (5.2)$$

or

$$(1 - \theta) \beta b(\lambda) - 1 = 0 \quad (5.3)$$

Now consider (5.2)

$$(\lambda + \mu)(\beta - \gamma - \lambda - \mu) + \beta\gamma = 0$$

$$= \frac{1}{2}\{(\beta - \mu) - (\gamma + \mu) \pm [(\beta - \mu) + (\gamma + \mu)]\}$$

i.e.

$$\lambda_1 = \frac{1}{2}\{(\beta - \mu) - (\gamma + \mu) + [(\beta - \mu) + (\gamma + \mu)]\}$$

$$= \frac{1}{2}\{2(\beta - \mu)\}$$

$$= \beta - \mu$$

and

$$\lambda_2 = \frac{1}{2}\{(\beta - \mu) - (\gamma + \mu) - [(\beta - \mu) + (\gamma + \mu)]\}$$

$$= \frac{1}{2}\{-2(\gamma + \mu)\}$$

$$= -(\gamma + \mu)$$

This shows that:

$$\lambda_1 < 0 \text{ if } \beta < \mu \quad (5.4)$$

$$\lambda_2 < 0 \quad (5.5)$$

In order to investigate the nature of the root of the transcendental equation (5.3), we first of all consider $b(\lambda)$

from equation (4.7)

$$b(\lambda) = \int_0^T \exp\left\{-\int_0^\tau (\lambda + \sigma(s))ds\right\} d\tau$$

As applied in [2], the result of Bellman and Cooke [3] is used next to analyze equation (5.3) for stability or otherwise of the zero equilibrium state. Let equation (5.3) takes the form:

$$J_1(k) = (1 - \theta)\beta(\mu + \tan \frac{\pi}{2k}) - \mu(\mu + \tan \frac{\pi}{2k})$$

$$- (1 - \theta)\beta\mu \exp\left(\frac{2Tk}{\pi} \ln \left|\cos \frac{\pi}{2k}\right| - \mu T\right) \quad (5.6)$$

So the origin will be stable when $\beta < \mu$ and $J_1(k) < 0$.

Using Mathcad, hypothetical parameter values were used to generate a table of values for $J_1(k)$, so as to verify the result of the analysis. Some of the values obtained are presented in the table 5.1 below:

$$\delta = 0.3, \theta = 0.4, T = 10$$

K	J ₁ (k)		J ₁ (k)		J ₁ (k)		J ₁ (k)	
	$\beta = 0.15 \quad \mu = 0.25$		$\beta = 0.14 \quad \mu = 0.28$		$\beta = 0.12 \quad \mu = 0.36$		$\beta = 0.45 \quad \mu = 0.15$	
0.1	0.1215588	S	0.1571033	S	0.2779776	S	-0.0018466	I
0.2	0.0823893	S	0.1116380	S	0.2140536	S	-0.0127084	I
0.3	0.0692963	S	0.0962944	S	0.1921092	S	-0.0155930	I
0.4	0.0646251	S	0.0908006	S	0.1842136	S	-0.0164622	I
0.5	0.0628980	S	0.0887692	S	0.1813038	S	-0.0167103	I
0.6	0.0622382	S	0.0879954	S	0.1802092	S	-0.0167532	I
0.7	0.0619732	S	0.0876868	S	0.1797842	S	-0.0167307	I
0.8	0.0618570	S	0.0875532	S	0.1796092	S	-0.0166911	I
0.9	0.0617986	S	0.0874873	S	0.1795292	S	-0.0166504	I
1.0	0.0617640	S	0.0874489	S	0.1794866	S	-0.0166135	I

S and I implies Stability and Instability respectively.

Table 5.1

From table 5.1 above, it can be seen that :

1. $J_1(k) > 0$ when $\beta < \mu$
2. $J_1(k) < 0$ when $\beta > \mu$

Note however that the result presented in table 5.1 above is for $\delta = 0.3, \theta = 0.4$; the profile remains the same when these values range from 0 to 1.

5.2 Stability of the Non-zero Equilibrium State

In order to analyze the non-zero state for stability, we shall similarly apply the result of Bellman and Cooke [3] to equation (4.9), taking it in the form

$$H_2(\lambda) = \{(\lambda + \mu)(\beta - \gamma - \mu - \lambda - \alpha z) + \gamma \beta\} \{(\alpha x + (1 - \theta)\beta)b(\lambda) - 1\} - \alpha z(\lambda + \mu)(\theta \beta - \alpha x)b(\lambda) = 0 \quad (5.7)$$

If we set $\lambda = iw$, we have that

$$H_2(iw) = F_2(w) + iG_2(w) \quad (5.8)$$

The condition for $\text{Re } \lambda < 0$, will then be given by the inequality

$$F_2(0)G_2'(0) - F_2'(0)G_2(0) > 0 \quad (5.9)$$

$$\begin{aligned}
 F_2(0) = & \mu(\beta - \gamma - \mu - \alpha z)(\alpha x + \beta + \theta\beta)\bar{\pi} \\
 & - \mu(\beta - \gamma - \mu - \alpha z) \\
 & + \gamma\beta(\alpha x + \beta + \theta\beta)\bar{\pi} \\
 & - \gamma\beta \\
 & - \alpha z\mu(\theta\beta - \alpha x)\bar{\pi}
 \end{aligned} \tag{5.10}$$

$$G_2(0) = 0 \tag{5.11}$$

$$F_2'(0) = 0 \tag{5.12}$$

$$\begin{aligned}
 G_2'(0) = & -\mu(\beta - \gamma - \mu - \alpha z)(\alpha x + \beta + \theta\beta)A \\
 & + (\beta - \gamma - 2\mu - \alpha z)(\alpha x + \beta + \theta\beta)\bar{\pi} \\
 & - (\beta - \gamma - 2\mu - \alpha z) \\
 & - \gamma\beta(\alpha x + \beta + \theta\beta)A \\
 & + \alpha z\mu(\theta\beta - \alpha x)A \\
 & - \alpha z(\theta\beta - \alpha x)\bar{\pi}
 \end{aligned} \tag{5.13}$$

Since $G_2(0) = 0$ and $F_2'(0) = 0$, the inequality (4.32) then gives

$$F(0)G'(0) > 0 \tag{5.14}$$

Let

$$J_2(k) = F(0)G'(0) \tag{5.15}$$

So the non-zero state will be stable when

$$J_2(k) > 0 \tag{5.16}$$

Using Mathcad, hypothetical parameter values were used to generate a table of values for $J_2(k)$, so as to verify the result of the analysis. Some of the values obtained are presented in the table 5.2 below:

$\alpha = 0.3, \delta = 0.3, \gamma = 0.45, \theta = 0.4, T = 10$

K	J ₂ (k)		J ₂ (k)		J ₂ (k)		J ₂ (k)	
	$\beta = 0.44 \quad \mu = 0.22$		$\beta = 0.29 \quad \mu = 0.13$		$\beta = 0.45 \quad \mu = 0.15$		$\beta = 0.15 \quad \mu = 0.25$	
0.1	-0.0005130	I	-0.0002484	I	-0.0000034	I	-0.0007557	I
0.2	-0.0000264	I	-0.0000109	I	0.0006221	S	-0.0004799	I
0.3	0.0001202	S	0.0000320	S	0.0009467	S	-0.0004027	I
0.4	0.0001727	S	0.0000484	S	0.0010693	S	-0.0003779	I

0.5	0.0001877	S	0.0000533	S	0.0011054	S	-0.0003694	I
0.6	0.0001893	S	0.0000539	S	0.0011099	S	-0.0003665	I
0.7	0.0001866	S	0.0000532	S	0.00111042	S	-0.0003655	I
0.8	0.0001829	S	0.0000521	S	0.0010960	S	-0.0003652	I
0.9	0.0001793	S	0.0000510	S	0.0010879	S	-0.0003652	I
1.0	0.0001761	S	0.0000500	S	0.0010807	S	-0.0003653	I

S and I imply Stability and Instability respectively.

Table 5.2

From table 5.2 above, it can be seen that:

1. $J_2(k) < 0$ when $\beta < \mu$
2. $J_2(k) < 0$ when $\beta > \mu$ and k is low
3. $J_2(k) > 0$ when $\beta > \mu$ and k is high

Note however that the result presented in table 5.2 above is for $\alpha=0.5$, $\delta = 0.3$, $\gamma=0.55$, $\theta = 0.4$; the profile remains the same when these values range from 0 to 1.

6. Conclusion and Recommendation

6.1 Conclusion

From the above observations, it can be seen that the zero equilibrium state which is the state of population extinction will be stable when the birth rate is unusually less than the death rate, i.e. when there is a way of speedily replenishing the population and lower number of child per family, which will eventually threatened a nation's population.

However, in situations where the birth rate is greater than the death rate and the application of antiretroviral drugs (k) is low, the non-zero equilibrium state will be unstable. But a high level of k revealed stability of the non-zero state, which is the state of population sustenance. Hence, we can conclude that once the virus is introduced into a population, the application of anti-retroviral drugs can at best slow down the eventual extinction of that population.

6.2 Recommendation

1. Controlling HIV requires our collective global commitment—governmental, societal, and personal by:
 - (i) Ensuring access for all to life-sustaining drugs, so that HIV-positive parents may provide and care for their children.

- (ii) Reducing AIDS-related stigma and discrimination, so that more people will get tested for HIV and receive prevention counseling
 - (iii) Respecting and enforcing the rights of women, so that they may control their bodies, reject unwanted sexual advances, and insist upon the use of condoms to protect against HIV infection
 - (iv) Ending modern-day slavery and reducing the spread of HIV by eradicating human sex trafficking
2. Further research work can be carried out such as the economic, social, or political effect of HIV/AIDS in national development with computer simulation.

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Simulation of temperature distribution and solidification fronts in squeeze cast commercially pure aluminium

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Abstract

This paper simulates the casting temperature of squeeze cast aluminium metal while varying squeeze cast parameters. The effects of applied pressures, die temperatures, delay time and retention time of pressure application on the temperature distribution and solidification fronts of squeeze cast aluminium was simulated. The simulation of temperature distribution was for both the cast metal and steel mould. A model was developed based on the heat transfer equations coupled with internal heat generation due to pressure applications and moving phase change energy balance equations. The computer simulation was used to track the temperature distribution and solidification fronts with time during solidification of molten aluminium in a steel mould. The result shows an increase in solidifying temperature with time as values of applied pressure increase. Longer pressure retention time and shorter delay time result in a higher peak solidifying temperature during squeeze casting of molten aluminium. The solidification time increases with increase in die pre-heat temperature while shorter delay times leads to higher peak solidifying temperature. The comparison between the simulated and the experimental results shows a fairly good agreement with the simulated temperatures slightly higher.

Keyword: solidification temperature, applied pressure, die pre-heat, squeeze casting

Mathematics Subject Classification 2000:35K05 & 35Q80

1. Introduction

Casting is the process of melting and pouring of molten metal into the formed mould cavity, which on solidification takes the shape of the formed shape. Squeeze casting operation involves pouring liquid molten metal in a steel mould cavity and compressing it under pressure as it solidifies. This process offers advantage where large and intricate shapes, which could not have been economically formed in one piece by either forging or welding could be obtained. Through squeeze casting, the molten metal solidifies rapidly leading to a high degree of undercooling, with the formation of small grains. Maleki et al (2006) discovered that products of squeeze casting produce considerable improvements in the macrostructure and hardness of the samples. Sand casting cools slowly, due to the insulating properties of the sand mould. Squeeze casting solidifies quickly because of the contact of the molten metal with the metal mould as

described by Higgins (1983). Heat is rapidly dissipated to the steel mould in contact with the molten metal, which is convected out at the outer surface of the steel mould.

The commercially pure aluminium metal used for this research work finds extensive use in the building, manufacturing and process industries, both as a material of construction and household goods.

Radlbeck (2004) submitted that remelting used aluminium requires only 5 per cent of the energy needed to produce primary metal. Thus, rather than contributing to society's growing waste problem, aluminium can be remelted and reformed to produce a new generation of building parts. Nearly 40 per cent of all aluminium in use is remelted metal as Radlbeck, C et al (2004) concluded. In general, however, aluminium products do not need to be protected by organic coatings used to safeguard some alternative materials. They therefore offer a source of good metal which can be recycled without any pre-processing.

Squeeze cast products of aluminium are of improved mechanical properties and could be given heat treatment. Aluminium is indeed replacing the more expensive brass in many applications.

2.0 THEORETICAL ANALYSIS

An algorithm was developed to monitor the solidification front and temperature distribution of squeeze cast aluminium. Heat transfer equations were generated at both the cast region and interfaces. The partial differential equations generated were solved using finite difference method (fdm). In the numerical equations generated, squeeze-casting parameters such as applied pressure, die pre-heat temperature, delay and retention times of the applied pressure were considered.

2.1 Assumptions made

1. Heat transfer in the cast zone is due to conduction with convection heat transfer occurring at the outer surface of the steel mould.
2. One dimensional heat transfer process is assumed.
3. Considering the symmetrical nature of the cast specimen, solidification process was assumed symmetrical and only one half of the specimen's thickness is analysed.
4. The bottom of the squeeze casting rig is lagged and the heat losses to the atmosphere is small and neglected.
5. Density of the molten and solidified aluminium metal is assumed to be the same and independent of temperature.
6. Thermal conductivity and specific heat of cast aluminium metal are dependent on the cast temperatures of the metal.

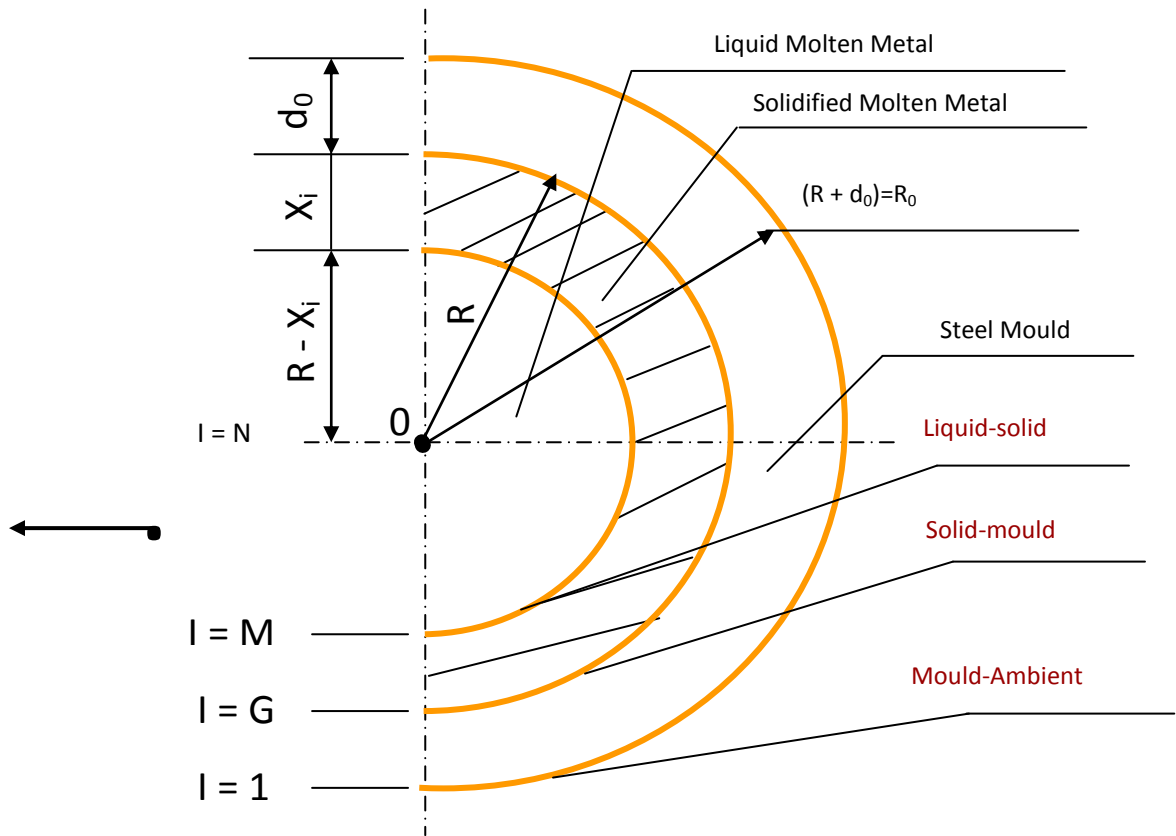


Figure 1. Schematic representation of solidification front

2.2 Governing equations

The governing equations are heat transfer equations due mainly to conduction and convection in one-dimension with phase change boundary effects and energy balances at the interfaces. The steel mould, the solidified metal and the liquid molten metal portions were discretized separately (see figure 1) and the rate of change with time defined.

(a) In the steel mould,

Within the steel mould, there is pure conduction,

$$\rho_{st} C_{st} \frac{\partial T_{st}}{\partial t} = K_{st} \left[\frac{\partial^2 T_{st}}{\partial r^2} + \frac{1}{r_{st}} \frac{\partial T_{st}}{\partial r} \right] \quad 1$$

this equation is applicable within the region defined by;

$$R \leq r_{st} \leq (R + d_0)$$

The finite difference form of equation (1) is;

$$T_i^{j+1} = \left[1 - \frac{2\delta\alpha_{st}}{d_{st}^2} - \frac{\delta\alpha_{st}}{r_{st}d_{st}} \right] T_i^j + \left[\frac{\delta\alpha_{st}}{d_{st}^2} + \frac{\delta\alpha_{st}}{r_{st}d_{st}} \right] T_{i+1}^j + \frac{\delta\alpha_{st}}{d_{st}^2} T_{i-1}^j \quad 2$$

where,

$$r_{st} = R + \frac{d_0(G-I)}{(G-1)} \quad 2a$$

$$d_{st} = \frac{d_0}{(G-I)} \quad 2b$$

$$\alpha_{st} = \frac{K_{st}}{\rho_{st} C_{st}} \quad 2c$$

$$I = 1, 2, 3, \dots, (G-1)$$

(b) In the solidified metal region,

$$\rho_s C_s \frac{\partial T_s}{\partial t} = K_s \left[\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r_s} \frac{\partial T_s}{\partial r} \right] \quad 3$$

within the region defined by;

$$(R - X_r^j) \leq r_s \leq R$$

the boundary condition;

$$T_s = T_{MM} = 660^\circ C \quad 3a$$

A finite difference form of solidified molten metal portion from equation (3) becomes;

$$T_i^{j+1} = \left[1 - \frac{(I-G)}{(M-G)} \frac{(X_r^{j+1} - X_r^j)}{d_s} - \frac{2\delta\alpha_s}{d_s^2} - \frac{\delta\alpha_s}{r_s d_s} \right] T_i^j + \left[\frac{(I-G)}{(M-G)} \frac{(X_r^{j+1} - X_r^j)}{d_s} + \frac{\delta\alpha_s}{d_s^2} + \frac{\delta\alpha_s}{r_s d_s} \right] T_{i+1}^j + \frac{\delta\alpha_s}{d_s^2} T_{i-1}^j \quad 5$$

where,

$$r_s = R - \frac{X_r^j(I-G)}{(M-G)} \quad 5a$$

$$d_s = \frac{X_r^j}{(M-I)} \quad 5b$$

$$\alpha_s = \frac{K_s}{\rho_s C_s} \quad 5c$$

$$I = G+1, G+2, G+3, \dots, (M-1)$$

(c) Within the liquid metal region,

$$\rho_L C_L \frac{\partial T_L}{\partial t} = K_L \left[\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r_L} \frac{\partial T_L}{\partial r} \right] \quad 6$$

this equation is applicable within the region defined by;

$$0 \leq r_L \leq (R - X_r^j)$$

with the boundary condition;

$$K_s \frac{\partial T_s}{\partial r} = 0 \quad 6a$$

$$r = 0 \quad 6b$$

$$T_L = T_p = 720^\circ C \quad 6c$$

Finite difference formulation of equation (6) is;

$$T_i^{j+1} = \left[1 - \frac{(I-M)}{(N-M)} \frac{(X_r^{j+1} - X_r^j)}{d_L} - \frac{2\delta\alpha_L}{d_L^2} - \frac{\delta\alpha_L}{r_L d_L} \right] T_i^j + \left[\frac{(I-M)}{(N-M)} \frac{(X_r^{j+1} - X_r^j)}{d_L} + \frac{\delta\alpha_L}{d_L^2} + \frac{\delta\alpha_L}{r_L d_L} \right] T_{i+1}^j + \frac{\delta\alpha_L}{d_L^2} T_{i-1}^j \quad 7$$

where,

$$r_L = (R - X_r^j) \frac{(I-M)}{(N-M)} \quad 7a$$

$$d_L = \frac{(R - X_r^j)}{(N-I)} \quad 7b$$

$$\alpha_L = \frac{K_L}{\rho_L C_L} \quad 7c$$

$$I = M, M+1, M+2, M+3, \dots, (N-1)$$

(d) At the phase change boundary condition,

$$\rho_L L_f \frac{dX_r^j}{dt} = K_L \frac{\partial T_L}{\partial r} - K_s \frac{\partial T_s}{\partial r} \quad 8$$

where,

$$r = R - X_r^j \quad 8a$$

The finite difference formulation is;

$$X_r^{j+1} = X_r^j - \left[\frac{\delta K_L}{\rho_L L_f h_{PL}} + \frac{\delta K_S}{\rho_L L_f h_{PS}} \right] T_i^j + \frac{\delta K_L}{\rho_L L_f h_{PL}} T_{i+1}^j + \frac{\delta K_S}{\rho_L L_f h_{PS}} T_{i-1}^j \quad 9$$

$$\text{where,} \quad h_{PL} = \frac{(R - X_r^j)}{(N - M)} \quad 9a$$

$$h_{PS} = \frac{X_r^j}{(M - G)} \quad 9b$$

(e) Energy balances equations

The principle of energy balance is derived from equating the amount of heat loss by one phase to the amount of heat gained by another adjacent phase occurring at the interfaces.

i. Steel mould-atmosphere interface (I = 1);

$$\frac{\partial T_{st}}{\partial t} = \frac{2K_{st}}{\rho_{st} C_{st} d_{st}} \frac{\partial T_{st}}{\partial r} - \frac{2H^*}{\rho_{st} C_{st} d_{st}} (T_i^j - T_\infty) \quad 10a$$

The finite difference formulation of equation (10a) is;

$$T_i^{j+1} = \left[1 - \frac{2\delta K_{st}}{d_{st}^2 \rho_{st} C_{st}} \right] T_i^j + \frac{2\delta K_{st}}{d_{st}^2 \rho_{st} C_{st}} T_{i+1}^j - \frac{2\delta H^*}{d_{st} \rho_{st} C_{st}} (T_i^j - T_\infty^i) \quad 10b$$

ii. In the solidified metal-steel mould interface (I = G);

$$K_S \frac{\partial T_s}{\partial r} + \frac{1}{2} \rho_S C_S d_S \frac{\partial T_s}{\partial t} = K_{st} \frac{\partial T_{st}}{\partial r} + \frac{1}{2} \rho_{st} C_{st} d_{st} \frac{\partial T_{st}}{\partial t} \quad 11a$$

Expressing equation (11a) in finite difference form;

$$T_i^{j+1} = a \left[\frac{2\delta K_S}{d_S} + d_S \rho_S C_S + \rho_S C_S \frac{(I - G)}{(M - G)} (X_r^{j+1} - X_r^j) + \frac{2\delta K_{st}}{d_{st}} - d_{st} \rho_{st} C_{st} \right] T_i^j \\ + a \left[\rho_S C_S \frac{(I - G)}{(M - G)} (X_r^{j+1} - X_r^j) - \frac{2\delta K_S}{d_S} \right] T_{i+1}^j - \frac{2a\delta K_{st}}{d_{st}} T_{i-1}^j \quad 11b$$

where,

$$a = \frac{1}{(d_S \rho_S C_S - d_{st} \rho_{st} C_{st})} \quad 11c$$

iii. In the liquid metal – solidified metal interface, (I = M);

$$K_L \frac{\partial T_L}{\partial r} - \frac{1}{2} \rho_L C_L d_L \frac{\partial T_L}{\partial t} = K_S \frac{\partial T_S}{\partial r} + \frac{1}{2} \rho_S C_S d_S \frac{\partial T_S}{\partial t} \quad 12a$$

THE FINITE DIFFERENCE FORMULATION OF EQUATION (12A) IS;

$$T_i^{j+1} = b \left[d_L \rho_L C_L - \rho_L C_L \frac{(I-M)}{(N-M)} (X_r^{j+1} - X_r^j) - \frac{2\delta K_L}{d_L} + d_S \rho_S C_S - \frac{2\delta K_S}{d_S} + \right] T_i^j$$

$$+ b \left[\rho_S C_S \frac{(I-G)}{(M-G)} (X_r^{j+1} - X_r^j) \right] T_{i-1}^j + b \left[\frac{2\delta K_S}{d_S} - \rho_S C_S \frac{(I-G)}{(M-G)} (X_r^{j+1} - X_r^j) \right] T_{i-1}^j + b \left[\frac{2\delta K_L}{d_L} + \rho_L C_L \frac{(I-M)}{(N-M)} (X_r^{j+1} - X_r^j) \right] T_{i+1}^j \quad 12b$$

where,

$$b = \frac{1}{(d_S \rho_S C_S + d_L \rho_L C_L)} \quad 12c$$

(f) First time analysis

At the first time analysis, the mould cavity is assumed filled with the molten metal before pressure is applied. The equation is expressed as;

$$\rho_L C_L \frac{dT_L}{dt} = K_L \left[\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r_{FL}} \frac{\partial T_L}{\partial r} \right] \quad 13a$$

Finite difference form of equation (13a) is;

$$T_i^{j+1} = \left[1 - \frac{2\delta \alpha_L}{d_{FL}^2} - \frac{\delta \alpha_L}{r_{FL} d_{FL}} \right] T_i^j + \left[\frac{\delta \alpha_L}{d_{FL}^2} + \frac{\delta \alpha_L}{r_{FL} d_{FL}} \right] T_{i+1}^j + \frac{\delta \alpha_L}{d_{FL}^2} T_{i-1}^j \quad 13b$$

where,

$$r_{FL} = \frac{(N-I)}{(N-G)} R \quad 13c$$

$$d_{LF} = \frac{R}{(N-G)} \quad 13d$$

$$I = G+1, G+2, G+3, \dots, (N-1)$$

(g) Completion of solidification

At the completion of solidification, the cast metal becomes solidified and the equation is defined by;

$$\rho_S C_S \frac{\partial T_L}{\partial t} = K_S \left[\frac{\partial^2 T_S}{\partial r^2} + \frac{1}{r_{CS}} \frac{\partial T_S}{\partial r} \right] \quad 14a$$

Finite difference form of equation (14a) at completion of solidification becomes;

$$T_i^{j+1} = \left[1 - \frac{2\delta\alpha_s}{d_{CS}^2} - \frac{\delta\alpha_s}{r_{CS}d_{CS}} \right] T_i^j + \left[\frac{\delta\alpha_s}{d_{CS}^2} + \frac{\delta\alpha_s}{r_{CS}d_{CS}} \right] T_{i+1}^j + \frac{\delta\alpha_s}{d_{CS}^2} T_{i-1}^j \quad 14b$$

$$\text{where;} \quad r_{CS} = \frac{(N-I)}{(N-G)} R \quad 14c$$

$$d_{CS} = \frac{R}{(N-G)} \quad 14d$$

$$I = G+1, G+2, G+3, \dots, (N-1)$$

2.3 Stability criterion

The stability criteria of the finite difference equations are defined such that the coefficients of T_i^j in the partial differential equation do not contribute negatively to the Finite Deference Equations.

2.4 Effect of pressure application

With pressure application on the cast metal, an internal energy Δq was generated within the solidified molten metal, the effect of which is temperature rise (see figure 2). The internal energy generated resulted from the plastic strain energy and frictional energy at the interfaces.

In the works of Franklin and Das (1984), pressure is best applied five minutes after pouring of molten metal into the mould cavity. Yang (2007), noted that the shorter the solidification time, the higher is the value of impact energy in squeeze cast aluminium. The governing heat transfer equation that applies takes the form of solidified metal as described by White (1991) and Ozisik (1985);

$$\frac{dT_s}{dt} = \alpha_s \frac{\partial^2 T_s}{\partial r^2} + \frac{\alpha_s}{r_{CS}} \frac{\partial T_s}{\partial r} + \frac{\Delta q}{\rho_s C_s} \quad 15a$$

where,

$$\nabla q = \nabla q_p + \nabla q_f \quad 15b$$

Δq -internal energy generated due to pressure application,

Δq_p -energy due to plastic strain within the solidified molten metal material,

Δq_f -frictional energy generated during pressure application,

$$\nabla q_f = \nabla q_{fp} + \nabla q_{fm} \quad 15c$$

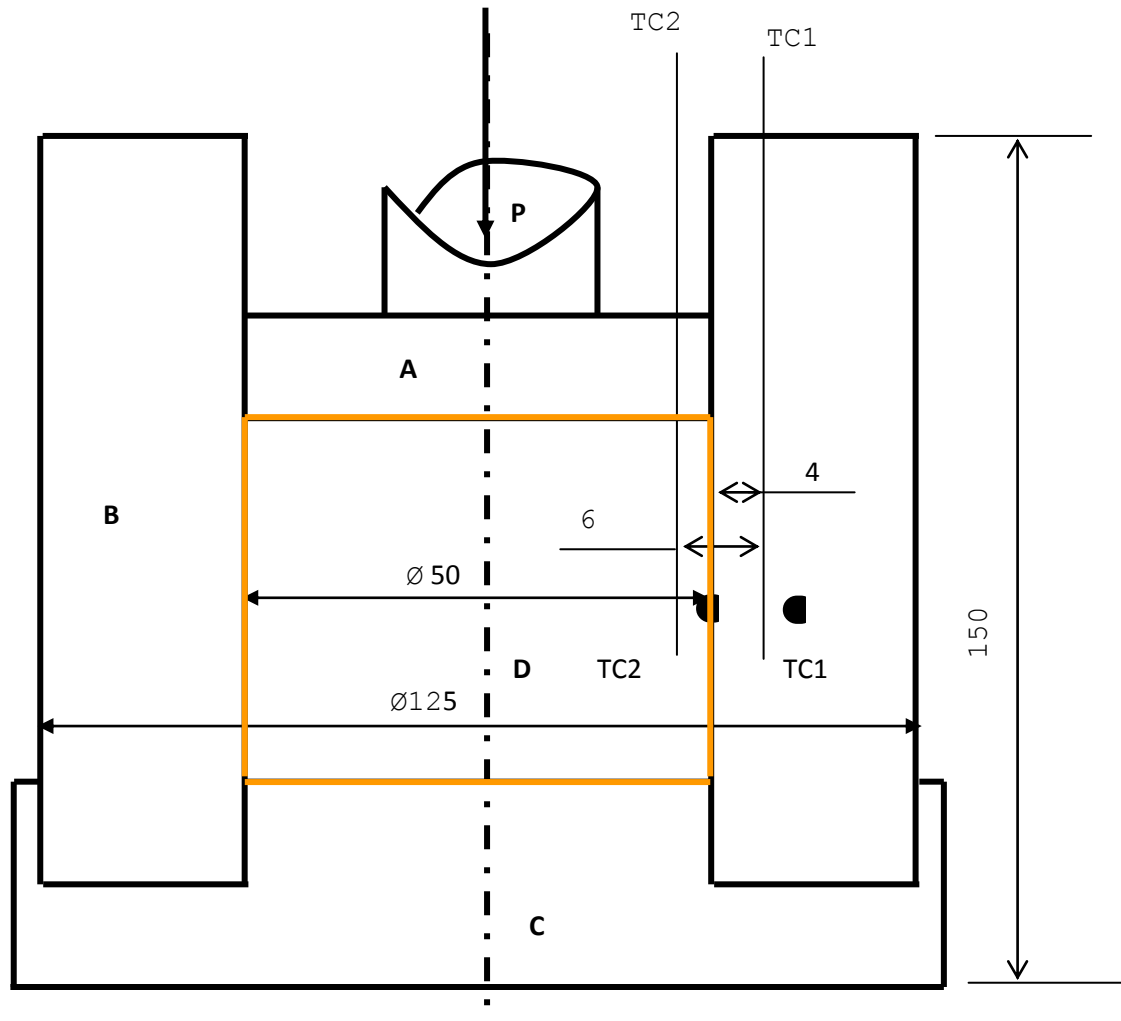


Figure 2. Schematic diagram of squeeze casting test rig

A-Punch, B-Cylindrical steel mould, C-Steel base, D-Molten aluminium cavity, TC1&TC2-Thermocouples

Δq_{fp} -frictional energy due to punch / solidified metal interface,

Δq_{fm} -frictional energy due to steel mould cylindrical surface - solidified metal interface.

Therefore,

$$\nabla q = \nabla q_p + \nabla q_{fp} + \nabla q_{fm} \quad 15d$$

Applying upper bound theory as presented by Abdul (1985), the internal energy generated by the application of pressure on the cast metal is given by;

$$\nabla q = \frac{\pi \bar{\sigma}_T}{\sqrt{3}} \left[2 \int_0^R \int_h^{h_0} \sqrt{\left(3 + \frac{r_0^4}{r^4} \right)} r_0 dr dh + m V_0 \int_h^{h_0} \left(1 - \frac{h_c}{h_0} \right) r_0 dh + \frac{2m V_0}{h_c} \int_{R_0}^R (r_0^2 - r^2) dr \right] \quad 16a$$

The cast specimen height, h_c is shown to be pressure dependent as determined by Aweda (2008) and the relationship h_c is expressed thus:

$$h_c = -0.00007P_i + 0.036833 \quad 16b$$

The plastic flow stress, σ_f is dependent on both the applied pressure, P_i and die temperature T_M , and expressed from data obtained by Aniyi et al (1996) to give equation (16c) as;

$$\sigma_T = 0.244P_i - 0.0405T_M + 18.614 \quad 16c$$

Where, h_c -cast specimen height,

σ_f -plastic flow stress,

P_i -applied pressure,

T_M -die temperature.

The finite difference form of equation (15a) with the energy change being converted into temperature change is:

$$T_i^{j+1} = \left[1 - \frac{2\delta\alpha_s}{d_{CS}^2} - \frac{\delta\alpha_s}{r_{CS}d_{CS}} \right] T_i^j + \left[\frac{\delta\alpha_s}{d_{CS}^2} + \frac{\delta\alpha_s}{r_{CS}d_{CS}} \right] T_i^j + \frac{\delta\alpha_s}{d_{CS}^2} T_{i-1}^j + \nabla T \quad 17a$$

where,

$$\nabla T = \frac{\nabla q \beta}{J C_P \rho V_{ol} \rho_S C_S} \quad 17b$$

where,

β -percentage of the deformation energy transformed into heat energy (90%)

ρ -density of workpiece

C_S -specific heat of workpiece

J_h -mechanical equivalent of heat (4185J/kal)

∇q -power dissipated due to pressure application

V_{ol} -volume of workpiece,

2.5 Computational Methods

The casting temperatures were monitored numerically at positions I=G and I=M see figure 1. The set of heat governing equations and boundary conditions obtained in the finite difference forms were solved numerically by using the explicit technique.

In the computational analysis, it was assumed that a thin solidified layer starts forming at the inner cylindrical surface of the steel mould (liquid metal/steel mould interface). This was followed by another

thin solidified layer at the inner surface of the formed solidified layer in the steel mould. This procedure of solidification continues in this manner and grows inwards into the core, $I=N$, of the cast metal until the whole molten region was 90% solidified. This was then followed by rapid heat abstraction from the solidified molten metal by pure conduction.

As thin film of solidified layer was formed in each case, stability of the solution was obtained before the formation of another thin solidified layer. The formed solidified layer was re-decretized into new nodal points and computed to determine the new temperature distributions in the liquid metal, the solidified metal, the steel mould portions and in the interfaces.

At each time step, stability conditions were evaluated at each nodal point as described by Shampire (1994). The smallest of the allowable time steps evaluated from the heat transfer equations thus formed the maximum allowable time step used for the entire domain. Iteration began only when the steel mould cavity was filled with the required quantity of molten aluminium metal. Thus at time $t = 0$, the molten aluminium metal was at its peak temperature of 720°C (i.e. super-heat of 60°C). At this time, the whole region was in a single phase, which is the liquid phase. At time interval $t + \Delta t$, thin film of solidified layer began to form. The effects of squeeze casting parameters such as applied pressure, die pre-heat temperature and delay time of pressure application are taken into account in the computation. The programme was written in Quick basic version 45 and run on a microcomputer, Pentium IV (80GB and 512 MB of RAM).

3.0 Experimental procedures

Two chromel-alumel thermocouples, tc1, positioned on the side of the cylindrical steel mould monitored the heating temperatures of the steel mould, and tc2, positioned 2mm into the cast aluminium metal in the cylindrical surface monitored the solidifying temperatures of the cast metal (see figure 2). The thermocouples were connected to the chart plotter through a cold junction maintained at 0°C . The punch was lowered to close the die and the vega compression machine, model utm 3c, serial no 1061 with a capacity of 89,000 n, was actuated to compress the solidifying aluminium metal. During solidification of the cast aluminium in the steel mould, the solidification and heating temperatures with times were recorded on the chart recorder by thermocouples tc2 and tc1 positioned in the cast metal and the steel mould respectively (see figure 2). Varying magnitudes of loads were applied through the punch on the cast metal under different delay times.

Squeeze cast aluminium specimens were produced while varying die pre-heat temperatures from room temperature to 300°C . The die heating process was carried out, using three electric heater rods (100watts each) that were inserted into the steel mould and connected to a.c supply. The required die temperatures were set and controlled, using the bimetallic thermostat installed on the steel mould by monitoring the die

temperature through thermocouples connected to the die base to indicate the die temperatures from a voltage readout.

4.0 Results and Discussions

4.1 Effect of pressure

Figure 3 shows typical temperature versus time curves for solidifying molten aluminium metal and steel mould respectively without the application of pressure. The solidification temperatures obtained 20 seconds after pouring of molten metal were 698.95°C and 656.00°C for numerical and experimental procedures respectively.

Typical result obtained under pressure application is shown in figure 4, indicating the temperatures versus times curve generated for solidifying molten aluminium metal. With the application of pressure, the temperatures obtained 20 seconds after pouring of molten metal were 712.51°C and 680°C at a pressure of 85.86 MPa for numerical and experimental methods respectively (see figure 4).

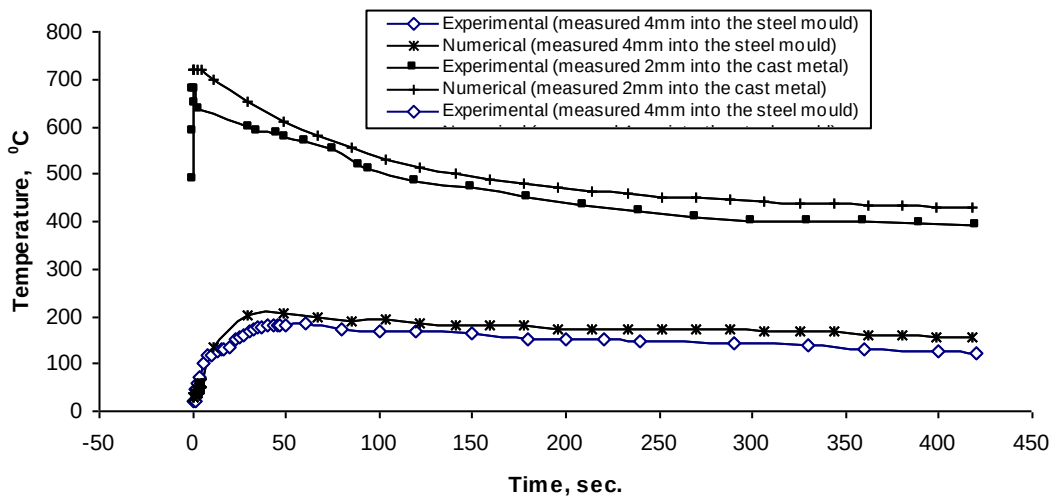


Figure 3 Comparison of experimental measured temperatures with numerical values of aluminium metal without pressure application ($P=0$, $T_M=30^{\circ}\text{C}$)

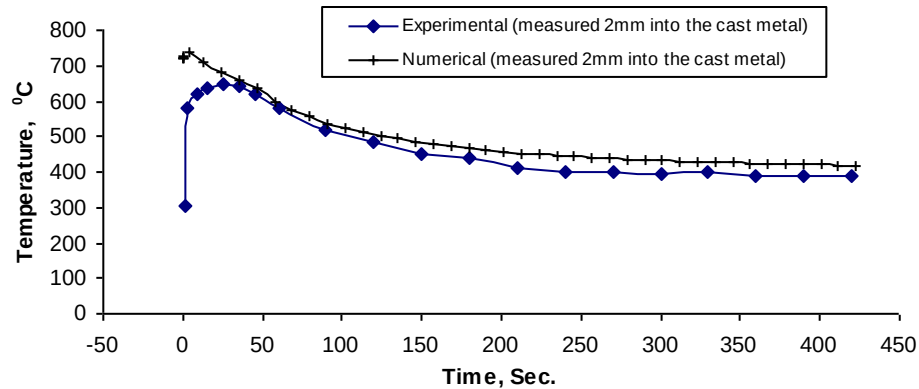


Figure 4 Typical comparison of the effect of applied pressure on the solidification temperature of aluminium metal (P=85.86MPa)

4.2 Effects of Die Heating

Effect of die pre-heat temperatures on the solidification temperatures versus times curves of aluminium under varying die temperatures is shown in figure 5. Under an applied pressure of 85.86MPa, the peak solidification temperatures are found to be higher with increase in die temperatures. At die temperature of 150°C, for example, the maximum obtainable temperature through simulation is 709.44°C as compared to 662.03°C obtained experimentally. At die temperature of 300°C the solidification temperatures are 716.72°C and 684.65°C for numerical and experimental values respectively. With higher die pre-heat temperatures, the peak cast metal temperature becomes higher, that leads to a decrease in the solidification rates as seen in figure 5.

The effect of die pre-heat temperatures on the time of solidification is shown in figure 6. The figure shows an increase in the solidification time with an increase in the die pre-heat temperature. For die temperature at room temperature, the time of solidification is 4.15 seconds. For die maintained at temperatures of 100 and 250°C the time of solidification become 4.70 and 6.48 seconds respectively.

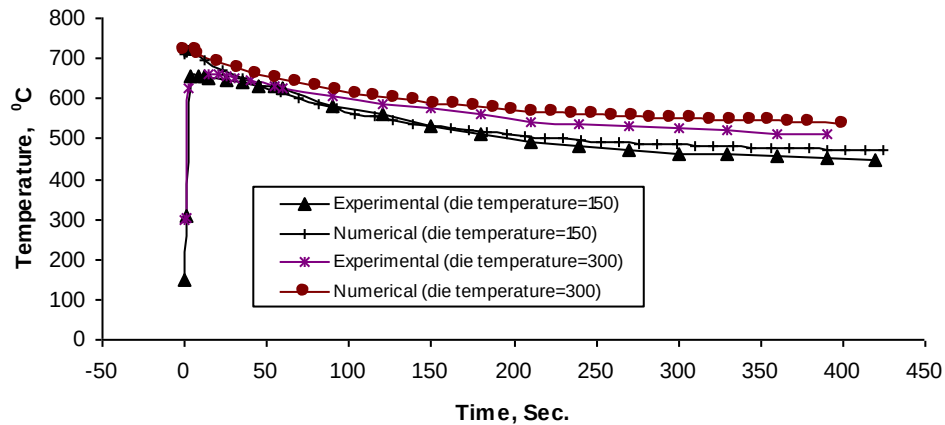


Figure 5 Typical comparison of experimental measured aluminium cast solidification temperatures with numerical values with die heating (P = 0)

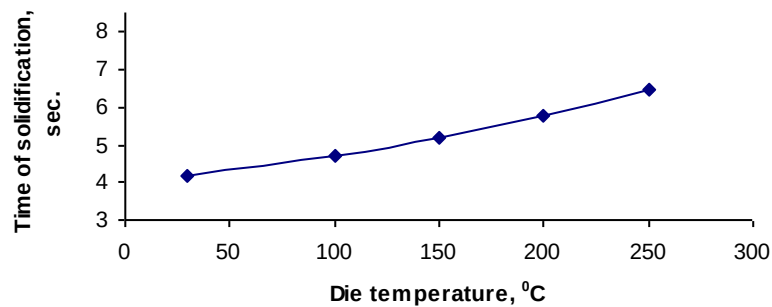


Figure 6 Effect of die heating on the time of solidification

4.3 Effect of Delay Times

Figure 7 shows the effects of delay times, (i.e. time before pressure application), on the solidification temperature versus time curves of molten aluminium metal, for situation without die heating and pressure retention time of 55 seconds under applied pressure of 85.86Mpa. For a delay time of 10 seconds (pressure applied 10 seconds after pouring of molten metal), the solidification temperatures rise from 687.890C to 710.66⁰C in 62.62 seconds for die at room temperature. For a delay time of 20 and 30 seconds the cast metal attains maximum temperature of 695.54⁰C in 73.71 seconds and 682⁰C in 90.03 seconds respectively. Regardless of the delay time, the effect of application of pressure becomes less significant after about 200 seconds of pouring molten aluminium as figure 6 reveals.

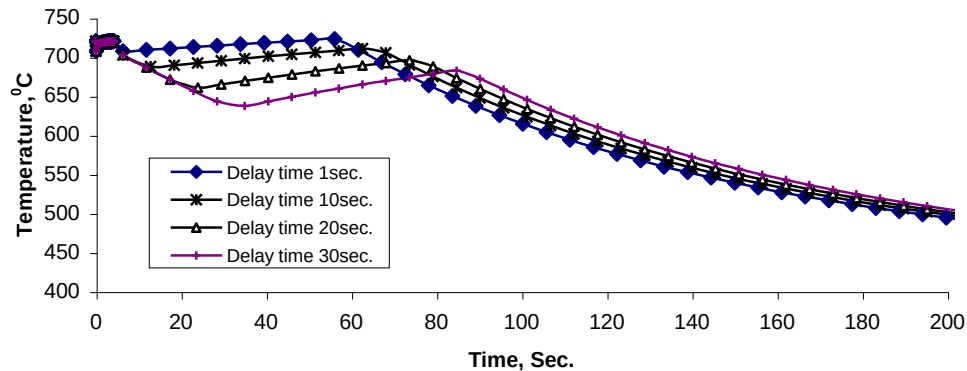


Figure 7 Effects of delay times on the solidification temperatures of aluminium metal, 2mm into the cast metal with die at room temperature and pressure application ($T_M=30^{\circ}\text{C}$, $P=85.86\text{Mpa}$)

5.0 CONCLUSION

From the discussions of results, the following conclusion could be made:

1. Increase in the applied pressure leads to a corresponding increase in the peak solidifying temperature attained,
2. Increase in the die pre-heat temperature raises the solidifying time,
3. Shorter delay times corresponds to higher peak solidifying temperatures,
4. The time of solidification of molten aluminium is dependent on the die temperature and independent of pressure application.
5. The comparison between the predicted and experimental values of solidification and cooling temperatures versus times indicates that the predicted results are in close agreement with the experimental results.

6.0 Reference

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COMPARATIVE METHODS OF DETERMINATION OF DEMAND RESPONSES BASED ON THE SALES OF GRAINS IN BAUCHI METROPOLIS, NIGERIA

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ABSTRACT

Parametric methods and non-parametric methods were used to determine the demand response of various grains sold within Bauchi. It was discovered that some demand responses were best described as prices increases in the market the quantity sold per unit price increase, an indication of perfect relation of the market situations, and spurious or nonsensical situations.

Keywords: Correlation coefficient, Response, Surface, Spurious, Sales

Mathematics Subject Classification 2000:91B02 & 91B26.

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INTRODUCTION

In the physical and social sciences one is often presented with the problem of inferring a formal relationship between certain variables based on experimentally obtained data, Adamu, *et al.*, (2007e). The simplest scenario involves two variables, for example, p and q , data points $[q_i, p_i]$, $i = 1, 2, \dots, n$, and the assumption that there is a curve that in some sense best fits the data, Adamu, *et. al.*, (2007d). According to Adamu *et al.*, (2007b), the idea is to find a first degree polynomial, $f(q) = a + bq$, with the property that $f(q_i)$ provides a “good approximation” to p_i for each $i = 1, 2 \dots n$.

The quality of the fit is determined by the differences, $p_i - f(q_i)$, $i = 1, 2, n$, between the observed and predicted values. Adamu *et al.* (2006b), stated that one natural approach is to seek $\underline{a}$ and $\underline{b}$ (and hence f) so

as to minimize $\sum_i^n [p_i - f(q_i)]$. However, for statistical reasons, best explained by Gupta and Gupta, (2004), a preferable measure is

$$\sum_i^n [p_i - f(q_i)]^2 \quad \dots \quad (1)$$

A choice of f that minimizes (1) is called a *least-square linear or non-linear fit* to the data, and the resulting approximating curve is referred to as the *curve or line of regression of p on q , or q on p* , (Wright, (1999)).

THE OBJECTIVES

In general, the present study seeks to undertake an empirical analysis of the sales of grains in Bauchi metropolis by the use of several approximation methods. However, the following are the specific objectives of the paper:

- 1) To generate preferable market situations base on the sales of grains.
- 2) To provide a quantitative framework for the prediction, with mathematical precision of the future sales of grains within the so-called markets in Bauchi metropolis and beyond.

METHODOLOGY

Area of Study

The Bauchi metropolis consist seven major grains markets, Adamu, *et al.*, (2007e), and these are: Bakaro, Central, Kasuwan Mata (Railway Market), Muda-Lawal (Matori), Sabuwar Kasuwa (Tudun Alkali), Wunti (Tarkunya) and Yelwan-Tudu. In the markets mentioned, there are grains sellers and data was generated by the use of questionnaire, weekly, throughout the period of September, 2002 – September, 2003.

MATHEMATICAL ANALYSIS OF THE METHOD

THEOREM 1:

If (q_i, p_i) $1 \leq i \leq n$, are data points with at least two of $q_1, q_2, \dots, q_n$ distinct, then there are unique real numbers a_o, b_o with the property that

$$\sum_i^n [p_i - (a_o + b_o q_i)]^2 \subseteq \sum_i^n [p_i - (a + b q_i)]^2 \quad \text{for all } a, b \in \mathbb{R}. \text{ Moreover, if}$$

$$P = \begin{bmatrix} P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix}, A = \begin{bmatrix} 1 & q_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & q_n \end{bmatrix} \text{ and } Z = \begin{bmatrix} a \\ b \end{bmatrix} \text{ then } \begin{bmatrix} a_o \\ b_o \end{bmatrix} \text{ is the solution of } AZ = \text{Proj}_j(P), \text{ where } \Phi = \Delta(A).$$

Proof:

The theorem 1 indicates that $Z_o = \begin{bmatrix} a_o \\ b_o \end{bmatrix}$ can be obtained by projecting P on Φ and then solving, $AZ =$

$\text{Proj}_\Phi(P)$, but there is a more direct route.

$$P - AZ_o = P - \text{Proj}_\Phi(P) = \text{Proj}_{\Phi^\perp}(P) \in \Phi^\perp$$

for each $Z \in \mathbb{R}^2$, $AZ \in \Phi$, so

$$\Delta = (AZ) \cdot (P - AZ_o) = (AZ)^T (P - (AZ_o)) = Z^T (A^T P - A^T AZ_o) = Z \cdot (A^T P - A^T AZ_o)$$

Thus, $A^T P - A^T A Z_o$, being orthogonal to every element of $\mathbb{R}^2$, is Δ , that is Z_o is the solution of

$$(A^T A) Z = A^T Y \quad \dots \quad (2)$$

The equations (2) in the system are called normal equations for Z_o .

RESUTLS AND DISCUSSIONS

Table 1: Correlation Coefficients of Response Surfaces for the Sales of Cowpea

Response method	surface	Correlation coefficients	Market situations for the sales of cowpea
Gaussian		0.15241	As prices increase in the market the quantity sold per unit price increases.
Gompertz		0.1251631	As prices increase in the market the quantity sold per unit price increases.
Lagrangian		1.00	Indicate perfect relation of the market situation.
Polynomial		0.316099122	As prices increase in the market the quantity sold per unit price increases.
Rational		2.0852350	Spurious or non-sensical situation
Truncated Series	Fourier	5.0208413	Spurious or non-sensical situation

Table 2: Correlation Coefficients of Response Surfaces for the Sales of Millet

Response method	surface	Correlation coefficients	Market situations for the sales of cowpea
Gaussian		0.05308476	As prices increase in the market the quantity sold per unit price increases.
Gompertz		0.52691881	As prices increase in the market the quantity sold per unit price increases.
Lagrangian		1.00	Indicate perfect relation of the market situation.
Polynomial		0.68322814	As prices increase in the market the quantity sold per unit price increases.
Rational		0.64864662	As prices increase in the market the quantity sold per unit price increase
Truncated Series	Fourier	10.25267859	Spurious or non-sensical situation

Table 3: Correlation Coefficients of Response Surfaces for the Sales of Rice

Response method	surface	Correlation coefficients	Market situations for the sales of cowpea
Gaussian		0.33369131	As prices increase in the market the quantity sold per unit price increases.
Gompertz		0.4104639	As prices increase in the market the quantity sold per unit price increases.
Lagrangian		1.00	Indicate perfect relation of the market situation.
Polynomial		0.51057301	As prices increase in the market the quantity sold per unit price increases.
Rational		3.37699897	Spurious or non-sensical situation
Truncated Series	Fourier	6.34920108	Spurious or non-sensical situation

Table 4: Correlation Coefficients of Response Surfaces for the Sales of Sorghum

Response	surface	Correlation	Market situations for the sales of cowpea
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method	coefficients	
Gaussian	0.27743282	As prices increase in the market the quantity sold per unit price increases.
Gompertz	0.25817505	As prices increase in the market the quantity sold per unit price increases.
Lagrangian	1.00	Indicate perfect relation of the market situation.
Polynomial	0.43853288	As prices increase in the market the quantity sold per unit price increases.
Rational	0.28076146	As prices increase in the market the quantity sold per unit price increases.
Truncated Fourier Series	5.61610217	Spurious or non-sensical situation

In all the approximation methods considered for the determination of an empirical formula for the sales of grains in Bauchi metropolis, the polynomial approximation method is the best fit with prices increasing as the quantity sold per unit price increasing, respectively and symmetrically.

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Mathematical modeling in Education, Social Sciences and Culture



The Meta-Heuristics of Global Financial Risk Management in the Eyes of the Credit Squeeze: Any Lessons for Modelling Emerging Financial Markets?

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Abstract

A globally challenging problem of current interest is the credit squeeze, whose further developments from 2007 up and till now has led to the (near) collapse of financial institutions, notably Bear Stearns, Lehman Brothers, American Insurance Group (AIG) in the USA, among others in UK, Iceland and Europe. Even though the credit squeeze is global in its reach, its impact is **regionally differentiated** among the triad of financial markets – developed markets of Europe and USA (in which it is most severe), emerging markets of Asia and the BRIC countries (Brazil, Russia, India and China), and emerging markets of Sub-Sahara Africa and the Middle East (in which it is less severe). Global communities of academics, practitioners, national and international economic development and financial regulatory bodies, are intensely debating approaches to prevent such events happening in the future. This paper **collates** some of the key ideas emerging from this debate, in order to develop an **agenda for mathematical modelling** of the financial markets, congruent with sounder **quantitative financial risk management**. The main focus of the paper is on **lessons** to be learned from the crisis for effective management of emerging financial markets of Sub-Sahara Africa. **Combining perspectives** from such disciplines as financial engineering, economic history, mathematical modelling and simulation, stochastic modelling and bank financial management, among others, the paper is essentially a **meta-heuristic exploration** of **what went wrong** and **what needs to happen**, individually within financial institutions and collectively among stakeholder communities, in order to avoid and mitigate the impact of such events. **Illustrations** of how the ideas in the paper could inform changes in **quantitative risk modelling** and **financial investments** are also provided in the paper.

Key words:

Quantitative risk management, *derivatives*, *mathematical modelling/simulation*, stochastic processes, *bank financial management* .

Mathematics Subject Classification 2000:01A67 & 97B10

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1. Introduction

We aim in this introduction to reiterate and justify the key ideas stated in the abstract to this paper and provide a roadmap to the paper, noting that the scope of the paper is too broad to be accommodated effectively in a single piece. We have therefore arranged the mosaic of ideas as a ***paper system*** made up of three parts.

Part I is a depository of the literature in which the dominant issues are stated and explored; Part II leans on Part I to illustrate the affect of the credit crisis on ***firm-level*** (essentially ***micro-economic***) investment decisions, and also to offer possible solutions (ways out of the crisis). Part III is a rejoinder to Professor Soludo's recently presented inaugural lecture (as a professor of economics) at the University of Nigeria, Nsukka, which enables us to develop complementary ***policy-level*** (essentially ***macro-economic***) solutions – the term ***solutions*** is our shorthand for ***research agenda*** and ***overall action plans*** for managing the crisis.

Starting from the abstract, we have attempted to improve readability of this paper system by highlighting in bold italics the key concepts involved, you will notice that the presentation of the paper(s) is anchored on these concepts, for economy of space and succinctness of the message.

We have stated in the abstract that the impact of the financial crisis, though global, is ***regionally differentiated*** among different types of markets – emerging versus developed markets – hence the lessons to be learned from the crisis have to reckon with those differences. As we draw those lessons, we focus attention on the ***emerging markets*** of Sub-Sahara Africa and the Middle East, we will particularly use the ***Nigerian financial system*** as a case in point. Our approach in the scripting of the papers is to collate the different debates and ideas contributed by communities of academics, professionals, policymakers (including the Central Bank of Nigeria (CBN) Governor, Professor Charles Chukwuma Soludo, as in Part III of the paper system) and explore the ***implications*** of those ideas for devising ***ways out of the crisis*** (that is, avoiding and lessening its impact).

This approach is essentially a '***wisdom of the crowds***' approach, in which it is felt that crafting solutions on the basis of pooled evidence from knowledgeable others, under certain generally obtainable conditions, leads to better decisions than those reached by lonely contemplation, Surowiecki (2005) and Ball (2004). Of necessity, we ***combine perspectives*** from a number of disciplines in our discussions of the crisis, because the crisis is inherently ***multi-issue*** and ***multi-disciplinary*** – it is in order to do adequate justice to some of these perspectives, and offer a ***richer intellectual foundation*** for the reader to build own ideas on, that we have the three parts to the paper system.

A number of the key concepts drop in and out of the discourse appropriately at different points; for example, concepts like ***quantitative financial risk management, financial derivatives, mathematical modelling, simulation, bank financial management and meta-heuristics*** feature repeatedly in the arguments, but our focus is irrevocably on lessons and solutions that fall out of the exploration of the concepts.

It is important to expatiate on the meaning of and justification for a meta-heuristics approach to presenting ideas in the papers. Traditionally, in most of mathematical modelling, emphasis is usually placed on the rigour or '***taxingness***' of the mathematics, but a holistic blend of ***mathematical rigour, commonsense, intuition, evidence*** and ***deep interpretation*** of results is more usefully required in handling big decisions, especially under conditions of ***complexity*** and ***uncertainty***. It is this holistic approach that we refer to as meta-heuristics. Normally, we say heuristics, see for example Gigerenzer (2000), Goldratt & Cox (1992), but the discussions in this paper involve the heuristics of heuristics (that is, reasoning about other people's reasoning with some purposes in view), hence the term meta-heuristics.

For example, suppose we want to explore the impact of the credit crisis on the conduct of business by an **individual firm**, say **Zenith Bank plc**, then we have to fully explore the **business model** according to which the bank monetizes its offerings, the totality of cogent views by experts and management about the possible effects of the crisis on banking, particular ways these effects travel across different investment schemes e.g. portfolio theory and management, risk management, possibly using derivatives, the macro-economic policy scaffolding, as with interest rates, government regulations, monetary and fiscal policy, behaviour of competitors, bank client behaviour in light of the crisis, among a host of other considerations. Such decision making is therefore meta-heuristic.

The organization of this paper is as follows. Section 2 examines why we emphasize emerging markets, essentially summarizing those differences between the features of various markets that make for the differential impact of the global financial crisis on the markets, as well as what the differences mean for *thinking around the crisis*.

Section 3 offers a foundational literature that sets out the key debates about the crisis, earlier mentioned, and again what the debates suggest for **possible solutions**. This part touches on such dimensions as the genesis and development of the financial crisis, links to economic crisis (Wall Street versus Main Street effects), market participants' roles, investment strategies, need for a new form of financial analysis, information requirements for different types of modelling and decision making, government reactions, among other perspectives.

Section 4 is a deeper excursion into the nature and causes of financial **market bubbles** (which the crisis evidences), the discussions flow mainly from Surowiecki (2005) and Shiller (2008), which encompass ideas central to an **intuitive understanding** of the crisis.

Section 5 collates the suggested solutions arising from the discussions, in anticipation of Part II of the paper system, which develops the solutions further. Section 6 concludes the paper.

As already mentioned, Part II of the paper, in addition to further exploring the solutions, illustrates the **theoretical** and **practical modelling agenda**, around the needs of the three main types of **market participants** – households, firms and government, mostly firms, and mainly using **banks** as examples. The aim is to substantiate the affects of the crisis on investment and financial risk management, based on concepts such as portfolio theory and management (PTM), integrated financial risk management framework, quantitative financial risk management (QFRM), including the special use of derivatives, stress testing and scenario building, what-ifs, heuristics, complex systems, especially as they arise from the tightening of the constraints sets in decision making, via new regulations, new global financial architecture, new business models, etc.

Part III completes the loop by using the rejoinder to Soludo's paper as way to also discuss **macro-economic solutions**.

2. Why emerging markets?

Kohers et al (2006) show that the literature on financial markets makes a distinction between developed and emerging markets, which are based on their **differentiating characteristics**. The developed markets are generally assumed to be **more efficient** compared with developing or emerging markets. Emerging markets are characterized by high levels of volatility of prices or returns. Finance theory suggests that the **higher volatility** typically translates to **higher expected returns**.

It is known that in recent years emerging markets have attracted the investing interests of global investors. Even though there are some evidence to suggest that these markets are becoming more integrated into the

global capital markets, they still differ from developed markets in their high liquidity risk, limited availability of high quality and large capitalization shares. They are also more likely to be dominated by a few stocks by relative market capitalization e.g. the banking sector in the Nigerian Stock Market (NSM).

Following Islam & Watanapalachaikul (2005), p. 3, Bekaert & Harvey (2002), Kohers et al (2006) and Hassan et al (2003), we summarises the characteristics of emerging markets thus:

- Investment returns are **not normally** distributed; they are **typically skewed** and have **fat tails**
- Compared with developed markets, emerging markets have a high degree of country risk (including political risk, economic risk and financial risk) linked to currency devaluations, failed economic plans, financial shocks and capital market reforms
- With regards to **portfolio diversification**, it seems promising to include financial assets in emerging markets into stock portfolios, since their very low correlations with well developed markets reduces overall portfolio risk
- The markets are characterized by thin trading, low liquidity, less informed traders or traders with **incomplete** or **unreliable information**. The markets are therefore relatively shallow, not deep enough to enable them to be as efficient as developed markets.

Similarly the characteristics of African markets as discussed by United Nations Commission for Africa (UNECA) (2007), Yartey & Adjasi (2007), Chukwuogor (2008) and Moss et al (2007) can be summarized as follow:

- African markets are small, with few listed companies and low market capitalization; Egypt, Nigeria, South African and Zimbabwe are the exceptions with listed companies of 792, 207, 403 and 79, respectively
- African stock markets suffer from **low liquidity**; that notwithstanding, the stock markets continue to perform remarkably well in terms of return on investment. Liquidity as measured by turnover ratio is as low as 0.02 percent in Swaziland compared with about 29 percent in Mexico. Low liquidity implies that it will be harder to support a local market with its own trading system, market analysis, brokers and the like, because the business volume would simply be too low. The NSM has almost outgrown such problems.
- The markets are not yet well integrated with regional and global markets (as the emerging markets of Brazil, Russia, India and China (BRIC) countries), and have a range of capacity and technology constraints; interestingly, the Nigerian Stock Market is now digitalized and therefore weans itself somewhat from acute technological constraints. It should be noted that the **regional integration** of capital markets in Africa offers a solution to this situation, especially for the smaller economies. Overall, to accelerate **capital market development**, governments need to improve the capacity of all stakeholders, invest in infrastructure and promote **good governance**.
- Compared to China for example, the African markets are characterized by political economies with low levels of saving and limited private capital flows, so the investment ratios in Sub-Sahara African (SSA) countries are lower than in other developing regions
- While African markets are relatively small in comparison to developed markets of US, Europe and Japan or even mid-sized emerging markets, they are not out of line with the global norms, given the size of their host economies.

Some notes on the Nigerian Stock Market (NSM) in particular and the kind of lessons to be learnt from the present crisis

The NSM is an interesting case because it is one of the four large African stock markets, as noted by the African Development Bank (AFDB, 2007). According to the IMF (2008), 'Nigeria's recent private sector-led growth and vibrant capital markets, with potentials for investors have placed it in a league of eight sub-Saharan African countries (outside South Africa), heading towards **emerging market statuses**. Nigeria is going through a rigorous programme of financial restructuring and seems as result of the positive reforms

to be relatively insulated from the current financial crisis, apart from the non-integration with the global financial markets to a large extent.

These facts warrant a study of the performance issues and characteristics of the NSM pre-and post-reforms, and now in light of the global financial crisis. We can see from the above samples of notes from the literature that lessons to be learnt from the global financial crisis for the SSA markets should be more in terms of **preparing the markets** to avoid and/or survive such shocks in the future, when the interests from foreign investors moves them towards similar characteristics as obtain in the developed markets, where the impact of the crisis is much more severe.

Also, anticipating **further deepening** of the emerging SSA markets as they mature, we need to use the lessons learnt to construct better ways of using **advanced financial engineering tools** e.g. derivatives for effective risk hedging in the markets, away from the approach adopted in the developed markets as of today.

Fundamental research agenda?

We think that in order to afford policymakers and investors a better understanding of the detailed characteristics and investment features of the SSA (emerging) markets, there is an urgent need for a baseline full characterization of calibration for investment decision making of the markets. This characterization is akin to the way the human and other mammalian genomes were mapped. We need to have that DNA-type mapping of the inherent features of the markets, firstly at an overall market index level of play, secondly at ramified levels of play within the sub-markets (niches) and thirdly at levels that measure the coupling of the markets, amongst the sub-markets and with other African markets.

For example, with the NSM, we need to do such characterizations that can afford the Central Bank of Nigeria (CBN), the Nigerian Stock Exchange (NSE), the banks, insurance companies, private investors and households, a deeper understanding of how the market performs over time at the aggregate market level, for the different sections e.g. financials (banks as a special category, being the dominant sector), communications industry (a nascent growth area), agriculture (a regrettably neglected, smaller area, with huge potentials for rebalancing the economy away from a mono-petro dollar economy) and other sectors. This basket of results mimics the DNA of the markets of which robust and enduring policy can be built and on which informed investment decisions by firms and households can be undertaken. As noted in Part III of this paper, we have put some **PhD students** on this task, the first topic looks at **Stochastic Modelling of Emerging Financial and Stock Markets: A Case Study of the Nigerian Stock Market**. It covers the remit briefly described here and uses predominantly tools from empirical finance (financial econometrics) and quantitative financial economics.

We have also instituted a **big theme research programme**, at the National Mathematical Centre (NMC) as coordinating national organ, to encompass at the core this characterization and calibration work and address the issue of change in the global financial and economic system. This programme of studies is called **Studies in Quantitative Finance, Financial Risk Management and Change in Global Financial Markets with a Focus on Emerging Markets of Sub-Saharan Africa**.

The rationale for this study is the fact that the differentiating characteristics of the triad of markets – emerging markets of the BRIC countries, emerging markets of African & Middle East and developed markets of US, Europe and Japan – require a study devoted to discovering what changes, if needed, will be made to the prevailing set of theories, in order to better model the **stylized facts** of the emerging markets. Our assessment of the manpower needs for this study is provided in Part III of this paper, Part II offers a few more technical details on the work.

3. Literature review

In this section we cover the remit indicated in the introduction, such as key debates about the crisis, and again what the debates suggest for **possible solutions**, the genesis and development of the financial crisis, its links to economic crisis (Wall Street versus Main Street effects), market participants' roles, investment strategies, need for a new form of financial analysis, information requirements for different types of modelling and decision making, government reactions, among other perspectives.

We do not necessarily take these items in the same order as listed, but hope that at appropriate times in the literature review the ideas will surface. In a sense, our dip in the literature base is random, but targeted at connecting the indicated range of ideas.

The genesis and development of the crisis

We lean on a profoundly clear discussion of the genesis and development of the crisis offered in Shiller (2008), complemented by additional facts from the daily grill of events mainly between 2007 and now, as gleaned from the financial media, notably the Financial Times. This weave establishes our wisdom of the crowds approach to learning about the crisis.

Shiller argues, p. 1 that the subprime crisis is at its core a result of **speculative bubbles** in the US housing market, which began to burst in 2006 and has now produced market ruptures mainly in the form of **financial failures** and a **global credit crunch**. We can easily mention such (near) failures as the investment banks Bear Stearns and Lehman Brothers, American Insurance Group (AIG) of the US, Northern Rock Bank, Royal Bank of Scotland (RBS), Halifax Bank of Scotland (HBOS) of the UK and others in Europe, which further notes in this literature will throw up. Importantly, the crisis has led to fundamental societal changes – of consumer habits, values, relatedness of man to man, etc. These are the types of changes that the above mentioned research programme should address, using a mix of disciplines, quantitative and qualitative e.g. psychology, sociology of markets, experimental finance and economics, decision markets, mathematical modelling and simulation, among others – see a fuller list in Part III of the paper system (Ezepue & Solarin 2008c).

Shiller notes, p. 2 that the social fabric is difficult to measure and 'is easily overlooked in favour of smaller, more discrete, elements and details. But the social fabric is indeed at risk and should be central to our attention as we respond to the subprime crises. Consider that the financial system and the stock market are a huge chunk of the socio-economic fabric, and Shiller really argues for a coming together of minds in undertaking the kind of big-theme studies that measures and maps this chunk of the fabric, instead of the individual limited-focus research traditionally undertaken by academic researchers mostly in pursuit of promotions to professorships via journal publications. Furthermore, Shiller, pp, 2-3, makes the key point that:

'It is time to recognize what has been happening and to take fundamental steps to restructure the institutional foundations of the housing and financial economy. This means taking both short-run steps to alleviate the crisis and making longer-term changes that will inhibit the development of bubbles, stabilize the housing and larger financial markets, and provide greater financial security to households and businesses, all the while allowing new ideas to drive financial innovation'.

This the precise goal driving all that Professor Soludo was saying in his inaugural lecture [and we are convinced that he could have further explicated in the keynote paper – *The Prospects of Quantitative Analysis and Financial Engineering in the 21st Century Banking and Financial Market* – had we had the good luck of having him in our midst], and this paper system squarely addresses, including all the wave of international responses across the globe.

It is about enshrining **social purpose** in finance and economics, including related fields used in understudying them e.g. mathematical sciences. It is about the need to temper '**animal spirits**' that characterize the behaviour of analysts, managers and financial intelligentsia, with mathematical

contemplation, heuristics and reasoning that is informed by **deep research**. This will help to reduce incidences of **moral hazard** and rebuild **trust** amongst financial clients and financial institutions and their management.

Henceforth, we summarize all the main ideas that are explored in Shiller, with their implications for research and action planning, we enjoin the reader to spend more timbre with the booklet for more details on the reasoning.

1. Having noted that part of the subprime problem has been the over-promotion of homeownership amongst the American households, some of whom could not meet up with the mortgage payments (especially adjustable-rate mortgages), Shiller argues (pp. 5-6) that, for all its advantages, homeownership is not the ideal housing arrangement for all people in all circumstances. The implication of this fact for research and investments in emerging markets is that Nigerian and African banks should be sufficiently responsible to create financial products that match the affordability and behaviours of different segments of their clientele. Banks that do this on the basis of far more detailed understanding of their customer bases (using advanced marketing analytics) than is currently the case, will be in stronger compliance and business profitability positions, compared to their peers.
2. The credit crisis has contaminated other sectors of the American **real economy** besides housing e.g. credit cards, automobile loans, credit rating of municipal bond insurers and markets for corporate loan obligations (p.8). The implication is that understanding the degree of coupling amongst sub-markets and business lines of markets and firms, as indicated in the above research notes, is a sine qua non for business success of firms.
3. The ultimate cause of the global financial crisis is the psychology of the real estate bubble (and other bubbles that preceded it e.g. the stock market bubble of the 1990-mid 2000s). The housing bubble which generated the subprime crisis ultimately grew as big as it did because the wider society, financial and non-financial, does not yet fully understands the mechanism behind speculative bubbles and could not therefore effectively manage its onset and development. There is a sense in which irrational exuberance (irrational public enthusiasm) for housing lies at the heart of the bubble and all other bubbles. What typically happens with bubbles is that this enthusiasm fuels purchase of financial products e.g. mortgage-backed assets at prices way above what the fundamental valuations of those assets would suggest, financial resources are increasingly allocated to such resources in the continuing expectation of plenty, which further fuels the asset prices, in a vicious cycle of price escalation, that must ultimately be corrected, either in a soft landing, if properly managed, or in a crash as in this aces. Hence, the social psychology of investing should be properly researched within a context deep re-appraisal of the lessons from economic history, over a wide range of asset classes, and mindful of the idiosyncrasies of particular markets. This is in our view a viable **doctoral research idea**. For instance in emerging markets of SSA, typically the NSM, it is necessary to refract firm investment policies, securitization of loans and financial products on the lenses of such research lessons. It is also necessary to mount a campaign of **mass financial training and literacy** of investors towards realizing a global financial citizenship in which the investors have the basic ability to understand the plethora of independent financial advice received from stoke brokers, banks and other investment groups. The case of microfinance in Nigeria makes this recommendation or **action plan** especially relevant. Moreover, bubbles are akin to cyclones or epidemics whose developmental are associated with complex systems ideas, tipping points, etc. Hence, **non-linear** and **complexity theory-based** mathematical and computational modeling tools should be used in researching the potential impact of such phenomena. This argument is remorselessly explored further in Part III of this paper system.
4. Whilst the emotions of the moment would suggest a retreat from financial innovation, Shiller, p. 10 argues that this is really an ‘opportunity to redouble our efforts to rethink and improve our risk management institutions, the frameworks that undergird our increasingly sophisticated financial sector’. We can add that this is a reasonable argument and that to succeed in such re-evaluation of the ways and means of **financial innovation** in an emerging market like Nigeria requires fundamentally the kind of baseline characterization of the system suggested in the foregoing notes. Specifically, aspects of that doctoral research programme (including MSc/Mphil/specialist MBA projects that are

theoretical and applied to contingent circumstances witnessed in particular firms) should be beamed on **advanced financial risk management**. Realizing that the stylized facts of these markets involve non-normal, asymmetric, fat-tailed risk distributions, we require such techniques as **extreme value methods (EVM)** in statistics to feature in the research, among other advanced tools e.g. stochastic processes for dependent phenomena, derivatives, volatility models, etc., see for example Ezepue (2007), Franke et al (2008), Chiarella & Khomin (2007), Cuthbertson & Nitzsche (2005), Beran & Ocker (2001), Avramov & Chorda (2006) Aggarwal et al (1999), Hong et al (2006), Peng et al (2005), Yu et al (2006), Yu & Meyer (2006), among other references covering the advanced techniques required.

5. As Shiller argues that the crisis calls for institutional reforms (also expressly the thesis argument in Professor Soludo's inaugural lecture), he uses a powerful train-track metaphor to convey the overriding importance of getting the track (institutions) right, if we can ever expect the train (financial system and markets) to run smoothly. A **potential (doctoral, if deepened appropriately)** research topic that comes to mind here is: **The nexus between deep characterization and calibration of financial and stock markets (in Nigeria) and financial reforms and policymaking** (may be specialized to, say, domestic monetary and fiscal policy, etc). The **requisite skills set** for this type of research are articulated in the rejoinder to Soludo's inaugural (Part III of this paper system). The sense in which we conceive of such characterization work is to see that what we do on the basis of informed research and reasoning, regarding the future of our financial system, passes **beyond quick fixes**.
6. Related to this issue of institutional reforms is a set of **specific solutions** preferred by Shiller (2008), pp. 20-27). We select the long-term solutions which appeal more to our emerging market needs; in any case the short-term solutions are precisely what global financial managers are currently doing. The key long-run solution is for the financial system, its governance and investment managers to get the risk management aspects of financial play properly addressed. This reinforces our earlier suggestion for **prioritizing risk training** in the country, from medium to advanced risk management levels, in banks, financial institutions generally, non-financial institutions, etc. We think we can approach this **civil society action plan** from the standpoint of regular seminars, summer schools, training workshops (SSTWs) at such national centres as the NMC, within the specific organizations for custom-built training that uses their in-house data confidentially and more formally through MSc, MBA and PhD training that are risk-focused. One area of immediate need here is to get our financial players to become familiar with and confident about using derivatives to hedge risks optimally, and appropriately understanding the risk profiles of new financial products they create.
7. Another specific solution is to extend the scope of financial markets to **cover a wider array of economic risks** – real estate, new futures markets, etc. Again, three baseline study of stock market characterization comes to the rescue in that the macro-financial environments against which risks are judged is then clearer to the producer developers. The third related solution is to **create retail financial instruments** – including income-based and continuous-workout mortgages, home equity insurance – in order to assure greater security to consumers. For us in emerging markets, we need to find proxies for housing in the domestic economy and use them as bases for the product creation, including the financial engineering end of the product creation process. This required fundamental research into the 'persona' and financial attributes of different segments of the society for which the products are targeted, an interplay between strategic marketing research, planning and financial engineering. If you want a working title for a doctoral research in this direction, it would be something like:

The Nexus Among Financial Product Origination, Strategic Marketing Planning and Financial Engineering and Bank Financial Management: A Case Study of XYZ.

Section 4 is a deeper excursion into the nature and causes of financial market bubbles (which the crisis evidences), the discussions flow mainly from Surowiecki (2005) and Shiller (2008), which encompass ideas central to an *intuitive understanding* of the crisis. Section 5 collates the suggested solutions arising from the discussions, in anticipation of Part II of the paper system which develops the solutions further. Section 6 concludes the paper.

Digression on how to organize the Meta-Heuristics paper (111108 2pm exactly)

Reading the FT article, Bofa captures Merrill's thundering herd, I happened up on a nice way to cast the narrative on the financial crisis in the NMC conference – hence a nice way to structure the paper system:

1. Financial crisis {financial risks, market participants, regulatory frameworks, etc.}
2. Economic crisis {the macroeconomic sphere, microeconomic (firm-level) spheres, monetary and fiscal policies, and other lines of thought from Soludo's lecture}
3. Investment strategies {portfolio theory and management, derivatives, bank financial management}
4. Anticipating government moves e.g. Obama regime, budgets, global shifts etc and modelling for policy and investments by different market participants}
5. Information requirements for different regimes of modelling and decision-making, Kassim's paper, etc}
6. New organizational plans to prepare emerging markets for the new realities {labour market dynamics and need for a radically new educational approach for finance training, education and collaborations amongst stakeholders – Project FINE, Project NEW BOOKS, NIAFER, CUREEN, FONAAID, etc.}
7. Putting it Altogether – Action Plans as above
8. Conclusion
9. Appendix or Part II – A rejoinder to Professor Soludo's lecture, with specific thinking around all his questions, including further ideas from Shiller, ...
10. A call for working groups to be set up around all the initiatives deriving from all parts of the paper system, especially planning towards the text on mathematical modelling and a similar conference and text on the Credit Squeeze and FINE to hold in 2009, etc.

It is clear that this plan can be flexed to accommodate the key points in the abstract of the paper and that a paper system with independent but linked parts with their own sub-abstracts is a good way forward. Presenting them will require at least an hour using a concept-based power point presentation slides and materials.

To start a fusion of the highlights of the paper system in the abstract and the above plan, we list the intended headings of the paper as follows – after this we incubate the thinking in the brain and trust the right brain to submit the combined plan to us, well before 'chalking the modules of the paper system' together commences.

4. Introduction – background to the credit squeeze covering briefly the entire plan and key points in the abstract
5. Why emphasis on emerging markets? Literature review that starts with this as in Taib's PhD work and as in the key references he made available to me (consider including him in some parts of the paper for fairness? Differences amongst the triad of markets as indicated and implications for work in African, Nigeria and a case study approach, including notes from the ADB, etc.
6. Wisdom of the crowds work (with governing literature from this topic as in Surowiecki) collating viewpoints from the global communities of academics, practitioners, national and international economic development experts, regulatory bodies (CBNs, with special emphases on Nigeria developed from Soludo's lecture and cross-referenced to a fuller rejoinder to the lecture as a separate part of the paper system, as indicated above)
7. An agenda for multidisciplinary research to be sculpted into each section of the paper for pulling together into action plans at the end of the paper system (and captured in the main paper itself, with details cross-referred to the parts) – global perspectives

It is my gut feel that this paper will literally write itself if we stay on message by using this latter plan as the **main outline** and using the earlier one as a **narrative device** that provides meaningful subheadings in the drama! Indeed, that is how we will develop (or vomit) the paper from the mosaic of research notes.

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Notice that it is good to capture most of the key points in the notes under the appropriate headings in **big literature review** section in Part I following the introduction. That can be followed by sections that develop the theoretical and practical research directions, while Parts II demonstrates the modelling suggestions on selected areas of finance and economics as indicated, as well as the rejoinder to Professor Soludo's paper.

For now, it seems logically superior to make the rejoinder to Soludo's paper the second part, so that Part III will reflect illustrate all the insights from both Parts I & II in the exemplifications. That is it! (Even this twist is belated as the rejoinder is already called Part III, we simply need to leave matters the way they are and ensure each part is as complete and independent as possible).

Finally, it might be helpful to give both **references** and **bibliography** as Kassim did in his paper, but to justify it e.g. need to indicate the sort of governing literature that should be bought by the NMC in founding the NIAFER, and for use by researchers and students at NMC and Nigerian HEIs. This way, some references in the FRM text could be included and the stocking of such texts in the NMC recommended as an action plan, alongside **developing indigenous texts** of such class.

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Rejoinder to Professor Soludo's Inaugural Lecture entitled Financial Globalization and Domestic Monetary Policy: Whither the Economics for the 21st Century?, submitted to the *NMC-COMSATS International Conference on Mathematical Modelling of Some Global Challenging Problems in the 21st Century*, 26-30 November, 2008, National Mathematical Centre, Kwali Abuja, Nigeria

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ROLE OF ENGINEERING AND SCIENCE IN SUSTAINABLE DEVELOPMENT IN THE 21ST CENTURY

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ABSTRACT

Advances made in industrial technology are the major elements for sustainable development, where the engineers and scientists are the main groups responsible for such industrial progress.

Sustainable development presents us all with the challenges of living in ways are compatible with the long-term constraints imposed by the finite carrying capacity of the closed system which is planet earth. The challenge of meeting human development needs while protecting the earth's life support systems confronts scientists, technologists, policy-makers, and communities from local to global levels. Clean technology is an approach to process, selection, designing and operation which combines for instance, a conventional chemical engineering with some of these system-based environmental management tools. The application of chemical engineers for example, requires to take on a significantly different role, using their professional expertise to work with people from other disciplines and with the lay public. The contribution of chemical engineering for instance to the formation of Nigeria energy policy provides an example of the importance of this role. This paper discusses the vital important role of engineering and science as agents of social change and the need to develop different set of skills, which might make the profession more attractive to potential new recruits.

Mathematics Subject Classification 2000:00-02 & 97B10

1. ENGINEERING AND SCIENCE

•Engineering is the discipline and profession of applying technical and scientific knowledge and utilizing natural laws and physical resources in order to design and implement materials, structures, machines devices, systems and processes that safely realize a desired objective and meet specified criteria.

•**Engineering** is a broad discipline, which is often broken down into several sub-disciplines; some of these disciplines are categorized as follows:

•**Aerospace engineering:** Deals with the design of aircraft, spacecraft and related topics.

•**Chemical engineering:** Deals with the conversion of raw materials into usable commodities and the optimization of flow systems, especially separations.

• **Civil engineering:** The design and constructs of public and private works, such as infrastructure, bridges and buildings.

- **Electrical engineering:** The design of electrical systems, such as transformers, as well as electronic goods.
- **Mechanical engineering:** The design of physical or mechanical systems, such as engines, power trains, kinematics chains and vibration isolation equipment.

1.1 ENGINEERS THEN USE THEIR KNOWLEDGE OF SCIENCE, MATHEMATICS AND APPROPRIATE EXPERIENCE TO FIND SUITABLE SOLUTION TO A PROBLEM

Engineering is considered a branch of applied mathematics and science, creating an appropriate mathematical model of a problem allows them to analyze it (Sometimes definitively), and to test potential solutions.

Science (From Latin Scientia, meaning “knowledge” or knowing”)is the effort to discover, and increase human understanding of how the physical world works, through controlled methods scientists use observable physical evidence of natural phenomena to collect data and analyze this information to explain how things work.

Such methods include experimentation that tries to simulate natural phenomena under controlled conditions and through experiments.

Knowledge in science is gained through research.

The impossibility of separating the nomenclature of a science from the science itself is owing to this, that every branch of physical science must consist of three things. These are:

- The series of facts which are the objects of the science,
- The ideas which represent these facts and
- The words by which these ideas are expressed.

1.2 RELATIONSHIP BETWEEN ENGINEERING AND SCIENCE

- Scientists study world as it is while engineers create the world that has never been.
- There exists an overlap between the science and Engineering practice: in engineering, one applies science.
- Both areas of endeavor rely on accurate observation of materials and phenomena.
- Both use mathematics and classification criteria to analyze and communicate observations.
- Scientists are expected to interpret their observations and to make expert recommendations for practical action based on those interpretations.
- Scientists and engineers make up less than 5% of the population but create up to 50% of the GDP.

2. EFFECTIVE SYSTEMS

The efforts to mobilize science and Engineering for sustainability are more likely to be effective when they manage Boundaries between knowledge and action in ways that simultaneous enhance the salience, credibility, and legitimacy of the information they produce.

We characterize the three functions that contributed most to such boundary management” as” communication,” “translation”, and “mediation”

(i) Communication: Active, iterative, and inclusive communication between experts and decision makers proves crucial to system that mobilize knowledge that is seen as salient, credible, and legitimate in the world of action. Also, the ability to mobilize knowledge for action was also reduced when communication was infrequent or occurred only at the outset of an assessment.

(ii) Translation: Mutual understanding between experts and decision makers is often hindered by jargon, language, experiences, and presumptions about what constitutes persuasive argument. Systems mobilize knowledge for action by translation that facilitates mutual comprehension in the face of such differences.

(iii) Mediation: It appears to be most important in facilitating the legitimacy of efforts to mobilize science and Engineering for sustainability while retaining adequate levels of salience and credibility to multiple actors. Mediation worked in our cases by enhancing the legitimacy of the process through increasing transparency, bringing all perspectives to the table, providing rules of conduct, and establishing criteria for decision making.

3. ENERGY SECTOR (CASE STUDY OF NIGERIA)

Energy has a major impact on every aspect of modern life. It plays a vital role in the development of our Nation.

The table below shows the Nigeria’s Energy Resources as at 2005:

Wind	2-4m/s at 10m height	
Solar Radiation	3.5 – 7.0KWh/m ² -day	
Crop Residue	83million tones/year	
Animal Waste	61million	
Fuel wood	13,071,464 Hectares	
Small Hydropower	3,500MW	
Large Hydropower	11,250MW	
Coal and Lignite	4 billion tonnes	2788.1
Tar Sands	30 billion barrels of oil equivalent	4216.0
Natural Gas	187.44 trillion Scf	4549.3
Crude Oil	35.2 billion barrels	4787.2
ENERGY RESOURCES PHCN’S installed generation capacity is about 6,000MW The generation stations currently supply less than 3000MW to the nation grid. Supply falls short of demand, availability is erratic and the level of accessibility is poor. Coal was for many years, the choice fuel for power generation and for driving locomotive engines.		
		ENERGY UNIT MILLION TOE
		RESERVES

- Coal production began in 1916 with annual product output of 24.511 tonnes and peaked in 1959 with an output of 905.397 tonnes per annum and ceased during the 1966 – 1970 Civil war.
- After the hostilities, production commenced and peaked again in 1972 at 323.001 tonnes per annum, thereafter, it began to decline by 1998, Coal production has declined to 21,940 tonnes per annum.
- Oil exploitation by six major multi-national companies (Shell, Elf, Texaco, Mobil, Agip and Chevron) is in joint venture (JV) with the NNPC.
- In indigenous Nigerian enterprises (Alfred Jame Petroleum Ltd, AMNI Int. petroleum Dev. Co. Ltd, Summit oil Int. Ltd and Yinka Folawinyo Petroleum) are also engaged in the upstream oil activities.
- Natural gas is managed by the Nigerian Gas Company (NGC)
- There are four (4) refineries (old and new port Harcourt, Kaduna, and Warri) managed as subsidiaries of the NNPC, with a total installed capacity of 445,000 barrel/day (bpd).
- Our Engineers and Scientists can make use of their different set of skills to manage these Energy Resources properly to satisfy the need of the Nation or its Consumers.

4. MAINTENANCE OF INFRASTRUCTURE

Maintenance of the infrastructure is of vital importance as it is one of the Role of Engineering and Science in Sustainable Development.

•Maintenance is defined as:

- The art of keeping the facility or equipment in a good working condition.
- The work done in order to keep restore or improve every part of the facility or equipment and its functions to currently acceptable standard.

Among the three types of maintenance systems identified that is, preventive maintenance, Corrective maintenance and Breakdown maintenance, the preventive maintenance is of more important..

- The preventive maintenance is based on the adage, which says: “prevention is better than cure” or a stitch in time save mine.”
- A good machine may keep running or good equipment may remain functional with minor faults, only for it to break down when put on full blast.
- This is Murphy's Law in action, the law says: if something can go wrong, it will do so at the worst possible moment”. This is what makes preventive maintenance more important.

5. CHALLENGES

- Lack of required equipments for conducting practical in our schools.
- Lack of required set of skills for Engineers and Scientists.

- Financial constraint affecting Engineering and Science projects.
- High level of poverty in the country.
- Poor maintenance attitude.
- In -efficient use of Energy Resource.
- In adherence to Energy Policy.

6. RECOMMENDATIONS

- Government should ensure manpower development of the Engineers and Scientists.
- The nation should ensure sustainable industrial production.
- Government should ensure the efficient use of Energy resources in the Country.
- The nation should provide enabling environment for local production.
- Government should encourage good attitude to maintenance work, putting into considerations the following factors. These factors are knowledge of the timing and content of the activity. Wherever the timing is known, we operate preventive maintenance, which is recommended. Wherever the content is known, we operate corrective maintenance. Wherever they both (Timing and Content) are not known, we operate Breakdown maintenance, which is deprecated.
- Science and Engineering project should not be limited to students of higher learning, but should also be extended to secondary levels.

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10. Musa D. Abudullahi: Maintenance of Infrastructure for Effective Science, Technology and Mathematics Education, (A paper delivered at the Federal Science Equipment Centre, Laboratory workshop, 2007).

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