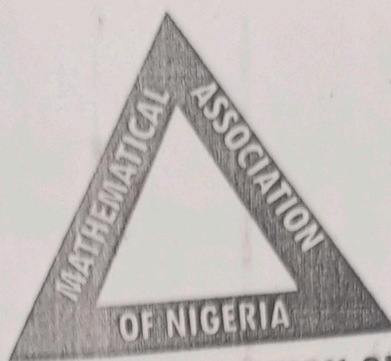




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Table of Contents

1.	Yahya, W. B., Adeniyi, I. A., Olayemi, R. M., Dauda, K. A., Oloredo, K. O., Banjoko, A. W., Garba, M.K. and Ezenweke, C. P.: On the Efficiency of Maximum Likelihood and Restricted Maximum Likelihood Estimators for Linear Mixed Effects Models	1
2.	Gegele, O. A, Taiwo, O. A, Uwaheren, O. A and Etuk, M. O.: Computational Methods for Higher Order Linear and Non -Linear Integro - Differential Equations by Spline Collocation	23
3.	Sagir, A.M.: Towards an Improved a Neuro-Fuzzy Classifier for Medical Diagnosis	33
4.	Agada, P.O., Agada, I.O., and Odia, J.: Modeling the Sequence of Wet and Dry Days Conditioned on the Occurrence of an Extreme Weather Condition in Port Harcourt, Nigeria.	41
5.	Mohammed, A. and Ifeanyi, M. C.: Extension of Measure Functions over Sigma-Algebra on Finite Matrix Set to Finite Rhorix Set	53
6.	Yakusak, N.S. and Adeniyi, R.B.: Three Off-Steps Hybrid Falkner - Type Block Method for Solution of Second Order Initial Value Problems	78
7.	Mohammed, A. & Balarabe, M.: Formulation of Cardinality of Green's Relations in Semigroup of all Rhotrices over Finite Field of Prime Order	87
8.	Ajisope M.O.,Rauf K., Anthonio Y. O., Kanu R.U. and Omolehin J.O.: Extension of Variants of Hardy Integral inequality	94
9.	Uwaheren, O. A. and Taiwo, O.A.: Numerical Solution of Multi-Order Fractional Differential Equations Using Orthogonal Polynomial Basis	98
10.	Ali M.S.:Synthesis of Fuzzy Optimization by Pseudogoal Function	104
11.	Ojo, T.O., Ogunleye, T.A., Olawuwo, S. and Adeleke, M.O.: On the Optimal Grain Yield of Cowpea Varieties in the three Senatorial Zones of Osun State, Nigeria	110
12.	Abdullahi M.A., Mohammed A., Aliyu D.H., Computational Study on application of Nanoparticles as a Drug Carrier on the Wall of Stenosed Arteries	118
13.	Akande, K.B. and Ibrahim, M.O.: Application of Differential Transformation Method in Solving Optimal control of Terrorism Model	127
14.	Funmilayo, H.O. and Moses, S.D.: Free Convection Flow of dissipative non-Newtonian fluid along a vertical plate in a porous medium	134
15.	Aliyu, D.H., Abdullahi, M.A., Rakiya M.K.A.: CRT-Based Access Control in Big Data	144
16.	Oahimire, J.I., Lelea, O.L. and Olajuwon, B.I.:Effects of Chemical Reaction and Thermal Radiation on Heat and Mass Transfer of Mhd Flow in the Presence of Thermal Diffusion and Heat Source	151
17.	Ibrahim, A.M. and Mukaila, I.F.: Subreflexive Property of Fuzzy Relation	163
18.	Audu, A. and Adewara, A.A.: A Factor-type Estimator with known functions of Auxiliary variable in Single-phase sampling	171
19.	Baraya, M.Y. and Sani, B.: A single-item inventory model for delayed deteriorating items with inventory level dependent demand rate and time-dependent holding cost	178
20.	Lukman L., Yusuf M. K. and Shehu A.A.: Sensitivity Analysis on Soft Drinks Produced By Coca-Cola Bottling Company Challawa Plant Kano	193

COMPUTATIONAL METHODS FOR HIGHER ORDER LINEAR AND NON – LINEAR INTEGRO – DIFFERENTIAL EQUATIONS BY SPLINE COLLOCATION

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ABSTRACT

In this paper, we present two Spline collocation methods namely standard cubic Spline and non – polynomial Spline collocation to solve third and fourth order linear and non – linear integro – differential equations. Newton Kantorovich scheme was used to linearize the non- linear term in the case of non linear equation and this leads to an iterative procedure. The resulting system of linear algebraic equations are then solved using MAPLE 13. The methods are applied to a few examples to illustrate the accuracy and effectiveness of the methods.

Key words: Cubic Spline, Non – polynomial Spline, Higher order integro – differential equation, Collocation.

1.0 INTRODUCTION:

Integro – differential equations play important role in many branches of linear and non-linear functional analysis and their application in elasticity, engineering, mathematical physics and contact mixed problem (Farshid and Elhami, (2013)). Because of the importance of integro – differential equations to modern mathematics and coupled with the fact that this class of equations are rarely solved analytically (Taiwo, Jimoh and Bello 2014), researchers in recent time have employed various numerical methods to solve integro – differential equations among which are Laplace decomposition methods, Pade approximant wavelet, Homotopy perturbation method (Bahuguna, Mayan and Pandey, 2009; Abdul-majid, 2011. Others are Legendre polynomial (Abbas and Mehdi 2010; Edalatpanah and Abdolmaleki, 2014, Chebyshev wavelet (Barzkar, et. al 2012); (Gamze, et. al 2012); (Maleknejad and Attary 2012), Taylor's series method (Eke and Jackreece, 2011); Dariania and Ivaz, 2008) and Haar wavelet (Fayyaz and Azram, 2013); Suha and Amal, 2012).

For more accurate results, the Spline approximation methods has been the focus of researchers in more recent time for the solution of ordinary and partial differential equations; integral equations and integro – differential equations (Navnit, 2012); Jalil and Reza (2009); Taiwo and Gegele (2014); Arezoo (2012); Ahlberg, Nilson and Walsh (1976); Gegele (2004); Zarebnia, Hoshyar and Sedaghati (2011)).

We focus on the application of standard function and non linear polynomial Spline functions to develop numerical methods for obtaining smooth approximations to the solution of higher order linear and non linear integro – differential equations of the form

$$\sum_{r=0}^n P_r(x) y^{(n)}(x) + \int_a^b k(x,t) y(t) dt = f(x) \quad (1)$$

With the initial conditions

$$y^{(z)}(0) = \beta_z, \quad z = 0, 1, 2, \dots, n-1 \quad (2)$$

where $P_r(x)$ are constants or constant coefficients are, $k(x,t)$, $y(t)$, $g(x)$ and $f(x)$ are given both continuous functions; a , b and β_z , are constants, n is the order of the derivative and $y(x)$ is the unknown function to be determined

Similarly, the n^{th} order non- linear Fredholm integro – differential equation is written as

$$\sum_{r=0}^n P_r(x) y^{(n)}(x) + \int_a^b k(x,t) f(y(x)) dt = f(x) \quad (3)$$

With the initial conditions

$$y^{(z)}(0) = \beta_z, \quad z = 0, 1, 2, \dots, n-1 \quad (4)$$

Where $f(y(x))$ is a non linear function of x

2.0 Standard Cubic Spline Method:

In this section the idea developed in Taiwo and Gegele, (2014) was modified to solve linear and non-linear Fredholm – integro differential equations.

The standard Cubic Spline approximation defined in the interval $[a, b]$ such that

$$s_i(x) = \frac{(x_i - x)^3}{6h} M_{i-1} + \frac{(x - x_{i-1})^3}{6h} M_i + \frac{1}{h} \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (x_i - x) + \frac{1}{h} \left(y_i - \frac{h^2}{6} M_i \right) (x - x_{i-1}), i=1, 2, \dots, N \quad (5)$$

with recursive relation gives

$$h^2(M_{i+1} + 4M_i + M_{i-1}) = 6(y_{i+1} - 2y_i + y_{i-1}) \quad i=1, \dots, N-1$$

Where $M_i = s''(x_i)$, $y_i = s(x_i)$

The direct approach method formulated in Ogunlaran (2012) was used to develop numerical scheme for the solution of third and fourth order Fredholm integro – differential equations, the approach are as follows:

$$y'(x_i) = \frac{1}{2h}(y_{i+1} - y_{i-1}) + \frac{h}{12}(M_{i+1} - M_{i-1}) \quad (7)$$

$$y''(x_i) = M_i \quad (8)$$

$$y'''(x_i) = \frac{1}{2h}(M_{i+1} - M_{i-1}) \quad (9)$$

$$y^{(4)}(x_i) = \frac{1}{h^2}(M_{i+1} - 2M_i + M_{i-1}) \quad (10)$$

Substituting equations (9) – (10) into equation (1), we obtained

$$P_0(x)y_i + P_1(x) \left[\frac{1}{2h}(y_{i+1} - y_{i-1}) + \frac{h}{12}(M_{i+1} - M_{i-1}) \right] + P_2(x)M_i + P_3(x) \left[\frac{1}{2h}(M_{i+1} - M_{i-1}) \right] + P_4(x) \left[\frac{1}{h^2}(M_{i+1} - 2M_i + M_{i-1}) \right] + \int_a^b k(x, t) \left[\frac{(t_i - t)^3}{6h} M_{i-1} + \frac{(t - t_{i-1})^3}{6h} M_i + \frac{1}{h} \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (t_i - t) + \frac{1}{h} \left(y_i - \frac{h^2}{6} M_i \right) (t - t_{i-1}) \right] dt = f(x) \quad (11)$$

Now we approximate the integral part of equation (11) to obtain

$$P_0(x)y_i + P_1(x) \left[\frac{1}{2h}(y_{i+1} - y_{i-1}) + \frac{h}{12}(M_{i+1} - M_{i-1}) \right] + P_2(x)M_i + P_3(x) \left[\frac{1}{2h}(M_{i+1} - M_{i-1}) \right] + P_4(x) \left[\frac{1}{h^2}(M_{i+1} - 2M_i + M_{i-1}) \right] + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} k(x, t) \left[\frac{(t_i - t)^3}{6h} M_{i-1} + \frac{(t - t_{i-1})^3}{6h} M_i + \frac{1}{h} \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (t_i - t) + \frac{1}{h} \left(y_i - \frac{h^2}{6} M_i \right) (t - t_{i-1}) \right] dt = f(x), i=1, 2, \dots, N-1 \quad (12)$$

Setting $t = t_{i-1} + ph$ and collocating at $x_i = a + \left(\frac{b-a}{N} \right) i, i=1, 2, \dots, N-1$ in equation (12) simplified to

$$P_0(x)y_i + P_1(x) \left[\frac{1}{2h}(y_{i+1} - y_{i-1}) + \frac{h}{12}(M_{i+1} - M_{i-1}) \right] + P_2(x)M_i + P_3(x) \left[\frac{1}{2h}(M_{i+1} - M_{i-1}) \right] + P_4(x) \left[\frac{1}{h^2}(M_{i+1} - 2M_i + M_{i-1}) \right] + h \sum_{i=1}^N \int_0^1 k(x_i, t_{i-1} + ph) \left[\frac{(1-p)^3}{6} M_{i-1} + \frac{p^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-p) + \left(y_i - \frac{h^2}{6} M_i \right) p \right] dp = f(x_i), i=1, 2, \dots, N-1 \quad (13)$$

Equations (6) and (13) constitute a system of $2N$ linear algebraic equations in $(2N + 2)$ unknowns; hence we require two more equations which are obtained by using $M_0 = 0$ and $M_N = 0$. This system of linear equations is solved to obtain the unknowns M_i and $y_i, i = 0, 1, 2, \dots, N$.

2.1 The Third Order linear Fredholm Integro – differential Equation,
for $n = 3$ in equation (13), we obtained

$$\frac{1}{2h} (M_{i+1} - M_{i-1}) + h \sum_{i=1}^N \int_0^1 k(x_i, t_{i-1} + ph) \left[\frac{(1-p)^3}{6} M_{i-1} + \frac{p^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-p) + \left(y_i - \frac{h^2}{6} M_i \right) p \right] dp = f(x_i), \quad i = 1, 2, \dots, N-1 \quad (14)$$

Equations (6) and (14) constitute system of $2N$ linear algebraic equations in $(2N + 2)$ unknowns; hence we required two more equations which are obtained by using $M_0 = 0$ and $M_N = 0$. Thus system of linear equations are solved to obtain the unknowns.

2.2 The Fourth Order linear Fredholm Integro – differential Equation. i.e $n = 4$

$$\frac{1}{h^2} (M_{i-1} - 2M_i + M_{i+1}) + h \sum_{i=1}^N \int_0^1 k(x_i, t_{i-1} + ph) \left[\frac{(1-p)^3}{6} M_{i-1} + \frac{p^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-p) + \left(y_i - \frac{h^2}{6} M_i \right) p \right] dp = f(x_i), \quad i = 1, 2, \dots, N-1 \quad (15)$$

Equations (6) and (15) together with $M_0 = 0$ and $M_N = 0$ form system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns

2.3 Non- linear Fredholm Integro – differential Equation

We applied Newton – Kantorovich linearization scheme to the integral part of equation (3) because it contains the non- linear term; hence we to obtain

$$\sum_{r=0}^n P_r(x) y^{(r)}(x) + \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[\frac{(t_i - t)^3}{6h} M_{i-1} + \frac{(t - t_{i-1})^3}{6h} M_i + \frac{1}{h} \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (t_i - t) + \frac{1}{h} \left(y_i - \frac{h^2}{6} M_i \right) (t - t_{i-1}) \right] dt + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x) \quad (16)$$

Equation (16) is the linearized form of equation (3).

Setting $t = t_{i-1} + qh$ and collocating at $x_j = a + \left(\frac{b-a}{N} \right) j, j = 1, 2, \dots, N-1$, we have

$$\sum_{r=0}^n P_r(x) y^{(r)}(x) + \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[\frac{(1-q)^3}{6} M_{i-1} + \frac{q^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-q) + \left(y_i - \frac{h^2}{6} M_i \right) q \right] dq + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_j) \quad (17)$$

Equations (6) and (17) together with $M_0 = 0$ and $M_N = 0$ form a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns. This system of linear equations is solved to obtain the unknowns.

2.4 The third order non linear Fredholm integro – differential equation,
for $n = 3$, we have

$$\frac{1}{2h} (M_{i+1} - M_{i-1}) +$$

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[\frac{(1-q)^3}{6} M_{i-1} + \frac{q^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-q) + \left(y_i - \frac{h^2}{6} M_i \right) q \right] dq$$

$$+ \left(\frac{\partial k(x_n, t_n, y_n)}{\partial x} (x - x_n) + y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_i) \quad (18)$$

Equations (6) and (18) together with $M_0 = 0$ and $M_N = 0$ form a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns. This system of linear equation is solved to obtain the unknowns.

2.5. The Fourth Order linear Fredholm Integro - differential Equation, for $n = 4$ to obtained

$$\frac{1}{h^2} (M_{i+1} - 2M_i + M_{i-1}) +$$

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[\frac{(1-q)^3}{6} M_{i-1} + \frac{q^3 h^2}{6} M_i + \left(y_{i-1} - \frac{h^2}{6} M_{i-1} \right) (1-q) + \left(y_i - \frac{h^2}{6} M_i \right) q \right] dq$$

$$+ \left(\frac{\partial k(x_n, t_n, y_n)}{\partial x} (x - x_n) + y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_i) \quad (19)$$

Equations (6) and (19) together with $M_0 = 0$ and $M_N = 0$ form system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns. These systems of linear equations are solved to obtain the unknowns.

3.0 Non - polynomial Spline Method: In this section the non - polynomial Spline function developed in Taiwo and Gegele (2014) and Arezoo (2012) were modified and applied to solve third and fourth order linear and non linear Fredholm integro - differential equations.

The non- polynomial Spline function $s(x_i)$ for all subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \dots, N-1$ is defined as

$$s_i(x) = a_i + b_i(x - x_i) + c_i \sin \tau(x - x_i) + d_i \cos \tau(x - x_i) \quad (20)$$

Where a_i, b_i, c_i and d_i are to be determined and τ is a free parameter. After necessary manipulation, we obtain

$$a_i = y_i - \frac{M_i}{\tau^2}, b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau \theta}, c_i = \frac{M_i \cos \theta - M_{i+1}}{\tau^2 \sin \theta} \text{ and } d_i = -\frac{M_i}{\tau^2} \quad (21) \text{ Su}$$

Substitute equation (21) in equation (20) to obtain

$$s_i(x) = y_i - \frac{M_i}{\tau^2} - \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) \right\} (x - x_i) + \frac{1}{\tau^2 \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau(x - x_i)$$

$$- \frac{M_i}{\tau^2} \cos \tau(x - x_i) \quad i = 0, 1, \dots, N-1 \quad (22)$$

with recursive equation

$$y_{i-1} - 2y_i + y_{i+1} = h^2 [\alpha \alpha_{i-1} + 2\beta \beta_i + \alpha M_{i-1}] \quad (23)$$

Where $M_i = s''(x_i)$, $y_i = s(x_i)$ and $\theta = \tau h$

Putting equation (21) in equation (1), we obtain

$$\sum_{i=0}^n P_i(x) y''(x) + \int_0^1 k(x, t) \left[\left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) \right\} (t - t_i) + \frac{1}{\tau^2 \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau(t - t_i) - \frac{M_{i+1}}{\tau^2} \cos \tau(t - t_i) \right] dt = f(x) \quad (24)$$

Approximating equation (24) gives

$$\sum_{i=1}^N P_i(x) y''(x) + \sum_{i=1}^N \int_{x_i}^{x_{i+1}} k(x_i, t) \left\{ y_i + \frac{M_i}{\tau} + \left[\frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (t - t_i) \right] \right\} \\ - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (t - t_i) - \frac{M_{i+1}}{\tau^2} \cos \tau (t - t_i) \Big] dt = f(x) \quad (25)$$

Setting $x = x_i + qh$ in equation (25) and collocating at $x_i = a + \frac{(b-a)}{N} i$, $i = 1, 2, \dots, N-1$ yield

$$\sum_{i=1}^N P_i(x) y''(x) + \sum_{i=1}^N \int_{x_i}^{x_{i+1}} k(x_{i+1}, t_i + qh) \left\{ y_i + \frac{M_i}{\tau} + \left[\frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (qh) \right] \right\} \\ - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (qh) - \frac{M_{i+1}}{\tau^2} \cos \tau (qh) \Big] dq = f(x_i), i = 1, 2, \dots, N \quad (26)$$

Equations (23) and (26) together with $M_0 = 0$ and $M_N = 0$ constitute a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns

3.2 Third Order linear Fredholm integro – differential Equation.

Let $n = 3$, we have

$$\sum_{i=1}^N P_i(x) y'''(x) + \sum_{i=1}^N \int_{x_i}^{x_{i+1}} k(x_{i+1}, t_i + qh) \left\{ y_i + \frac{M_i}{\tau} + \left[\frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (qh) \right] \right\} \\ - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (qh) - \frac{M_{i+1}}{\tau^2} \cos \tau (qh) \Big] dq = f(x_i), i = 1, 2, \dots, N \quad (27)$$

Equations (23) and (27) together with $M_0 = 0$ and $M_N = 0$ constitute system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns

3.3 Fourth Order linear Fredholm integro – differential Equation.

Let $n = 4$, we obtained

$$\sum_{i=1}^N P_i(x) y''''(x) + \sum_{i=1}^N \int_{x_i}^{x_{i+1}} k(x_{i+1}, t_i + qh) \left\{ y_i + \frac{M_i}{\tau} + \left[\frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (qh) \right] \right\} \\ - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (qh) - \frac{M_{i+1}}{\tau^2} \cos \tau (qh) \Big] dq = f(x_i), i = 1, 2, \dots, N \quad (28)$$

Equations (23) and (28) together with $M_0 = 0$ and $M_N = 0$ constitute a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns

3.4 Non Linear Integro differential Equation

Now we apply Newton – Kantorovich linearization scheme to the integral part of equation (3)

Equation (3) contains the non- linear term.

$$\sum_{i=1}^N P_i(x) y''(x) + \sum_{i=1}^N \int_{x_i}^{x_{i+1}} \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[y_i + \frac{M_i}{\tau} + \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (t - t_i) + \right. \right. \\ \left. \left. - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (t - t_i) - \frac{M_{i+1}}{\tau^2} \cos \tau (t - t_i) \right\} \right] dt + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - \right. \\ \left. \frac{\partial^2 k(x_n, t_n, y_n)}{\partial x^2} (x - x_n)^2 \right) \left[y_i + \frac{M_i}{\tau} + \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) (t - t_i) + \right. \right. \\ \left. \left. - \frac{1}{\tau \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau (t - t_i) - \frac{M_{i+1}}{\tau^2} \cos \tau (t - t_i) \right\} \right] dt = f(x) \quad (29)$$

Setting $x = x_i + qh$ in equation (29) and collocating at $x_i = a + \frac{(b-a)}{N} i$, $i = 1, 2, \dots, N-1$ yield

$$\sum_{r=0}^n P_r(x) y^{(r)}(x) + \sum_{i=1}^N \int_0^1 \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[y_i + \frac{M_i}{t^2} + \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) \right\} (qh) + \frac{1}{\tau^2 \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau(qh) - \frac{M_i}{t^2} \cos \tau(qh) \right] dt + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_i) \quad (30)$$

Equations (23) and (30) together with $M_0 = 0$ and $M_N = 0$ constitute a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns

3.5 Third Order Non-linear Fredholm integro – differential Equation. i.e $n = 3$.

$$\sum_{r=0}^3 P_r(x) y^{(r)}(x) + \sum_{i=1}^N \int_0^1 \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[y_i + \frac{M_i}{t^2} + \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) \right\} (qh) + \frac{1}{\tau^2 \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau(qh) - \frac{M_i}{t^2} \cos \tau(qh) \right] dt + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_i) \quad (31)$$

Equations (23) and (31) together with $M_0 = 0$ and $M_N = 0$ constitute a system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns.

3.6 Fourth order non linear Fredholm integro – differential equation. i.e $n = 4$.

$$\sum_{r=0}^4 P_r(x) y^{(r)}(x) + \sum_{i=1}^N \int_0^1 \frac{\partial k(x_n, t_n, y_n)}{\partial y} \left[y_i + \frac{M_i}{t^2} + \left\{ \frac{1}{h} (y_{i+1} - y_i) + \frac{1}{\tau \theta} (M_{i+1} - M_i) \right\} (qh) + \frac{1}{\tau^2 \sin \theta} (M_i \cos \theta - M_{i+1}) \sin \tau(qh) - \frac{M_i}{t^2} \cos \tau(qh) \right] dt + \left(\partial k(x_n, t_n, y_n) + (x - x_n) \frac{\partial k(x_n, t_n, y_n)}{\partial x} - y_n \frac{\partial k(x_n, t_n, y_n)}{\partial y} \right) = f(x_i) \quad (32)$$

Equations (23) and (32) together with $M_0 = 0$ and $M_N = 0$ constitute system of $(2N + 2)$ linear algebraic equations in $(2N + 2)$ unknowns.

3.0 Numerical Examples:

In this section numerical examples were used to demonstrate the performance of the scheme presented for linear and non linear third and fourth order Fredholm integro – differential equations.

Example 1: Consider the third order linear Fredholm integro differential equation

$$y'''(x) = e - 2 - x + (3 + x)e^x + \int_0^1 (x - t)y(t) dt$$

with the initial conditions

$$y(0) = 0, y'(0) = 1, y''(0) = 2$$

The exact solution is $y(x) = x e^x$

Example 2: Consider the third order non - linear Fredholm integro differential equation

$$y'''(x) = 8e^{2x} - \frac{1}{16}e^x + \frac{1}{16} \int_0^1 e^{x \cdot 4t} y^2(t) dt$$

with the initial conditions

$$y(0) = 1, y'(0) = 2, y''(0) = 4$$

The exact solution is $y(x) = e^{2x}$

Example 3: Consider the fourth order linear Fredholm integro differential equation

$$y^{(iv)}(x) = e^x + (1-e)x - 2 + \int_0^1 (x+2t)y''(t)dt$$

with the initial conditions

$$y(0)=1, y'(0)=1, y''(0)=1$$

The exact solution is $y(x) = e^x$

Example 4: Consider the fourth order non - linear Fredholm integro differential equation

$$y^{(iv)}(x) = e^x + 1 - e + \int_0^1 e^{-t} y^2(t)dt$$

with the boundary conditions

$$y(0)=1, y''(0)=1, y(1)=e, y''(1)=e$$

The exact solution is $y(x) = e^x$

Table 1: Numerical Results for example 1 when $N = 10$

x	Exact solution	Standard Cubic Spline	Non polynomial Spline		
			$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0.0	0.000000000	0.000000000	0.000000001	0.000000000	0.000000000
0.1	0.1105170918	0.1105291068	0.1105273528	0.1105272298	0.1105271008
0.2	0.2442805516	0.2442969226	0.2442930246	0.2442922736	0.2442908206
0.3	0.4049576424	0.4049767634	0.4049740614	0.4049718284	0.4049708604
0.4	0.5967298792	0.5967513362	0.5967488312	0.5967470922	0.5967452502
0.5	0.8243606355	0.8243868705	0.8243834165	0.8243804015	0.8243787655
0.6	1.0932712800	1.0933011040	1.0932962960	1.0932946370	1.0932923890
0.7	1.4096268951	1.4096599140	1.4096566000	1.4096539150	1.4096515230
0.8	1.7804327423	1.7804343690	1.7804339760	1.7804337570	1.7804337510
0.9	2.2136428002	2.2136450140	2.2136447760	2.2136445360	2.2136441990
1.0	2.7182818281	2.7182818284	2.7182818282	2.7182818281	2.7182818281

Table 2: Numerical Results for example 2 when $N = 10, n = 5$

x	Exact solution	Standard Cubic Spline	Non Polynomial Spline		
			$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.221402758	1.221547958	1.221534898	1.221522158	1.221505428
0.2	1.491824698	1.492018518	1.492001718	1.491992868	1.491967908
0.3	1.822118800	1.822349870	1.822313120	1.822298990	1.822290900
0.4	2.225540928	2.225823088	2.225762508	2.225760808	2.225748108
0.5	2.718281828	2.718605008	2.718551648	2.718529458	2.718513648
0.6	3.320116923	3.320127137	3.320127096	3.320127015	3.320126926
0.7	4.055199967	4.055214748	4.055212283	4.055211832	4.055210692
0.8	4.953032424	4.953051667	4.953049830	4.953046816	4.953045408
0.9	6.049647464	6.049670680	6.049669631	6.049669441	6.049668834
1.0	7.389056099	7.389056099	7.389056099	7.389056099	7.389056099

Table 3: Numerical Results for example 3 when $N = 10$

x	Exact solution	Standard Cubic Spline	Non Polynomial Spline		
			$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105195050	1.105194646	1.105192825	1.105191541
0.2	1.221402758	1.221430934	1.221429689	1.221428544	1.221426670
0.3	1.349855808	1.349889980	1.349888561	1.349886153	1.349884956
0.4	1.491824698	1.491859914	1.491857677	1.491855786	1.491854570
0.5	1.648758195	1.648760642	1.648759027	1.648758195	1.648755599

Table

0	1.822118800	1.822120251	1.822120122	1.822120079	1.822119993
0.1	2.013752707	2.013754528	2.013754409	2.013754304	2.013754086
0.2	2.225540928	2.225543164	2.225543121	2.225542909	2.225542594
0.3	2.459603111	2.459605744	2.459605704	2.459605490	2.459605169
0.4	2.718281828	2.718281822	2.718281825	2.718281827	2.718281828

Table 4: Numerical Results for example 4 when $N = 10, n = 5$

X	Exact solution	Standard Cubic Spline	Non Polynomial Spline		
			$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105172240	1.105172115	1.105171932	1.105171920
0.2	1.221402758	1.221404520	1.221404271	1.221404147	1.221403953
0.3	1.349858808	1.349860971	1.349860801	1.349860618	1.349860440
0.4	1.491824698	1.491827154	1.491826914	1.491826817	1.491826673
0.5	1.648721271	1.648724164	1.648723994	1.648723865	1.648723599
0.6	1.822118800	1.822118907	1.822118906	1.822118903	1.822118901
0.7	2.013752707	2.013752848	2.013752837	2.013752821	2.013752813
0.8	2.225540928	2.225541120	2.225541110	2.225541098	2.225541066
0.9	2.459603111	2.459603333	2.459603330	2.459603315	2.459603307
1.0	2.718281828	2.718281828	2.718281828	2.718281828	2.718281828

Table

Table 5: Error of methods for Example 1

X	Standard Cubic Spline	Non Polynomial Spline		
		$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0	2.0000E-10	1.0000E-10	0	0
0.1	1.2015E-05	1.0261E-05	1.0138E-05	1.0009E-05
0.2	1.6371E-05	1.2473E-05	1.1722E-05	1.0269E-05
0.3	1.9121E-05	1.6419E-05	1.4186E-05	1.3218E-05
0.4	2.1457E-05	1.8952E-05	1.7213E-05	1.5371E-05
0.5	2.6235E-05	2.2781E-05	1.9766E-05	1.8130E-05
0.6	2.9824E-05	2.5016E-05	2.3357E-05	2.1109E-05
0.7	3.3019E-05	2.9705E-05	2.7020E-05	2.4628E-05
0.8	1.6273E-06	1.2341E-06	1.0148E-06	1.0089E-06
0.9	2.2140E-06	1.9758E-06	1.7362E-06	1.3992E-06
1.0	3.0000E-10	2.0000E-10	0	0

Table

Table 6: Error of methods for Example 2

X	Standard Cubic Spline	Non Polynomial Spline		
		$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0	0	0	0	0
0.1	1.4520E-04	1.3214E-04	1.1940E-04	1.0267E-04
0.2	1.9382E-04	1.7702E-04	1.6817E-04	1.4321E-04
0.3	2.3107E-04	1.9432E-04	1.8019E-04	1.7210E-04
0.4	2.8216E-04	2.2158E-04	2.1988E-04	2.0718E-04
0.5	3.2318E-04	2.6982E-04	2.4763E-04	2.3182E-04
0.6	1.0214E-05	1.0173E-05	1.0092E-05	1.0003E-05
0.7	1.4781E-05	1.2316E-05	1.1865E-05	1.0725E-05
0.8	1.9243E-05	1.7406E-05	1.4392E-05	1.2984E-05
0.9	2.3216E-05	2.2167E-05	2.1977E-05	1.9370E-05
1.0	0	0	0	0

Table

Table 7: Error of methods for Example 3

X	Standard Cubic Spline	Non Polynomial Spline		
		$\alpha = \frac{1}{8}, \beta = \frac{1}{8}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0	0	0	0	0
0.1	2.4132E-05	2.3728E-05	2.1907E-05	2.0623E-05
0.2	2.8176E-05	2.6931E-05	2.5786E-05	2.3912E-05
0.3	3.1172E-05	2.9753E-05	2.7345E-05	2.6148E-05
0.4	3.5216E-05	3.2979E-05	3.1088E-05	2.9872E-05
0.5	3.9371E-05	3.7756E-05	3.6924E-05	3.4328E-05
0.6	1.4512E-06	1.3218E-06	1.2791E-06	1.1930E-06
0.7	1.8211E-06	1.7019E-06	1.5967E-06	1.3785E-06
0.8	2.2362E-06	2.1932E-06	1.9810E-06	1.6659E-06
0.9	2.6335E-06	2.598E-06	2.3786E-06	2.0581E-06
1.0	6.0000E-10	3.0000E-10	1.0000E-10	0

Table 8: Error of methods for Example 4

X	Standard Cubic Spline	Non Polynomial Spline		
		$\alpha = \frac{1}{6}, \beta = \frac{1}{3}$	$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$	$\alpha = \frac{1}{8}, \beta = \frac{3}{8}$
0	0	0	0	0
0.1	1.3217E-06	1.1973E-06	1.0137E-06	1.0021E-06
0.2	1.7625E-06	1.5126E-06	1.3891E-06	1.1947E-06
0.3	2.1632E-06	1.9931E-06	1.8103E-06	1.6320E-06
0.4	2.4561E-06	2.2163E-06	2.1189E-06	1.9752E-06
0.5	2.8934E-06	2.7235E-06	2.5936E-06	2.3281E-06
0.6	1.0745E-07	1.0593E-07	1.0327E-07	1.0135E-07
0.7	1.4115E-07	1.2982E-07	1.1428E-07	1.0574E-07
0.8	1.9213E-07	1.8213E-07	1.6957E-07	1.3820E-07
0.9	2.2156E-07	2.1871E-07	2.0386E-07	1.9633E-07
1.0	0	0	0	0

Conclusion:

Spline collocation techniques have been presented for the approximate solution of higher order linear and non – linear integro differential equations. These methods have been extended to solve boundary value problems (example 4), since $M_0 = M_N = 0$ can also be used as boundary conditions (see Burden and Faires, 1993). A comparison with the exact solution reveals that the present methods are very effective and reliable (tables 5 – 8). We observed that the Non- Polynomial Spline method at different values of α and β performs better than the Standard Cubic Spline method. We can conclude that the proposed methods give close approximation to the exact solution.

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