Parameter Estimation and Reliability, Hazard Functions of Gompertz Burr Type XII Distribution

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Abstract

This paper introduces and explores a new compound distribution named the Gompertz Burr Type XII (GOBXII) distribution. Its statistical properties were established where the unknown parameters of the new model were estimated by using the method of MLE (maximum likelihood estimation).

The simulation method has been used as a one of manners of operation research for evaluating the performance of the estimated distribution parameters. Its superiority over other compound distributions like the Gompertz Weibull distribution, Gompertz Lomax distribution and Beta Burr Type X was illustrated with the aid of real life data sets.**S**imulation studies

were also conducted to assess the performance and behavior of the parameters which consider unknown.

Keywords: Burr Type XII distribution, Estimation, Generalized models, Gompertz distribution, Mathematical Statistics, Statistical Properties, Simulation

1.0Introduction

There are different forms of the Burr distribution; the common ones are the Burr III (BIII), Burr X (BX) and Burr XII (BXII) distributions. In recent times, the BX distribution has received appreciable usage in distribution theory. In the literature, there exist the one parameter shape (BX1) and the two-parameter (BX) distributions. In particular, the BX1 distribution and the well-known Rayleigh distribution are special cases of the BX distribution. Further details about the BX distribution are available in Merovci et al., [1] and the references therein. It is worthy of note that the inverted (or inverse) version of the BX distribution has been developed by Ravikumar and Kantam [2]. Also, some other extensions of the distribution have been developed recently, the Gamma Burr Type X (GBX) distribution (Khaleel et al., [3]) and Beta Burr X (BBX) distribution (Merovci et al., [1]) are notable examples.

Meanwhile, our focus in this research is based on the BXII distribution because it has some other interesting and important models like the Weibull distribution and the Logistic distribution as special cases. Also, it has not yet been rigorously studied unlike the BX distribution. Besides, it is also of two types; two-parameter Burr XII and the three-parameter Burr XII distributions.

More useful details on this are available in Titterington et al., [5] and Paranaiba et al., [6].

An extension of the BXII distribution has previously been studied by Paranaiba et al., [6] which resulted into the Beta Burr XII (BBXII) distribution. On its application to real data sets, it provided a better fit than both the BXII, Beta Weibull (BW) and Log-logistic (LoL) distributions. Another extension is available in Al-Khazaleh [6] where the Transmuted Burr XII (TBXII) distribution was defined.

The c.d.f and p.d.f of the BXII distribution are:

$$G(\mathrm{t}) = 1 - (1 + t^{\vartheta})^{-\rho} \qquad ; t, \vartheta, \rho > 0 \tag{1}$$

and,

$$g(\mathrm{t}) = \rho\vartheta t^{\vartheta-1}\ (1 + t^{\vartheta})^{-\rho-1} \qquad ; t, \vartheta, \rho > 0 \tag{2}$$

Where; ρ and ϑ are the shape parameters $G(\mathrm{t})$ and $g(\mathrm{t})$ are the Cdf and pdf respectively.

Now, there are several families of generalized distribution in the literature; detailed examples of such are readily available in Alizadeh et al., [12], Cordeiro et al., [11], Oguntunde et al., [9] and Owoloko et al., [8]. However, the focus of this research is to explore and generalize the BXII distribution using the family of distribution developed by Alizadeh et al., [12]. The aim is to develop a compound distribution that would provide more flexibility than the existing BXII distribution and some other important distributions in the literature. Also, some basic statistical properties of the resulting Gompertz Burr Type XII distribution (GOBXII for short) would be derived and established. Applications to real life data would be provided to assess its flexibility and a simulation study would be conducted.

This research is outlined and written in the following manner: in section 2, the researchers derived the densities of the GOBXII distribution and its statistical properties, in section 3, the simulation's results studies are presented and discussed, while in section 4, the real life application data showing the potentials of the GOBXII distribution over some other models is presented.

2.0 The Gompertz Burr Type XII Distribution

The densities of the Gompertz-G family of distribution are:[12]

$$F(\mathrm{t}) = 1 - e^{\frac{\omega}{\varphi}\{1-[1-G(t)]^{-\varphi}\}} \qquad ; \ \omega, \varphi > 0 \tag{3}$$

and;

$$f(\mathrm{t}) = \omega g(t)[1 - G(t)]^{-\varphi-1} e^{\frac{\omega}{\varphi}\{1-[1-G(t)]^{-\varphi}\}} \qquad ; \ \omega, \varphi > 0 \tag{4}$$

respectively.

Where; $G(\mathrm{t})$ and $g(\mathrm{t})$ are the Cdf and pdf of the parent distribution and ω , φ are additional shape parameters. Now, to derive the Cdf of the GOBXII distribution, the density in Equation (1) is inserted into Equation (3) to give:

$$F(\mathrm{t}) = 1 - e^{\frac{\omega}{\varphi}\left\{1-[1+t^{\vartheta}]^{\rho\varphi}\right\}} \qquad ;t,\omega,\varphi,\rho,\vartheta > 0 \tag{5}$$

The corresponding pdf is however derived by substituting Equations (1) and (2) into Equation (4) to give:

$$f(\mathrm{t}) = \omega\rho\vartheta t^{\vartheta-1}[1+t^{\vartheta}]^{(\rho\varphi-1)}e^{\frac{\omega}{\varphi}\left\{1-[1+t^{\vartheta}]^{\rho\varphi}\right\}} \qquad ;t,\omega,\varphi,\rho,\vartheta > 0 \tag{6}$$

Where: ω,φ,ρ and ϑ are the shape parameters

Possible plots for the pdf of the GOBXII distribution are displayed in Figure 1:

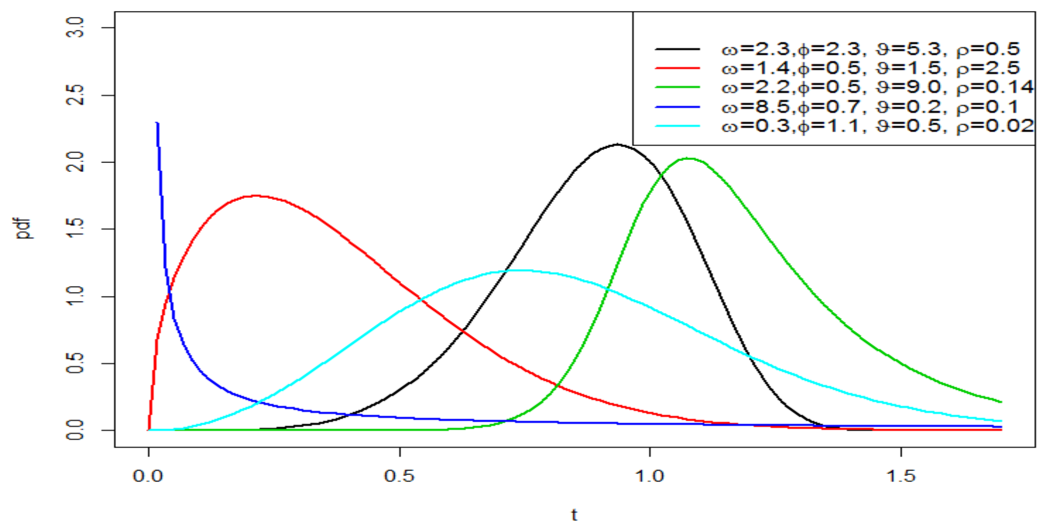

Figure 1: PDF of the GOBXII Distribution

From the plots in Figure 1, the shape of the GOBXII distribution is shown to unimodal (or inverted bathtub). When ω,ϑ and ρ small the shape be symmetric, While ω, ϑ and ρ be large that give us the shape with left skewed, whereas, ω, ϑ be large and ρ be small that give us the shape with right skewed.

Special Case(s):

When $\omega = \varphi = 1,$ the GOBXII distribution we have the BXII distribution.

2.1: Reliability Characteristics of the GOBXII Distribution

Here, explicit expressions for the reliability function, failure rate, reversed hazard function (Rhf), cumulative hazard function (Chf) and the odds function (Od.f) are derived.

2.1.1: Reliability (R)(or Survival Function (S(x)))

This is obtained from the relation:

$$\bar{F}(t) = 1 - F(t)$$

Hence, the GOBXII distribution of the (S(x)) is :

$$\bar{F}(\mathrm{t}) = e^{\frac{\omega}{\varphi}\left\{1-\left[1+t^{\vartheta}\right]^{\rho\varphi}\right\}} \qquad ; t, \omega, \varphi, \rho, \vartheta > 0 \qquad (7)$$

2.1.2: Hazard function (Hf)

This is obtained from:

$$h(t) = \frac{f(t)}{\bar{F}(t)}$$

Thus, the Hf of the GOBXII distribution is given by:

$$h(\mathrm{t}) = \omega\rho\vartheta t^{\vartheta-1}[1+t^{\vartheta}]^{(\rho\varphi-1)} \qquad ;t,\omega,\varphi,\rho,\vartheta > 0 \qquad (8)$$

Some plots for the Hf of the GOBXII distribution at various parameters are presented in Figure 2:

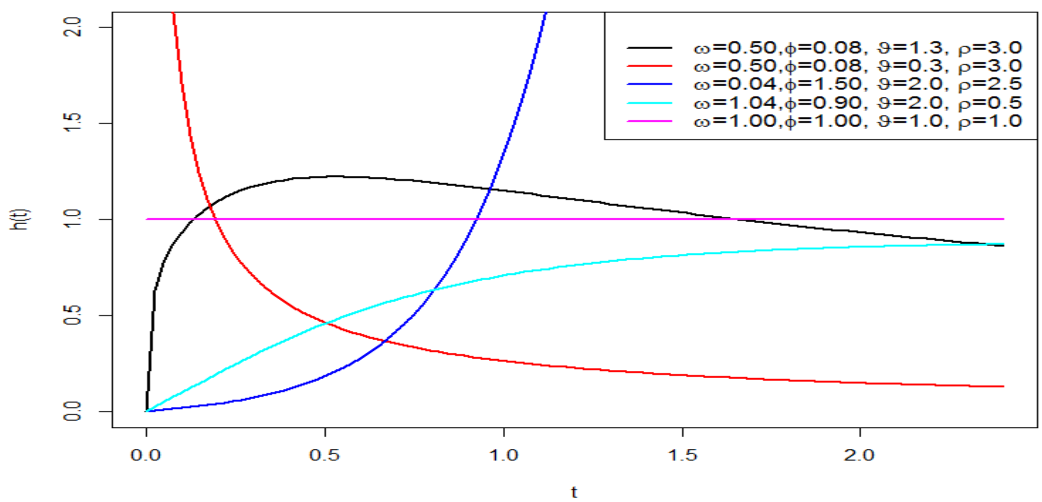

Figure 2: Hazard function of the GOBXII distribution

The plots in Figure 2 indicate that the shape of the failure rate for the GOBXII distribution could be decreasing, increasing, constant and inverted bathtub (this however depends on the parameter values).

2.1.3: The Reversed Hazard Function

This is obtained from [10]:

$$r(t) = \frac{f(t)}{F(t)}$$

Hence, the Rhf for the GOBXII distribution is given by:

$$r(t) = \frac{\omega\rho\vartheta t^{\vartheta-1}[1+t^{\vartheta}]^{(\rho\varphi-1)} e^{\frac{\omega}{\varphi}\left\{1-\left[1+t^{\vartheta}\right]^{\rho\varphi}\right\}}}{1-e^{\frac{\omega}{\varphi}\left\{1-\left[1+t^{\vartheta}\right]^{\rho\varphi}\right\}}} \qquad For \qquad t,\omega,\varphi,\rho,\vartheta > 0 \quad (9)$$

2.1.4: The Odds Function (Od.f)

This is obtained from:

$$O(t) = \frac{F(t)}{\bar{F}(t)}$$

Hence, the Od.f for the GoBXII distribution is given by:

$$O(t) = \frac{1- e^{\frac{\omega}{\varphi}\left\{1-\left[1+t^{\vartheta}\right]^{\rho\varphi}\right\}}}{e^{\frac{\omega}{\varphi}\left\{1-\left[1+t^{\vartheta}\right]^{\rho\varphi}\right\}}} \qquad ;\ t,\omega,\varphi,\rho,\vartheta > 0 \quad (10)$$

2.2: The Quantile Function (Qf) and Median

The Quantile function Qf is denoted by the inverse of the Cdf, then, the Qf of the GOBXII distribution is obtained as follows:

$$Q(u) = F^{-1}(u) = \left\{[1 - \frac{\varphi}{\omega}\log(1-u)]^{\frac{1}{\rho\varphi}} - 1\right\}^{\frac{1}{\vartheta}}, \qquad 0 < u < 1 \quad (11)$$

Where; u refers to a uniformly distributed with parameters 0 and 1. i.e., $u \sim$ Uniform (0, 1). This means that random samples can be obtained or simulated from the GOBXII distribution using the relation:

$$t = \left\{[1 - \frac{\varphi}{\omega}\log(1-U)]^{\frac{1}{\rho\varphi}} - 1\right\}^{\frac{1}{\vartheta}}$$
(12)

When the value of $u = 0.5$ substitutes into Equation (11), we have the median for the GOBXII distribution as follows:

$$Q(0.5) = \left\{[1 - \frac{\varphi}{\omega}\log(0.5)]^{\frac{1}{\rho\varphi}} - 1\right\}^{\frac{1}{\vartheta}}$$
(13)

Other quartiles or higher quartiles of the GOBXII distribution can similarly be obtained from the result in Equation (11) by means of substituting appropriate value(s) of (u). Simulating the GOBXII random variable is straightforward when U defined by as continuous uniform variable (0,1). By employing the method of invers transformation. The random variable t is given at Equation (12).,[4]

2.3: Parameter Estimation

Let $t_1, t_2, t_3, \dots, t_n$ represent a RS of size 'n' drawn from the GOBXII distribution with parameters ω, φ, ρ and ϑ. The method of (MLE) can be used to estimate the four unknown parameters and the likelihood function as follow:

$$f(t_1, t_2, t_3, \dots, t_n; \omega.\varphi.\rho.\vartheta) = \prod_{i=1}^{n}[\omega\rho\vartheta t^{\vartheta-1}[1+t_i^{\vartheta}]^{(\rho\varphi-1)} e^{\frac{\omega}{\varphi}\left\{1-\left[1+t_i^{\vartheta}\right]^{\rho\varphi}\right\}}] \qquad (14)$$

Hence, the log-likelihood function which can be denoted by $'l'$ is given by:

$$l = nlog\omega + nlog\rho + nlog\vartheta + (\vartheta - 1)\sum_{i=1}^{n} log(t_i) + (\rho\varphi - 1)\sum_{i=1}^{n} log\left(1 + t_i^{\vartheta}\right)$$

$$+ \sum_{i=1}^{n}\left\{\frac{\boldsymbol{\omega}}{\boldsymbol{\varphi}}[\mathbf{1} - (\mathbf{1} + \boldsymbol{t}_i^{\boldsymbol{\vartheta}})^{\boldsymbol{\theta\varphi}}]\right\}$$

(15)

The solution of $\frac{dl}{d\omega} = 0$, $\frac{dl}{d\varphi} = 0$, $\frac{dl}{d\rho} = 0$ and $\frac{dl}{d\vartheta} = 0$ results into the MLE of parameters $\omega. \varphi. \rho$ and ϑ respectively. However, these results cannot be obtained in closed form but numerical solutions can be obtained for the estimates using appropriate software. In particular, R software was used in this research to obtain the parameter estimates.

3.0 Simulation Studies

In this section, the procedures in performing the simulation studies are explained. For the sake of emphasis, the aim of the simulation is to investigate the behavior of the parameters of the GOBXII distribution at various parameter values using R software. RSs were generated from the GOBXII distribution with 1,000 iterations. Also, RSs of sizes n=25, 50, and 100 were further drawn. This procedure was repeated for (3) different cases whilst using different true parameter values (TPV). For the first case, the TPVs considered are $\omega = 0.5,$ $\varphi = 0.5,$ $\rho = 0.5$ and $\vartheta = 0.5$, for the second case, $\omega = 1,$ $\varphi = 1,$ $\rho = 1$ and $\vartheta = 1,$ and for the third case, $\omega = 2,$ $\varphi = 2,$ $\rho =2$ and $\vartheta =2$. Furthermore, means of the MLE, Bias and the Root Mean Square Error (RMSE) of these true parameters were obtained. Obviously, it is expected that as the sample size ‘n’ is being increased, the associated error due to the estimates would decrease. Tables 1, 2 and 3 give a clear picture of the simulation results.

Table 1: The Simulation result when parameters $\omega = 0.5,\ \varphi = 0.5,\ \ \rho = 0.5$ and $\vartheta = 0.5$

n=25				n=50				n=100			
Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE
ω	0.4546	-0.0454	0.1831	ω	0.4766	-0.0234	0.1342	ω	0.4966	-0.0034	0.1005
φ	0.5450	0.0450	0.2198	φ	0.5317	0.0317	0.1668	φ	0.5249	0.0249	0.1224
ϑ	0.5383	0.0383	0.1635	ϑ	0.5145	0.0145	0.1199	ϑ	0.4993	-0.0007	0.0902
ρ	0.5589	0.0589	0.2070	ρ	0.5330	0.0330	0.1404	ρ	0.5168	0.0168	0.0930

Table 2: The Simulation result when parameters $\omega = 1,\ \varphi = 1,\ \rho = 1$ and $\vartheta = 1$

n=25				n=50				n=100			
Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE

ω	0.9652	-0.0348	0.4130	ω	0.9815	-0.0185	0.3433	ω	1.0020	0.0020	0.2788
φ	1.0536	0.0536	0.5359	φ	1.0346	0.0346	0.4424	φ	1.0279	0.0279	0.3550
ϑ	1.0787	0.0787	0.2284	ϑ	1.0470	0.0470	0.1760	ϑ	1.0179	0.0179	0.1380
ρ	1.0523	0.0523	0.2695	ρ	1.0285	0.0285	0.2059	ρ	1.0143	0.0143	0.1512

Table 3: The Simulation result when $\omega = 2,\ \varphi = 2,\ \rho = 2$ and $\vartheta = 2$

n=25				n=50				n=100			
Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE	Par.	Means	Bias	RMSE
ω	2.0822	0.0822	0.9020	ω	2.1292	0.1292	0.8323	ω	2.1163	0.1163	0.6579
φ	2.0782	0.0782	0.9476	φ	2.0261	0.0261	0.8255	φ	2.0035	0.0035	0.6288
ϑ	2.2212	0.2212	0.5943	ϑ	2.1077	0.1077	0.4168	ϑ	2.0437	0.0437	0.2996
ρ	2.0997	0.0997	0.4230	ρ	2.0740	0.0740	0.3350	ρ	2.0453	0.0453	0.2473

Remark: It is very clear in Tables 1, 2 and 3 that as the sample size 'n' increases from 25 to 100, the RMSE associated with the estimates reduces. In the same way, the estimates appear closer to the TPVs; hence the bias generated is relatively small.

4.0 Application to Real Life Data

The GOBXII distribution is compared with the Gompertz Weibull (GoW) distribution, Gompertz Lomax (GoLo) distribution and BBX distribution with the aid of a real life data. The performances of these models are judged based on the Bayesian and Akaike's Information Criteria (BIC, AIC), Hannan & Quinn Information Criterion (HQIC) and Consistent Akaike Information Criterion (CAIC). Also, the Kolmogorov Smirnov (K-S) statistic is provided.

In this paper, the researchers employed the data of UK National Physical Laboratory which referee to the strengths of 1.5cm glass fibers in which used by Merovci et al., [1], Smith and Naylor [11] and Bourguinon et al., [12]. It consists of 63 observations and it has been used previously by and the observations are as follows:

The summary of the data is made available in Table 4:

Table 4: The Summary of dataset on Strength of Glass Fibers

N	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
63	0.550	2.240	1.507	1.590	0.1050575	-0.8999263	3.923761

The competing distributions are given in Table 5:

Table 5: Competing distributions

Distributions	Number of Parameters	Author(s)
GOBXII	Four	Proposed
GoW	Four	New
GoLo	Four	New
BBX	Four	Merovci et al., (2016)

Remark: The lower the AIC, CAIC, BIC and HQIC values posed by the distributions, the better the distribution. From Table 6, it can be observed that the GOBXII distribution performed better than the other distributions relied on its AIC, CAIC, BIC and HQIC values.

In Figure (3), the plots of all distributions considered against the empirical histogram of the observed data. Also, Figure (4) the Cdf of the competing distributions by the empirical plots for are compared in Figure 4. Table (6) referees to the competing distributions performances as follows:

Table 6: Table of Results

GOBXII distribution						
Parameter Estimates	AIC	CAIC	BIC	HQIC	K-S	p-value

$\hat{\omega} = 0.0482$ $\hat{\varphi} = 2.4765$ $\hat{\vartheta} = 0.8928$ $\hat{\rho} = 3.2441$	36.00162	36.69127	44.57416	39.37324	0.13188	0.22313
GoW distribution						
Parameter Estimates	AIC	CAIC	BIC	HQIC	K-S	p-value
$\hat{\omega} = 0.2285$ $\hat{\varphi} = 0.0096$ $\hat{\vartheta} = 0.7949$ $\hat{\rho} = 5.6121$	38.37694	39.06659	46.94948	41.74856	0.15198	0.10888
GoLo distribution						
Parameter Estimates	AIC	CAIC	BIC	HQIC	K-S	p-value

$\hat{\omega} = 0.0045$ $\hat{\varphi} = 8.1790$ $\hat{\vartheta} = 0.5069$ $\hat{\rho} = 1.5158$	37.00548	37.69513	45.57802	40.3771	0.15422	0.09987
BBX distribution						
Parameter Estimates	AIC	CAIC	BIC	HQIC	K-S	p-value
$\hat{\omega} = 0.41769$ $\hat{\varphi} = 76.8410$ $\hat{\vartheta} = 0.4671$ $\hat{\rho} = 6.7248$	37.42564	38.1153	45.99818	40.79727	0.15511	0.09644

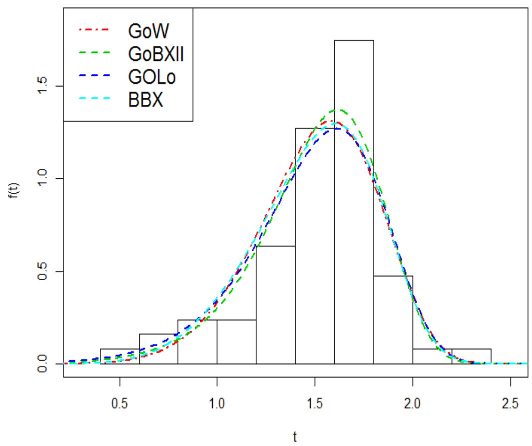

Figure 3: Plots showing all the distributions

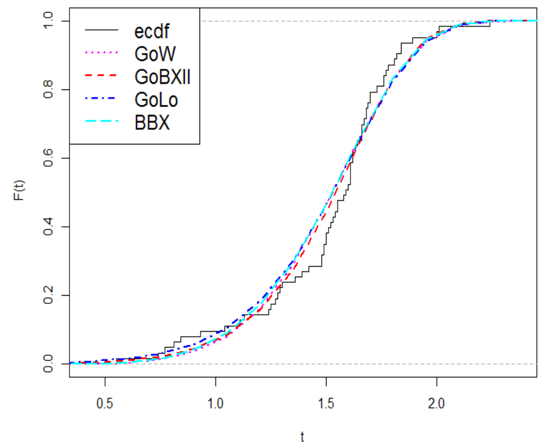

Figure 4: Plots showing the empirical Cdfs against the empirical histogram of the observed data.

Remark: It can be seen in Figures 3 and 4 that the GOBXII distribution provides a better fit than the GoW, GoLo and BBX distributions. The probability plots for the various competing distributions are displayed in Figures 5 to 8

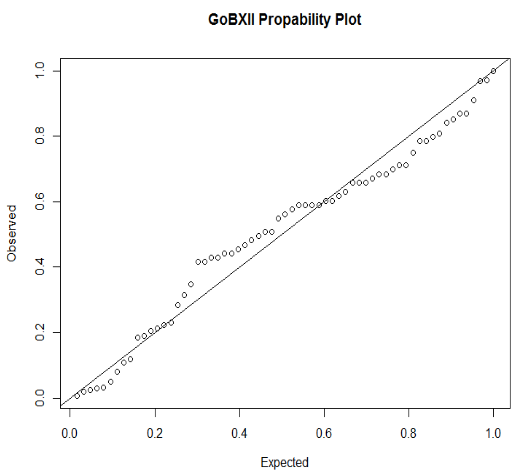

Figure 5: Probability Plot for GOBXII distribution

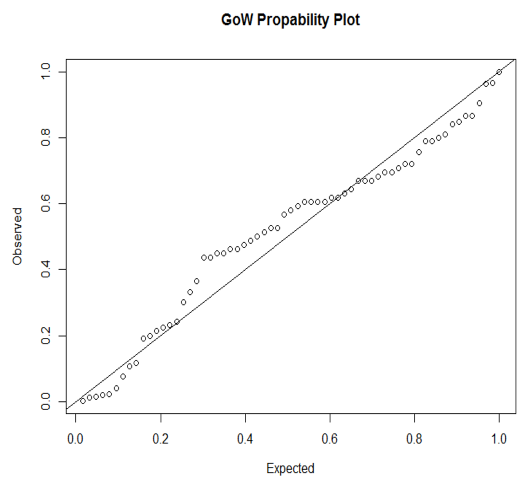

Figure 6: Probability Plot for GoW distribution

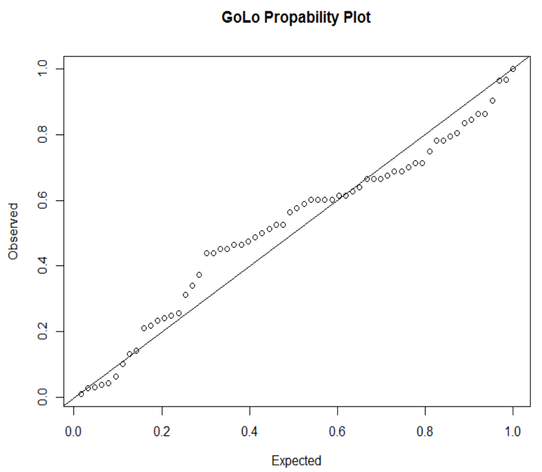

Figure 7: Probability Plot for GoLo distribution

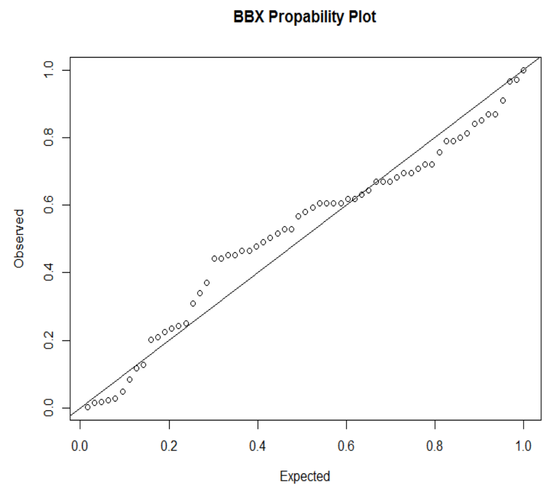

Figure 8: Probability Plot for BBX distribution

5.0 Conclusion

The GOBXII distribution has been successfully developed and its various statistics properties have also been established. The distribution could be useful in modeling real life events with constant, decreasing, increasing and inverted bathtub failure rates. The simulation study conducted show that the estimates of the GOBXII distribution is close to the TPVs and the error generated by these parameters reduces with increased sample sizes. The GOBXII distribution would serve as a good competitor as it performed better than the GoW, GoLo and BBX distributions.

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