

On Gröbner Bases via Alternating Semigroup

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Abstract

Let $\Omega_\eta = \{1, 2, \dots, \eta\}$ be a finite set ordered in standard way. This paper presents an in-depth investigation into the application of Gröbner bases within the framework of alternating semigroups (A_η^ζ) . The study specifically focuses on the exploration of Gröbner bases in the context of A_η^ζ , which forms a subset of the symmetric inverse semigroup (C_η) . Key results derived concerning monomial orderings, including Lexicographic (LEX), Graded Lexicographic (GRLEX), and Graded Reverse Lexicographic (GRVLEX) orders. Monomials are degenerated, and computational techniques employed to achieve the outcomes. The broader significance lies in the potential applications of these results in areas such as combinatorial optimization, symbolic computation, and automated theorem proving. By demonstrating the practical utility of monomial degenerations and computational techniques, this research opens new avenues for efficient problem-solving in both theoretical and applied mathematical domains. A notable observation is that every graded lexicographic ordering of $M(A_\eta^\zeta)$ is a lexicographic order, whereas the converse does not hold. Additionally, $M(A_\eta^\zeta)$ adheres to graded reverse lexicographic order if and only if the index of the first entry of the first monomial is smaller than that of the second monomial. Some classical examples with their graphical representations included to substantiate the theoretical results.

1. Introduction

A Gröbner basis is a set of multivariate nonlinear polynomials that reduces many fundamental problems in mathematics into simple solutions.

It is a particular kind of generating set of an ideal in a polynomial ring $R = K[x_1, x_2, \dots, x_n]$ where K indicate a field and R represent the polynomial ring in the indeterminate $x_1, x_2, \dots, x_n$ over the field K .

A monomial is a single term in a polynomial that consists of a product of variables, each raised to a non-negative integer exponent, and multiplied by a constant coefficient:

$$c \cdot x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} ; \alpha \in \mathbb{Z}.$$

Monomial orderings have a significant impact on computational algebra, especially within the realm of Gröbner basis computations. They dictate how polynomials are structured and manipulated, consequently influencing the efficiency and accuracy of algorithms. In the context of Lexicographic Order (LEX), monomials are evaluated according to the organization of their respective variables specifically. It contrasts by first assessing the variable with the greatest exponent and it is widely utilized in various computational algebraic procedures due to its simplicity and effectiveness.

The Graded Lexicographic Order (GRLEX) extends the concept of lexicographic order by initially evaluating the total degree of monomials, and subsequently applying lexicographic ordering to monomials with equal degrees. Within the context of GRLEX,

monomials are predominantly arranged based on their total degrees; in cases where the total degrees coincide, lexicographic order can then be employed. It proves advantageous in situations where the total degree of polynomials carries significance, particularly in contexts like homogenization.

The Graded Reverse Lexicographic Order (GRVLEX), on the other hand, is a variation of GRLEX that utilizes lexicographic ordering in a reversed manner.

Monomial sorted initially based on their total degree, where the lexicographic order is applied from the last variable to the first in cases where total degrees are the same. It proves to be particularly beneficial for eliminating higher-degree terms during reduction procedures in Gröbner basis computations, thereby leading to the generation of more succinct and effective bases.

Many authors in various directions have expanded the theory of Gröbner base and it has many applications in branches of mathematics such as commutative algebra, homological algebra, differential algebra, algebraic geometry and graph theory. It provides powerful tools for tackling problems related to polynomial equations and algebraic structure. Recently [20] gave a counter example to the claim that; final polynomials always appear in a lexicographic Gröbner basis of a certain ideal.

More so, [14] have worked on commutative polynomials rings over fields, residue class rings modulo polynomial ideal, congruence and algebraic equations. In [15], a Gröbner basis theory for divide-and-conquer

recurrences was investigated and [13] generalized the work of [18] together with the work of [8] titled Grobner-Shirshon basis for monomials semiring over D-A rings.

For standard concept on Grobner bases theory see [[1], [7], [15], [11] and [9]].

A semigroup is an algebraic structure consisting of a non-empty set Γ together with an associative binary operation. A transformation of T is a function from T to itself. Transformation semigroups are one of the most fundamental mathematical objects in theoretical computer science, where properties of language depend on the algebraic properties of various transformation semigroups related to them.

Let $\Omega_\eta = \{1, 2, \dots, \eta\}$. The map $\psi: \text{Dom}\psi \subseteq \Omega_\eta \rightarrow \text{Im}\psi \subseteq \Omega_\eta$ is said to be full or total, if $\text{Dom}\psi = \Omega_\eta$ partial if $\text{Dom}\psi \subseteq \Omega_\eta$ or else it is called strictly partial denoted by T_η , P_η , and C_η respectively. These are the three essential part of transformation semigroups which were introduced by [17].

The set of all partial transformation on n -object form a semigroup under the usual composition of transformation.

The idea of an even permutation generalized via path notation, to one-to-one partial transformation and [19] called the semigroups C_η charts

while [10], [21] named it subpermutations.. A transformation ψ in C_η said to be even if it can be expressed as a product of an even number of transpositional or a product of any number of circuits/paths of odd length. The

set of all even charts denoted by A_η^ζ on Ω_η defined as Alternating Semigroups. Some researchers have done several works on this class of transformation semigroup see [[2], [3], [4], [5] and [6]].

2. Research Methodology

In the domain of algebra and algebraic geometry, Grobner bases occupy a pivotal position in the resolution of polynomial equation systems and the comprehension of the algebraic varieties' geometry. In alternating semigroups, the specialized examination of the theoretical structure of Grobner bases is feasible and the computational techniques employed in the calculation of it over (A_η^ζ) firstly explored. Then,

The study utilizes computational tools and techniques such as Buchberger's Algorithm for constructing Gröbner bases, alongside methods for implementing and analysing the monomial orderings such as Lexicographic Order (LEX),

Graded Lexicographic Order (GRLEX), and Graded Reverse Lexicographic Order (GRVLEX), together with their significance in the computations of Grobner bases examined. Additionally, symbolic computation software MATLAB, is employed to perform complex algebraic manipulations, execute algorithmic computations, and visualize graphical representations of classical examples. These tools ensure the reproducibility of results and facilitate a systematic exploration of algebraic structures within alternating semigroups.

The following Definitions and Theorem used to obtain the results.

Theorem 2.1: A monomial ordering on A_η^ζ

is a total ordering on the variables $x_1, x_2, \dots, x_\eta$, (i.e there are no elements which are not comparable) with a relation $>$ on the set of monomials of $R = K[x_1, x_2, \dots, x_\eta]$ such that (i) $x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma \quad \forall x^\gamma$

(ii) $x^\alpha > 1 \quad \forall x^\alpha \neq 1$.

Definition 2.2: A Lexicographic order (LEX) is an ordering $A >_{lex} B$ if and only if the index of the first left-most nonzero entry of the monomial

$A - B$ is positive.

Definition 2.3: A Graded Lexicographic order (GRLEX) is an ordering $A >_{grlex} B$ if and only if $A >_{lex} B$ and $|A| > |B|$ or $(|A| = |B|)$ Where $|A|$ and $|B|$ denote the summation of the index of variables in the monomial A and B

Definition 2.4: A Graded Reverse Lexicographic order (GREVLEX) is an ordering $A >_{grelex} B$ if and only if the index of the $|A| > |B|$ or $(|A| = |B|)$ and the index of the right-most nonzero entry in $A - B$ is negative.

3. Results

Theorem 3.1: Let $S = A_\eta^\zeta$ and $M(A_\eta^\zeta)$ represent the set of monomials in A_η^ζ , then the monomials in $M(A_\eta^\zeta)$ follow the Lexicographic (LEX) order if the index of the

first non-zero leftmost of entry of any monomial in $M(A_\eta^\zeta)$ equals η .

Proof: Consider any elements A_η^ζ

for $n \geq 2$ having at least two monomials

$A = x^\alpha y^\beta$ and $B = x^\gamma y^\delta$ for $\alpha \geq \beta, \gamma \geq \delta$ holds.

Now, by Definition 2.2 $\alpha > \gamma$

implies $\alpha - \gamma > 0$. Then, it follows that for each value of n in $M(A_\eta^\zeta)$, $\alpha > \gamma$

and $\alpha = \eta$. It is clearly shown that

$A, B \in M(A_\eta^\zeta)$ and $A >_{lex} B$

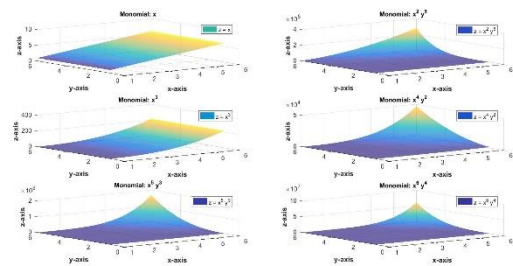
Example 3.2: Consider $\emptyset \in (A_6^\zeta)$ such that $\emptyset = (1)[64253]$ with monomial functions;

$$M(\emptyset) = x, x^2y^5, x^3, x^4y^2, x^5y^3, x^6y^4$$

Then by virtue of Definition (2.2)

$$x^6y^4 >_{lex} x^5y^3 >_{lex} x^4y^2 >_{lex} x^3 >_{lex} x^2y^5 >_{lex} x$$

Graphical illustration,



The output containing six subplots, each representing one of the monomials:

- $x, x^2y^5, x^3, x^4y^2, x^5y^3$, and x^6y^4 .

- Legends are displayed in the top-right corner of each plot.
- Labels and titles enhance clarity.

In the above graphs, the lexicographic ordering (LEX) of the monomials is related to how they increase in terms of their exponents, as illustrated by these surfaces. Monomials with higher powers of x and y dominate the graph in terms of their magnitude.

Example 3.3: Consider $\lambda \in (A_7^\zeta)$

such that $\lambda = (1234567)$ with monomial functions;

$$M(\lambda) = xy^2, x^2y^3, x^3y^4, x^4y^5, x^5y^6, x^6y^7, x^7y$$

Implies

$$x^7y >_{lex} x^6y^7 >_{lex} x^5y^6 >_{lex} x^4y^5, \\ >_{lex} x^3y^4 >_{lex} x^2y^3 >_{lex} xy^2$$

Theorem 3.4: Let Y, Ω be any two monomials in $M(A_\eta^\zeta)$. Then, every

$$M(A_\eta^\zeta) \text{ are GRLEX if } Y >_{lex} \Omega$$

$$\text{and } \sum_{i=1}^2 \alpha_i = \sum_{j=1}^2 \beta_j$$

Proof:. Consider $Y, \Omega \in M(A_\eta^\zeta)$ defined as $Y = x_1^{\alpha_1} y_1^{\alpha_2}$ and

$$\Omega = x_2^{\beta_1} y_2^{\beta_2}$$

Now, suppose $\alpha_1 < \alpha_2$ implies $\alpha_1 - \alpha_2 < 0$ thus, $Y \not>_{lex} \Omega$ which is a contradiction by Definition (2.2).. Thus, there exists $\alpha_1 > \alpha_2$ implies that $Y >_{lex} \Omega$. More so, since

$\alpha_i \beta_j \in \aleph$ and $Y \not>_{lex} \Omega$ then, by Definition (2.3) $Y \not>_{Grlex} \Omega$

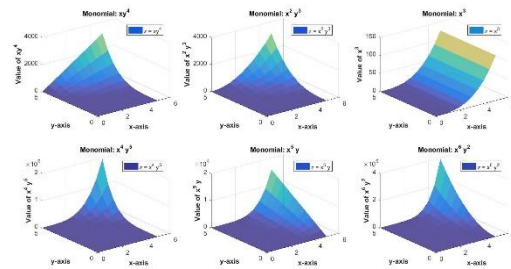
Example 3.5: Suppose $\mu \in (A_6^\zeta)$ such that $\mu = (145)(623)$ with monomial functions;

$$M(\mu) = xy^4, x^2y^3, x^3, x^4y^5, x^5y, x^6y^2$$

$$\text{Let } A = x^2y^3, B = xy^4 \quad \forall AB \in M(\mu)$$

Since $\alpha_1 > \beta_1$ $\alpha_1 > \alpha_2$ then, $A >_{lex} B$
Thus, $|A| = |B|$ implies $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, hence $A >_{Grlex} B$

Graphically,



Each graph illustrates a specific monomial's behaviour in the 3D space of variables x and y .

The figure contains six subplots, each subplot demonstrates the contribution of x and y to the respective monomial, providing a visual understanding of their effects. The graphical representation provides a clear visualization of how different monomials behave based on their structure and how they relate to one another under the GRLEX.

The steepness and shape of each surface illustrate the impact of the variables' exponents on the monomial's value, emphasizing the differences in growth rates.

Example 3.6: Consider $\omega \in (A_9^\zeta)$ such that

$$\omega = (1\ 2\ 8\ 3)(5\ 7\ 4\ 6)(9)$$

with monomial functions;

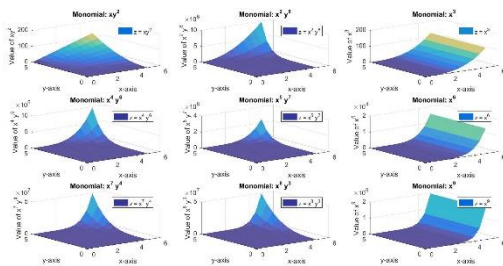
$$M(\omega) = xy^2, x^2y^8, x^3, x^4y^6, x^5y^7, x^6, x^7y^4, x^8y^3, x^9.$$

Now let $A = x^4y^6, B = x^2y^8, = x^8y^3$ and $D = x^7y^4 \forall A, B, C, D \in M(A_9^S)$

Then by virtue of Definition 2.3 and Theorem 3.4

$$A >_{Grlex} B \text{ and } C >_{Grlex} D$$

Graphically,



As the exponents of x and y increase, the rate of growth of the surface increases, making the plot steeper. Monomials with higher powers of y , such as x^2y^8 or x^4y^6 , show rapid increases in height with respect to y leading to steeper surfaces along the y -axis. The function with the highest exponent of x , such as x^9 , shows the steepest growth with respect to x , but the monomials with high exponents in both x and y (like x^5y^7) produce surfaces that grow rapidly in both directions.

Example 3.7: Consider $\tau \in (A_{18}^S)$ such that $\tau = (18\ 12\ 16\ 9\ 14\ 5\ 1)(13\ 4\ 17\ 11\ 2\ 10\ 6\ 7\ 3)$ with monomial functions;

$$M(\tau) = xy^2, x^2y^{10}, x^3y^{13}, x^4y^{17}, x^5y, x^6y^7, x^7y, x^7y^3, x^8, x^9y^{14}, x^{10}y^6, x^{11}y^2, x^{12}y^{16}, x^{13}y^4, x^{14}y^5, x^{15}, x^{16}y^9, x^{17}y^{11}, x^{18}y^{12}$$

Then by virtue of Theorem (3.4) the

$$\begin{aligned} \text{monomial } x^{10}y^6 &>_{Grlex} x^3y^{13}, \\ x^{11}y^2 &>_{Grlex} x^6y^7 \quad \text{and} \\ x^{17}y^{11} &>_{Grlex} x^{12}y^{16} \end{aligned}$$

Corollary 3.8: Every Graded lexicographic order of monomial in alternating semigroup are Lexicographic

order but the converse is not true.

Theorem 3.9: Let $\Phi \Psi \in M(A_\eta^S)$ such that $\Phi = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots x_n^{\alpha_i}$ and $\Psi = y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3} \dots y_m^{\beta_j}$.

Then, $\Phi >_{GREVLEX} \Psi$ if and only if

$$\alpha_i < \beta_j \text{ for } i, j \in \mathbb{Z}$$

Proof: Assuming $\alpha_i > \beta_j$ then

$$\alpha_i - \beta_j = 0 \text{ implies } A \not>_{GREVLEX} B$$

i.e $\alpha_i > \beta_j$ or $\alpha_i = \beta_j$. Conversely, if

$$\alpha_1 \geq \beta_1, \alpha_2 \geq \beta_2, \dots, \alpha_{i-1} \geq \beta_{j-1}$$

and $\alpha_i < \beta_j$. Then $\alpha_i - \beta_j < 0$ follow by Definition (2.4), hence, the result shows that $\Phi >_{GREVLEX} \Psi$ for any $\Phi, \Psi \in M(A_\eta^S)$

Example 3.10: Consider $\mu \in (A_6^S)$ such that $\mu = (145)(623)$ with monomial functions; $M(\mu) = xy^4, x^2y^3, x^3, x^4y^5, x^5y, x^6y^2$ Let $A = x^2y^3, B = xy^4 \forall A, B \in M(\mu)$ Then

by virtue of Definition (2.3) and Theorem (3.9) implies

$$|A| = |B| \text{ hence, } A >_{\text{GREVLEX}} B$$

Example 3.11: Consider $\omega \in (A_9^\zeta)$ such that

$$\omega = (1\ 2\ 8\ 3)(5\ 7\ 4\ 6)(9)$$

with monomial functions;

$$M(\omega) = xy^2, x^2y^8, x^3, x^4y^6, x^5y^7, x^6, x^7y^4, x^8y^3, x^9$$

Then by virtue of Definition 2.3 and Theorem 3.9

The monomial $x^2y^8 >_{\text{Grvlex}} x^4y^6$ and $x^7y^4 >_{\text{Grvlex}} x^8y^3$

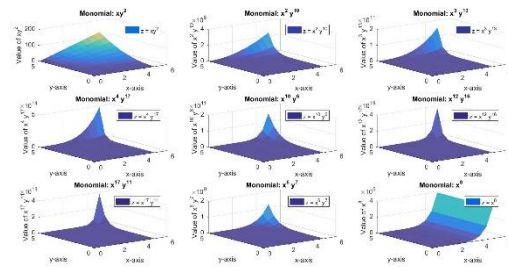
Example 3.12: Consider $\tau \in (A_{18}^\zeta)$ such that $\tau = (18\ 12\ 16\ 9\ 14\ 5\ 1)(13\ 4\ 17\ 11\ 2\ 10\ 6\ 7\ 3)$

with monomial functions;

$$M(\tau) = xy^2, x^2y^{10}, x^3y^{13}, x^4y^{17}, x^5y, x^6y^7, x^7y, x^7y^3, x^8, x^9y^{14}, x^{10}y^6, x^{11}y^2, x^{12}y^{16}, x^{13}y^4, x^{14}y^5, x^{15}, x^{16}y^9, x^{17}y^{11}, x^{18}y^{12}$$

Then by virtue of Definition (2.3), the monomial $x^{10}y^6 >_{\text{Grvlex}} x^3y^{13}$, $x^{11}y^2 >_{\text{Grvlex}} x^6y^7$ and $x^{17}y^{11} >_{\text{Grvlex}} x^{12}y^{16}$

Graphical illustration,



The figure illustrates various monomials, with specific attention to their growth rates and contributions to lexicographic comparisons under the Grvlex order. Key plots highlight the relationships like $x^{10}y^6 >_{\text{Grvlex}} x^3y^{13}$ showcasing the dominance of higher powers of x and y in the specified order.

4. Discussion

In this paper, we have shown that any monomial in alternating semigroup $M(A_\eta^\zeta)$ are lexicographic order, if the index of the first non-zero left most entry equal to η as shown in the Theorem 3.1 and illustrated in the Example 3.2 and 3.3. More so, any two monomials in $M(A_\eta^\zeta)$ are graded lexicographic order, if it is lexicographic order and the sum of the index of the first monomial equal to the sum of the index of the second monomial as shown in Theorem 3.4 and as illustrated in Examples 3.5, 3.6 and 3.7 respectively. Finally, we have also observed that, every graded lexicographic order of $M(A_\eta^\zeta)$ are lexicographic order but the converse is not true. Furthermore, $M(A_\eta^\zeta)$ are graded reverse lexicographic order if and only if the index of the first most entry of first monomial less than the index of the first most entry of the second monomial as shown in Theorem 3.9 and illustrated in Examples 3.10, 3.11 and

3.12 respectively. Based on this finding, we strongly believe that many other research can still be carried out on Grobner bases through other transformation semigroups such as full, partial and other subsemigroups of transformations.

5. Conclusion

This study examines the concept of Grobner bases through the lens of alternating semigroups, shedding new light on their computational aspects and theoretical underpinnings. It was uncovered the new insights into the algebraic structure and the findings not only contribute to advancing the theoretical understanding of Grobner bases but also offer practical implications for various applications in computational mathematics, cryptography, and computer algebra systems. Moreover, this study also opens up avenues for further investigation into the interplay between semigroup theory and algebraic geometry, potentially leading to the development of algorithms that are more efficient and computational techniques.

Declaration

Competing Interests. The authors declare that they have no competing interests.

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Author's Contributions: IF originated the foundational concept, engaged in the theoretical examination of Gröbner bases within alternating semigroups, and aided in the composition of the introduction and methodology sections. OK emphasized computational dimensions, encompassing derivations of monomial ordering,

corroborations, and illustrative examples, and was responsible for drafting the discussion and results segments. GN oversaw the research effort, contributed to the overarching study design and theoretical framework, and offered substantial revisions, particularly concerning the abstract and conclusion. GR executed a comprehensive literature review, imparted valuable insights regarding the applications of Gröbner bases, and enhanced the methodological approach. AD participated in the theoretical exploration, provided examples, and undertook the final manuscript revisions, thereby ensuring the coherence and synthesis of both theoretical and computational methodologies.

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