EFFECTS OF PERTURBATIONS IN THE CORIOLIS AND CENTRIFUGAL FORCES ON THE STABILITY OF GENERALIZED PHOTO-GRAVITATIONAL RESTRICTED THREE-BODY PROBLEM

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Certification

This is to certify that the research reported in this thesis was carried out by JAIYEOLA, Sefinat Bola with matriculation number 13/68DR025 in the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria, for the award of Doctor of Philosophy Degree in Mathematics.

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Dedication

This thesis is dedicated to Almighty Allah (the most gracious, the beneficent and the most merciful) and to the memory of my beloved mother, Late Alhaja F. M. Khadija Akere.

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Abstract

The study of classical Restricted Three-body Problem (RTBP) and its generalizations have been of major interest to researchers over the years. This is due to the rise in the need for accuracy in determining astrometric positions which would help to reveal some peculiarities of components of motion and draw conclusions on the stability of space vehicles to be launched. This has led to the necessity of considering all possible physical properties (oblateness/triaxiality, radiation pressure, Poynting-Robertson (PR) drag, perturbing forces etc.) that affect the motion of particles in space. The effect of perturbations in the coriolis and centrifugal forces on the stability of the generalized photo-gravitational RTBP has been a major focus of investigations. However, the effect under the influence of the PR-drag from both oblate bodies has received little or no attention. Therefore, the aim of this research work was to investigate how perturbations in the coriolis (ε) and centrifugal forces (ε') affect the stability of the triangular libration points of the RTBP when the primaries were considered to be oblate, radiating with PR-drag effects. The objectives of this study were to: (i) determine the effect of PR-drag on the stability of the libration of the generalized RTBP; (ii) investigate the effects of ε and ε' on the stability of the generalized RTBP in the linear sense; (iii) establish the periodic orbit: period of oscillation, orientation and semi-axes of the proposed system; and (iv) verify the results obtained using astrophysical data for the Kruger 60 and RXJ0450, 1-5658 binary systems.

The Hamiltonian and Lagrangian methods were employed to establish the relevant equations of motion, obtain the triangular libration points and investigate their stability using Murray's and Routh $\&$ Hurwitz's criteria and verifying the results for the two binary systems using, MATLAB and Microsoft Excel Mathematical softwares.

The findings from this study showed that the:

- generalized system was unstable around the triangular libration points due to the presence of the PR-drag effect from both bodies;
- presence of the parameter of the stabilizing factor, (ε), in the roots of the characteristics equation does not change the instability of the system around the libration points;
- period for the growth of the particle oscillation is dependent on the PR-drag parameter only, in the linear sense;
- orientation and length of semi- axes are dependent on all the perturbing parameters; and
- change in the values of ε and ε' affects the values of the libration points and roots of the characteristics equations computed for the two binary but does not satisfy the criteria for stability.

The study concluded that the system remained unstable even with the significant influence of perturbations due to the strong destabilizing effect of the PR-drag force. This work as a generalization of the classical case and the work of others, is therefore recommended to

serve as a form of reference to achieving more interesting and vital results in Space Dynamics and also an added value to designers of space crafts and aerospace agencies.

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1 General Introduction

1 Background to the Study

Space dynamics is a branch of astronomy which considers the study of celestial mechanics and control applied to spacecraft and natural objects. The initial goal of celestial mechanics was to explain the motion of the sun, the moon, and the planet. However, mathematical methods of celestial mechanics find several applications such as the determination of the dynamics of the planet, asteroids, comets, artificial satellites and the design of orbit for interplanetary travels. celestial mechanics plays a vital role in all areas of life. Space research has brought about lots of progress in, satellite telecommunications, weather forecasting, targeting missiles, tourism, exploration of mineral resources, defense and security and much more.

Nigeria is amongst the nations that have shown interest in space science. The Nigerian Government have launched five satellites since her intention to venture into space research was first made known at an inter-governmental meeting in Addis Ababa in 1976. The first was the world-wide Disaster Monitoring Constellation (DMC) System, the Nigeria Sat-1 built by a United Kingdom-based satellite technology company, Surrey Space Technology Limited (SSTL Ltd). It has a mass of 100 kg, carries an optical imaging payload which uses green, red and near-infrared bands equivalent to Landsat TM+ bands 2, 3 and 4 and 32 m ground resolution with an exceptionally wide swath width of over 640 km. NigeriaSat-1 was launched by Kosmos-$3M$ rocket from Russian Plesetsk spaceport on 27 September 2003.

The second Nigerian satellite to be placed into orbit was the NigComSat-1, a communications satellite owned and operated by Nigerian Communication Satellite limited. This was launched on 13th May, 2007 from the Xichang satellite launch centre in China with the aim of providing rural internet access. This satellite was a total loss because it lost both of its solar arrays and was switched off.

The NigeriaSat-2 and NigeriaSat-X of mass, 300 kg each, built to replace the NigeriaSat-1, were launched on the $17th$ of August, 2011. The NigComSat-1R was built to replace the lost NigComSat-1 and launched by China on 19th December, 2011 with no cost to Nigeria.

More recently are the NigComSat-2 and NigComSat-3 launched in 2012 and 2013 respectively. The NigeriaSAT-1-dual-aimed military/civil Earth monitoring satellite with synthetic aperture radar was launched in 2015. The main reason for these activities is to use space acquired information to understand and manage our environment and natural resources.

Isaac Newton was the first to study the motion of particles moving under the influence of a mutual gravitational force of attraction. His effort resulted in oral descriptions and geometrical sketches. He unified the three laws of motion (the law of inertia, the law of conservation of momentum and the law of action and reaction) and proved that these laws govern both earthly and celestial mechanics. Using Huygens results on centripetal acceleration, Hooke and Wren concluded that this diminishes as the inverse square of the distance. They identified this force as the same force that makes objects fall near the surface of the earth and hence succeeded in computing the orbits of celestial bodies using the inverse square law.

Based on Euler extension of Newton's laws of motion from particles to rigid bodies and the reformulation of these laws by Lagrange and Hamilton, it was possible to solve problems that seem complicated in space dynamics. Like the two-body, three-body, four-body and N-body problems.

The two-body problem describes the motion of two bodies of finite masses moving under the influence of a mutual gravitational force of attraction. For example, the solar system, which consists of the sun and its nine planets. This problem has been solved completely. The three-body problem studies the motion of three bodies of finite masses attracting it each other in pairs under a mutual gravitational force of attraction. An example of this is the sun, earth and moon relation.

Due to the complexity of solving the three-body problem, Mathematicians studied a special case, the Restricted Three-body problem (RTBP). This is one of the most important components of Space Dynamics which has captured the attention of many researchers over the years. The problem is restricted in the sense that it describes the motion of a third body with negligible mass moving in the plane of two massive bodies, called the primaries, such that its motion does not influence their motion. A typical example is a spacecraft moving between planets or the satellites orbiting the planets.

The history of RTBP began with Euler and Lagrange through their lunar theories. Euler's introduction of the rotational (synodic) coordinate system brought about his major accomplishment which led to the discovery of the Jacobian integral by Jacobi. These integrals connect the magnitude of the velocity vector of the body to its location. Hill described the motion of the moon using these integrals. Poincare initiated the analytical methods which are his highest theoretical accomplishment. He considered the study of the Periodic Orbit as the only means by which the unsolvable problems of three-body system can be approached. He also emphasized the importance of the periodic orbit and suggested it be used as a reference orbit. This was adopted by many prominent researchers using ellipse and variational orbit.

The classical problem assumes that the primaries are spherical. Due to the advancement in astrophysical studies, the true nature (oblateness, triaxiality, surface area light, force order than the gravitational force, coriolis and centrifugal forces, atmospheric drag, solar wind e.t.c.) of the planet and extrasolar bodies became clear.

In recent times all these properties are taken into consideration in describing the motion and stability of satellites (both natural and artificial) and other planetary bodies. The Poynting-Robertson drag effect which is the effect of electromagnetic radiation on the moving spherical body was first discovered and studied by Poynting (1903) and Robertson (1937). The effect of this drag force can not be over emphasized.

In order to solve problems in celestial mechanics exactly in an Earth-bound reference frame, the Coriolis and the centrifugal forces must be introduced. Specifically, when objects in the inertial frame are transformed to a rotating frame of reference the coriolis and centrifugal forces appear. These forces are weak compared to most typical forces in everyday life.

The coriolis force acts to the left of any object moving in a circular motion in the clockwise direction relative to a rotating reference frame. It causes a deflection known as the 'coriolis effect' while the centrifugal force acts outward in the radial direction and it is proportional to the distance from the axis of the rotating frame. The effect is quite small but generally more noticeable only for motions occurring over large distances and long period of time.

In the case of a distant star observed from a rotating spacecraft in the reference frame co-rotating with the spacecraft, the star appears to move along a circular trajectory around the spacecraft, hence the resultant force of centrifugal and coriolis force must be taken into account. Here the magnitude of the coriolis force is twice that of the centrifugal force.

The periodic orbits is an important topic in celestial mechanics that can not be left untouched because it provides vital information on the orbits or spin of the particles. The study of of the periodic orbit in the framework of the generalized RTBP putting all the perturbing forces (oblateness, triaxality, radiation due to pressure, PR-drag, perturbations in the coriolis and centrifugal forces) into account All these properties exhibited by planetary bodies brought about many modifications in the formulation and study of the stability of the RTBP. Also, the recent increase of the accuracy of ground-based astronautic observation of asteroids makes it very essential to consider these properties.

2 Statement of the Problem

In the classical case, the RTBP were found to have five equilibrium or libration points, three of which are collinear points (L_1, L_2, L_2) located along the axis connecting the primaries and the other two (L_4, L_5) forms triangular points which are symmetrical with respect to this axis. The collinear points were found to be unstable while the triangular points are stable for the mass values $0 < \mu \leq \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$ where μ_c is the critical mass value.

With the presence of Poynting-Robertson drag force, it was found that six unstable libration points existed at most in which the sixth point is located out of the plane of motion.

In this research work, the effects of small perturbations in the Coriolis and Centrifugal forces on the stability of the libration points in the generalized RTBP when the primaries are radiating with PR-drag effect is investigated. The problem is generalized in the sense that the effects of the perturbing forces are studied when the primaries are considered to be oblate spheroid. The results obtained have been further verified using the Kruger- 60 and $RXJ\,0450,1-5658$ binary system as a model.

3 Aim and Objectives of the Research

The main aim of this research work is to investigate the effects of small perturbations in the Coriolis (ε) and centrifugal (ε') forces on the stability of the triangular libration points of the RTBP when both primaries are considered to be oblate spheroid as well as sources of radiation with PR-drag from the primaries. The objectives of the study are to:

- determine the effect of PR-drag on the stability of the libration of the generalized RTBP;
- investigate the effects of small perturbations in the Coriolis and centrifugal forces on the stability of the generalised photo-gravitational RTBP in the linear sense;
- establish the Periodic orbit: period of oscillation, orientation and semi-axes of the proposed system; and
- verify the results obtained using astrophysical data for the Kruger 60 and $RXJ\,0450,1-5658$ binary systems.

4 Significance of the Study

Space dynamics is an important component of Space Science and Technology program. It is one of the central problems in space Science.

The rise in the need for accuracy in determining astrometric positions and radiation influence on celestial bodies led to the necessity to take into account the non-sphericity of the bodies, phase angle, surface area light, perturbing and drag forces. This would help to plan the launching and control of space vehicle would reveal some peculiarities of components of motion and to draw the conclusion on their stability. The verification of this result on the Kruger 60 and $RXJ\,0450{,}1-5658$ binary systems shows its significance when launching a space vehicle in their vicinity. This work, therefore, would be of great importance to the Space And Research Agencies.

5 Scope of the Study

This research work has only considered how small perturbations in the Coriolis and Centrifugal forces affects the linear stability of the triangular libration of the circular RTBP under the combined influence of the oblateness, radiation pressure force and PR-drag force from both primaries. Other important and interesting aspects of RTBP such as other shapes (eg.triaxiality), orbit (elliptic), non-linear form, the collinear points etc have not been considered.

6 Research Methodology

In order to achieve the objectives mentioned above, the

- Hamiltonian and Lagrangian method was employed to establish the relevant equations of motion;
- triangular libration points were obtained and their stability investigated using Murray's and Routh $\&$ Hurwitz's criteria; and
- results were verified for the Kruger-60 and $RXJ\,0450{,}1-5658$ binary systems using MATLAB and Microsoft Excel Mathematical software.

7 Definition of Terms

Here are some basic definitions and concepts used in this work.

Definition 1.7.1

The velocity of a particle of mass m moving at a distance $\vec{r}(x, y, z)$ from the origin at time t is given in vector form as

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = lim \frac{\Delta \vec{r}}{\Delta t} \tag{1}$$

Definition 1.7.2

The angular velocity of a body rotating about it axes with an angle say θ is

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Where $\vec{\omega} = \dfrac{d\theta}{dt}$ is the angular velocity vector. For a rigid body (that is, has invariable shape and size) the angular velocity is

$$\vec{v}' = \vec{v} + \vec{\omega} \times \vec{r}' \tag{2}$$

Where, $\vec{v}$ is the velocity due to a fixed axes

$\vec{\omega} \times \vec{r}'$ is the velocity due to rotating axes.

The component of the velocity $\dot{r}$ in the direction of the moving axes OX' and OY' are ($\dot{x} - \omega y$, $\dot{y} + \omega x$)

Definition 1.7.3

The acceleration for the particle described in definition $(1.7.1)$ is

$$\frac{d\vec{v}}{dt} = \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} \tag{3}$$

Definition 1.7.4

The momentum L is the product of the mass of the body and it velocity which is represented as

$$L = m\dot{\vec{r}} = m\frac{d\vec{r}}{dt} = (m\frac{dx}{dt}, m\frac{dy}{dt}, m\frac{dz}{dt}) \tag{4}$$

Definition 1.7.5

The force acting on a particle is the product of the mass, m of the body and its acceleration which is

$$m\frac{d\vec{v}}{dt} = m\ddot{\vec{r}} = m\frac{d^2\vec{r}}{dt^2} = (m\frac{d^2x}{dt^2}, m\frac{d^2y}{dt^2}, m\frac{d^2z}{dt^2})$$

This is according to Newton's law of motion.

For two masses m_1 and m_2 separated by a distance, r , by Newton's law of gravitation,

$$F = \frac{Gm_1m_2}{r^2} \tag{5}$$

where G is the mutual gravitational constant.

Definition 1.7.6

The Energy E is given by

$$E = T + V = constant$$

where

$T = \dfrac{1}{2}m\dot{r}^2$ is the Kinetic Energy (energy due to motion)

and

$$V = \int F.dr = -\frac{Gm_1m_2}{r} \tag{6}$$

In the case of motion of a close satellite about a non-spherical planet, the potential is formed such that

$V = V_0 + R$

Where V_0 is the potential function due to the point mass of the two-body problem and R is the potential due to any other attracting masses in the system or to the arbitrary shape of the planet about which the body revolves.

Definition 1.7.7

The circular restricted three-body problem is said to describe the motion of a third body of infinitesimal mass, m attracted by two bodies of finite masses m_1 and m_2, known as the primaries moving around their center of mass in a circular orbit under the influence of their mutual gravitational attraction. Its motion does not influence their motion but it is affected by theirs. Aside the sun, the heaviest of all planets, Jupiter, moves around the Sun in a circle. There is a group of tiny planets, the Trojan asteroid whose motion is controlled principally by the sun and Jupiter. The motion of the Trojan asteroid is described by the restricted three body problems with the sun and Jupiter as primaries.

Definition 1.7.8

The radiation force F_p changes with distance by the same law as the gravitational force of attraction F_g but acts in opposite direction. This result in a reduction in the effective mass of a particle. This resulting force on the particle is given by

$$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g \tag{7}$$

where $q = 1 - \dfrac{F_p}{F_g}$ is the mass reduction factor such that $0 < (1-q) << 1$, for a particle expressed in terms of particle radius (a), density (δ) and solar radiation pressure efficiency factor (κ) (in CGS units)

$$q = 1 - \frac{5.6\times 10^{-}5}{a\delta}\kappa(Radzievsky, 1950)$$

Since q is assumed to be a constant, it is adequate to neglect the solar radiation flood fluctuations and shadow effect of a planet.

For the primaries of masses m_1 and m_2, the mass reduction factors are denoted by q_1 and q_2 respectively.

Definition 1.7.9

Oblateness is the measure of non-sphericity or the degree of flattening of the primaries. The coefficient for this can be measured with the expression given below as,

$$A_i = \frac{ae_i^2 - ap_i^2}{5R_i^2} \qquad \text{(McCuskey, 1963)} \tag{8}$$

where ae_i, ap_i $(i = 1,2)$ are the equatorial and polar radii for primaries respectively and R_i is the distance between the primaries

Definition 1.7.10

The Poynting-Robertson effect, also known as, Poynting-Robertson Drag named after John Henry Poynting and Howard Robertson is a process by which solar radiation causes meteors and dust grain orbiting a star to lose angular momentum relative to their orbit. This causes dust that is small enough to be affected by this drag. Robertson used a precise relativistic treatment of the first order in the ratio of the velocity of the particle to the speed of light $\frac{\bar{v}}{c}$ and the expression for the net drag force which opposes the direction of motion is

$$\vec{F} = F_P\left[\frac{\vec{r}}{r} - \frac{\vec{v}}{c}\frac{\vec{r}}{r}\frac{\vec{r}}{r} - \frac{\vec{v}}{c}\right] \tag{9}$$

where,

$$F_P = \frac{3Lm}{16\pi r^2 \rho s c}$$

denotes the measure of radiation pressure, $\bar{r}$ the position vector of a particle with respect to the radiation source, $\bar{v}$ is the corresponding velocity, c is the speed of light, L is the luminosity of the radiating body, m is the mass of the particle, ρ is the density of the particle, s is the cross section of the particle.

The first term expresses the radiation pressure effect, the second represents the Doppler shifts owing to the motion of the particle and the third is due to the absorption and subsequent re-emission of part of the radiation. The last two terms of equation (1.9) constitute the PR-drag effect.

Definition 1.7.11

Figure 1: The synodic coordinate relative to the sidereal coordinate system,**Source: Szebehely (1967a)**

The sidereal (fixed) or inertial coordinate system has zero acceleration since there is no identifiable force produced in this state. It is associated with the inertial frame of reference. The synodic or rotating coordinate system is accelerated since it is not fixed. There are forces associated with the rotating reference frame, which are called the artificial forces.

Definition 1.7.12

Assuming a body is moving at a constant velocity with respect to an inertial frame, here no net force acts on it. If the body is viewed from a frame of reference which is accelerating (in this case rotating). In general, the body is no longer observed to move with constant velocity and it appears as though a force is acting on it. This force is called an "effective" or "fictitious" force. The acceleration due to such a force is caused solely by the motion of the observer. To describe the motion of a particle relative to a body that is rotating with respect to an inertial frame is complicated but can be made relatively easy by the non-inertial, artificial forces; the Coriolis (fictitious correction force) and Centrifugal (outward force) forces. These forces have been introduced in an artificial manner as a result of an arbitrary requirement to write an equation which resembles Newton's equation since the Newton's equation is only valid in an inertia frame of reference. The vector sum of the centrifugal and the Coriolis force is the total fictitious force given by

$$\vec{F}_{eff} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{w}\times\vec{v} \tag{10}$$

Centrifugal

Coriolis

where m is the mass of the object, $\vec{\omega}$ is its angular velocity of the rotating frame, $\vec{r}$ the position vector and $\vec{v}$ the corresponding velocity as seen in the rotating frame.

Definition 1.7.13

A body is said to be at libration or Equilibrium point if it is stationary at that point or it is in a steady state. From mathematical point of view, if a dynamical system is in a state of equilibrium it remain in that state as $t \to \infty$. Precisely, considering a system of ordinary differential equation

$$\dot{x} = X(x) \tag{11}$$

with a point $x = a$

Where,

$x = (x_1, x_2, \ldots, x_n)$ and $X = (X_1, X_2 \ldots, X_n)$

A solution $x(t)$ is called the equilibrium or libration point when $x(t_o) = a$ is a solution of the equation $X(x) = 0$, that is $x = 0$ is the libration point

From the physical point of view, the equilibrium solutions represent points where the force acting on the third body in the rotating system is balanced. That is, if the body is given a little displacement, it oscillates and returns to the same point when time elapses the it is stable, otherwise unstable.

Definition 1.7.14

The third body is also said to be stable near one of the equilibrium or libration points if given a small displacement with small velocity, it oscillates and returns to the same point when time elapses, otherwise unstable. The solution $x = a$ or the point a is said to be stable if for any given $\varepsilon < 0$ there exit a $\delta(\varepsilon) > 0$ such that when the disturbances satisfy

$$|x(t_0) - a| \leq \delta$$

then for all $t > t_0$,

$$|x(t) - a| < \varepsilon$$

Otherwise the equilibrium point or libration point $x = a$ is unstable.

For a Linear System

If the system of differential equation given in equation (1.11) is re-written in the form

$$\dot{x} = Ax + f(x)$$

where A is a constant matrix and $f(x)$ is a vector function such that

$$\frac{|f(x)|}{|x|} \to \infty$$

as $|x| \to 0$ for all $f > 0$

then the linearized system of equation is

$$\dot{x} = Ax \qquad (12)$$

The stability condition for $t \geq t_0$ of the linearized system stated as:

- if the roots of the characteristic equation of A are complex which
  - have all negative real part then the libration points are stable similarly for multiple roots.
  - have any positive real parts then the libration points are unstable. This is also valid for multiple roots.

- for pure imaginary roots, the motion is oscillatory and the solution is stable though not asymptotically stable. If these are multiple roots the solution contains mixed (period and secular) terms and the libration point is unstable.
- if the roots are real and all negative, the solution is stable. If any of the roots are positive, then the point is unstable. This is also true for multiple roots.

Routh and Hurwitz Criteria for Stability

The Routh and Hurwtz criteria is a mathematical test used to determine the nature of the roots of a characteristic polynomial of a linear system and to make conclusions on the stability of the system without solving directly. It is given as

Let $P(x)$ be a linear homogeneous characteristics equation of order n, given by

$$P(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \tag{13}$$

where $a_1, \ldots, a_n$ are real constant coefficients of the polynomial. Using the coefficients, the Hurwitz's determinants are defined as

$$D_1 = |a_1|, D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}, D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$D_j = \begin{vmatrix} a_1 & a_3 & a_5 & . & . & . & a_{2j-1} \\ 1 & a_2 & a_4 & . & . & . & a_{2j-2} \\ 0 & a_1 & a_3 & . & . & . & a_{2j-3} \\ 0 & 1 & a_2 & . & . & . & a_{2j-4} \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & . & . & . & a_j \end{vmatrix} \tag{14}$$

where $a_i = 0$ for $i > n$

All the roots of the characteristic polynomial above, would have negative real part if and only if all the $D_j > 0$, $j = 1, \ldots, n$. Consequently, the polynomial $P(x)$ is stable as $t \to \infty$

Definition 1.7.15

The discriminant of a polynomial with real coefficient gives more information about the properties and nature of the roots of the polynomial without actually solving it. The resultant (Res) or determinant of a matrix known as the Sylvester matrix is used to obtain the discriminant for higher polynomials. This matrix is associated with two uni-variate (one variable) polynomial and is defined as

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0$ then the Sylvester matrix associated with p and q is the $(m+n)\times(m+n)$ matrix given as

$$S_{p,q} = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & . & . & . & a_2 & a_1 & a_0 & 0 & . & . & . & 0 \\ 0 & a_n & a_{n-1} & . & . & . & a_3 & a_2 & a_1 & 0 & . & . & . & 0 \\ 0 & 0 & a_n & . & . & . & a_4 & a_3 & a_2 & 0 & . & . & . & 0 \\ . \\ . \\ . \\ 0 & 0 & 0 & a_n & a_{n-1} & a_{n-2} & . & . & . & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_n & b_{n-1} & b_{n-2} & . & . & . & b_2 & b_1 & b_0 & 0 & . & . & . & 0 \\ 0 & b_n & b_{n-1} & . & . & . & b_3 & b_2 & b_1 & 0 & . & . & . & 0 \\ 0 & 0 & b_n & . & . & . & b_4 & b_3 & b_2 & 0 & . & . & . & 0 \\ . \\ . \\ . \\ 0 & 0 & 0 & b_n & b_{n-1} & b_{n-2} & . & . & . & b_4 & b_3 & b_2 & b_1 & b_0 \end{pmatrix} \tag{15}$$

The discriminants of the polynomial, $p(x)$ defined by Δ_p is described using the Sylvester matrix in equation (1.15) as the quotient of the determinant of $p(x)$ and its derivative, $p'(x)$ by a_n. Therefore,

$$\Delta_p = \frac{1}{a_n} det(p(x), p'(x)) \tag{16}$$

Generally, if:

- $\Delta_p > 0$, then there are $2k$ pairs of complex conjugate roots and $n-4k$ real roots for some integer k such that $0 \le k \le \frac{n}{4}$;
- $\Delta_p < 0$, then there $2k+1$ pairs of complex conjugate roots and $n-4k-2$ real roots in which they are all different, for some k such that $0 \le k \le \frac{n-2}{4}$;
- $\Delta_p = 0$ then there exist at least 2 roots which coincide and could either be real or complex.

Definition 1.7.16

The mean motion n of the massive bodies m_1 and m_2 obtained from Kepler's third law which state that "the square of the period of planets or binary stars is proportional to the cube of the semi-major axis," was given by

$$n^2 = \frac{G(m_1+m_2)}{r^3} = \frac{1(m_1+m_2)}{rm_1m_2}\frac{\partial V}{\partial r} \tag{17}$$

Definition 1.7.17

A dynamical system is said to be periodic when the same configuration is repeated at a regular interval of time. For instance, in a sidereal or inertial (fixed) coordinate system, the two body problem will have a solution which is repeated after a period of

$$T_{sidereal} = \frac{2\pi}{n} \tag{18}$$

where n is the mean motion.

And for the synodic (rotating) coordinate system, periodic motion occur when $n = \frac{p}{q}$ where p and q are integers and n is a rotational number. The period is

$$T_{synodic} = |p| T_{sidereal} = \frac{2\pi p}{n} = 2\pi q$$

The elliptic motions are always periodic in the sidereal system but not necessarily in the rotating frame of reference, while the circular motions are always periodic in both systems. In the three- body problems, a solution is said to be periodic if the mutual distance of the bodies are periodic function of time.

8 Organization of the Thesis

This thesis comprises of five chapters, references and appendices.

In chapter one, the background, aim and objectives, methodology, justification, scope and organization of the study were introduced. Chapter two reveals a review of relevant literature under various headings (radiation pressure, oblateness, PR-drag, Coriolis and Centrifugal forces and the periodic motion) of the classical RTBP. Chapter three is divided into three sections in which the equations of motion, equations of the coordinate of the libration points and the stability around these points were obtained for the study of the problem in section one (the effects of PR-drag force and oblateness on the stability of the triangular libration points)and section two (the effects of the Coriolis and Centrifugal forces on the stability of generalized RTPB) while section three established the equations of the periodic motion around L_4. Chapter four presents the analysis of the results obtained in chapter three while in cchapter five the summary, conclusion and recommendation were given.

2 Literature Review

1 Introduction

In space dynamics, an understanding of the near-earth objects (NEOs) is essential for resolving the relationships between asteroids, comets and meteorites. They are the smallest solar system bodies observable because of their proximity to the earth. They display certain physical properties such as:

-Possession of irregular shapes

-Possession of small perturbation forces other than the gravitational force of attraction.

-The ability of the asteroids to emit radiation due to their surface light.

In view of these properties, many papers that generalize the classical Restricted Three Body Problem (RTBP) have been published. These generalization made the problem more realistic by incorporating the force of radiation pressure, oblateness/triaxiality and Poynting-Robertson (PR) drag effect.

The related literature and research works were reviewed and discussed under various generalizations in this chapter as follows;

2 Classical Case

Duboshin (1958) studied the motion of RTBP and established the rational equations of motion and later studied the circular RTBP which showed that the collinear libration points existed and that the triangular points make an isosceles triangle with the primaries and continued by studying the motion of three rigid bodies whose elementary particles act upon each other according to arbitrary laws of forces along the straight line joining them. Szebehely $(1967a)$ discovered that the classical RTBP possesses three collinear (L_1, L_2, L_2) which were found to be unstable and two triangular points (L_4, L_5) which are stable for $0 < \mu \le \mu_c$ where μ is the mass parameter and $\mu_c = 0.03852...$ is the critical mass value. Sengupta and Singla (2002) similarly analyzed the stability of the classical RTBP by formulating the equation of motion using the Langrange's-Hamiltonian technique.

Here, the primaries were assumed to be spherical and other forces (radiation pressure, solar wind, Poynting-Robertson (PR) drag, Coriolis and Centrifugal forces etc) other than the gravitational force of attraction were not put into consideration in establishing the existence and stability of the libration points.

3 Effect of Radiation Pressure

In the real sense, the planetary bodies: planets and dwarf planets, natural and artificial satellites, asteroids, comets and meteorite exhibit different properties that affect the motion of a particular system, thereby leading to a change in the general solution.

One of the immediate generalization of the RTBP was the study of the photo-gravitational effects. For small particles like asteroids and binary stars, light can cause a significant change in the altitude and direction of motions over a large period when rotating relative to the sun. This energy radiated from the celestial bodies known as the photo-gravitational effect was put into consideration in establishing the stability of RTBP. Radzievskii (1950), was the first to formulate the problem. He studied the linear stability of the problem and obtained the five libration points. Kunitsyn and Perezhogin (1978) studied the stability in the Lyapunov sense with one of the primaries radiating. Mignard (1982) explored the Astronomical applications of the stability problem by looking into the influence of radiation pressure from the sun in the planet-satellite-particle system. Simmons, Mc Donald and Brown (1985) gave the complete solution of the RTBP. They also discussed the existence and linear stability of the equilibrium points for all values of radiation pressure from both radiating bodies for all values of mass ratio. Kumar and Choudry (1987) examined the stability of triangular libration points when the attracting primaries are radiating under the non-resonance cases. It was discovered that the motion will be stable for all values of the mass value, μ and mass reduction factor due to radiation pressure, q_1, q_2. Kunitsyn and Polyathara (1995) investigated the photo-gravitational effect on the infinitesimal mass from both primaries. Khasan (1996) obtained the collinear and triangular libration points for the averaged equations of motion of the elliptic photo-gravitational RTBP and their stability is studied to a first approximation. Kunitsyn (2000,2001) investigated the stability of the relative equilibrium positions (collinear libration points) of the circular photo-gravitational RTBP, in which a point is passively experiencing the Newtonian gravitational force from the main bodies (stars) which also experience forces of light pressure from each of them and analyzed previously obtained conditions of stability from new perspective.

The collinear points were found to be unstable for all values of the mass ratio, μ while the non-collinear points are stable and form isosceles triangles due to the radiation effect from either or both of the primaries on the RTBP.

4 Effect of Oblateness

The classical RTBP further modified by considering the unusual shape of the planetary bodies since all the planetary bodies were observed flattened due to their rotation around the sun. The measure of these flattening is known as Oblateness. Many researchers have studied the effect of oblateness on the stability of libration points for various system. McCuskey (1963) established the equation for obtaining the Oblateness coefficients, $A_i, i = 1,2$ and consequently, the force due to Oblateness is given by

$$\overline{F} = Gm\left[\frac{m_i \vec{r}_i}{r_i^3} + \frac{3m_i A_i \vec{r}_i}{2r_i^5}\right], \ (i = 1,2) \quad (AbdulraheemandSingh, 2006) \tag{19}$$

Where G is the gravitational constant, m the mass of infinitesimal mass and $r_i = (i = 1,2)$ is the distance between the primaries m_i and the infinitesimal body. Vidyakin (1974) established the location of the libration points and studied their stability in the Lyapunov sense when both primaries are oblate spheroids with their equatorial plane coinciding with the plane of motion. Sharma and Subbarao (1976,1979) studied the of RTBP when one of the primaries is an oblate spheroid. They established that the decrease in the range of stability was due to oblateness. Bhatnagar and Khanna (1999) considered the smaller primaries to be triaxial with one of its axes of symmetry coinciding with the plane of motion. Abouelmagd (2012) observed that there still exist five equilibrium points for which due to oblateness, the triangular points deviate from its positions but does not influence the motion of the system in the x-y plane, in the linear sense. Arrendondo, Gui and Stocia (2012) investigated the linear stability numerically using $J^{(1)}$ and $J^{(2)}$ parameters.

5 Effects of Oblateness and Radiation Pressure

The generalized RTBP were modified by different authors to investigate the effect of both oblateness and radiation pressure force on the stability of RTBP. Sharma (1982) studied the linear stability of triangular libration points of the restricted three body problem when the bigger primary is an oblate spheroid as well as a source of radiation. He generalized the study

(1987) by considering an oblate primary and radiating secondary. Ishwar and Singh (1999), Tsirogiannis, Doukos and Perdios (2006) also computed the lyapunov's orbit of a similar system. They discovered that oblateness of the primary and radiation of the secondary reduced the stability region of the triangular equilibrium points. like-wise, Shankaran , Ishwar, Chakraborty and Abdullah (2011), Jain *etal* (2013), Singh and Umar (2012,2013,2014) worked on related problems by considering the elliptic orbits.

Sharma, Taqvi and Bhatnagar (2001*a*,2001*b*) studied the stationary solutions of the planar RTBP when the primaries are triaxial rigid bodies as well as sources of radiation with one of the axes as the axis of symmetry and its equatorial plane coinciding of motion. They obtained five libration points: two triangular points which are stable for a certain range of mass value and three collinear points which are unstable.

6 Effect of the Poynting-Robertson Drag

The spectacular results of the effect of radiation pressure on RTBP prompted researchers to generalize previous works further by considering the radiation pressure force produced from the absorption and subsequent re-emission of sun rays striking small particles orbiting it thereby retarding the motion of the particle thus lowering the angular momentum and consequently spiral into the sun. This process is known as the Poynting-Robertson drag effect. Poynting (1903) was the first to consider this problem and later modified by Robertson (1937). He established the expression for the net drag force which opposes the direction of motion using a precise relativistic treatment of the first order in the ratio of the velocity of the particle to the speed of light. Colombo, Lautman and Shapiru (1996) studied the effect of radiation pressure and PR-drag on the RTBP. Chernikov (1970) and Schuerman (1980) established the existence of six libration points in which one lie out of the orbital plane. He found that due to the PR-drag effect, the triangular libration points are unstable. Following this discovery, several articles have been published related work. Murray (1994) explained the dynamical effect of drag in general in the planar circular RTBP. Liou, Zook and Jackson (1995) studied the effect of radiation, PR and solar wind drag in the RTBP. Ragos and Zafiropoulos (1995) established the equations of motion for when the primaries are radiating with PR-drag effect from the expression of the net force acting on the system. He studied this problem numerically and discovered that the collinear points deviate from the axis while the triangular points are no longer symmetrical. Lhotka and Celletti (2014) studied the effect of the PR-drag on the triangular Lagrangian points but in the spatial, elliptic RTBP. Raj and Ishwar (2017) obtained the diagonalizable Hamiltonian for the photogravitational RTBP with the PR-drag.

7 Effects of Oblateness and Poynting-Robertson Drag

Kushvah and Ishwar (2004) and Ishwar and Kushvah (2006) examined the linear stability of the generalised photo-gravitational RTBP when the smaller primary is considered to be oblate spheroid and the bigger one radiating with PR-drag. Das, Narang, Mahajan and Yuasa \$(2009)\$ worked on the out of plane equilibrium points of a passive micron size particle and examined their stability in the field of radiating binary stars. Lhotka and Celletti (2014) examined the effect of PR-drag on the triangular libration points in the framework of elliptic RTBP. This is an extension of Murray (1994) work. Singh and Amuda (2014) studied the photo-gravitational problem when the bigger primary is oblate and smaller a source of radiation with PR-drag, Singh, Taura and Joel (2014) using analytical and numerical methods, obtained the triangular libration points which were found to move towards the line joining the primaries in the presence of any of perturbations (such as oblateness up to J_4 of the less massive primary, electromagnetic radiation of the more massive primary and potential from the belt), except in the presence of oblateness up to J_4 where the points move away from the line joining the primaries and examined their linear stability. A practical application of their model is the study of the motion of a dust particle near a radiating star and an oblate body surrounded by a belt. Jaiyeola, AbdulRaheem and Titiloye (2016) extended their works to understand the effects of various perturbing factors on the dynamics of a particle orbiting the primaries. They concluded that the P-R drag renders unstable those libration points that are conditionally stable in the classical case. Lhotka, Celletti, and Gales (2016) investigated the effect of PR and solar wind drag on space debris. Narayan and Shrivastava (2013), Singh *etal* (2016), and many others have used various binary stars such as Prokyon, Kruger, RW-Monocerotis, Achird, Luyten, α Cen AB, Xi-Bootis, Algol etc to verify their results.

8 Effects the Coriolis and Centrifugal Forces

The study of the effects of small perturbation in the coriolis and centrifugal forces on the stability of libration points of the RTBP cannot be over-emphasized because of their peculiar nature.

The Classical RTBP has been generalized extensively by prominent researchers. Wintner (1941) showed that the stability of two equilateral points was due to the presence of the coriolis parameter in the equation of motion. Szebehely (1967b) considered similar problem keeping the centrifugal force constant and established for the triangular points a relation between the critical value of the mass parameter μ_c and the change ε in the coriolis (ε) force as

$$\mu_c = \mu_0 + \frac{16\varepsilon}{3\sqrt{69}}$$

and thus concluded that the coriolis force is a stabilizing force. Subbarao and Sharma (1975) showed that with oblateness the coriolis force is not always a stabilizing force. Bhatnagar and Hallan (1979) extended their work to include the centrifugal (ε') force and showed that the collinear points remain unstable while for the triangular points he obtained a relation

$$\mu_c = \mu_0 + \frac{4(16\varepsilon - 19\varepsilon')}{3\sqrt{69}}$$

which implies that the increase or decrease in the range of stability depends upon the points $(\varepsilon, \varepsilon')$. Singh and Iswhar (1984) investigated the effect of small perturbations in the coriolis and the centrifugal forces on the location of libration points in the RTBP with variable mass. In line with the other results they established that the range of stability of the triangular points increases or decreases depending on whether the perturbation point $(\varepsilon, \varepsilon')$ lies in either of the two parts in which $(\varepsilon, \varepsilon')$ plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$. Abdulraheem and Singh (2006) building upon previous works, studied the combined effects of small perturbations in the coriolis and centrifugal forces, radiation and oblateness on the stability of the libration points the RTBP and discovered that the collinear points remained unstable while the range of stability of the triangular points decreases as seen in the critical mass value μ_c obtained as

$$\mu_c = \mu_0 + \mu_p + \mu_b + \mu_r \tag{20}$$

where

$\mu_0 = \frac{1}{2}(1 - \sqrt{\frac{23}{27}})$ $\mu_p = 4(\frac{36\varepsilon - 19\varepsilon'}{27\sqrt{69}})$ . . . due to perturbations in the coriolis and centrifugal force.

$\mu_r = -\frac{2}{27\sqrt{69}}(1 - q_1) - \frac{2}{27\sqrt{69}}(1 - q_2)$ . . . due to radiation effect.

$\mu_b = -\frac{1}{9}(1 + \frac{13}{\sqrt{69}})_{A_1} + \frac{1}{9}(1 - \frac{13}{\sqrt{69}})_{A_2}$ . . . effect due to oblateness.

Singh (2009,2011) investigated the non-linear stability of the triangular equilibrium points under the effects of small perturbations in the coriolis and the centrifugal forces, together with the effects of oblateness and radiation pressures of the primaries. Singh and Aminu (2014) examined the influence of small perturbations in the coriolis and centrifugal forces, both analytically and numerically on the stability of circular RTBP with PR-drag from both primaries. This was shown for the binary systems Luyten 726-8 and Kruger 60. Akere-Jaiyeola, Singh, AbdulRaheem and Braimah (2015) considered the effect of perturbations on the stability of the libration points on RTBP with a triaxial primary and radiating secondary. The range of stability were found to be affected by the perturbing parameter as seen in the relation for the critical mass value obtained as

$$\mu_c = \mu_0 + \mu_p + \mu_r + \mu_t \tag{21}$$

where

$\mu_0 = \frac{1}{2}(1 - \sqrt{\frac{23}{27}})$ classical

$\mu_p = 4(\frac{36\varepsilon - 19\varepsilon'}{27\sqrt{69}})$ due to perturbations

$\mu_r = -\frac{2}{27\sqrt{69}}(1 - q_1) - \frac{2}{27\sqrt{69}}(1 - q)$ due to radiation

$\mu_t = \frac{1}{2}(\frac{5}{6} + \frac{59}{9\sqrt{69}})\sigma_1 - \frac{1}{2}(\frac{19}{18} + \frac{85}{9\sqrt{69}})\sigma_2$ due to triaxiality.

Abouelmagd, Alhothuali, Guirao and Malaikah (2015) presented the graphical analysis for the variations of the angular frequencies for the periodic and secular RTBP under harmonic effect. Zoto (2015) investigated how the oblateness coefficient influence the nature of orbits in the RTBP and discovered that the it has a huge impact on the character of orbits. Singh and Omale (2015) determined the effect of small perturbations in coriolis and centrifugal forces on the axial equilibrium points and examined stability in Robe's circular RTBP when the hydrostatic equilibrium figure of the massive primary is an oblate spheroid; the shape of the less massive primary is a triaxial rigid body. It was discovered that the locations of the axial equilibrium points were only influenced by a small change in the centrifugal force and many other researchers have introduced and studied the effects of the coriolis and centrifugal forces, radiation pressure force, oblateness, on the stability of the RTBP. They observed that coriolis force has a stabilizing tendency while the centrifugal force, radiation pressure force and oblateness have destabilizing effect.

9 Periodic Orbit

In addition, the periodic orbits of the classical RTBP and its numerous generalizations have been studied extensively by researchers. Poincare (1897) gave the three definitions of the first kind for the periodic solution for an orbit in a synodic coordinate system in terms of its inclination as: zero for small mass value μ in a circular orbit, zero inclination for particles perturbing in a keplerian elliptic orbit and when the inclination are no longer zero. Arenstorf (1963) studied analytically the periodic orbit of the second kind in the planar RTBP. Barrar (1965) examined similar problem using Cartesian rectangular coordinates and Delaunay's canonical variables. This work was later extended to investigate the collision orbits as well. Szebehely $(1967a)$ discussed the periodic motion of a particle in the classical RTBP. Sharma (1976) in line with Barrar's method, considered the bigger primary to be oblate spheroid and established the existence periodic orbits of the first kind. Sharma (1981) modified this work to study the period orbit of the second kind. Sharma and Subbarao (1986) provided approximations to periodic solutions around the triangular libration points with an oblate massive primary. Elipe and Lara (1997) obtained various natural families of periodic orbits of the RTBP when the influence of the radiation pressure on the gravitational forces from the primaries are put in consideration. Sharma, Taqvi and Bhatnagar $(2001a, 2001b)$ established the existence of the long and short period, orientation and the semi-axes of the RTBP when the primaries are triaxial rigid bodies and sources as well as sources of radiation. Mittal, Ahmad and Bhatnagar (2009) examined the effect of oblateness on the periodic orbits around the Lagrangian points of the RTBP. Singh and Begha (2011) established the existence of the periodic orbits of the RTBP with oblate (massive) and triaxial (less massive) primaries. They deduced their period, orientation and eccentricities are influenced by the small perturbations in the coriolis and centrifugal forces, oblateness and triaxiality of the primaries. Singh and Haruna (2014) established the periodic orbits around the triangular libration points when the three bodies are considered to be oblate. Abouelmagd $et al\,(2015)$ determined the periodic structure of the RTBP considering the influence of the zonal harmonics parameters for the bigger primary. Singh, Narayan and Ishwar (2015) showed that oblateness, radiation pressure and eccentricity have a significant effect on the trajectories and stability of the infinitesimal mass around the libration points. Zoto (2015) investigated how the oblateness coefficient influence the nature of orbits in the RTBP and discovered that it has a huge impact on the character of orbits. Many of the previously mentioned researchers have also studied the periodic orbits. Recently Pushparaj and Sharma (2017) studied the Periodic orbits of the photo-gravitational RTBP using Poincare approach and found that the period of time of Jupiter decreases with increase in radiation pressure from the Sun while due to oblateness of Jupiter the period increases.

Due to the remarkable effects of all these perturbing (coriolis and centrifugal, shape of the primaries, radiation pressure, solar wind drag, Poynting-Robertson drag etc.)forces on the motion around the orbit of the satellite (both Natural and artificial), this research work hereby modify specifically, the works of Ragos and Zafiropoulos (1995), Ishwar and Kushvah (2006) and Singh and Amuda (2014) to achieve this new and interesting result.

3 Methodology and Results

In this chapter, the equations needed to study the effects of small perturbations in the coriolis and centrifugal forces on the stability of the generalized photo-gravitational RTBP with PR-drag force and the periodic motion around L_4 were established.

1 Effect Of Poynting-Robertson Drag And Oblateness On The Stability Of Restricted Three-Body Problem

1.1 The Equations of Motion

The energy (potential and kinetic) of the RTBP when the primaries are considered to be both oblate spheroid and source of radiation with PR-drag effect (in the absence of small perturbations in the coriolis and centrifugal forces) is obtained and to establish the equations of motion for the proposed system.

Figure 2: The primaries rotating with respect to the inertial frame of reference.
Source:Szebehely (1967a)

With reference to an inertial or fixed coordinates OXYZ, let $P(x, y, z)$, $S_1(-a, 0, 0)$ and $S_2(b, 0)$ be the coordinates of the infinitesimal body and primaries with masses m, m_1 and m_2 respectively. let r_1, r_2 be the distances between each of the primary and the infinitesimal while r is the distance between the primaries. Introducing a rotating coordinate system Oxyz with the origin O at the barycenter of the primaries in which the axis rotate relative to the inertial space with an angular velocity $\omega = nk$

The Net Potential

The net force on the infinitesimal body due to both primaries being oblate spheroid and radiating with PR-drag effect is

$$\overline{F} = \overline{F}_O + \overline{F}_{PR}$$

Where the subscripts O and PR indicate the force due to oblateness and PR-drag effect respectively.

Based on equations (1.9) and (2.1) the total force becomes

$$\bar{F} = -Gm\left[\frac{m_i \bar{r}_i}{r_i^3} + \frac{3m_i A_i \bar{r}_i}{2r_i^5}\right] + F_{p_i}\left\{\frac{\bar{r}_i}{r_i} - \frac{\bar{v}\bar{r}_i \bar{r}_1}{c r_1 r_1} - \frac{\bar{v}_1}{c}\right\}$$

by equation (1.2), (1.5) and (1.7) gives,

$$\begin{aligned}\bar{F} &= -Gm\left\{\frac{m_i \bar{r}_i}{r_i^3} + \frac{3m_i A_i \bar{r}_i}{2r_i^5}\right.\\ &\left. - \frac{m_i(1-q_i)}{r_i^2}\left[\frac{\bar{r}_i}{r_i} - \frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)\bar{r}_i)}{c r_i^2} - \frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)}{c}\right]\right\}\end{aligned}$$

or

$$\begin{aligned}\bar{F} &= -Gm\left\{\frac{m_i q_i}{r_i^3}\bar{r}_i + \frac{3m_i A_i}{2r_i^5}\bar{r}_i\right.\\ &\left. + \frac{m_i(1-q_i)}{r_i^2}\left[\frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)\bar{r}_i}{c r_i^2} + \frac{(\bar{r}_i + \bar{\omega}\times\bar{r}_i)}{c}\right]\right\}\end{aligned}$$

where,

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \bar{r}_1 = (x+a)\hat{i} + y\hat{j} + z\hat{i},\ \bar{r}_2 = (x-b)\hat{i} + y\hat{j} + z\hat{k}$$

$$\dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k},\ \bar{r}_1^{\,2} = (x+a)^2 + y^2 + z^2,\ \bar{r}_2^{\,2} = (x-b)^2 + y^2 + z^2,\ v = \dot{\bar{r}} + \omega\times\bar{r}$$

which is equivalent to

$$\bar{F} = \bar{F}_x\hat{i} + \bar{F}_y\hat{j} + \bar{F}_z\hat{k} \tag{22}$$

where,

$$\begin{aligned}F_x = &-\frac{Gmm_1(x+a)}{r_1^3} - \frac{Gmm_2(x-b)}{r_2^3} - \frac{3Gmm_1(x+a)A_1}{2r_1^5} - \frac{3Gmm_2(x-b)A_2}{2r_2^5}\\ &-(1-q_1)\frac{Gmm_1}{r_1^2}\left\{\frac{(x+a)}{r_1} - \frac{(x+a)}{cr_1^2}[\dot{x}(x+a) + y\dot{y} + z\dot{z}] - \frac{(\dot{x}-ny)}{c}\right\}\\ &-(1-q_2)\frac{Gmm_2}{r_2^2}\left\{\frac{(x-b)}{r_2} - \frac{(x-b)}{cr_2^2}[\dot{x}(x-b) + y\dot{y} + z\dot{z}] - \frac{(\dot{x}-ny)}{c}\right\}\end{aligned}$$

$$F_y = -\frac{Gmm_1 y}{r_1^3} - \frac{Gmm_2 y}{r_2^3} - \frac{3Gmm_1 A_1 y}{2r_1^5} - \frac{Gmm_2 A_2 y}{2r_2^5 y}$$

$$-(1-q_1)\frac{Gmm_1}{r_1^2}\left\{\frac{y}{r_1}-\frac{y}{cr_1^2}[\dot{x}(x+a)+y\dot{y}+z\dot{z}]-\frac{\dot{y}+n(x+a)}{c}\right\}$$

$$-(1-q_2)\frac{Gmm_2}{r_2^2}\left\{\frac{y}{r_2}-\frac{y}{cr_2^2}[\dot{x}(x-b)+y\dot{y}+z\dot{z}]-\frac{\dot{y}+n(x-b)}{c}\right\}$$

$$F_z=-\frac{Gmm_1 z}{r_1^3}-\frac{Gmm_2 z}{r_2^3}-\frac{3Gmm_1A_1z}{2r_1^5}-\frac{3Gmm_2A_2z}{2r_1^5}$$

$$-(1-q_1)\frac{Gmm_1}{r_1^2}\left\{\frac{z}{r_1}-\frac{z}{cr_1^2}[\dot{x}(x+a)+y\dot{y}+z\dot{z}]-\frac{\dot{z}}{c}\right\}$$

$$-(1-q_2)\frac{Gmm_2}{r_2^2}\left\{\frac{z}{r_2}-\frac{z}{cr_2^2}[\dot{x}(x-b)+y\dot{y}+z\dot{z}]-\frac{\dot{z}}{c}\right\}$$

Integrating equation (3.1) according to equation (1.6) the net potential of the system is

$$\bar{V}=\int\bar{F}d\bar{r}=\hat{i}\int F_x dx+\hat{j}\int F_y dy+\hat{k}\int F_z dz \tag{23}$$

where $F_x, F_y and F_z$ are given in equation (3.1).

The Kinetic Energy of the System

The kinetic of the system is given by

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)+mn(x\dot{y}-\dot{x}y)+\frac{1}{2}mn^2(x^2+y^2) \\ &= T_2+T_1+T_o \end{aligned} \tag{24}$$

where

$$\begin{aligned} T_2 &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \\ T_1 &= m(x\dot{y}-\dot{x}y) \\ T_o &= \frac{1}{2}mn^2(x^2+y^2). \end{aligned} \tag{25}$$

Let p_x, p_y and p_z be the generalized component of momentum then by equations (3.3) and (3.4), I get

$$\left.\begin{aligned} &p_x = \frac{\partial T}{\partial \dot{x}}=m(\dot{x}-ny),\ p_y=\frac{\partial T}{\partial \dot{y}}=m(\dot{y}+nx),\ p_z=\frac{\partial T}{\partial \dot{z}}=m\dot{z} \\ &and \\ &\dot{p}_x = m(\ddot{x}-n\dot{y}),\ \dot{p}_y=m(\ddot{y}+nx),\ \dot{p}_z=m\ddot{z} \end{aligned}\right\} \tag{26}$$

then, the Hamiltonian denoted by H is

$$H=\sum_1^n \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L$$

becomes

$$H = T_2 - T_o + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}mn^2(x^2 + y^2) + V \tag{27}$$

Using equation (3.5), equation (3.6) becomes,

$$H = \frac{m}{2}\left[\left(\frac{p_x}{m} + ny\right)^2 + \left(\frac{p_y}{m} - nx\right)^2 + \left(\frac{p_z}{m}\right)^2\right] - \frac{1}{2}mn^2(x^2 + y^2) + V$$

$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + n(p_x y - p_y x) + V \tag{28}$$

Using equation (3.6), the Hamilton's equations of motion given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}$$

and

$$\dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}$$

yields, (component wise system description)

$$\left.\begin{array}{ll} \dot{x} = & \frac{p_x}{m} + ny, \\ \dot{y} = & \frac{p_y}{m} - nx, \\ \dot{z} = & \frac{p_z}{m} \\ and & \\ \dot{p}_x = & -np_y - \frac{\partial V}{\partial x}, \\ \dot{p}_y = & -np_x - \frac{\partial V}{\partial y}, \\ \dot{p}_z = & -\frac{\partial V}{\partial z} \end{array}\right\} \tag{29}$$

Substituting equation (3.5) into (3.8), gives

$$m(\ddot{x} - n\dot{y}) = mn(\dot{y} + nx) - \frac{\partial V}{\partial x}$$

$$m(\ddot{y} + n\dot{x}) = -mn(\dot{x} - ny) - \frac{\partial V}{\partial y}$$

$$m\ddot{z} = \frac{-\partial V}{\partial z}$$

That is

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{1}{m}\frac{\partial V}{\partial x}$$

$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{1}{m}\frac{\partial V}{\partial y} \tag{30}$$

$$\ddot{z} = \frac{-1}{m}\frac{\partial V}{\partial z}$$

By equation (3.2) , equation (3.9) results to

$$\ddot{x} - 2n\dot{y} = n^2 x + (1-q_1)\frac{Gm_1}{r_1^2}\left\{\frac{(x+a)}{r_1} - (x+a)\frac{[\dot{x}(x+a)+y\dot{y}+z\dot{z}]}{cr_1^2} - \frac{(\dot{x}-ny)}{c}\right\}$$

$$+(1-q_2)\frac{Gm_2}{r_2^2}\left\{\frac{(x-b)}{r_2} - (x-b)\frac{\dot{x}(x-b)+y\dot{y}+z\dot{z}}{cr_2^2} - \frac{(\dot{x}-ny)}{c}\right\}$$

$$-\frac{Gm_1(x+a)}{r_1^3} - \frac{Gm_2(x-b)}{r_2^3} - \frac{3Gm_1(x+a)A_1}{2r_1^5} - \frac{3Gm_2(x-b)A_2}{2r_2^5}$$

$$\ddot{y} + 2n\dot{x} = n^2 y + (1-q_1)\frac{Gm_1}{r_1^2}\left\{\frac{y}{r_1} - y\frac{[\dot{x}(x+a)+y\dot{y}+z\dot{z}]}{cr_1^2} - \frac{\dot{y}+n(x+a)}{c}\right\} \tag{31}$$

$$+(1-q_2)\frac{Gm_2}{r_2^2}\left\{\frac{y}{r_2} - y\frac{\dot{x}(x-b)+y\dot{y}+z\dot{z}}{cr_2^2} - \frac{y+n(x-b)}{c}\right\}$$

$$-\frac{Gm_1 y}{r_1^3} - \frac{Gm_2 y}{r_2^3} - \frac{3Gm_1 A_1 y}{2r_1^5} - \frac{Gm_2 A_2 y}{2r_2^5 y}$$

$$\ddot{z} = (1-q_1)\frac{Gm_1}{r_1^2}\left\{\frac{z}{r_1} - z\frac{[\dot{x}(x+a)+y\dot{y}+z\dot{z}]}{cr_1^2} - \dot{z}\right\}$$

$$+(1-q_2)\frac{Gm_2}{r_2^2}\left\{\frac{z}{r_2} - z\frac{[\dot{x}(x-b)+y\dot{y}+z\dot{z}]}{cr_2^2} - \dot{z}\right\}$$

$$-\frac{Gm_1 z}{r_1^3} - \frac{Gm_2 z}{r_2^3} - \frac{3Gm_1 A_1 z}{2r_1^5} - \frac{3Gm_2 A_2 z}{2r_1^5}$$

Adopting the notation of Szebehely, the distance between the primaries along the x- axis is taken to be equal to one. The sum of the masses of the primaries is also assumed to be 1 so that if $m_2 = \mu$ then $m_1 = 1-\mu$ and the origin as the barycenter of the masses m_1 at $(-a.0,0)$ and m_2 at $(b,0,0)$ which implies that

$$m_1(-a) + m_2(b) = 0$$

$$-(1-\mu)a + \mu b = 0$$

$$\mu(a+b) = a$$

Since the distance between the primaries is assumed to be 1, then $a = \mu$ and $b = 1-\mu$

where, $\mu = \dfrac{m_2}{m_1 + m_2}$ is the mass ratio parameter.

The unit of time is chosen so as to make the gravitational constant G to be equal to unity. The speed of light c is given as $c = c_d$. Assuming that $q_i (i = 1,2)$ are constant (neglecting fluctuations in the beam of solar radiation and the effect of the planet shadow.). In the dimensionless synodic coordinate system, the equations of motion of the photo-gravitational RTBP, in the absence of small perturbations in the coriolis and centrifugal forces obtained in equation (3.10) becomes

$$\begin{aligned} &\ddot{x} - 2n\dot{y} = n^2 x + U_x \\ &\ddot{y} + 2n\dot{x} = n^2 y + U_y \\ &\ddot{z} = U_z \end{aligned} \tag{32}$$

where,

$$\begin{aligned} U_x \quad &= n^2 x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu q_2(x+\mu-1)}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1}{2r_1^5} \\ &- \frac{3\mu(x+\mu-1)A_2}{2r_2^5} - \frac{W_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[\dot{x}(x+\mu) + \dot{y}y + \dot{z}z] + \dot{x} - ny\right\} \\ &- \frac{W_2}{r_2^2}\left\{\frac{(x+\mu-1)}{r_2^2}[\dot{x}(x+\mu-1) + \dot{y}y + \dot{z}z] + \dot{x} - ny\right\} \end{aligned}$$

$$\begin{aligned} U_y \quad &= n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3(1-\mu)yA_1}{2r_1^5} - \frac{3\mu A_2 y}{2r_2^5} \\ &- \frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[\dot{x}(x+\mu) + \dot{y}y + \dot{z}z] + [\dot{y} + n(x+\mu)]\right\} \end{aligned}$$

$$- \frac{W_2}{r_2^2}\left\{\frac{y}{r_1^2}[\dot{x}(x+\mu-1) + \dot{y}y + \dot{z}z] + [\dot{y} + n(x+\mu-1)]\right\} \tag{33}$$

$$U_z = -\frac{(1-\mu)q_1 z}{r_1^3} - \frac{\mu q_2 z}{r_2^3} - \frac{3(1-\mu)zA_1}{2r_1^5} - \frac{3\mu A_2 z}{2r_2^5}$$
$$-\frac{W_1}{r_1^2}\left\{\frac{z}{r_1^2}[\dot{x}(x+\mu)+\dot{y}y+\dot{z}z]+\dot{z}\right\} - \frac{W_2}{r_2^2}\left\{\frac{z}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y+\dot{z}z]+\dot{z}\right\}$$

In the $x-y$ orbital plane (z=0) the equations of motion above takes the form

$$\ddot{x} - 2n\dot{y} = \Omega_x$$

$$\ddot{y} + 2n\dot{x} = \Omega_y \tag{34}$$

where,

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \mu\frac{(x+\mu-1)q_2}{r_2^3} - \frac{3(1-\mu)(x+\mu)A_1}{2r_1^5}$$
$$-\frac{3\mu(x+\mu-1)A_2}{2r_2^5} - \frac{W_1}{r_1^2}\left\{\frac{x+\mu}{r_1^2}[\dot{x}(x+\mu)+y\dot{y}]+\dot{x}-ny]\right\}$$
$$-\frac{W_2}{r_2^2}\left\{\frac{x+\mu-1}{r_2^2}[\dot{x}(x+\mu-1)+y\dot{y}]+\dot{x}-ny]\right\}$$

$$\Omega_y = n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \mu\frac{q_2}{r_2^3} - \frac{3(1-\mu)yA_1}{2r_1^5} - \frac{3\mu y A_2}{2r_2^5}$$
$$-\frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[\dot{x}(x+\mu)+y\dot{y}]+[\dot{y}+n(x+\mu)]\right\} \tag{35}$$
$$-\frac{W_2}{r_2^2}\left\{\frac{y}{r_2^2}[\dot{x}(x+\mu-1)+y\dot{y}]+[\dot{y}+n(x+\mu)]\right\}$$

$$r_1^2 = (x+\mu)^2 + y^2 \quad \text{and} \quad r_2^2 = (x+\mu-1)^2 + y^2 \tag{36}$$

$$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}, W_2 = \frac{\mu(1-q_2)}{c_d}, \tag{37}$$

The mean motion, n by equation (1.17) gives

$$n^2 = 1 + \frac{3A_1}{2} + \frac{3A_2}{2} \tag{38}$$

and is found not to be influenced by the mass reduction factor (q_1, q_2) due to radiation pressure and PR-drag, (W_1, W_2) effects, but only by the oblateness (A_1, A_2) coefficients. (Abdulraheem and Singh, 2006, Ishwar and Kushvah, 2006, Amuda and Singh 2014.)

The equations of motion (3.13) and (3.14) are affected by the radiation pressure, oblateness of the primaries and Poyntiing Robertson drag.

1.2 The Jacobi Integral

One of the implications of this Jacobi integral is that it allows the making of certain general qualitative statements concerning, the motion without actually solving the equations of motion which gives great importance to integral applicable unsolvable dynamical problems.

The angular velocity ω of the finite masses is constant because they move in a circular orbit and therefore the Hamiltonian is constant. By multiplying the first equation of equation (3.13) by $2\dot{x}$ and the second by $2\dot{y}$ and then adding, gives

$$2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 2(\dot{x}\Omega_x + \dot{y}\Omega_y)$$

which is equivalent to

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 2\frac{\partial\Omega^*}{\partial t} + 2(\dot{x}F_{PRx} + \dot{y}F_{PRy}) \tag{39}$$

where,

$$\Omega^* = \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1}{2r_1^3} + \frac{\mu A_2}{2r_2^3}$$

$$\begin{aligned} F_{PRx} &= -\frac{W_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[\dot{x}(x+\mu) + \dot{y}y] + \dot{x} - ny\right\} \\ &\quad -\frac{W_2}{r_2^2}\left\{\frac{(x+\mu-1)}{r_2^2}[\dot{x}(x+\mu-1) + \dot{y}y] + \dot{x} - ny\right\} \end{aligned}$$

$$\begin{aligned} F_{PRy} &= -\frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[(x+\mu)\dot{x} + \dot{y}] + \dot{y} + n(x+\mu)\right\} \\ &\quad -\frac{W_2}{r_2^2}\left\{\frac{y}{r_2^2}[\dot{x}(x+\mu-1) + \dot{y}y] + \dot{y} + n(x+\mu-1)]\right\} \end{aligned}$$

F_{PRx} and F_{PRy} are the partial derivatives of the PR-drag function with respect to x and y respectively. These are functions of the position and velocity.

Integrating equation (3.18) with respect to time t, yields

$$\dot{x}^2 + \dot{y}^2 = 2\Omega^* + 2\int(\dot{x}F_{PRx} + \dot{y}F_{PRy})dt - C$$

where the left hand-side is the square of the velocity of the infinitesimal body which cannot be negative and C is the constant of integration known as the Jacobi integral. The motion of the body is restricted to the region where

$$v^2 = 2\Omega^* + 2\int(\dot{x}F_{PRx} + \dot{y}F_{PRy})dt - C \geq 0 \tag{40}$$

This condition in equation (3.19) does not tell about the shape of the orbit but it determines the region where the particle could move. The equation of the zero velocity curves (ZVC) are given by

$$C = 2\Omega^*(x, y) \tag{41}$$

The curve C represent various regions of possible motion.

1.3 Location of the Triangular Libration Points

The libration points are the solution of the equations of motion in (3.13), when the velocity and acceleration are equal zero (i.e. $\dot{x}=\dot{y}=\ddot{x}=\ddot{y}=0$), therefore,

$$\left.\begin{aligned}\Omega_x = \; & n^2x-\frac{(1-\mu)(x+\mu)q_1}{r_1^3}-\mu\frac{(x+\mu-1)q_2}{r_2^3}-\frac{3(1-\mu)(x+\mu)A_1}{2r_1^5} \\ & -\frac{3\mu(x+\mu-1)A_2}{2r_2^5}+\frac{nW_1y}{r_1^2}+\frac{nW_2y}{r_2^2}=0 \\ \Omega_y = \; & n^2y-\frac{(1-\mu)q_1y}{r_1^3}-\mu\frac{q_2y}{r_2^3}-\frac{3(1-\mu)yA_1}{2r_1^5}-\frac{3\mu yA_2}{2r_2^5} \\ & -\frac{nW_1(x+\mu)}{r_1^2}-\frac{nW_2(x+\mu-1)}{r_2^2}=0\end{aligned}\right\} \tag{42}$$

The triangular libration points are the solutions of equations $(3,21)$ when $y\neq 0$.

In the absence of oblateness $(A_1,A_2=0)$ and the PR-drag $(W_1,W_2=0)$ the equation (3.21) above reduces to the photo gravitational RTBP and which is

$$r_1=q_1^{\frac{1}{3}} \text{ and } r_2=q_2^{\frac{1}{2}} \quad \text{(Kunitsynan dPolyakhov a,1995)}$$

Now, letting $|\alpha_1|<<1$ and $|\beta_1|<<1$ be small perturbation (due to the oblateness and PR-drag of the primaries) in r_1 and r_2 respectively. Then

$$r_1=q_1^{\frac{1}{3}}+\alpha_1 \text{ and } r_2=q_2^{\frac{1}{3}}+\beta_1$$

Furthermore, assuming that, $q_1=(1-w_1),\ q_2=(1-w_2)$, $(|w_1|,|w_2|<<1)$ so that

$$r_1=(1-w_1)^{\frac{1}{3}}+\alpha_1 \text{ and } r_2=(1-w_2)^{\frac{1}{3}}+\beta_1 \tag{43}$$

Putting equation (3.22) into (3.15), gives

$$r_1^2=[(1-w_1)^{\frac{1}{3}}+\alpha_1]^2=(x+\mu)^2+y^2$$

$$r_2^2=[(1-w_2)^{\frac{1}{3}}+\beta_1]^2=(x+\mu-1)^2+y^2$$

Solving these equations, r_1^2 and r_1^2 simultaneously for x and y, neglecting second and higher order terms of small quantities, then,

$$(x+\mu)^2+(x+\mu-1)^2=\left\{1-\frac{w_1}{3}+\alpha_1\right\}^2-\left\{1-\frac{w_2}{3}+\beta_1\right\}^2$$

$$2(x+\mu)-1=1-\frac{2w_1}{3}+2\alpha_1-1+\frac{2w_2}{3}-2\beta_1$$

which yields

$$x=\frac{1}{2}-\mu-\frac{w_1}{3}+\frac{w_2}{3}+\alpha_1-\beta_1 \tag{44}$$

or

$$x = x_o\left\{1+\frac{\alpha_1-\beta_1}{x_o}\right\} \tag{45}$$

where,

$$x_o = \frac{1}{2}-\mu-\frac{w_1}{3}+\frac{w_2}{3}$$

and,

$$y^2 = [(1-w_1)^{\frac{1}{3}}+\alpha_1]^2-\left[\frac{1}{2}-\frac{w_1}{3}+\frac{w_2}{3}+\alpha_1-\beta_1\right]^2$$

$$= 1-\frac{2w_1}{3}+2\alpha_1-\frac{1}{4}\left[1-\frac{4w_1}{3}+\frac{4w_2}{3}+4\alpha_1-4\beta_1\right]$$

$$= \frac{3}{4}\left(1-\frac{4w_1}{9}-\frac{4w_2}{9}+\alpha_1+\beta_1\right)$$

$$y = y_o\left[1+\frac{\alpha_1}{y_o^2}+\frac{\beta_1}{y_o^2}\right]^{\frac{1}{2}}$$

or

$$y = y_o\left(1+\frac{\alpha_1}{2y_o^2}+\frac{\beta_1}{2y_o^2}\right) \tag{46}$$

where

$$y_o = \pm\frac{\sqrt{3}}{2}\left(1-\frac{2w_1}{9}-\frac{2w_2}{9}\right) \tag{47}$$

Multiplying the first equation of equation (3.14) by y, the second by $(x+\mu)$ and then the first again by $(x+\mu-1)$ gives the homogeneous system of equation as

$$n^2xy-\frac{(1-\mu)q_1(x+\mu)y}{r_1^3}-\frac{\mu q_2(x+\mu-1)y}{r_2^3}-\frac{3(1-\mu)(x+\mu)A_1y}{2r_1^5}$$

$$-\frac{3\mu(x+\mu-1)A_2y}{2r_2^5}+\frac{nW_1y^2}{r_1^2}+\frac{nW_2y^2}{r_2^2}=0 \tag{48}$$

$$n^2(x+\mu)y-\frac{(1-\mu)q_1(x+\mu)y}{r_1^3}-\frac{\mu q_2(x+\mu)y}{r_2^3}-\frac{3(1-\mu)(x+\mu)A_1y}{2r_1^5}$$

$$-\frac{3\mu(x+\mu)A_2y}{2r_2^5}-\frac{nW_1(x+\mu)^2}{r_1^2}-\frac{nW_2(x+\mu)(x+\mu-1)}{r_2^2}=0 \tag{49}$$

$$n^2(x+\mu-1)y-\frac{(1-\mu)q_1(x+\mu-1)y}{r_1^3}-\frac{\mu q_2(x+\mu-1)y}{r_2^3}-\frac{3(1-\mu)(x+\mu-1)A_1 y}{2r_1^5}$$

$$-\frac{3\mu(x+\mu-1)A_2 y}{2r_2^5}-\frac{nW_1(x+\mu)(x+\mu-1)}{r_1^2}-\frac{nW_2(x+\mu-1)^2}{r_2^2}=0 \qquad (50)$$

By elimination method equations (3.27), (3.28) and (3.29) reduces to

$$n^2\mu y-\frac{\mu q_2 y}{r_2^3}-\frac{3\mu A_2 y}{2r_2^5}-nW_1-nW_2-\frac{nW_2(x+\mu-1)}{r_2^3}=0$$

and

$$n^2(1-\mu)y-\frac{(1-\mu)q_1 y}{r_1^3}-\frac{3(1-\mu)A_1 y}{2r_1^5}+nW_1+nW_2-\frac{nW_1(x+\mu)}{r_1^2}=0$$

or

$$\mu y\left[n^2-\frac{q_2}{r_2^3}-\frac{3A_2}{2r_2^5}\right]=nW_1+\frac{nW_2}{2}+\frac{nW_2}{2r_2^2}(r_1^2-1)$$

$$(1-\mu)y\left[n^2-\frac{q_1}{r_1^3}-\frac{3A_1}{2r_1^5}\right]=\frac{nW_1}{2}-nW_2-\frac{nW_1}{2r_1^2}(r_2^2-1) \qquad (51)$$

since $x+\mu=\dfrac{r_1^2-r_2^2+1}{2}$ and $x+\mu-1=\dfrac{r_1^2-r_2^2-1}{2}$

Using (3.17), (3.22), (3.24) and (3.25) in (3.30) and by considering only first order terms of small quantities $(|w_1|,|w_2|,|A_1|,|A_2|,|W_1|,|W_2|)$, gives

$$\mu y_o\left[1+\frac{3A_1}{2}+\frac{3A_2}{2}-\frac{(1-w_2)}{\left(1-\frac{w_2}{3}+\beta_1\right)^3}-\frac{3A_2}{2\left(1-\frac{w_2}{3}+\beta_1\right)^5}\right]\left[1+\frac{\alpha_1}{2y_o^2}+\frac{\beta_1}{2y_o^2}\right]$$

$$=nW_1+\frac{nW_2}{2}+\frac{nW_2\left[\left(1-\frac{w_1}{3}+\alpha_1\right)^2-1\right]}{2\left(1-\frac{w_2}{3}+\beta_1\right)^2}$$

and

$$(1-\mu)y_o\left[1+\frac{3A_1}{2}+\frac{3A_2}{2}-\frac{(1-w_1)}{\left(1-\frac{w_1}{3}+\alpha_1\right)^3}-\frac{3A_1}{2\left(1-\frac{w_1}{3}+\alpha_1\right)^5}\right]\left[1+\frac{\alpha_1}{2y_o^2}+\frac{\beta_1}{2y_o^2}\right]$$

$$= -n\frac{W_1}{2} - nW_2 - \frac{nW_1\left[\left(1-\frac{w_2}{3}+\beta_1\right)^2 - 1\right]}{2\left(1-\frac{w_1}{3}+\alpha_1\right)^2}$$

Simplifying, , gives

$$3\mu y_o\left[\frac{A_1}{2}+\beta_1\right]\left[1+\frac{\alpha_1}{2y_o^2}+\frac{\beta_1}{2y_o^2}\right] = nW_1 + \frac{nW_2}{2}$$

and,

$$3(1-\mu)y_o\left[\frac{A_2}{2}+\alpha_1\right]\left[1+\frac{\alpha_1}{2y_o^2}+\frac{\beta_1}{2y_o^2}\right] = -nW_2 - \frac{nW_1}{2}$$

which implies,

$$3\mu y_o\left[\frac{A_1}{2}+\beta_1\right] = nW_1 + \frac{nW_2}{2}$$

and

$$3(1-\mu)y_o\left[\frac{A_2}{2}+\alpha_1\right] = -nW_2 - \frac{nW_1}{2}$$

and yields,

$$\beta_1 = -\frac{A_1}{2} + \frac{nW_1}{3\mu y_o} + \frac{nW_2}{3\mu y_o}$$

and

$$\alpha_1 = -\frac{A_2}{2} - \frac{nW_1}{3(1-\mu)y_o} - \frac{nW_2}{3(1-\mu)y_o}$$

Using equation (3.26), α_1 and (β_1) becomes,

$$\alpha_1 = -\frac{A_2}{2} - \frac{nW_1}{3(1-\mu)\sqrt{3}} - \frac{2nW_2}{3(1-\mu)\sqrt{3}}$$

and (52)

$$\beta_1 = -\frac{A_1}{2} + \frac{2nW_1}{3\mu\sqrt{3}} + \frac{nW_2}{3\mu\sqrt{3}}$$

Substituting the values of α_1 and β_1 from equation (3.31) into equations (3.23) and (3.25), then simplifying givess,

$$x = \frac{1}{2} - \mu - \frac{w_1}{3} + \frac{w_2}{3} - \frac{A_2}{2} - \frac{nW_1}{3(1-\mu)\sqrt{3}} - \frac{2nW_2}{3(1-\mu)\sqrt{3}} + \frac{A_1}{2} - \frac{2nW_1}{3\mu\sqrt{3}} - \frac{nW_2}{3\mu\sqrt{3}}$$

and yields,

$$x = \frac{1}{2} - \mu - \frac{w_1}{3} + \frac{w_2}{3} + \frac{A_1}{2} - \frac{A_2}{2} - \frac{nW_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}} - \frac{nW_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}} \quad (53)$$

and

$$y=\pm\frac{\sqrt{3}}{2}\left(1-\frac{2w_1}{9}-\frac{2w_2}{9}\right)$$

$$+\frac{1}{\sqrt{3}}\left[-\frac{A_2}{2}-\frac{nW_1}{3(1-\mu)\sqrt{3}}-\frac{2nW_2}{3(1-\mu)\sqrt{3}}-\frac{A_1}{2}+\frac{2nW_1}{3\mu\sqrt{3}}+\frac{nW_2}{3\mu\sqrt{3}}\right]$$

$$\times\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}\right]^{-1}$$

or

$$y=\pm\frac{\sqrt{3}}{2}\left(1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}\right)+\frac{nW_1(2-3\mu)}{9\mu(1-\mu)}+\frac{nW_2(1-3\mu)}{9\mu(1-\mu)} \tag{54}$$

Equations (3.32) and (3.33) are the coordinates of the triangular points of the RTBP when the primaries are assumed to be oblate (A_1, A_2), radiating (w_1, w_2) with PR-drag (W_1, W_2) effect in the absence of perturbations in the coriolis and centrifugal forces

1.4 Stability of the Triangular Points

To determine the stability of the libration points, (x_*, y_*) is assumed to be the coordinate of the libration point and $|\xi|,|\eta|=1$, the small displacement in the points such that $x=x_*+\xi, y=y_*+\eta.$ is a point in the neighborhood of the libration point.

Therefore, the equations of motion of the systems in equation (3.13) becomes,

$$\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\Omega_x(x_*+\xi,b+\eta)\\ \ddot{\eta}+2n\dot{\xi}&=\Omega_y(y_*+\xi,b+\eta)\end{aligned} \tag{55}$$

and by series expansion, equation (3.34) gives

$$\ddot{\xi}-2n\dot{\eta}=\Omega_x^o+\Omega_{xx}^o\xi+\Omega_{xy}^o\eta+\Omega_{x\dot{x}}^o\dot{\xi}+\Omega_{x\dot{y}}^o\dot{\eta}+0(2)$$

$$\ddot{\eta}+2n\dot{\xi}=\Omega_y^o+\Omega_{yx}^o\xi+\Omega_{yy}^o\eta+\Omega_{y\dot{x}}^o\dot{\xi}+\Omega_{y\dot{y}}^o\dot{\eta}+0(2)$$

where $0(2)$ represents second and higher order terms in ξ and η. The superscript (o) indicates the second order partial derivatives are evaluated at the libration points.

At $\Omega_x=\Omega_y=0$, we have

$$\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\Omega_{xx}^o\xi+\Omega_{xy}^o\eta+\Omega_{x\dot{x}}^o\dot{\xi}+\Omega_{x\dot{y}}^o\dot{\eta}\\ \ddot{\eta}+2n\dot{\xi}&=\Omega_{yx}^o\xi+\Omega_{yy}^o\eta+\Omega_{y\dot{x}}^o\dot{\xi}+\Omega_{y\dot{y}}^o\dot{\eta}\end{aligned} \tag{56}$$

This is the variational equation of motion corresponding to the equations of motion in (3.13), considering only linear terms of η and ξ.

Suppose $\xi=Ae^{\lambda t}$ and $\eta=Be^{\lambda t}$ are the trial solutions to the variational equation. Then equation (3.35) is written as

$$(\lambda^2-\dot{\lambda}\Omega_{x\dot{x}}^o-\Omega_{xx}^o)A+[-(2n+\Omega_{x\dot{y}}^o)\lambda-\Omega_{\dot{x}y}^o]B=0$$

$$[(2n-\Omega^o_{y\dot{x}})\lambda-\Omega^o_{yx}]A+(\lambda^2-\Omega^o_{y\dot{y}}-\Omega^o_{yy})B=0$$

and implies

$$\begin{vmatrix} \lambda^2-\dot{\lambda}\Omega^o_{x\dot{x}}-\Omega^o_{xx} & -(2n+\Omega^o_{x\dot{y}})\lambda-\Omega^o_{\dot{x}y} \\ (2n-\Omega^o_{y\dot{x}})\lambda-\Omega^o_{yx} & \lambda^2-\Omega^o_{y\dot{y}}-\Omega^o_{yy} \end{vmatrix}=0$$

which on solving yields the characteristic equation corresponding to the variational equation of motion in (3.35)

$$\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d=0 \tag{57}$$

where

$$a=-(\Omega^o_{y\dot{y}}+\Omega_{x\dot{x}})$$

$$b=4n^2+\Omega^o_{y\dot{y}}\Omega^o_{x\dot{x}}-\Omega^o_{xx}-\Omega^o_{yy}-(\Omega^o_{x\dot{y}})^2$$

$$c=\Omega^o_{x\dot{x}}\Omega^o_{yy}+\Omega^o_{xx}\Omega^o_{y\dot{y}}+2n\Omega^o_{xy}-2n\Omega^o_{yx}-\Omega^o_{y\dot{x}}\Omega^o_{xy}-\Omega^o_{yx}\Omega^o_{x\dot{y}}$$

$$d=\Omega^o_{xx}\Omega^o_{yy}-\Omega^o_{yx}\Omega^o_{xy}$$

Differentiating equations (3.14) with respect to x, y, $\dot{x}$ and $\dot{y}$, resulted to the second partial derivatives given as

$$\begin{aligned}\Omega_{xx} = \quad & n^2-(1-\mu)q_1\left[\frac{1}{r_1^3}-\frac{3(x+\mu)^2}{r_1^5}\right]-\mu q_2\left[\frac{1}{r_1^3}-\frac{3(x+\mu-1)^2}{r_2^5}-\frac{3(1-\mu)}{2}\right] \\ & -\left[\frac{1}{r_1^5}-\frac{5(x+\mu)^2}{r_1^7}\right]A_1-\frac{3\mu}{2}\left[\frac{1}{r_2^5}-\frac{5(x+\mu-1)^2}{r_2^7}\right]A_2 \\ & -\frac{W_1}{r_1^4}\left\{(x+\mu)\dot{x}\left[2-\frac{4(x+\mu)^2}{r_1^2}\right]+\left[1-\frac{4(x+\mu)^2}{r_1^2}\right]y\dot{y}-2(x+\mu)(\dot{x}-ny)\right\} \\ & -\frac{W_2}{r_2^4}\left\{\left[2-\frac{4(x+\mu)^2}{r_1^2}\right]+(x+\mu-1)\dot{x}+\left[1-\frac{4(x+\mu-1)^2}{r_2^2}\right]y\dot{y}-2(x+\mu-1)(\dot{x}-ny)\right\}\end{aligned} \tag{58}$$

$$\begin{aligned}\Omega_{xy} \quad & =\frac{3(1-\mu)q_1(x+\mu)y}{r_1^5}+\frac{3\mu q_2(x+\mu-1)y}{r_2^5} \\ & +\frac{15(1-\mu)(x+\mu)A_1y}{2r_1^7}+\frac{15\mu(x+\mu-1)A_2y}{2r_2^7} \\ & -\frac{W_1}{r_1^4}\left\{\left[1-\frac{4(x+\mu)y^2}{r_1^2}\right](x+\mu)\dot{x}+\left[2-\frac{4y^2}{r_1^2}\right]y\dot{y}-2[\dot{y}+n(x+\mu)]y\right\} \\ & -\frac{W_2}{r_2^4}\left\{\left[1-\frac{4(x+\mu-1)y^2}{r_1^2}\right](x+\mu-1)\dot{x}+\left[2-\frac{4y^2}{r_1^2}\right]y\dot{y}-2[\dot{y}+n(x+\mu-1)]y\right\}\end{aligned} \tag{59}$$

$$
\begin{aligned}
\Omega_{yx} &= \frac{3(1-\mu)q_1(x+\mu)y}{r_1^5} + \frac{3\mu q_2(x+\mu-1)y}{r_2^5} \\
&+ \frac{15(1-\mu)(x+\mu)A_1 y}{2r_1^7} + \frac{15\mu(x+\mu-1)A_2 y}{2r_2^7} \\
&- W_1\left\{\left[\frac{1}{r_1^4} - \frac{4(x+\mu)^2}{r_1^6}\right]\dot{x}y - 4\left[\frac{(x+\mu)y^2\dot{y}}{r_1^6}\right] - \frac{2(x+\mu)\dot{y}}{r_1^4} + n\left[\frac{1}{r_1^2} - \frac{2(x+\mu)^2}{r_1^4}\right]\right\} \\
&- W_2\left\{\left[\frac{1}{r_2^4} - \frac{4(x+\mu-1)^2}{r_2^6}\right]x\dot{y} - \frac{4(x+\mu-1)\dot{y}y^2}{r_2^6} - \frac{2(x+\mu-1)\dot{y}}{r_2^4}\right. \\
&\left. + n\left[\frac{1}{r_2} - \frac{2(x+\mu-1)}{r_2^4}\right]\right\}
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\Omega_{yy} &= n^2 - (1-\mu)q_1\left[\frac{1}{r_1^3} - \frac{3y^2}{r_1^5}\right] - \mu q_2\left[\frac{1}{r_2^3} - \frac{3y^2}{r_2^5}\right] \\
&- \frac{3(1-\mu)}{2}\left[\frac{1}{r_1^5} - \frac{5y^2}{r_1^7}\right]A_1 - \frac{3\mu}{2}\left[\frac{1}{r_2^5} - \frac{5y^2}{r_2^7}\right]A_2 \\
&- W_1\left\{\left[\frac{1}{r_1^4} - \frac{4y^2}{r_1^6}\right](x+\mu)\dot{x} + \left[\frac{2}{r_1^4} - \frac{4y^2}{r_1^6}\right]y\dot{y} - 2\left[\frac{\dot{y}+n(x+\mu)}{r_1^4}\right]y\right\} \\
&- W_2\left\{\left[\frac{1}{r_2^4} - \frac{4y^2}{r_2^6}\right](x+\mu-1)\dot{x} + \left[\frac{2}{r_2^4} - \frac{4y^2}{r_2^6}\right]y\dot{y} - 2\left[\frac{\dot{y}+n(x+\mu-1)}{r_2^4}\right]y\right\}
\end{aligned} \tag{61}
$$

$$
\left.
\begin{aligned}
\Omega_{x\dot{x}} &= -W_1\left[\frac{(x+\mu)^2}{r_1^4} + \frac{1}{r_1^2}\right] - W_2\left[\frac{(x+\mu-1)^2}{r_2^4} + \frac{1}{r_2^2}\right] \\
\Omega_{x\dot{y}} &= -W_1\left[\frac{(x+\mu)}{r_1^4}y\right] - W_2\left[\frac{(x+\mu-1)}{r_2^4}y\right] \\
\Omega_{y\dot{x}} &= -\frac{W_1}{r_1^4}(x+\mu)y - \frac{W_2}{r_2^4}(x+\mu-1)y \\
\Omega_{y\dot{y}} &= -W_1\left[\frac{y^2}{r_1^4} + \frac{1}{r_1^2}\right] - W_2\left[\frac{y^2}{r_2^4} - \frac{1}{r_2^2}\right]
\end{aligned}
\right\} \tag{62}
$$

Evaluating the second partial derivatives in equations equations (3.37) - (3.41) at the libration points obtained in equations (3.32) and (3.33). Also using equations (3.17), (3.22) and (3.31) (neglecting second and higher order terms of small quantities) so that

$$
r_1 = 1 - \frac{w_1}{3} - \frac{A_2}{2} - \frac{nW_1}{3(1-\mu)\sqrt{3}} - \frac{2nW_2}{3(1-\mu)\sqrt{3}}
$$

$$
r_2 = 1 - \frac{w_2}{3} - \frac{A_1}{2} + \frac{2nW_1}{3\mu\sqrt{3}} + \frac{nW_2}{3\mu\sqrt{3}}
$$

$$x+\mu=\frac{1}{2}\left[1-\frac{2}{3}w_1+\frac{2w_2}{3}+A_1-A_2-\frac{2nW_1(2-\mu)}{3\mu(1-\mu)}-\frac{2nW_2(1+\mu)}{3\mu(1-\mu)}\right]$$

$$x+\mu-1=-\frac{1}{2}\left[1+\frac{2}{3}w_1-\frac{2w_2}{3}-A_1+A_2+\frac{2W_1(2-\mu)}{3\mu(1-\mu)}+\frac{2W_2(1+\mu)}{3\mu(1-\mu)}\right]$$

$$y=\pm\frac{\sqrt{3}}{2}\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{2W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$n=\left[1+\frac{3A_1}{2}+\frac{3A_2}{2}\right]^{\frac{1}{2}}=1+\frac{3A_1}{4}+\frac{3A_2}{4}$$

yields,

$$\Omega_{xx}^{o}=1+\frac{3A_1}{2}+\frac{3A_2}{2}-(1-\mu)(1-w_1)\left\{\left[1-w_1-\frac{A_2}{2}-\frac{nW_1}{3(1-\mu)\sqrt{3}}-\frac{2nW_2}{3(1-\mu)\sqrt{3}}\right]^{-3}\right.$$

$$-\frac{3}{4}\left[1-\frac{2}{3}w_1+\frac{2}{3}w_2+A_1-A_2-\frac{2nW_1(2-\mu)}{3\mu(1-\mu)}-\frac{2nW_2(1+\mu)}{3\mu(1-\mu)}\right]^2$$

$$\left.\left[1-\frac{w_1}{3}-\frac{A_2}{2}-\frac{nW_1}{3(1-\mu)\sqrt{3}}-\frac{2nW_2}{3\sqrt{3}(1-\mu)}\right]^{-5}\right\}$$

$$-\mu(1-w_2)\left\{\left[1-\frac{w_2}{2}-\frac{A_1}{2}+\frac{2nW_1}{3\mu\sqrt{3}}+\frac{nW_2}{3\mu\sqrt{3}}\right]^{-3}\right.$$

$$-\frac{3}{4}\left[1+\frac{2}{3}w_1-\frac{2}{3}w_2-A_1+A_2+\frac{2W_1(2-\mu)}{3\mu(1-\mu)}-\frac{2W_2(1+\mu)}{3\mu(1-\mu)}\right]^2$$

$$\left.\left[1-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2W_1}{3\mu\sqrt{3}}+\frac{nW_2}{3\sqrt{3}\mu}\right]^{-5}\right\}$$

$$-\frac{3(1-\mu)}{2}\left\{\left[1-w_1-\frac{A_2}{2}-\frac{W_1}{3(1-\mu)\sqrt{3}}-\frac{2W_2}{3(1-\mu)\sqrt{3}}\right]^{-5}\right.$$

$$-\frac{5}{4}\left[1-\frac{2}{3}w_1-\frac{2}{3}B_2+A_1-A_2-\frac{2W_1(2-\mu)}{3\mu(1-\mu)}-\frac{2W_2(1+\mu)}{3\mu(1-\mu)}\right]^2$$

$$\left.\left[1-\frac{w_1}{3}-\frac{A_2}{2}-\frac{W_1}{3(1-\mu)\sqrt{3}}-\frac{2W_2}{3(1-\mu)\sqrt{3}}\right]^{-7}\right\}A_1$$

$$-\frac{3\mu}{2}\left\{\left[1-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2W_1}{3\mu\sqrt{3}}+\frac{2nW_2}{3\mu\sqrt{3}}\right]^{-5}\right.$$

$$-\frac{5}{4}\left[1+\frac{2}{3}w_1-\frac{2}{3}w_2-A_1+A_2+\frac{2nW_1(2-\mu)}{3\mu(1-\mu)}+\frac{2nW_2(1+\mu)}{3\mu(1-\mu)}\right]^2$$

$$\left.\left[1-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2nW_1}{3\mu\sqrt{3}}+\frac{nW_2}{3\mu\sqrt{3}}\right]^{-7}\right\}A_2-\frac{\sqrt{3}}{2}W_1+\frac{\sqrt{3}}{2}W_2$$

$$\Omega_{xx}^{o}=1+\frac{3A_1}{2}+\frac{3A_2}{2}-(1-\mu)(1-w_1)\left\{1+w_1+\frac{3A_2}{2}+\frac{nW_1}{(1-\mu)\sqrt{3}}+\frac{2nW_2}{(1-\mu)\sqrt{3}}\right.$$

$$\left.-\frac{3}{4}\left[1+\frac{1}{3}w_1+\frac{4}{3}w_2+2A_1+\frac{A_2}{2}-\frac{W_1(8-9\mu)}{3\mu(1-\mu)\sqrt{3}}+\frac{W_2(4-6\mu)}{3\mu(1-\mu)\sqrt{3}}\right]\right\}$$

$$-\mu(1-w_2)\left\{1+w_2+\frac{3A_1}{2}+\frac{2W_1}{\mu\sqrt{3}}+\frac{W_2}{\mu\sqrt{3}}\right.$$

$$-\frac{3}{4}\left[1+\frac{4}{3}w_1+\frac{1}{3}w_2+\frac{A_1}{2}+2A_2+\frac{nW_1(2-6\mu)}{3\mu(1-\mu)\sqrt{3}}-\frac{nW_2(1-9\mu)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$\left.+\frac{3(1-\mu)A_1}{8}+\frac{3\mu A_2}{8}-\frac{\sqrt{3}}{2}nW_1+\frac{\sqrt{3}}{2}W_2n\right\}$$

$$\Omega_{xx}^{o}=\frac{3}{4}-\left(\frac{1}{2}-\frac{3\mu}{2}\right)w_1+\left(1-\frac{3}{2}\mu\right)w_2+\left(\frac{27}{8}-\frac{24\mu}{8}\right)A_1$$

$$+\left(\frac{3}{8}+\frac{24\mu}{8}\right)A_2-\frac{W_1(8-13\mu+\mu^2)}{4\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-11\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}} \tag{63}$$

$$\Omega_{xy}^{o}=\frac{3\sqrt{3}}{4}(1-\mu)(1-w_1)\left[1-\frac{2w_1}{3}+\frac{w_2}{3}+A_1-A_2-\frac{2nW_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}}-\frac{2W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{5w_1}{3}+\frac{5A_2}{2}+\frac{5nW_1}{3(1-\mu)\sqrt{3}}+\frac{10nW_2}{3(1-\mu)\sqrt{3}}\right]$$

$$\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2nW_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{2nW_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$-\frac{3\sqrt{3}}{4}\mu(1-w_2)\left[1+\frac{2w_1}{3}-\frac{2w_2}{3}-A_1+A_2+\frac{2nW_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}}+\frac{2nW_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{5w_2}{3}+\frac{5A_1}{2}-\frac{10W_1}{3\mu\sqrt{3}}-\frac{5W_2}{3\mu\sqrt{3}}\right]$$

$$\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2nW_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{2nW_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$+\frac{15\sqrt{3}}{8}(1-\mu)\left[1-\frac{2w_1}{3}+\frac{2w_2}{3}+A_1-A_2-\frac{2nW_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}}-\frac{2W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{7w_1}{3}+\frac{7A_2}{2}+\frac{7nW_1}{3(1-\mu)\sqrt{3}}+\frac{14nW_2}{3(1-\mu)\sqrt{3}}\right]$$

$$\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{2W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]A_1$$

$$-\frac{15\sqrt{3}}{8}\mu\left[1+\frac{2w_1}{3}-\frac{2w_2}{3}-A_1+A_2+\frac{2W_1(2-\mu)}{3\mu(1-\mu)\sqrt{3}}+\frac{2W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{7w_2}{3}+\frac{7A_1}{2}-\frac{14W_1}{3\mu\sqrt{3}}-\frac{7W_2}{3\mu\sqrt{3}}\right]$$

$$\left[1-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2nW_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{2W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]A_2-\frac{1}{2}W_1-\frac{W_2}{2}$$

$$\Omega_{xy}^{o}=\frac{3\sqrt{3}}{4}(1-\mu)\left[1-\frac{2w_1}{9}+\frac{4w_2}{9}+\frac{2}{3}A_1+\frac{7}{6}A_2-\frac{W_1(8-15\mu)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-18\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$-\frac{3\sqrt{3}}{4}\mu\left[1+\frac{4}{9}w_1-\frac{2}{9}w_2+\frac{7}{6}A_1+\frac{2}{3}A_2-\frac{W_1(14-18\mu)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(7-15\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$+\frac{15}{8}\sqrt{3}(1-\mu)A_1-\frac{15}{8}\sqrt{3}\mu A_2-\frac{W_1}{2}-\frac{W_2}{2}$$

$$\Omega_{xy}^{o}=\frac{\sqrt{3}}{4}\left[3-6\mu-\frac{2}{3}(1+\mu)w_1+\frac{2}{3}(2-\mu)w_2+\frac{1}{2}(19-26\mu)A_1+\frac{1}{2}(7-26\mu)A_2\right]$$

$$-\frac{W_1(8-31\mu+27\mu^2)}{12\mu(1-\mu)}-\frac{W_2(4-23\mu+27\mu^2)}{12\mu(1-\mu)}=\Omega_{yx}^{o} \qquad (64)$$

$$\Omega_{yy}^{o}=1+\frac{3}{2}A_1+\frac{3}{2}A_2-(1-\mu)(1-w_1)\left[1+w_1+\frac{3}{2}A_2+\frac{nW_1}{(1-\mu)\sqrt{3}}+\frac{2W_2}{(1-\mu)\sqrt{3}}\right]$$

$$-\frac{9}{4}\left[1-\frac{4w_1}{9}-\frac{4w_2}{9}-\frac{2A_1}{3}-\frac{2A_2}{3}+\frac{4W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{4W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{5w_1}{3}+\frac{5A_2}{2}+\frac{5nW_1}{3(1-\mu)\sqrt{3}}+\frac{10W_2}{3(1-\mu)\sqrt{3}}\right]$$

$$-\mu(1-w_2)\left[1+w_2+\frac{3}{2}A_1-\frac{2nW_1}{\mu\sqrt{3}}-\frac{nW_2}{\mu\sqrt{3}}\right]$$

$$-\frac{9}{4}\left[1-\frac{4w_1}{9}-\frac{4w_2}{9}-\frac{2A_1}{3}-\frac{2A_2}{3}+\frac{4nW_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{4W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{5w_2}{3}+\frac{5A_1}{2}-\frac{10W_1}{3\mu\sqrt{3}}-\frac{5W_2}{3\mu\sqrt{3}}\right]$$

$$-\frac{3}{2}(1-\mu)\left[1+\frac{5w_1}{3}+\frac{5A_2}{2}+\frac{5W_1}{3(1-\mu)\sqrt{3}}+\frac{10W_2}{3(1-\mu)\sqrt{3}}\right]$$

$$-\frac{15}{4}\left[1-\frac{4w_1}{9}-\frac{4w_2}{9}-\frac{2A_1}{3}-\frac{2A_2}{3}+\frac{4W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{4W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{7w_1}{3}+\frac{7A_2}{2}+\frac{7W_1}{3(1-\mu)\sqrt{3}}+\frac{14W_2}{3(1-\mu)\sqrt{3}}\right]A_1$$

$$+\frac{3}{2}\mu\left[1+\frac{5w_2}{3}+\frac{5A_1}{2}-\frac{10W_1}{3\mu\sqrt{3}}-\frac{5W_2}{3\mu\sqrt{3}}\right]$$

$$-\frac{15}{4}\left[1-\frac{4w_1}{9}-\frac{4\beta_2}{9}-\frac{2A_1}{3}-\frac{2A_2}{3}+\frac{4W_1(2-3\mu)}{9\mu(1-\mu)\sqrt{3}}+\frac{4W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]$$

$$\left[1+\frac{7w_1}{3}+\frac{7A_1}{2}-\frac{14nW_1}{3\mu\sqrt{3}}-\frac{7W_2}{3\mu\sqrt{3}}\right]A_2+\frac{\sqrt{3}}{2}W_1-\frac{\sqrt{3}}{2}W_2$$

$$\Omega^o_{yy}=1+\frac{3}{2}A_1+\frac{3}{2}A_2+\frac{5}{4}(1-\mu)\left[1+\frac{2}{5}w_1-\frac{4}{5}w_2-\frac{6}{5}A_1+\frac{21}{10}A_2\right.$$
$$\left.+\frac{W_1(8+\mu)}{5\mu(1-\mu)\sqrt{3}}+\frac{W_2(4+10\mu)}{5\mu(1-\mu)\sqrt{3}}\right]+\frac{5}{4}\mu\left[1-\frac{4}{5}w_1+\frac{2}{5}w_2+\frac{21}{10}A_1-\frac{6}{5}A_2\right.$$
$$\left.-\frac{W_1(14-10\mu)}{5\mu(1-\mu)\sqrt{3}}-\frac{W_2(7+\mu)}{5\mu(1-\mu)\sqrt{3}}\right]+\frac{33}{8}(1-\mu)A_1-\frac{33}{8}\mu A_2+\frac{\sqrt{3}}{2}W_1-\frac{\sqrt{3}}{2}W_2$$

$$\Omega^o_{yy}=\frac{9}{4}+\frac{1}{2}(1-3\mu)w_1-\frac{1}{2}(2-3\mu)w_2+\frac{33}{8}A_1+\frac{33}{8}A_2$$
$$+\frac{W_1(8-17\mu+5\mu^2)}{4\mu(1-\mu)\sqrt{3}}+\frac{W_2(4-7\mu-5\mu^2)}{4\mu(1-\mu)\sqrt{3}} \quad (65)$$

$$\left.\begin{aligned}\Omega^o_{x\dot{x}} &= -\frac{5}{4}W_1-\frac{5}{4}W_2\\ \Omega^o_{x\dot{y}} &= -\frac{\sqrt{3}}{4}W_1-\frac{\sqrt{3}}{4}W_2=\Omega^o_{y\dot{x}}\\ \Omega^o_{y\dot{y}} &= -\frac{7}{4}W_1-\frac{7}{4}W_2\end{aligned}\right\} \quad (66)$$

Therefore, substituting these values from equations (3.42) to (3.45) in equation (3.35) , neglecting second and higher terms of small quantities, yields,

$$a = -\left(-\frac{5}{4}W_1 - \frac{5}{4}W_2 - \frac{7}{4}W_1 - \frac{7}{4}W_2\right) = 3W_1 + 3W_2 \tag{67}$$

$$b = 4 + 6A_1 + 6A_2 + \left(\frac{\sqrt{3}}{4}W_1 + \frac{\sqrt{3}}{4}W_2\right)\left(-\frac{7}{4}W_1 - \frac{7}{4}W_2\right)$$

$$-\left[\frac{3}{4} - \frac{1}{2}(1-3\mu)w_1 + \frac{1}{2}(2-3\mu)w_2 + \left(\frac{27}{8} - \frac{24\mu}{8}\right)A_1 + \left(\frac{3}{8} + \frac{24\mu}{8}\right)A_2\right.$$

$$\left. - \frac{W_1(8-13\mu+\mu^2)}{4\mu(1-\mu)\sqrt{3}} - \frac{W_2(4-11\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}}\right]$$

$$-\left[\frac{9}{4} + \frac{1}{2}(1-3\mu)w_1 - \frac{1}{2}(2-3\mu)w_2 + \frac{33}{8}A_1 + \frac{33}{8}A_2 + \right.$$

$$\left.\frac{W_1(8-17\mu+5\mu^2)}{4\mu(1-\mu)\sqrt{3}} + \frac{W_2(4-7\mu-5\mu^2)}{4\mu(1-\mu)\sqrt{3}} - \left(\frac{\sqrt{3}}{4}W_1 + \frac{\sqrt{3}}{4}W_2\right)^2\right]$$

$$= 1 - \left(\frac{3}{2} - 3\mu\right)A_1 + \left(\frac{3}{2} - 3\mu\right)A_2 + \frac{W_1}{\sqrt{3}} - \frac{W_2}{\sqrt{3}}$$

$$b = b_o + b_1 \tag{68}$$

where’

$$b_o = 1,\; b_1 = -\left(\frac{3}{2} - 3\mu\right)A_1 + \left(\frac{3}{2} - 3\mu\right)A_2 + \frac{W_1}{\sqrt{3}} - \frac{W_2}{\sqrt{3}}$$

$$c = \left[\frac{9}{4} + \frac{1}{2}(1-3\mu)w_1 - \frac{1}{2}(2-3\mu)w_2 + \frac{33A_1}{8} + \frac{33A_2}{8} + \frac{W_1(8-15\mu+3\mu^2)}{4\mu(1-\mu)\sqrt{3}} + \right.$$

$$\left.\frac{W_2(4-7\mu-5\mu^2)}{4\mu(1-\mu)\sqrt{3}}\right]\left[-\frac{5}{4}W_1 - \frac{5}{4}W_2\right]$$

$$+\left[\frac{3}{4} - \frac{1}{2}(1-3\mu)w_1 + \frac{1}{2}(2-3\mu)w_2 + \left(\frac{27}{8} - \frac{24\mu}{8}\right)A_1 + \left(\frac{3}{8} - \frac{24\mu}{8}\right)A_2\right.$$

$$\left. - \frac{W_1(8-13\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}} - \frac{W_2(4-11\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}}\right]\left[\frac{-7}{4}W_1 - \frac{7}{4}W_2\right]$$

$$-\frac{2\sqrt{3}}{4}\left[\frac{-\sqrt{3}}{4}W_1+\frac{\sqrt{3}}{4}W_2\right]\left[3-6\mu-\frac{2}{3}(1+\mu)w_1+\frac{2}{3}(2-\mu)w_2+\frac{1}{2}(19-26\mu)A_1\right.$$

$$\left.+\frac{1}{2}(7-26\mu)A_2-\frac{W_1(8-13\mu+27\mu^2)}{3\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-23\mu+27\mu^2)}{3\mu(1-\mu)\sqrt{3}}\right]$$

$$-\frac{45}{16}W_1-\frac{45}{16}W_2-\frac{21}{16}W_1-\frac{21}{16}W_2+\left(\frac{3}{8}W_1-\frac{3}{8}W_2\right)(3-6\mu)$$

$$=-\left(3+\frac{9}{4}\mu\right)W_1-\left(\frac{21}{4}-\frac{9\mu}{4}\right)W_2 \tag{69}$$

$$\begin{aligned}d\ &=\frac{27}{16}\left[1-\frac{4}{9}(1-3\mu)w_1+\frac{4}{9}(2-3\mu)w_2+\left(\frac{19}{3}-4\mu\right)A_1+\left(\frac{7}{3}+4\mu\right)A_2\right.\\&\left.-\frac{W_1(16-22\mu-2\mu^2)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(8-26\mu+2\mu^2)}{9\mu(1-\mu)\sqrt{3}}\right]-\frac{3}{16}\left[9-36\mu+36\mu^2\right.\\&-\frac{4}{3}(3-3\mu-6\mu^2)w_1+\frac{4}{3}(6-15\mu+6\mu^2)w_2+(57-192\mu+156\mu^2)A_1\\&+(21-120\mu+156\mu^2)A_2-\frac{2W_1(24-141\mu+267\mu^2-162\mu^3)}{3\mu(1-\mu)\sqrt{3}}\\&\left.-\frac{2W_2(12-93\mu+219\mu^2-162\mu^3)}{3\mu(1-\mu)\sqrt{3}}\right]\\&=\frac{27}{4}\mu(1-\mu)+\frac{3}{2}\mu(1-\mu)w_1+\frac{3}{2}\mu(1-\mu)w_2+\frac{117\mu}{4}(1-\mu)A_1+\frac{117\mu}{4}(1-\mu)A_2\\&-\frac{W_1(210\mu-584\mu^2+324\mu^3)}{16\mu(1-\mu)\sqrt{3}}-\frac{W_2(108\mu+432\mu^2+324\mu^3)}{16\mu(1-\mu)\sqrt{3}}\end{aligned}$$

$$d=\frac{27}{4}\mu(1-\mu)+\frac{3}{2}\mu(1-\mu)w_1+\frac{3}{2}\mu(1-\mu)w_2+\frac{117\mu}{4}(1-\mu)A_1$$

$$+\frac{117}{4}\mu(1-\mu)A_2-\frac{27W_1(2-3\mu)}{4\sqrt{3}}-\frac{27W_2(1-3\mu)}{4\sqrt{3}} \tag{70}$$

Now, the four roots of the characteristic equation of the classical RTBP is

$$\lambda_n=\pm zi \quad (n=1,2,3,4) \tag{71}$$

where'

$$z^2=\frac{1}{2}\left\{1\pm\left[1-27\mu(1-\mu)\right]^{\frac{1}{2}}\right\} \quad \text{(Szebehely, 1967)} \tag{72}$$

Assuming that, due to oblateness and PR drag

$$\lambda=\lambda_n(1+\sigma_1+i\sigma_2)$$

is a solution for the equation (3.36), where σ_1, σ_2 are small real quantities.

Using equation (3.50), neglecting second and higher order terms of small quantities, equation (3.52) gives

$$\lambda = \pm(1+\sigma_1+i\sigma_2)zi = \pm[-\sigma_2+(1+\sigma_1)i]z \tag{73}$$

$$\lambda^2 = [-2(1+\sigma_1)ni-(1+\sigma_1)^2]z^2 = [-(1+2\sigma_1)-2\sigma_2 i]z^2$$

$$\lambda^3 = \pm\{3\sigma_2-(1+3\sigma_1)i\}z^3$$

$$\lambda^4 = [(1+4\sigma_1)+4\sigma_2 i]z^4$$

Putting these in equation (3.36) considering only first order terms of small quantities yields,

$$\begin{aligned}&[(1+4\sigma_1)+4\sigma_2 i]z^4 \pm a[3\sigma_2-(1+3\sigma_1)i]z^3 - b[(1+2\sigma_1)-2\sigma_2 i]z^2\\ &\pm c[-n+(1+\sigma_1)i]z+d=0.\end{aligned}$$

This implies

$$\begin{aligned}&(1+4\sigma_1)z^4 \pm 3a\sigma_2 z^3 - b(1+2\sigma_1)z^2 \pm c\sigma_2 z + d = 0\\ &4\sigma_2 z^4 \pm a(1+3\sigma_1)z^3 + 2\sigma_2 bz^2 \pm c(1+\sigma_1)z = 0\end{aligned} \tag{74}$$

since $b = b_o + b_1$, a, b_1, contain only components of small quantities

$$[(1+4\sigma_1)z^4 - b_o(1+2\sigma_1)z^2 + d] = 0$$

$$4\sigma_2 z^4 \pm az^3 + 2\sigma_2 b_o z^2 \pm cz = 0$$

From which,

$$\sigma_1 = -\frac{(z^4-b_o z^2+d)}{4z^4-2z^2 b_o} \tag{75}$$

and

$$\sigma_2 = \frac{\pm az^3 \mp cz}{4z^4+2b_o z^2} \tag{76}$$

where the values of a, b_o, c, d and z are given in equations $(3.46)-(3.49)$ and (3.51) respectively.

The motion around the triangular libration points is asymptotically stable only if $\sigma_2 \neq 0$ and the real part, $Re(\lambda)$ of the root are all negative.

Now, the real part of equation (3.52) using equation (3.55) is,

$$Re(\lambda) = \pm\sigma_2 z = \frac{c-az^2}{2(2z^2-b_o)}) \tag{77}$$

from (3.51)

$$z^2 = \frac{1}{2}\left\{1 \pm \left[1 \pm \left(\frac{27}{2}\mu(1-\mu)\right) - \frac{1}{24}(27\mu(1-\mu))^2 + \ldots\right]\right\}$$

Considering only first order term of the quantity μ,

$$z^2 = \frac{1}{2}\left\{1 \pm \left[1 - \frac{27\mu}{2}(1-\mu)\right]\right\} \tag{78}$$

so taking positive sign and $Re(\lambda) < 0$ then by equations (3.56) and (3.57),

$$Re(\lambda) = \left\{c - a\left[1 - \frac{27\mu(1-\mu)}{4}\right]\right\}\left\{2 - \frac{27\mu(1-\mu)}{2} - b_o\right\}^{-1} < 0$$

which implies

$$\left\{c - a\left[1 - \frac{27\mu(1-\mu)}{4}\right]\right\}\left\{1 + \frac{27\mu(1-\mu)}{2}\right\} < 0 \text{ since } b_o = 1$$

or

$$c\left[1 + \frac{27}{2}\mu(1-\mu)\right] - a\left[1 - \frac{27}{4}\mu(1-\mu) + \frac{27}{2}\mu(1-\mu)\right] < 0$$

and yields

$$c + \frac{27}{4}\mu(1-\mu)(2c-a) < a \tag{79}$$

On the other hand, taking negative sign

$$Re(\lambda) = \left[-c + \frac{a27}{4}\mu(1-\mu)\right]\left[1 - \frac{27}{2}\mu(1-\mu)\right]^{-1} < 0$$

which implies

$$-\left[c - \frac{a27}{4}\mu(1-\mu)\right]\left[1 + \frac{27}{2}\mu(1-\mu)\right] < 0$$

or

$$-\left[c + \frac{27}{4}\mu(1-\mu)(2c-a)\right] < 0$$

and yields

$$0 < c + \frac{27}{4}\mu(1-\mu)(2c-a) \tag{80}$$

from equation (3.58) and (3.59),

$$0 < c + \frac{27}{4}\mu(1-\mu)(2c-a) < a$$

as $\mu \to 0$

$$0 < c < a \qquad (Murray, 1994) \tag{81}$$

This inequality, according to Murray (1994) is the condition necessary for the stability of triangular libration points at L_4 and L_5.

2 Effects of Coriolis and Centrifugal Forces on the Stability of Generalised Photo-gravitational Restricted Three-Body Problem

2.1 The Equations of Motion

Introducing the parameters ε and ε' into the equations of motion obtained in chapter three, to represent small perturbations in the coriolis and centrifugal forces, using the parameter ϕ and ψ respectively such that

$$\begin{aligned} \phi &= 1+\varepsilon \quad |\varepsilon|=1 \\ \psi &= 1+\varepsilon' \quad |\varepsilon'|=1 \end{aligned} \tag{82}$$

The equations of motion (3.13) now becomes,

$$\begin{aligned} \ddot{x}-2n\phi\dot{y} &= n^2\psi x+\overline{U}_x \\ \ddot{y}+2n\phi\dot{x} &= n^2\psi y+\overline{U}_y \end{aligned}$$

where,

$$\begin{aligned} \overline{U}_x = &-\frac{(1-\mu)(x+\mu)q_1}{r_1^3}-\frac{\mu(x+\mu-1)q_2}{r_2^3} \\ &-\frac{3(1-\mu)(x+\mu)A_1}{2r_1^5}-\frac{3\mu(x+\mu-1)A_2}{2r_2^5} \\ &-\frac{W_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[\dot{x}(x+\mu)+\dot{y}y]+\dot{x}-ny\right\} \\ &-\frac{W_2}{r_2^2}\left\{\frac{(x+\mu-1)}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y]+\dot{x}-ny\right\} \end{aligned}$$

$$\begin{aligned} \overline{U}_y = &\left\{\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu q_2}{r_2^3}+\frac{3(1-\mu)A_1}{2r_1^5}+\frac{3\mu A_2}{2r_2^5}\right\}y \\ &-\frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[(x+\mu)\dot{x}+\dot{y}y]+\dot{y}+n(x+\mu)\right\} \\ &-\frac{W_2}{r_2^2}\left\{\frac{y}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y]+\dot{y}+n(x+\mu-1)\right\}(Jaiyeola\ et\ al, 2016) \end{aligned}$$

which is re-written as,

$$\begin{aligned} \ddot{x}-2n\phi\dot{y} &= \Omega_x = \Omega_x^*+F_{PRx} \\ \ddot{y}+2n\phi\dot{x} &= \Omega_y = \Omega_y^*+F_{PRy} \end{aligned} \tag{83}$$

where,

$$\Omega^* = \frac{n^2\psi}{2}(x^2+y^2)+\frac{(1-\mu)q_1}{r_1}+\frac{\mu q_2}{r_2}+\frac{(1-\mu)A_1}{2r_1^3}+\frac{\mu A_2}{2r_2^3} \tag{84}$$

$$\left.\begin{aligned}F_{PRx} &= -\frac{W_1}{r_1^2}\left\{\frac{(x+\mu)}{r_1^2}[\dot{x}(x+\mu)+\dot{y}y]+\dot{x}-ny\right\}\\ &\quad -\frac{W_2}{r_2^2}\left\{\frac{(x+\mu-1)}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y]+\dot{x}-ny\right\}\\ F_{PRy} &= -\frac{W_1}{r_1^2}\left\{\frac{y}{r_1^2}[(x+\mu)\dot{x}+\dot{y}y]+\dot{y}+n(x+\mu)\right\}\\ &\quad -\frac{W_2}{r_2^2}\left\{\frac{y}{r_2^2}[\dot{x}(x+\mu-1)+\dot{y}y]+\dot{y}+n(x+\mu-1)]\right\}\end{aligned}\right\} \tag{85}$$

F_{PRx} and F_{PRy} are the partial derivatives of the PR-drag function with respect to x and y respectively. These are functions of the position and velocity.

The equations of motion (3.63) and (3.64), which are modifications of those obtained in equation (3.13) shows the presence the parameter for small perturbations in the coriolis (ϕ) and centrifugal (ψ) forces.

2.2 The Jacobi Integral

To obtain the Jacobi Integral for the equations of motion obtained above, the first equation of (3.62) is multiplied by $2\dot{x}$, the second by $2\dot{y}$ and then added up to give,

$$2\dot{x}\ddot{x}+2\dot{y}\ddot{y}=2(\dot{x}\Omega_x^*)+2(\dot{x}F_{PRx}+\dot{y}F_{PRy})$$

which implies,

$$\frac{d}{dt}(\dot{x}^2+\dot{y}^2)=2\frac{\partial\Omega^*}{\partial t}+2(\dot{x}F_{PRx}+\dot{y}F_{PRy})$$

F_{PRx} and F_{PRy} are given in equations (3.64) and (3.55) above

Integrating this with respect to time t, yields

$$\dot{x}^2+\dot{y}^2=2\Omega^*+2\int(\dot{x}F_{PRx}+\dot{y}F_{PRy})dt-C$$

where the left hand-side is the square of the velocity of the infinitesimal body which cannot be negative and C is the constant of integration known as the Jacobi integral. The motion of the body is restricted to the region where

$$v^2=2\Omega^*+2\int(\dot{x}F_{PRx}+\dot{y}F_{PRy})dt-C\geq 0 \tag{86}$$

This condition determines the region where motion would take place and not the shape of the orbit. The equation of the Zero Velocity Curves (ZVC) are given by

$$C=2\Omega^*(x,y) \tag{87}$$

The curve C represent various regions of possible motion.

2.3 Location of The Triangular Libration Points

The triangular libration points are the solutions of equations $\Omega_x=\Omega_y=0$ when $\dot{x}=\dot{y}=\ddot{x}=\ddot{y}=0$ and $y\neq 0$. From equations (3.62) - (3.64)

$$\left.\begin{aligned}
\Omega_x &= \Omega_x^* + F_{PRx} \\
&= n^2\psi x - \frac{(1-\mu)(x+\mu)q_1}{r_1^3} - \frac{\mu(x+\mu-1)q_2}{r_2^3} \\
&\quad - \frac{3(1-\mu)(x+\mu)A_1}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2}{2r_2^5} + \frac{nW_1 y}{r_1^2} + \frac{nW_2 y}{r_2^2} = 0 \\
\Omega_y &= \Omega_y^* + F_{PRy} \\
&= [n^2\psi - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} - \frac{3(1-\mu)A_1}{2r_1^5} - \frac{3\mu A_2}{2r_2^3}]y \\
&\quad - \frac{nW_1(x+\mu)}{r_1^2} - \frac{nW_1(x+\mu-1)}{r_2^2} = 0
\end{aligned}\right\} \tag{88}$$

In the absence of the radiation, oblateness and PR-drag (i.e. $n = q_1 = q_2 = 1, W_1 = W_2 = A_1 = A_2 = 0$) equations (3.68) gives,

$$r_1 = r_2 = \frac{1}{\psi^{\frac{1}{3}}}$$

Now, assuming that due to the presence of radiation, oblateness and PR-drag,

$$r_1 = \frac{1}{\psi^{\frac{1}{3}}}(1+\alpha_2) \textit{ and } r_2 = \frac{1}{\psi^{\frac{1}{3}}}(1+\beta_2), \quad |\alpha_2| = 1, \ |\beta_2| = 1 \tag{89}$$

Substituting equation (3.68) in (3.15) and solving simultaneously, neglecting second and higher order terms of small quantities, the equation of the coordinates, x and y in terms α and β is given as,

$$\left.\begin{aligned}
x &= \frac{1}{2} - \mu + \frac{1}{\psi^{\frac{1}{3}}}(\alpha_2 - \beta_2) \\
y &= \frac{\sqrt{4-\psi^{\frac{2}{3}}}}{2\psi^{\frac{1}{3}}}\left[1 + \frac{2\psi^{\frac{1}{3}}}{4-\psi^{\frac{2}{3}}}(\alpha_2 + \beta_2)\right]
\end{aligned}\right\} \tag{90}$$

Also multiplying the first equation in equations (3.68) by y, the second by $(x+\mu)$ and then the first again by $(x+\mu-1)$ produces the following equations;

$$\begin{aligned}
n^2\psi xy &- \frac{(1-\mu)(x+\mu)q_1 y}{r_1^3} - \frac{\mu(x+\mu-1)q_2 y}{r_2^3} \\
&- \frac{3(1-\mu)(x+\mu)A_1 y}{2r_1^5} - \frac{3\mu(x+\mu-1)A_2 y}{2r_2^5} \\
&+ \frac{nW_1 y^2}{r_1^2} + \frac{nW_2 y^2}{r_2^2} = 0
\end{aligned} \tag{91}$$

$$n^2\psi(x+\mu)y \quad -\frac{(1-\mu)(x+\mu)q_1 y}{r_1^3}-\frac{\mu(x+\mu-1)q_2 y}{r_2^3} -\frac{3(1-\mu)(x+\mu)A_1 y}{2r_1^5}-\frac{3\mu(x+\mu-1)A_2 y}{2r_2^5} -\frac{nW_1(x+\mu)^2}{r_1^2}-\frac{nW_2(x+\mu-1)(x+\mu)}{r_2^2}=0 \tag{92}$$

$$n^2\psi(x+\mu-1)y \quad -\frac{(1-\mu)(x+\mu-1)q_1 y}{r_1^3}-\frac{\mu(x+\mu-1)q_2 y}{r_2^3} -\frac{3(1-\mu)(x+\mu-1)A_1 y}{2r_1^5}-\frac{3\mu(x+\mu-1)A_2 y}{2r_2^5} -\frac{nW_1(x+\mu)(x+\mu-1)}{r_1^2}-\frac{nW_2(x+\mu-1)^2}{r_2^2}=0 \tag{93}$$

By elimination method, the equations (3.70), (3.71) and (3.72) reduces to

$$\left.\begin{aligned} \left[n^2\psi-\frac{q_2}{r_2^3}-\frac{3A_2}{2r_2^5}\right]\mu y &= nW_1+\frac{nW_2}{2}+\frac{nW_2}{2r_2^2}(r_1^2-1) \\ \left[n^2\psi-\frac{q_1}{r_1^3}-\frac{3A_1}{2r_1^5}\right](1-\mu)y &= -\frac{nW_1}{2}-nW_2-\frac{nW_1}{2r_1^2}(r_2^2-1) \end{aligned}\right\} \tag{94}$$

taking $q_1=(1-w_1)$, $q_2=(1-w_2)$, $|w_i|=1(i=1,2)$ and substituting the values for n, r_1, r_2 x *and* y from equations (3.17), (3.68), and (3.69) into equation (3.73) above, considering only linear terms of $w_1, w_2, A_1, A_2, W_1, W_2$, yields,

$$\alpha_2 \quad =-\frac{w_1}{3\psi^{\frac{1}{3}}}-\frac{(1-\psi^{\frac{2}{3}})A_1}{2\psi^{\frac{1}{3}}}-\frac{A_2}{2\psi^{\frac{1}{3}}}-\frac{[W_1(2-\psi^{\frac{2}{3}})+2W_2]}{3(1-\mu)\psi\sqrt{4-\psi^{\frac{2}{3}}}}$$

$$\beta_2 \quad =-\frac{w_2}{3\psi^{\frac{1}{3}}}-\frac{A}{2\psi^{\frac{1}{3}}}-\frac{(1-\psi^{\frac{2}{3}})A_2}{2\psi^{\frac{1}{3}}}+\frac{[2W_1+W_2(2-\psi^{\frac{2}{3}})]}{3\mu\psi\sqrt{4-\psi^{\frac{2}{3}}}}$$

By putting these values of α_2 and β_2 in equation (3.69) gives,

$$
\left.
\begin{aligned}
x &= \frac{1}{2}-\mu-\frac{w_1}{3\psi^{\frac{2}{3}}}+\frac{w_2}{3\psi^{\frac{2}{3}}}+\frac{A_1}{2}-\frac{A_2}{2}-\frac{[W_1(2-\mu\psi^{\frac{2}{3}})+W_2(2-\psi^{\frac{2}{3}}+\mu\psi^{\frac{2}{3}})]}{3\mu(1-\mu)\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}} \\
y &= \pm\frac{\sqrt{4-\psi^{\frac{2}{3}}}}{2\psi^{\frac{1}{3}}}\left\{1-\frac{2}{4-\psi^{\frac{2}{3}}}\left[\frac{w_1}{3}+\frac{w_2}{3}+\frac{(2-\psi^{\frac{2}{3}})A_1}{2}+\frac{(2-\psi^{\frac{2}{3}})A_2}{2}\right.\right. \\
&\quad \left.\left.-\left[\frac{W_1(2-\mu(4-\psi^{\frac{2}{3}}))+W_2(2-\psi^{\frac{2}{3}}-\mu(4-\psi^{\frac{2}{3}}))}{3\mu(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right]\right]\right\}.
\end{aligned}
\right\} \tag{95}
$$

Equations (3.74) are the coordinates of the triangular libration points, $L_4(x,+y)$ and $L_5(x,-y)$. Putting equation (3.61) in (3.74) neglecting product of ε' with other small quantities $(|w_1|,|w_2|,|A_1|,|A_2|,|W_1|,|W_2|)$ the coordinates become

$$
\left.
\begin{aligned}
x_p &= \frac{1}{2}-\mu-\frac{w_1}{3}+\frac{w_2}{3}+\frac{A_1}{2}-\frac{A_2}{2}-\frac{[W_1(2-\mu)+W_2(1+\mu)]}{3\mu(1-\mu)\sqrt{3}} \\
y_p &= \pm\frac{\sqrt{3}}{2}\left[1-\frac{4\varepsilon'}{9}-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{[W_1(2-3\mu)+W_2(1-3\mu)]}{9\mu(1-\mu)\sqrt{3}}\right]
\end{aligned}
\right\} \tag{96}
$$

where the subscript p indicates the presence of perturbations in the centrifugal forces. In order to appreciate the impact of the centrifugal force on the location of the libration points, the product of ε' with the small quantity parameters is further considered, taking only the first order terms in ε'. The coordinate are obtained as

$$
\left.
\begin{aligned}
x_p^* &= \frac{1}{2}-\mu-\frac{w_1(3-2\varepsilon')}{9}+\frac{w_2(3-2\varepsilon')}{9}+\frac{A_1}{2}-\frac{A_2}{2} \\
&\quad -\frac{W_1[18-22\varepsilon'-\mu(9-5\varepsilon')]}{27\mu(1-\mu)\sqrt{3}}-\frac{W_2[9-17\varepsilon'+\mu(9-5\varepsilon')]}{27\mu(1-\mu)\sqrt{3}} \\
y_p^* &= \pm\frac{\sqrt{3}}{2}\left\{1-\frac{4\varepsilon'}{9}-\frac{2(9-2\varepsilon')w_1}{81}-\frac{2(9-2\varepsilon')w_2}{81}-\frac{(9-8\varepsilon')A_1}{27}-\frac{(9-8\varepsilon')A_2}{27}\right. \\
&\quad \left.+\frac{W_1[18-14\varepsilon'-\mu(27-54\varepsilon')]}{81\mu(1-\mu)\sqrt{3}}+\frac{W_2[9-11\varepsilon'-\mu(27-23\varepsilon')]}{81\mu(1-\mu)\sqrt{3}}\right\}.
\end{aligned}
\right\} \tag{97}
$$

2.4 Stability of the Triangular Libration Points

Assuming also that, (x_0, y_0) is the coordinate of the triangular libration points and $\xi,\eta << 1$ are the small displacements such that $(x_0+\xi, y_0+\eta)$ is a point in the vicinity of (x_0, y_0), with velocity component $(\dot{\xi},\dot{\eta})$. Then substituting these in the equations of motion (3.62) and using the Taylor series expansion produces,

$$
\begin{aligned}
\ddot{\xi}-2n\phi\dot{\eta} &= \Omega_x^0+[\xi\Omega_{xx}^0+\eta\Omega_{xy}^0]+\dot{\xi}\Omega_{x\dot{x}}^0+\dot{\eta}\Omega_{x\dot{y}}^0+0(2)\\
\ddot{\eta}+2n\phi\dot{\xi} &= \Omega_y^0+\xi\Omega_{yx}^0+\eta\Omega_{yy}^0+\dot{\xi}\Omega_{y\dot{x}}^0+\dot{\eta}\Omega_{y\dot{y}}^0+0(2)
\end{aligned}
$$

where $0(2)$ represents the second and higher order terms, the superscript $(^0)$ indicate that the partial derivatives are evaluated at the libration points which implies that $\Omega_x^0=\Omega_y^0=0$

The equation above give the variational equation of motion corresponding to equations of motion as,

$$
\begin{aligned}
\ddot{\xi}-2n\phi\dot{\eta} &= \xi\Omega_{xx}^0+\eta\Omega_{xy}^0+\dot{\xi}\Omega_{x\dot{x}}^0+\dot{\eta}\Omega_{x\dot{y}}^0\\
\ddot{\eta}+2n\phi\dot{\xi} &= \xi\Omega_{yx}^0+\eta\Omega_{yy}^0+\dot{\xi}\Omega_{y\dot{x}}^0+\dot{\eta}\Omega_{y\dot{y}}^0
\end{aligned}
\tag{98}
$$

Let $\xi=Ae^{\lambda t}$, $\eta=Be^{\lambda t}$ be the trial solution of equation (3.77), the we can write

$$
\begin{aligned}
&(\lambda^2-\lambda\Omega_{x\dot{x}}^0-\Omega_{xx}^0)A+[-(2n\phi+\Omega_{x\dot{y}}^0)\lambda-\Omega_{xy}^0]B=0\\
&[(2n\phi-\Omega_{y\dot{x}}^0)\lambda-\Omega_{yx}^0]A+(\lambda^2-\lambda\Omega_{y\dot{y}}^0-\Omega_{yy}^0)B=0
\end{aligned}
$$

solving this,

$$
\begin{vmatrix}
\lambda^2-\lambda\Omega_{x\dot{x}}^0-\Omega_{xx}^0 & -(2n\phi+\Omega_{x\dot{y}}^0)\lambda-\Omega_{xy}^0\\
(2n\phi-\Omega_{y\dot{x}}^0)\lambda-\Omega_{yx}^0 & \lambda^2-\lambda\Omega_{y\dot{y}}^0-\Omega_{yy}^0
\end{vmatrix}=0
$$

yields the characteristic equation corresponding to the variational equation of motion (3.77) as,

$$
\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d=0 \tag{99}
$$

where

$$
\left.
\begin{aligned}
a &= -(\Omega_{x\dot{x}}^0+\Omega_{y\dot{y}}^0)\\
b &= 4n^2\phi^2+2n\phi\Omega_{x\dot{y}}^0-2n\phi\Omega_{y\dot{x}}^0-\Omega_{x\dot{y}}^0\Omega_{y\dot{x}}^0+\Omega_{x\dot{x}}^0\Omega_{y\dot{y}}^0-\Omega_{xx}^0-\Omega_{yy}^0\\
c &= \Omega_{x\dot{x}}^0\Omega_{yy}^0+\Omega_{xx}^0\Omega_{y\dot{y}}^0-2n\phi\Omega_{yx}^0-\Omega_{yx}^0\Omega_{x\dot{y}}^0+2n\phi\Omega_{xy}^0-\Omega_{\dot{x}y}^0\Omega_{y\dot{x}}^0\\
d &= \Omega_{xx}^0\Omega_{yy}^0-\Omega_{yx}^0\Omega_{xy}^0
\end{aligned}
\right\}
\tag{100}
$$

Differentiating equations (3.67) with respect to $x, y, \dot{x}, \dot{y}$ respectively and evaluating the second order partial derivatives at libration points using equations (3.61), (3.68), (3.74), so that

$$
r_1=\frac{1}{\psi^{\frac{1}{3}}}\left\{1-\frac{w_1}{3}-\frac{(1-\psi^{\frac{2}{3}})A_1}{2}-\frac{A_2}{2}-\frac{W_1(2-\psi^{\frac{2}{3}})+2W_2}{3(1-\mu)\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right\}
$$

$$
r_2=\frac{1}{\psi^{\frac{1}{5}}}\left\{1-\frac{w_2}{3}-\frac{A_1}{3}-\frac{(1-\psi^{\frac{2}{3}})A_2}{2}+\frac{2W_1+W_2(2-\psi^{\frac{2}{3}})}{3\mu\psi^{\frac{2}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}\right\}
$$

and

$$
\begin{aligned}
\Omega_{xx}^{0} \quad &= \frac{3}{4}\psi^{\frac{5}{3}} - \frac{[(2\psi-\psi^{\frac{5}{3}})-\mu(4\psi-\psi^{\frac{5}{3}})]}{2}w_1 \\
&+\frac{[2\psi-\mu(4\psi-\psi^{\frac{5}{3}})]}{2}w_2+\frac{3}{8}(9\psi^{\frac{5}{3}}-8\mu\psi^{\frac{5}{3}})A_1+\frac{3}{8}(\psi^{\frac{5}{3}}+8\mu\psi^{\frac{5}{3}})A_2 \\
&-\frac{W_1[8\psi^{\frac{5}{3}}-\mu(16\psi^{\frac{1}{3}}-3\psi^{\frac{5}{3}})+\mu^2(4\psi-3\psi^{\frac{5}{3}})]}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}} \\
&+\frac{W_2[(8\psi^{\frac{1}{3}}-4\psi)-\mu(16\psi^{\frac{1}{3}}-8\psi+3\psi^{\frac{5}{3}})-\mu^2(4\psi-3\psi^{\frac{5}{3}})]}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}} \\
&=\frac{3}{4}+\frac{5}{4}\varepsilon'-\frac{1}{2}(1-3\mu)w_1+\frac{1}{2}(2-3\mu)w_2+\frac{3}{8}(9-8\mu)A_1+\frac{3}{8}(1+8\mu)A_2 \\
&-\frac{W_1(8-13\mu+\mu^2)}{4\mu(1-\mu)\sqrt{3}}-W_2\frac{(4-11\mu-\mu^2)}{4\mu(1-\mu)\sqrt{3}}
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
\Omega_{xy}^{0} \quad &= \frac{3\psi^{\frac{4}{3}}\sqrt{4-\psi^{\frac{2}{3}}}}{4}\left\{1-2\mu-\frac{[(8-8\psi^{\frac{2}{3}}+2\psi^{\frac{4}{3}})-\mu(4\psi^{\frac{2}{3}}-2\psi^{\frac{4}{3}})]w_1}{3\psi^{\frac{2}{3}}(4-\psi^{\frac{2}{3}})}\right. \\
&+\frac{[(8-4\psi^{\frac{2}{3}})-\mu(4\psi^{\frac{2}{3}}-2\psi^{\frac{4}{3}}]w_2}{3\psi^{\frac{2}{3}}(4-\psi^{\frac{2}{3}})}+\frac{[(24-5\psi^{\frac{2}{3}})-\mu(32-6\psi^{\frac{2}{3}})]A_1}{2(4-\psi^{\frac{2}{3}})} \\
&\left.+\frac{[(8-\psi^{\frac{2}{3}})-\mu(32-6\psi^{\frac{2}{3}})]A_2}{3\psi^{\frac{2}{3}}(\psi-\psi^{\frac{2}{3}})}\right\} \\
&-\frac{W_1[(16-8\psi^{\frac{2}{3}})-\mu(56\psi^{\frac{2}{3}}-28\psi^{\frac{4}{3}}+3\psi^{2})+\mu^2(48\psi^{\frac{2}{3}}-24\psi^{\frac{4}{3}}+3\psi^{2})]}{4\mu(1-\mu)(4-\psi^{\frac{2}{3}})} \\
&-\frac{W_2[(16-16\psi^{\frac{2}{3}}+4\psi^{\frac{4}{3}})-\mu(40\psi^{\frac{2}{3}}-20\psi^{\frac{4}{3}}+3\psi^{2})+\mu^2(48\psi^{\frac{2}{3}}-24\psi^{\frac{4}{3}}+\psi^{2})]}{4\mu(1-\mu)(4-\psi^{\frac{2}{3}})} \\
&=1-2\mu+\frac{11}{9}(1-2\mu)\varepsilon'-\frac{2}{9}(1+\mu)w_1+\frac{2}{9}(2-\mu)w_2+\frac{1}{6}(19-26\mu)A_1 \\
&+\frac{1}{6}(7-26\mu)A_2-\frac{W_1(8-31\mu+27\mu^2)}{9\mu(1\mu)\sqrt{3}}-\frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)} \\
&=\Omega_{yx}^{0}
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
\Omega^0_{yy} \quad &= \frac{(12\psi - 3\psi^{\frac{5}{3}})}{4} + \frac{[(4\psi - 2\psi^{\frac{5}{3}}) - \mu(8\psi - 2\psi^{\frac{5}{3}}]w_1}{4} \\
&- \frac{[4\psi - \mu(8\psi - 2\psi^{\frac{5}{3}})]w_2}{4} + \frac{3}{8}(12\psi - \psi^{\frac{5}{3}})A_1 + \frac{3}{8}(12\psi - \psi^{\frac{5}{3}})A_2 \\
&+ \frac{W_1[8\psi^{\frac{1}{3}} - \mu(16\psi^{\frac{1}{3}} + 4\psi - 3\psi^{\frac{5}{3}}) + \mu^2(8\psi - 3\psi^{\frac{5}{3}})]}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}} \\
&+ \frac{W_2[(8\psi^{\frac{1}{3}} - 4\psi) - \mu(16\psi^{\frac{1}{3}} - 12\psi + 3\psi^{\frac{5}{3}}) - \mu^2(8\psi - 3\psi^{\frac{1}{3}})]}{4\mu(1-\mu)\sqrt{4-\psi^{\frac{2}{3}}}} \\
&= \frac{9}{4} + \frac{7}{4}\varepsilon' + \frac{1}{2}(1-3\mu)w_1 - \frac{1}{2}(2-3\mu)w_2 + \frac{33}{8}A_1 + \frac{33}{8}A_2 \\
&+ \frac{W_1(8-17\mu+5\mu^2)}{4\mu(1-\mu)\sqrt{3}} + \frac{W_2(4-7\mu-5\mu^2)}{4\mu(1-\mu)\sqrt{3}}
\end{aligned}
\tag{103}
$$

$$
\left.
\begin{aligned}
\Omega^0_{x\dot{x}} \quad &= -\frac{\psi^{\frac{2}{3}}}{4}(4+\psi^{\frac{2}{3}})(W_1+W_2) \\
\Omega^0_{x\dot{y}} \quad &= \frac{-\psi\sqrt{4-\psi^{\frac{2}{3}}}}{4}(W_1-W_2) = \Omega^0_{y\dot{x}} \\
\Omega^0_{y\dot{y}} \quad &= -\frac{\psi^{\frac{2}{3}}}{4}(8-\psi^{\frac{2}{3}})(W_1+W_2)
\end{aligned}
\right\}
\tag{104}
$$

Now substituting these values from (3.80) to (3.83) in (3.79) , gives the coefficient of the characteristics equation as

$$
\left.
\begin{aligned}
a =\ & 3\psi^{\frac{2}{3}}(W_1+W_2) \\
b =\ & \frac{16\psi^2-12\psi}{4}+\frac{3}{8}(16\psi^2-12\psi-8\psi^{\frac{5}{3}}+8\mu\psi^{\frac{5}{3}})A_1 \\
& +\frac{3}{8}(16\psi^2-12\psi-8\mu\psi^{\frac{5}{3}})A_2+\frac{W_1\psi}{\sqrt{4-\psi^{\frac{2}{3}}}}+\frac{W_2\psi}{\sqrt{4-\psi^{\frac{2}{3}}}} \\
c =\ & -\frac{3\psi^{\frac{5}{3}}}{4}[4+\mu\psi^{\frac{2}{3}}(4-\psi^{\frac{2}{3}})]W_1-\frac{3\psi^{\frac{5}{3}}}{4}[(4+4\psi^{\frac{2}{3}}-\psi^{\frac{4}{3}})-\mu\psi^{\frac{4}{3}}(4-\psi^{\frac{2}{3}})]W_2 \\
d =\ & \frac{9\psi^{\frac{8}{3}}}{4}\mu(1-\mu)(4+\psi^{\frac{2}{3}})+\frac{2\psi^{\frac{8}{3}}}{3}\mu(1-\mu)(2-\psi^{\frac{2}{3}})w_1 \\
& +\frac{3\psi^{\frac{8}{3}}}{2}\mu(1-\mu)(2-\psi^{\frac{2}{3}})w_2+\frac{9\psi^{\frac{8}{3}}}{4}\mu(1-\mu)(16-3\psi^{\frac{2}{3}})A_1 \\
& +\frac{9\psi^{\frac{8}{3}}}{4}\mu(1-\mu)(16-3\psi^{\frac{2}{3}})A_2-\frac{W_1[(84\psi^2-30\psi^{\frac{8}{3}})-\mu(144\psi^2-72\psi^{\frac{8}{3}}+9\psi^{\frac{10}{3}})]}{4\sqrt{4-\psi^{\frac{2}{3}}}} \\
& -\frac{W_2[(60\psi^2-42\psi^{\frac{8}{3}}+9\psi^{\frac{10}{3}})-\mu(144\psi^2-72\psi^{\frac{8}{3}}+9\psi^{\frac{10}{3}})]}{4\sqrt{4-\psi^{\frac{2}{3}}}}
\end{aligned}
\right\} \quad (105)
$$

using equation (3.61) neglecting second and higher order of small quantities and product of ε' with $(w_1, w_2, A_1, A_2, W_1, W_2)$, equation (3.84) takes the form

$$
\left.
\begin{aligned}
a =\ & 3(W_1+W_2)>0 \\
b =\ & 1+8\varepsilon-3\varepsilon'-(3/2-3\mu)A_1+(3/2-3\mu)A_2+\frac{W_1}{\sqrt{3}}-\frac{W_2}{\sqrt{3}}>0 \\
c =\ & \frac{-3}{4}(4+3\mu)W_1-\frac{3}{4}(7-3\mu)W_2<0 \\
d =\ & \frac{27}{4}\mu(1-\mu)+\frac{33}{2}\mu(1-\mu)\varepsilon'+\frac{3}{2}\mu(1-\mu)w_1+\frac{3}{2}\mu(1-\mu)w_2 \\
& +\frac{117}{4}\mu(1-\mu)A_1+\frac{117}{4}\mu(1-\mu)A_2-\frac{W_1(54-81\mu)}{4\sqrt{3}}-\frac{W_2(27-81\mu)}{4\sqrt{3}}>0
\end{aligned}
\right\} \quad (106)
$$

These are all constant coefficients which are seen to depend on the parameters of small perturbations in the coriolis and centrifugal forces, (ε, ε'), oblateness (A_1, A_2), radiation pressure force (w_1, w_2) with PR- drag force (W_1, W_2). And for $0<\left|\varepsilon,\varepsilon',w_1,w_2,A_1,A_2,W_1,W_2\right|=1$, the coefficients $a>0$, $b>0$, $c<0$, $d>0$

According to Routh and Hurwitz's criteria for stability, a characteristics equation would have all negative real roots if all the $D's_i$, (the Hurwitz's determinants) have positive values. Therefore,

$$D_1 = a > 0$$

$$D_2 = \begin{vmatrix} a & c \\ 1 & b \end{vmatrix} = ab - c > 0$$

$$D_3 = \begin{vmatrix} a & 0 & 0 \\ 1 & b & d \\ 0 & a & c \end{vmatrix} = a(bc - ad) - c^2$$

$$D_4 = \begin{vmatrix} a & c & 0 & 0 \\ 1 & b & d & 0 \\ 0 & a & c & 0 \\ 0 & 1 & b & d \end{vmatrix} = a\begin{vmatrix} b & d & 0 \\ a & c & 0 \\ 1 & b & d \end{vmatrix} - c\begin{vmatrix} 1 & d & 0 \\ 0 & c & 0 \\ 0 & b & d \end{vmatrix} = abcd - (ad)^2 - c^2 d$$

Even with the nature of the values given in (3.85), it would difficult to predict the nature of the $D's_i (i = 1,2,3,4)$. Therefore their values have been computed for the binary systems: Kruger-60 and RXJ0450,1-5856 in table 4 and table 5.

3 Effects of Perturbations on the Periodic Orbit of the Generalized Restricted Three-Body Problem

It is important to study the periodic orbit of a system in order to obtain a complete information about the orbit of a non-integrable dynamical system and as the time, $t \to \infty$, the behaviour of the solution cannot be predicted.

The effect of small perturbations in the coriolis and centrifugal forces have been examined on some of the periodic elements (the period of oscillation, orientation, semi-major and semi-minor axes) of the oblate RTBP under the effect of the PR-drag force from the primaries.

3.1 The Critical Mass

The critical mass value μ_c is expected to exist when the discriminants vanishes, that is, $\Delta = 0$.

Now from the Sylvester equation (1.6), the discriminant of a polynomial with constant coefficient of degree four

$$a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$$

is

$$\Delta = \frac{1}{a_4}\begin{vmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{vmatrix}$$

therefore, the discriminants for the characteristics equation (3.78) gives

$$\begin{aligned} \Delta \quad &= (a^2b^2c^2 - 4a^3c^3 - 4b^3c^2 + 18abc^3 - 27c^4 + 256d^3) \\ &+ d(-4a^2b^3 + 18a^3bc + 16b^4 - 80ab^2c - 6a^2c^2 + 144bc^2) \\ &+ d^2(-27a^4 + 144a^2b - 128b^2 - 192ac) \end{aligned}$$

considering only first order term of small quantities and since a and c, given in equation (3.85), are functions of $|W_1| << 1, |W_2| << 1$, then the equation above reduces to

$$\Delta = 256d^3 - 128b^2d^2 + 16b^4d \qquad (107)$$

which gives,

$$\begin{aligned}\Delta \quad &= 256\left[\frac{27}{4}\mu(1-\mu)\right]^3\left[1+\frac{22}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2+\frac{13}{3}A_1+\frac{13}{3}A_2-\frac{W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}\right.\\
&\left.-\frac{W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right]^3-128\left[\frac{27}{4}\mu(1-\mu)\right]^2\left[1+8\varepsilon-3\varepsilon'-(\frac{3}{2}-3\mu)+(\frac{3}{2}-3\mu)A_1\right.\\
&\left.+(\frac{3}{2}-3\mu)A_2+\frac{W_1}{\sqrt{3}}-\frac{W_2}{\sqrt{3}}\right]^2\left[1+\frac{22}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2+\frac{13}{3}A_1+\frac{13}{3}A_2-\frac{W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}\right.\\
&\left.-\frac{W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right]^2+16\left[\frac{27}{4}\mu(1-\mu)\right]\left[1+8\varepsilon-3\varepsilon'-(\frac{3}{2}-3\mu)A_1+(\frac{3}{2}-3\mu)A_2+\frac{W_1}{\sqrt{3}}\right.\\
&\left.-\frac{W_2}{\sqrt{3}}\right]^4\left[1+\frac{22}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2+\frac{13}{3}A_1+\frac{13}{3}A_2-\frac{W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}-\frac{W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right].\end{aligned}$$

Simplifying, considering only first order terms of small quantities, results to

$$\begin{aligned}\Delta \quad &= 256\left[\frac{27}{4}\mu(1-\mu)\right]^3\left[1+\frac{22}{3}\varepsilon'+\frac{2}{3}w_1+\frac{2}{3}w_2+13A_1+13A_2-\frac{3W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}\right.\\
&\left.-\frac{3W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right]-128\left[\frac{27}{4}\mu(1-\mu)\right]^2\left[1+16\varepsilon-6\varepsilon'-(3-6\mu)A_1+(3-6\mu)A_2\right.\\
&\left.+\frac{2W_1}{\sqrt{3}}-\frac{2W_2}{\sqrt{3}}\right]\left[1+\frac{44}{9}\varepsilon'+\frac{4}{9}w_1+\frac{4}{9}w_2+\frac{26}{3}A_1+\frac{26}{3}A_2-\frac{2W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}\right.\\
&\left.-\frac{2W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right]+16\left[\frac{27}{4}\mu(1-\mu)\right]\left[1+32\varepsilon-12\varepsilon'-(6-12\mu)A_1\right.\\
&\left.+(6-12\mu)A_2+\frac{4W_1}{\sqrt{3}}-\frac{4W_2}{\sqrt{3}}\right]\left[1+\frac{22}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2+\frac{13}{3}A_1+\frac{13}{3}A_2\right.\\
&\left.-\frac{W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}-\frac{W_2(1-3\mu)}{\mu(1-\mu)\sqrt{3}}\right].\end{aligned}$$

or

$$\left.\begin{aligned}
\Delta \quad &= 256\left[\frac{27}{4}\mu(1-\mu)\right]^3\left[1+\frac{22}{3}\varepsilon'+\frac{2}{3}w_1+\frac{2}{3}w_2+13A_1+13A_2\right.\\
&\left.-\frac{3W_1(2-3\mu)}{\mu(1-\mu)\sqrt{3}}-\frac{3W_2(1-3\mu)}{\mu(1-\mu\sqrt{3})}\right]\\
&-128\left[\frac{27}{4}\mu(1-\mu)\right]^2\left[1+16\varepsilon-\frac{10}{9}\varepsilon'+\frac{4}{9}w_1+\frac{4}{9}w_2+\left(\frac{17}{3}+6\mu\right)A_1\right.\\
&\left.+\left(\frac{35}{3}-6\mu\right)A_2-\frac{W_1(4-8\mu+2\mu^2)}{\mu(1-\mu\sqrt{3})}-\frac{W_2(2-4\mu-2\mu^2)}{\mu(1-\mu)\sqrt{3}}\right]\\
&+16\left[\frac{27}{4}\mu(1-\mu)\right]\left[1+32\varepsilon-\frac{86}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2-\left(\frac{5}{3}-12\mu\right)A_1\right.\\
&\left.+\left(\frac{31}{3}-12\mu\right)A_2-\frac{W_1(2-7\mu+4\mu^2)}{\mu(1-\mu)\sqrt{3}}+\frac{W_2(1+\mu-4\mu^2)}{\mu(1-\mu)\sqrt{3}}\right]
\end{aligned}\right\} \tag{108}$$

Here, the discriminant Δ is a function of the mass parameter μ and other perturbing factors $(\varepsilon,\varepsilon',w_1,w_2,A_1,A_2,W_1 and W_2)$. Δ is studied in the interval $0\le\mu\le\frac{1}{2}$.

If $\mu=0$, then $\Delta=0$

This implies that the discriminant varnishes at this point and since the critical mass value, μ_c is expected to exist when $\Delta=0$ therefore

$$\mu_c=\mu=0 \tag{109}$$

and if $\mu=\frac{1}{2}$, then equation (3.87) becomes

$$\begin{aligned}
\Delta \quad &= 256\left[\frac{27}{16}\right]^3\left[1+\frac{22}{3}\varepsilon'+\frac{2}{3}w_1+\frac{2}{3}w_2+13A_1+13A_2-\frac{3W_1(\frac{1}{2})}{\frac{1}{4}\sqrt{3}}-\frac{3W_2(\frac{-1}{2})}{\frac{1}{4}\sqrt{3}}\right]\\
&-128\left[\frac{27}{16}\right]^3\left[1+16\varepsilon-\frac{10}{9}\varepsilon'+\frac{4}{9}w_1+\frac{4}{9}w_2+\left(\frac{17}{2}+3\right)A_1+\left(\frac{35}{3}-3\right)A_2\right.\\
&\left.-\frac{W_1(\frac{1}{2})}{\frac{1}{2}\sqrt{3}}-\frac{W_2(-\frac{1}{2})}{\frac{1}{4}\sqrt{3}}\right]+16\left[\frac{27}{16}\right]^3\left[1+32\varepsilon-\frac{86}{9}\varepsilon'+\frac{2}{9}w_1+\frac{2}{9}w_2\right.\\
&\left.-\left(\frac{5}{3}-6\right)A_1+\left(\frac{31}{3}-6\right)A_2-\frac{W_1(-\frac{1}{2})}{\frac{1}{4}\sqrt{3}}-\frac{W_2(\frac{1}{2})}{\frac{1}{4}\sqrt{3}}\right]
\end{aligned}$$

and gives

$$\Delta = 27\left[\frac{529}{16} - 184\varepsilon + \frac{24449}{72}\varepsilon' + \frac{1771}{72}w_1 + \frac{1771}{72}w_2 + \frac{23023}{48}A_1 + \frac{23023}{48}A_2 - \frac{1955}{8\sqrt{3}}W_1 + \frac{1955}{8\sqrt{3}}W_2\right] > 0 \tag{110}$$

This shows that when $\mu = \frac{1}{2}$, $\Delta > 0$ and it implies that the solution of the characteristics equation (3.78) would consist of both real and complex conjugate roots (secular terms) and the critical mass value $\mu_c = \mu = 0$ does not exist in the interval $0 \le \mu \le \frac{1}{2}$ and hence the triangular libration point is unstable.

Now the roots of the dynamical system is obtained below.

In the absence of perturbations in the coriolis and centrifugal forces, radiation pressure, oblateness and PR-drag effect from both primaries, the characteristics equation of motion obtained in equation (3.78) using equation (3.85) reduces to that of the classical case. This was found to be,

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

Assuming that $\lambda^2 = \Lambda$ this equation gives a quadratic equation

$$\Lambda^2 + \Lambda + \frac{27}{4}\mu(1-\mu) = 0$$

by which solving gives the four roots of the classical characteristic equation of motion as,

$$\lambda = \lambda_n = \pm zi \qquad (n = 1,2,3,4) \tag{111}$$

where

$$z^2 = \frac{1}{2}\{1 \mp [1 - 27\mu(1-\mu)]^{\frac{1}{2}}\} \quad (Szebehely, 1967) \tag{112}$$

Assuming that due to small perturbations in the coriolis and centrifugal forces, oblateness, radiation with PR-drag effect the solutions of equation (3.78) are

$$\lambda = \lambda_n(1 + \sigma_1 + i\sigma_2) = \pm[-\sigma_2 + (1+\sigma_1)i]z \tag{113}$$

where $\sigma_1, \sigma_2 \in \mathrm{R}$ are small quantities.

By substituting equation (3.90) and its multiples in equation (3.78), neglecting product of small quantities and comparing coefficients of the real and imaginary part gives

$$\sigma_1 = \frac{-z^4 + bz^2 - d}{2z^2(2z^2-1)} \; and \; \sigma_2 = \frac{\pm az^3 \mp cz}{2z^2(2z^2-1)} \tag{114}$$

where the values of a, b, c, d and z are given in the equation (3.85) and (3.91)

Therefore using these values above in equation (3.92), gives the roots of the characteristic equation of the perturbed system as

$$\begin{aligned} \lambda_{1,2} &= \pm\gamma\{-az^3 + cz + i[3z^4 - (2-b)z^2 - d]\} \\ \lambda_{3,4} &= \pm\gamma\{az^3 - cz + i[3z^4 - (2-b)z^2 - d]]\} \end{aligned} \tag{115}$$

where $\gamma = \dfrac{1}{2z^2(2z^2-1)}$

The roots, $\lambda_i, i = 1,2,3,4$ are functions of the constants coefficients (a,b,c,d) obtained in equation (3.85). These are seen to be dependent on the parameters of the small perturbations in the coriolis and centrifugal forces, oblateness, mass reduction factor due radiation pressure and PR-drag force. This shows that the root is influenced by the aforementioned factors.

3.2 The Period of Motion

The folding time (T) for the growth of the particle oscillation about the libration points L_4 and L_5 is given by

$$1/T = nRe(\lambda) \qquad (Schuerman, 1980)$$

using equation (3.94)

$$1/T = n\gamma(\mp az^3 \pm cz) = \frac{\mp nz(az^2 - c)}{2z^2(2z^2-1)}$$

and

$$T = \frac{2z(2z^2-1)}{\mp n(az^2-c)} \tag{116}$$

T is always positive for at least one choice of sign in the equation (3.95) above which imply that the particles have no stable equilibrium solution when the PR-drag effect is put into consideration. This is based on the fact that in the linear sense, a and c are pure functions of W_1, W_2, the PR- drag parameter and not of the perturbations in the coriolis and centrifugal forces.

So it is more convenient to evaluate T when applied to the solar system. In this case, the terms containing $(1-\mu)$ can be ignored. Thus putting the values n, a, c and z from equations (3.17), (3.85) and (3.91), the equation (3.95) gives,

$$\begin{aligned} T &= \frac{2}{n(a-c)} \\ &; \frac{2}{\frac{3}{4}n[W_1(8+3\mu) + W_2(11-3\mu)]} \\ &; \frac{2}{3n[W_1(11-3(1-\mu)) + W_2(8+3(1-\mu)\mu)]} \\ &; \frac{8}{3n[11W_1 + 8W_2]} \end{aligned} \tag{117}$$

The equation (3.97) is the equation for the Period of oscillation. In the linear sense, the time of oscillation is seen to depend only on the parameters of mass reduction factor and the PR-drag force but not on the parameters of the mass value, oblateness and small perturbations in the coriolis and centrifugal forces.

3.3 The Orientation

By Taylor Series expansion, the expression Ω around L_4 given in equation (3.63) and (3.64) is

$$\begin{aligned}\Omega(x,y,\dot{x},\dot{y}) &= \Omega^0 + \xi\Omega_x^0 + \eta\Omega_y^0 + \dot{\xi}\Omega_{\dot{x}}^0 + \dot{\eta}\Omega_{\dot{y}} + \frac{1}{2}(\xi^2\Omega_{xx}^0 + \eta^2\Omega_{yy} + \dot{\xi}^2\Omega_{\dot{x}\dot{x}} \\ &+ \dot{\eta}^2\Omega_{\dot{y}\dot{y}} + 2\xi\eta\Omega_{xy} + 2\xi\dot{\xi}\Omega_{x\dot{x}} + 2\xi\dot{\eta}\Omega_{x\dot{y}} + 2\dot{\xi}\eta\Omega_{\dot{x}y} + 2\dot{\xi}\dot{\eta}\Omega_{\dot{x}\dot{y}}) + 0(3)\end{aligned} \tag{118}$$

where $0(3)$ are the third and higher order terms.

At libration points

$$\Omega_x^0 = \Omega_y^0 = \dot{\xi} = \dot{\eta} = 0$$

therefore equations (3.97) reduces to

$$\Omega = \Omega^0 + \frac{1}{2}\xi^2\Omega_{xx}^0 + \xi\eta\Omega_{xy}^0 + \frac{1}{2}\eta^2\Omega_{yy}^0 + 0(3) \tag{119}$$

from equation (3.63) and (3.64),

$$\begin{aligned}\Omega &= \frac{n^2\psi}{2}(x^2+y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1}{2r_1^2} + \frac{\mu A_2}{2r_2^2} \\ &+ W_1\left[\frac{(x+\mu)\dot{x}+y\dot{y}}{2r_1^2} - n\arctan\left(\frac{y}{x+\mu}\right)\right] \\ &+ W_2\left[\frac{(x+\mu-1)\dot{x}+y\dot{y}}{2r_2^2} - n\arctan\left(\frac{y}{x+\mu-1}\right)\right]\end{aligned}$$

which is written as

$$\begin{aligned}\Omega &= \frac{n^2\psi}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1}{2r_2^2} + \frac{\mu A_2}{2r_2^2} \\ &+ W_1\left[\frac{(x+\mu)\dot{x}+y\dot{y}}{2r_1^2} - n\arctan\left(\frac{y}{x+\mu}\right)\right] \\ &+ W_2\left[\frac{(x+\mu-1)\dot{x}+y\dot{y}}{2r_2^2} - n\arctan\left(\frac{y}{x+\mu-1}\right)\right]\end{aligned} \tag{120}$$

Evaluating equation (3.99) above at libration point using equations (3.17), (3.68) and (3.75), neglecting second and higher order terms of small quantities,

$$
\begin{aligned}
r_1 &= 1-\frac{\varepsilon'}{3}-\frac{w_1}{3}-\frac{A_2}{2}-\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}} \\
r_2 &= 1-\frac{\varepsilon'}{3}-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2W_1+W_2}{3\mu\sqrt{3}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega^0 &= \frac{1}{2}(1+\varepsilon')\left(1+\frac{3A_1}{2}+\frac{3A_2}{2}\right)\left\{(1-\mu)\left[1-\frac{\varepsilon'}{3}-\frac{w_1}{3}-\frac{A_2}{2}-\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}}\right]^2\right. \\
&\left.-\mu\left[1-\frac{\varepsilon'}{3}-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2W_1+W_2}{3\mu\sqrt{3}}\right]^2\right\} \\
&+(1-\mu)(1-w_1)\left[1-\frac{\varepsilon'}{3}-\frac{w_1}{3}-\frac{A_2}{2}-\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}}\right]^{-1} \\
&+\mu(1-w_2)\left[1-\frac{\varepsilon'}{3}-\frac{w_2}{3}-\frac{A_1}{2}+\frac{2W_1+W_2}{3\mu\sqrt{3}}\right]^{-1} \\
&+\frac{1}{2}(1-\mu)\left[1-\frac{\varepsilon'}{3}-\frac{w_1}{3}-\frac{A_2}{2}-\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}}\right]^{-2}A_1 \\
&+\frac{1}{2}\mu\left[1-\frac{\varepsilon'}{3}-\frac{w_1}{3}-\frac{A_1}{2}+\frac{2W_1+W_2}{3\mu\sqrt{3}}\right]^{-2}A_2 \\
&-W_1\arctan\left\{\sqrt{3}\left[1-\frac{4}{9}\varepsilon'-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu)+2W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]\right. \\
&\left.\times\left[1-\frac{2w_1}{3}+\frac{2w_2}{3}+A_1-A_2-\frac{2W_1(2-\mu)+2W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]^{-1}\right\} \\
&-W_2\arctan\left\{-\sqrt{3}\left[1-\frac{4\varepsilon'}{9}-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu+2W_2(1-3\mu))}{9\mu(1-\mu)\sqrt{3}}\right]\right. \\
&\left.\times\left[1+\frac{2w_1}{3}-\frac{2w_2}{3}-A_1+A_2+\frac{2w_1(2-\mu)+2W_1(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]^{-1}\right\}
\end{aligned}
$$

simplifying

$$
\begin{aligned}
\Omega^0 \quad &= \frac{1}{2}(1+\varepsilon')\left(1+\frac{3A_1}{2}+\frac{3A_2}{2}\right)\left\{(1-\mu)\left[1-\frac{2\varepsilon'}{3}-\frac{2w_1}{3}-A_2-\frac{2W_1+4W_2}{3(1-\mu)\sqrt{3}}\right]\right.\\
&\left.+\mu\left[1-\frac{2\varepsilon'}{3}-\frac{2w_2}{3}-A_1+\frac{4W_1+2W_2}{3\mu\sqrt{3}}\right]\right\}+(1-\mu)(1-w_1)\left[1+\frac{\varepsilon'}{3}+\frac{w_1}{3}+\frac{A_2}{2}\right.\\
&\left.+\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}}\right]+\mu(1-w_2)\left[1+\frac{\varepsilon'}{3}+\frac{w-2}{3}+\frac{A_1}{2}-\frac{4W_1+2W_2}{3\mu\sqrt{3}}\right]\\
&+\frac{1}{2}(1-\mu)\left[1+\frac{2\varepsilon'}{3}+\frac{2w_1}{3}+A_2+\frac{2W_1+4W_2}{3(1-\mu)\sqrt{3}}\right]A_1+\frac{1}{2}\mu\left[1+\frac{2\varepsilon'}{3}+\frac{2w_2}{3}+A_1-\frac{4W_1+2W_2}{3\mu\sqrt{3}}\right]A_2\\
&-W_1\arctan\left\{\sqrt{3}\left[1-\frac{4}{9}\varepsilon'-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu)+2W_2(1-3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]\right.\\
&\left.\times\left[1+\frac{2w_1}{3}-\frac{2w_2}{3}-A_1+A_2+\frac{2W_1(2-\mu)+2W_2(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]\right\}\\
&-W_2\arctan\left\{-\sqrt{3}\left[1-\frac{4\varepsilon'}{9}-\frac{2w_1}{9}-\frac{2w_2}{9}-\frac{A_1}{3}-\frac{A_2}{3}+\frac{2W_1(2-3\mu+2W_2(1-3\mu))}{9\mu(1-\mu)\sqrt{3}}\right]\right.\\
&\left.\times\left[1-\frac{2w_1}{3}+\frac{2w_2}{3}+A_1-A_2-\frac{2w_1(2-\mu)+2W_1(1+\mu)}{3\mu(1-\mu)\sqrt{3}}\right]\right\}
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
\Omega^0 \quad &= \frac{1}{2}(1-\mu)\left[1+\frac{\varepsilon'}{3}-\frac{2w_1}{3}+\frac{3A_1}{2}+\frac{A_2}{2}-\frac{2W_1+4W_2}{3(1-\mu)\sqrt{3}}\right]\\
&+\frac{1}{2}\mu\left[1+\frac{\varepsilon'}{3}-\frac{2w_2}{3}+\frac{A_1}{2}+\frac{3A_2}{2}+\frac{4W_1+2W_2}{2\mu\sqrt{3}}\right]\\
&+(1-\mu)\left[1+\frac{\varepsilon'}{3}-\frac{2w_1}{3}+\frac{A_2}{2}+\frac{W_1+2W_2}{3(1-\mu)\sqrt{3}}\right]+\mu\left[1+\frac{\varepsilon'}{3}-\frac{2w_2}{3}+\frac{A_1}{2}-\frac{2W_1+W_2}{3\mu\sqrt{3}}\right]\\
&+\frac{1}{2}(1-\mu)A_1+\frac{1}{2}\mu A_2-W_1\arctan(\sqrt{3}+s)-W_2\arctan(-\sqrt{3}+t)
\end{aligned}
$$

or

$$
\begin{aligned}
\Omega^0 \quad &= \left[\frac{3}{2}+\frac{1}{2}\varepsilon'-(1-\mu)w_1-\mu w_2+\frac{1}{4}(5-2\mu)A_1+\frac{1}{4}(3+2\mu)A_2\right]\\
&-W_1\arctan\left(\sqrt{3}+s\right)-W_2\arctan\left(-\sqrt{3}+t\right)
\end{aligned}
\tag{121}
$$

where

$$
\left.\begin{aligned}
s &= -\sqrt{3}\left[\frac{4}{9}\varepsilon' - \frac{4}{9}w_1 + \frac{8}{9}w_2 + \frac{4}{3}A_1 - \frac{2}{3}A_2 - \frac{4W_1(4-3\mu)+8W_2}{9\mu(1-\mu)\sqrt{3}}\right] \\
t &= \sqrt{3}\left[\frac{4}{9}\varepsilon' + \frac{8}{9}w_1 - \frac{4}{9}w_2 - \frac{2}{3}A_1 + \frac{4}{3}A_2 + \frac{8W_1+4W_2(1+3\mu)}{9\mu(1-\mu)\sqrt{3}}\right]
\end{aligned}\right. \tag{122}
$$

$$
|s|,|t| = 1
$$

Using equations (3.80) - (3.82) , (3.100) and (3.101) in (3.98)

$$
\begin{aligned}
\Omega ={}& \left[\frac{3}{2} + \frac{1}{2}\varepsilon' - (1-\mu)w_1 - \mu w_2 + \frac{1}{4}(5-2\mu)A_1 + \frac{1}{4}(3+2\mu)A_2\right. \\
& \left. - W_1 \arctan(\sqrt{3}+s) - W_2 \arctan(-\sqrt{3}+t)\right] \\
& + \xi^2\left[\frac{3}{8} + \frac{5}{8}\varepsilon' - \frac{1}{4}(1-3\mu)w_1 + \frac{1}{4}(2-3\mu)w_2 + \frac{3}{16}(9-8\mu)A_1 + \frac{3}{16}(1+8\mu)A_2\right. \\
& \left. - \frac{W_1(8-13\mu+\mu^2)}{8\mu(1-\mu)\sqrt{3}} - W_2\frac{(4-11\mu-\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right] \\
& + \frac{3\sqrt{3}}{4}\xi\eta\left[1-2\mu + \frac{11}{9}(1-2\mu)\varepsilon' - \frac{2}{9}(1+\mu)w_1 + \frac{2}{9}(2-\mu)w_2 + \frac{1}{6}(19-26\mu)A_1\right. \\
& \left. + \frac{1}{6}(7-26\mu)A_2 - \frac{W_1(8-31\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}} - \frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)}\right] \\
& + \eta^2\left[\frac{9}{8} + \frac{7}{8}\varepsilon' + \frac{1}{4}(1-3\mu)w_1 - \frac{1}{4}(2-3\mu)w_2 + \frac{33}{16}A_1 + \frac{33}{16}A_2\right. \\
& \left. + \frac{W_1(8-17\mu+5\mu^2)}{8\mu(1-\mu)\sqrt{3}} + \frac{W_2(4-7\mu-5\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right]
\end{aligned}
$$

Introducing the variables $\bar{\xi}$ and $\bar{\eta}$ by the transformation

$$
\left.\begin{aligned}
\xi &= \bar{\xi}\cos\alpha - \bar{\eta}\sin\alpha \\
\eta &= \bar{\xi}\sin\alpha + \bar{\eta}\cos\alpha
\end{aligned}\right\} \tag{123}
$$

which is equivalent to a rotation of the coordinate system by an angle α , the quadratic form becomes,

$$
\begin{aligned}
\overline{\Omega} \quad &= \left[\frac{3}{2}+\frac{1}{2}\varepsilon'-(1-\mu)w_1-\mu w_2+\frac{1}{4}(5-2\mu)A_1+\frac{1}{4}(3+2\mu)A_2\right.\\
&\left.-W_1\arctan(\sqrt{3}+s)-W_2\arctan(-\sqrt{3}+t)\right]\\
&+\left[\frac{3}{8}+\frac{5}{8}\varepsilon'-\frac{1}{4}(1-3\mu)w_1+\frac{1}{4}(2-3\mu)w_2+\frac{3}{16}(9-8\mu)A_1+\frac{3}{16}(1+8\mu)A_2\right.\\
&\left.-\frac{W_1(8-13\mu+\mu^2)}{8\mu(1-\mu)\sqrt{3}}-W_2\frac{(4-11\mu-\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right]\left(\overline{\xi}^2\cos^2\alpha+\overline{\eta}^2\sin^2\alpha-2\overline{\xi}\overline{\eta}\sin\alpha\cos\alpha\right)\\
&+\frac{3\sqrt{3}}{4}\left[1-2\mu+\frac{11}{9}(1-2\mu)\varepsilon'-\frac{2}{9}(1+\mu)w_1+\frac{2}{9}(2-\mu)w_2+\frac{1}{6}(19-26\mu)A_1+\frac{1}{6}(7-26\mu)A_2\right.\\
&\left.-\frac{W_1(8-31\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)}\right]\left[\overline{\xi}^2\sin\alpha\cos\alpha-\overline{\eta}^2\sin\alpha\cos\alpha+\overline{\xi}\overline{\eta}(\cos^2\alpha-\sin^2\alpha)\right]\\
&+\left[\frac{9}{8}+\frac{7}{8}\varepsilon'+\frac{1}{4}(1-3\mu)w_1-\frac{1}{4}(2-3\mu)w_2+\frac{33}{16}A_1+\frac{33}{16}A_2\right.\\
&\left.+\frac{W_1(8-17\mu+5\mu^2)}{8\mu(1-\mu)\sqrt{3}}+\frac{W_2(4-7\mu-5\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right]\left(\overline{\xi}^2\sin^2\alpha+\overline{\eta}^2\cos^2\alpha+2\overline{\xi}\overline{\eta}\sin\alpha\cos\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Omega} \quad &= \left[\frac{3}{2}+\frac{1}{2}\varepsilon'-(1-\mu)w_1-\mu w_2+\frac{1}{4}(5-2\mu)A_1+\frac{1}{4}(3+2\mu)A_2\right.\\
&\left.-W_1\arctan(\sqrt{3}+s)-W_2\arctan(-\sqrt{3}+t)\right]\\
&+\xi^2\left[\frac{3}{8}+\frac{5}{8}\varepsilon'-\frac{1}{4}(1-3\mu)w_1+\frac{1}{4}(2-3\mu)w_2+\frac{3}{16}(9-8\mu)A_1+\frac{3}{16}(1+8\mu)A_2\right.\\
&\left.-\frac{W_1(8-13\mu+\mu^2)}{8\mu(1-\mu)\sqrt{3}}-W_2\frac{(4-11\mu-\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right]\\
&+\frac{3\sqrt{3}}{4}\xi\eta\left[1-2\mu+\frac{11}{9}(1-2\mu)\varepsilon'-\frac{2}{9}(1+\mu)w_1+\frac{2}{9}(2-\mu)w_2+\frac{1}{6}(19-26\mu)A_1\right.\\
&\left.+\frac{1}{6}(7-26\mu)A_2-\frac{W_1(8-31\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)}\right]\\
&+\eta^2\left[\frac{9}{8}+\frac{7}{8}\varepsilon'+\frac{1}{4}(1-3\mu)w_1-\frac{1}{4}(2-3\mu)w_2+\frac{33}{16}A_1+\frac{33}{16}A_2\right.\\
&\left.+\frac{W_1(8-17\mu+5\mu^2)}{8\mu(1-\mu)\sqrt{3}}+\frac{W_2(4-7\mu-5\mu^2)}{8\mu(1-\mu)\sqrt{3}}\right]
\end{aligned}
$$

Choosing α such that the coefficients of $\overline{\xi}\overline{\eta}=0$

$$
\begin{aligned}
\overline{\Omega} \quad &= \left[\frac{3}{2} + \frac{1}{2}\varepsilon' - (1-\mu)w_1 - \mu w_2 + \frac{1}{4}(5-2\mu)A_1 + \frac{1}{4}(3+2\mu)A_2 \right. \\
&\quad \left. - W_1 \arctan(\sqrt{3}+s) - W_2 \arctan(-\sqrt{3}+t) \right] \\
&\quad + \left\{ \frac{3}{8}\cos^2\alpha + \frac{9}{8}\sin^2\alpha + \frac{3\sqrt{3}}{4}(1-2\mu)\sin\alpha\cos\alpha \right. \\
&\quad + \left[\frac{5}{8}\cos^2\alpha + \frac{7}{8}\sin^2\alpha + \frac{11\sqrt{3}}{12}\sin\alpha\cos\alpha \right]\varepsilon' \\
&\quad + \left[-\frac{1}{4}(1-3\mu)\cos^2\alpha + \frac{1}{4}(1-3\mu)\sin^2\alpha - \frac{\sqrt{3}}{6}(1+\mu)\sin\alpha\cos\alpha \right] w_1 \\
&\quad + \left[\frac{1}{4}(2-3\mu)\cos^2\alpha - \frac{1}{4}(2-3\mu)\sin^2\alpha + \frac{\sqrt{3}}{6}(2-\mu)\sin\alpha\cos\alpha \right] w_2 \\
&\quad + \left[\frac{3}{16}(9-8\mu)\cos^2\alpha + \frac{33}{16}\sin^2\alpha + \frac{\sqrt{3}}{8}(19-26\mu)\sin\alpha\cos\alpha \right] A_1 \\
&\quad + \left[\frac{3}{16}(1+8\mu)\cos^2\alpha + \frac{33}{16}\sin^2\alpha + \frac{\sqrt{3}}{8}(7-26\mu)\sin\alpha\cos\alpha \right] A_2 \\
&\quad + \frac{W_1}{\mu(1-\mu)\sqrt{3}} \left[-\frac{1}{8}(8-13\mu+\mu^2)\cos^2\alpha + \frac{1}{8}(8-17\mu+5\mu^2)\sin^2\alpha \right. \\
&\quad \left. - \frac{\sqrt{3}}{12}(8-31\mu+27\mu^2)\sin\alpha\cos\alpha \right] + \frac{W_2}{\mu(1-\mu)\sqrt{3}} \left[-\frac{1}{8}(4-11\mu-\mu^2)\cos^2\alpha \right. \\
&\quad \left. \left. + \frac{1}{8}(4-7\mu-5\mu^2)\sin^2\alpha - \frac{\sqrt{3}}{12}(4-23\mu+27\mu^2)\sin\alpha\cos\alpha \right] \right\} \overline{\xi}^2
\end{aligned}
$$

$$+\left\{\frac{3}{8}\sin^2\alpha+\frac{9}{8}\cos^2\alpha\frac{3}{4}(1-2\mu)\sin\alpha\cos\alpha\right.$$

$$+\left[\frac{5}{8}\sin^2\alpha+\frac{7}{8}\cos^2\alpha-\frac{11\sqrt{3}}{12}(1-2\mu)\sin\alpha\cos\alpha\right]\varepsilon'$$

$$+\left[-\frac{1}{4}(1-3\mu)\sin^2\alpha+\frac{1}{4}(1-3\mu)\cos^2\alpha+\frac{\sqrt{3}}{6}(1+\mu)\sin\alpha\cos\alpha\right]w_1$$

$$+\left[\frac{1}{4}(2-3\mu)\sin^2\alpha-\frac{1}{4}(2-3\mu)\cos^2\alpha-\frac{\sqrt{3}}{6}(2-\mu)\sin\alpha\cos\alpha\right]w_2$$

$$+\left[\frac{3}{16}(9-8\mu)\sin^2\alpha+\frac{33}{16}\cos^2\alpha-\frac{\sqrt{3}}{8}(19-26\mu)\sin\alpha\cos\alpha\right]A_1$$

$$+\left[\frac{3}{16}(1+8\mu)\sin^2\alpha+\frac{33}{16}\cos^2\alpha-\frac{\sqrt{3}}{8}(7-26\mu)\sin\alpha\cos\alpha\right]A_2$$

$$+\frac{W_1}{\mu(1-\mu)\sqrt{3}}\left[-\frac{1}{8}(8-13\mu+\mu^2)\sin^2\alpha+\frac{1}{8}(8-17\mu+5\mu^2)\cos^2\alpha\right.$$

$$\left.+\frac{\sqrt{3}}{12}(8-31\mu+27\mu^2)\sin\alpha\cos\alpha\right]+\frac{W_2}{\mu(1-\mu)\sqrt{3}}\left[-\frac{1}{8}(4-11\mu-\mu^2)\sin^2\alpha\right.$$

$$\left.\left.+\frac{1}{8}(4-7\mu-5\mu^2)\cos^2\alpha+\frac{\sqrt{3}}{12}(4-23\mu+27\mu^2)\sin\alpha\cos\alpha\right]\right\}\bar{\eta}^2$$

Using the trigonometric identities

$$\cos^2\alpha+\sin^2\alpha=1,\ \sin\alpha\cos\alpha=\frac{\sin 2\alpha}{2}\ \textit{and}\ \cos^2\alpha-\sin^2\alpha=\cos 2\alpha$$

the new quadratic form becomes

$$\overline{\Omega}=p\bar{\xi}^2+q\bar{\eta}^2+r \qquad (124)$$

where

$$\begin{aligned}
p \quad &= \frac{3}{8}+\frac{3}{4}\sin^2\alpha+\frac{3\sqrt{3}}{8}(1-2\mu)\sin 2\alpha+\left[\frac{5}{8}+\frac{1}{4}\sin^2\alpha+\frac{11\sqrt{3}}{24}(1-2\mu)\sin 2\alpha\right]\varepsilon' \\
&-\left[\frac{1}{4}(1-3\mu)\cos 2\alpha+\frac{\sqrt{3}}{2}(1+\mu)\sin 2\alpha\right]w_1+\left[\frac{1}{4}(2-3\mu)\cos 2\alpha+\frac{\sqrt{3}}{2}(2-\mu)\sin 2\alpha\right]w_2 \\
&+\left[\frac{27}{16}+\frac{3}{8}\sin^2\alpha-\frac{3}{2}\mu\cos^2\alpha+\frac{\sqrt{3}}{16}(19-26\mu)\sin 2\alpha\right]A_1 \\
&+\left[\frac{3}{16}+\frac{15}{8}\sin^2\alpha+\frac{3}{2}\mu\cos^2\alpha+\frac{\sqrt{3}}{16}(7-26\mu)\sin^2\alpha\right]A_2 \\
&-\frac{W_1}{\mu(1-\mu)}\sqrt{3}\left[\frac{1}{8}(8-13\mu+\mu^2)\cos\alpha+\frac{1}{2}\mu(1-\mu)\sin^2\alpha+\frac{\sqrt{3}}{24}(8-31\mu+27\mu^2)\sin 2\alpha\right] \\
&-\frac{W_2}{\mu(1-\mu)\sqrt{3}}\left[\frac{1}{8}(4-11\mu-\mu^2)\cos 2\alpha-\frac{1}{2}\mu(1-\mu)\sin^2\alpha+\frac{\sqrt{3}}{24}(4-23\mu+27\mu^2)\sin 2\alpha\right]
\end{aligned} \tag{125}$$

$$\begin{aligned}
q \quad &= \frac{3}{8}+\frac{5}{8}\cos^2\alpha-\frac{3}{8}(1-2\mu)\sin\alpha+\left[\frac{5}{8}+\frac{1}{4}\cos^2\alpha-\frac{11\sqrt{3}}{24}(1-2\mu)\sin 2\alpha\right]\varepsilon' \\
&+\left[\frac{1}{4}(1-3\mu)\cos 2\alpha+\frac{\sqrt{3}}{12}(1+\mu)\sin 2\alpha\right]w_1-\left[\frac{1}{4}(2-3\mu)\cos 2\alpha+\frac{\sqrt{3}}{12}(2-\mu)\right]w_2 \\
&+\left[\frac{27}{16}+\frac{3}{8}\cos^2\alpha-\frac{3}{2}\mu\sin^2\alpha-\frac{3}{16}(19-26\mu)\sin 2\alpha\right]A_1 \\
&+\left[\frac{3}{16}+\frac{15}{8}\cos^2\alpha+\frac{3}{2}\mu\sin^2\alpha-\frac{\sqrt{3}}{16}(7-26\mu)\sin 2\alpha\right]A_2 \\
&+\frac{W_1}{\mu(1-\mu)}\sqrt{3}\left[\frac{1}{8}(8-13\mu+\mu^2)\cos 2\alpha-\frac{1}{2}\mu(1-\mu)\cos^2\alpha+\frac{\sqrt{3}}{24}(8-31\mu+27\mu^2)\sin 2\alpha\right] \\
&+\frac{W_2}{\mu(1-\mu)}\sqrt{3}\left[\frac{1}{8}(4-11\mu-\mu^2)\cos 2\alpha+\frac{1}{2}\mu(1-\mu)\cos^2\alpha+\frac{\sqrt{3}}{24}(4-23\mu+27\mu^2)\sin^2\alpha\right]
\end{aligned} \tag{126}$$

and

$$\begin{aligned}
r \quad &= \frac{3}{4}+\frac{1}{2}\varepsilon'-(1-\mu)w_1-\mu w_2+\frac{1}{4}(5-2\mu)A_1+\frac{1}{4}(3+2\mu)A_2 \\
&-W_1\arctan(\sqrt{3}+s)-W_2\arctan(-\sqrt{3}+t)
\end{aligned} \tag{127}$$

s and t are given in equation (3.101)

Now, setting the coefficients of $\bar{\xi}\,\bar{\eta}=0$, gives

$$\left[\frac{3}{4}+\frac{1}{4}\varepsilon'+\frac{1}{2}(1-3\mu)w_1-\frac{1}{2}(2-3\mu)w_2+\frac{3}{4}(1+4\mu)A_1+\frac{3}{8}(5-4\mu)A_2\right.$$
$$\left.+\frac{W_1(8-15\mu+3\mu^2)}{4\mu(1-3\mu)\sqrt{3}}+\frac{W_2(4-9\mu-3\mu^2)}{4\mu(1-\mu)\sqrt{3}}\right]\sin 2\alpha$$
$$+\frac{3}{4}\sqrt{3}\left[1-2\mu+\frac{11}{9}(1-2\mu)\varepsilon'-\frac{2}{9}(1+\mu)w_1+\frac{2}{9}(2-\mu)w_2+\frac{1}{6}(19-26\mu)A_1\right.$$
$$\left.+\frac{1}{6}(7-26\mu)A_2-\frac{W_1(8-31\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)}\right]\cos 2\alpha=0$$

which implies

$$\tan 2\alpha \quad =-\frac{3}{4}\sqrt{3}\left[1-2\mu+\frac{11}{9}(1-2\mu)\varepsilon'-\frac{2}{9}(1+\mu)w_1+\frac{2}{9}(2-\mu)w_2\right.$$
$$+\frac{1}{6}(19-26\mu)A_1+\frac{1}{6}(7-26\mu)A_2-\frac{W_1(8-31\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}}$$
$$\left.-\frac{W_2(4-23\mu+27\mu^2)}{9\mu(1-\mu)\sqrt{3}}\right]\left[\frac{3}{4}+\frac{1}{4}\varepsilon'+\frac{1}{2}(1-3\mu)w_1-(2-\mu)w_2\right.$$
$$\left.+\frac{3}{8}(1+2\mu)A_1+\frac{3}{8}(5-12\mu)A_2+\frac{W_1(8-15\mu+3\mu^2)}{4\mu(1-\mu)\sqrt{3}}+\frac{W_2(4-9\mu+3\mu^2)}{4\mu(1-\mu)\sqrt{3}}\right]^{-1}$$

Simplifying this, considering only first order term of small quantities, gives

$$\tan 2\alpha \quad =-\sqrt{3}\left[1-2\mu+\frac{8}{9}(1-2\mu)\varepsilon'-\frac{4}{9}(2-7\mu+9\mu^2)w_1\right.$$
$$+\frac{4}{9}(4-11\mu+9\mu^2)w_2+\frac{4}{3}(2-4\mu+3\mu^2)A_1-\frac{4}{3}(1-2\mu+3\mu^2)A_2$$
$$\left.-\frac{W_1(32-124\mu-72\mu^2-18\mu^3)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(16-74\mu+99\mu^2-18\mu^2)}{9\mu(1-\mu)\sqrt{3}}\right]$$

or

$$\alpha \quad =\frac{1}{2}\arctan Q \tag{128}$$

where

$$Q \quad =-\sqrt{3}\left[1-2\mu+\frac{8}{9}(1-2\mu)\varepsilon'-\frac{4}{9}(2-7\mu+9\mu^2)w_1\right.$$
$$+\frac{4}{9}(4-11\mu+9\mu^2)w_2+\frac{4}{3}(2-4\mu+3\mu^2)A_1-\frac{4}{3}(1-2\mu+3\mu^2)A_2$$
$$\left.-\frac{W_1(32-124\mu-72\mu^2-18\mu^3)}{9\mu(1-\mu)\sqrt{3}}-\frac{W_2(16-74\mu+99\mu^2-18\mu^2)}{9\mu(1-\mu)\sqrt{3}}\right]$$

The orientation α of the orbit is seen to be dependent on the small perturbation in the centrifugal force, mass reduction factor, oblateness and the PR-drag force due to the presence of their parameters.

3.4 The Semi-axes

Using the equation (3.103), the Jacobian constant in equation (3.66), $C = 2\Omega(x, y)$ becomes

$$C = 2(p\bar{\xi}^2 + q\bar{\eta}^2 + r) \tag{129}$$

where p, q and r are given in equations (3.104), (3.105) and (3.106) respectively.

The equation (3.108) is rewritten to give a corresponding equations of an ellipse as

$$\frac{\bar{\xi}^2}{\left[\left(\frac{C-2r}{2p}\right)^{\frac{1}{2}}\right]^2} + \frac{\bar{\eta}^2}{\left[\left(\frac{C-2r}{2q}\right)^{\frac{1}{2}}\right]^2} = 1$$

and the length of the semi-major (a') and semi-minor (b') are

$$a' = \left[\frac{C-2r}{2p}\right]^{\frac{1}{2}} \; and \; b' = \left[\frac{C-2r}{2q}\right]^{\frac{1}{2}} \; respectively$$

Using equations (3.100) and (3.106), knowing that $C = 2\Omega$, the equation of the length of the semi-major, a' and semi-minor b' axes, reduces to

$$a' = \left[\xi^2 + \frac{q}{2p}\eta^2\right]^{\frac{1}{2}} and \; b' = \left[\frac{p}{2q}\xi^2 + \eta^2\right]^{\frac{1}{2}} \tag{130}$$

where p and q are given in equations (3.106) and (3.108).

The value of the semi axes depends on the direction of motion, (α), the mass parameter, (μ), centrifugal force, (ε), mass reduction factor due to radiation pressure force, (w_1, w_2), oblateness, (A_1, A_2), and the PR-drag force, (W_1, W_2). This is due to the presence of the aforementioned parameters.

4 Analysis and Discussion

1 Effect Of Poynting-Robertson Drag And Oblateness On The Stability Of Restricted Three-Body Problem

This research work considered the effect of small perturbations in the coriolis and centrifugal on the stability of the RTBP, specifically when the primaries are considered to be oblate spheroid, radiating with PR-drag force.

In order to achieve this, the effect of PR-drag on the stability of oblate, photo-gravitational RTBP was investigated first. The equations of motion of the infinitesimal body under the influence of mass reduction factor due to radiation (w_1, w_2), oblateness (A_1, A_2) and Poyting Robertson Drag (W_1, W_2) of both primaries were obtained and given by the equations (3.13) to (3.17) (i.e. the absence of small perturbations in the coriolis and centrifugal forces). The presence of these parameters in the equations, shows that its motion is affected by the perturbing factors.

The coordinates, $(x,\pm y)$ of the triangular libration point L_4 and L_5 are given in equations (3.32) and (3.33). They are also seen to depend on the mass ratio, μ and the aforementioned parameters.

In the absence of the parameters $(w_1 = w_2 = A_1 = A_2 = W_1 = W_2 = 0)$ equations (3.32) and (3.33) reduces to the classical RTBP (Szbehely, 1967). When both primaries are radiating and spherical ($w_1 \neq 0$, $w_2 \neq 0$, $A_1 = A_2 = W_1 = W_2 = 0$) the equations coincides with those of Kunitsyn and Perezhogin (1979) and others. when oblateness of both primaries are considered $(w_1 = w_2 = 0, A_1 \neq 0, A_2 \neq 0, W_1 = W_2 = 0)$ the result agrees with those of Vidyakin (1974). When both primaries are radiating with PR effect in ($w_1 \neq 0, w_2 \neq 0, A_1 = A_2 = 0, W_1 \neq 0 \; and \; W_2 \neq 0$) the equations are in agreement with those of Ragos and Zafiropoulos (1995).

When the smaller primaries is oblate and the bigger primary is considered as radiating with PR effect ($w_1 \neq 0, w_2 = 0, A_1 = 0, A_2 \neq 0, W_1 \neq 0 \; and \; W_2 = 0$) the result agrees with those of Kushvah and Ishwar (2006a,b). When the bigger primary is oblate and the smaller primary is radiating with PR effect ($w_1 = 0, w_2 \neq 0, A_1 \neq 0, A_2 = 0, W_1 = 0 \; and \; W_2 \neq 0$) the equations (3.32) and $(3.33$) agrees with those of Singn and Amuda (2014). The variation in the value of coordinate points can be seen in the computation for the kruger-60 $(\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_2 = 0.99996)$ binary system shown in the Table 1 below.

Table 4.1: Effects of Radiation, Oblateness and PR-drag on the location of the triangular points for Kruger-60 Binary

Case	w_1	w_2	A_1	A_2	W_1	W_2
Case (1)	0	0	0	0	0	0
Case (2)	0	0	0.01	0	0	0
Case (3)	0	0	0	0.02	0	0
Case (4)	0	0	0.01	0.02	0	0
Case (5)	0.00008	0	0	0	0	0
Case (6)	0	0.00004	0	0	0	0
Case (7)	0.00008	0.00004	0	0	0	0
Case (8)	0.00008	0.00004	0.01	0.02	0	0
Case (9)	0.00008	0.00004	0	0	$1.01045E-9$	$3.28067E-10$
Case (10)	0.00008	0	0	0.02	$1.01045E-9$	0
Case (11)	0	0.00004	0.01	0	0	$3.28067E-10$
Case (12)	0.00008	0.00004	0.01	0.02	$1.01045E-9$	$3.28067E-10$

This table shows that the coordinate of the triangular libration points deviate from the result of the classical case due to the presence of these perturbing factors.

When $w_1 = w_2 = 0, A_1 = A_2 = 0$ and $W_1 = W_2 = 0,$

then $a=0, b=b_o=1$, $d=\dfrac{27}{4}\mu(1-\mu)$ and the roots of the characteristics equation (3.36) obtained gives

$$\lambda^2=\frac{-1\pm\sqrt{1-27\mu(1-\mu)}}{2} \quad \text{(Szebehely, 1967)}$$

and for stable motion $1>27\mu(1-\mu)$ which implies $\mu<0.0385$. This reduces to the case of the classical RTBP.

When $A_1=0, A_2=0, W_1=W_2=0$ then $a=0, b=1, c=0$ and

$$d=\frac{3}{2}\mu(1-\mu)\left[\frac{9}{2}+w_1+w_2\right] \text{ (Schuerman, 1979)}$$

This gives the case of photo gravitational RTBP.

When $w_1=0, A_1=A_2=0$ and $W_1=0$. Then

$$a=3W_2, b=1-\frac{W_2}{\sqrt{3}}, c=-\left(\frac{21}{4}-\frac{9}{4}\mu\right)W_2$$

and

$$d=\frac{27\mu(1-\mu)}{4}+\frac{3}{2}\mu(1-\mu)w_1-\frac{27W_1(1-3\mu)}{4\sqrt{3}}$$

$$Re(\lambda)=\frac{3\left[\dfrac{1}{2}\pm\left[\dfrac{1}{4}-\dfrac{27}{4}\mu(1-\mu)\right]^{\frac{1}{2}}-\left(\dfrac{21}{4}-\dfrac{9}{4}\mu\right)\right]}{\pm 4\left[\dfrac{1}{4}-\dfrac{27}{4}\mu(1-\mu)\right]^{\frac{1}{2}}}.$$

This agrees with the result of Schuerman (1979). When $w_1\neq 0$, $w_2=0$, $A_1=0$, $A_2\neq 0$, $W_1\neq 0$ and $W_2=0$,

then $a=3W_1$, $b=1+\left(\dfrac{3}{2}-3\mu\right)A_2+\dfrac{W_1}{\sqrt{3}}$, $c=-\left(3+\dfrac{9}{4}\mu\right)W_1$ and

$$d=\frac{27}{4}\mu(1-\mu)+\frac{3}{2}\mu(1-\mu)w_1+\frac{117}{4}\mu(1-\mu)A_2-\frac{27W_1(2-3\mu)}{4\sqrt{3}}$$

(Ishwar and Kushvah, 2006)

When $w_1=0, w_2\neq 0, A_1\neq 0, A_2=0, W_1=0$ and $W_2\neq 0$ then

$$a=3W_2, b=1-\left(\frac{3}{2}-3\mu\right)A_1-\frac{W_2}{\sqrt{3}}, c=-\left(\frac{21}{4}-\frac{9\mu}{4}\right)W_2$$

and

$$d=\frac{27}{4}\mu(1-\mu)+\frac{3}{2}\mu(1-\mu)w_2+\frac{117}{4}\mu(1-\mu)A_1-\frac{W_2(27-81\mu)}{a\sqrt{3}}$$

(Singh and Amuda, 2014)

The roots of the characteristics equation (3.36) corresponding to the variational equation (3.35) was given by equations (3.52), (3.54) and (3.55) where values of the coefficients of the characteristics equation, a, b, c and d are given in equations (3.46) to (3.49). These equations all depend on the parameters of the PR-drag force and other perturbing factors.

According to Murray (1994), the inequality in equation (3.60), is the necessary condition for the stability of triangular libration points at L_4 and L_5. But by equation (3.48) as $\mu \to 0$

$$c = -\left(3W_1 + \frac{21}{4}W_2\right) < 0 \quad (W_1, W_2 \geq 0).$$

This contradicts the Murray's condition for stability. So, due to oblateness, radiation pressure and PR-drag effects from both primaries the motion remains unstable in the linear sense.

2 Effects of Coriolis and Centrifugal Forces on the Stability of Generalised Photo-gravitational Restricted Three-Body Problem

The effects of small perturbations in the coriolis and centrifugal forces on the stability of the triangular Libration points of the RTBP in the presence of oblateness, radiation and P-R drag effects was studied.

The equations of motion in equations (3.62) and (3.63) which are modifications of those obtained in equations (3.13) and (3.14) are seen to possesses a force function (3.64) which is dependent on the parameter ε' of the centrifugal force. The equation of the Zero Velocity Curve (ZVC) (3.66) is a function of the force function in equation (3.64) and consequently the value of the ZVC also depends on the parameter ε'.

The equations (3.74) and (3.75) are the coordinates of the triangular libration points, $L_4(x,+y)$ and $L_5(x,-y)$. They are seen to be influenced by the small perturbation in the centrifugal force due to the presence of the parameter ψ $(\psi = 1+\varepsilon', |\varepsilon'|=1)$. Furthermore, to appreciate the impact of the centrifugal force on the location of these points, the product of ε' with the small quantity parameters is considered, taking only the first order terms in ε'. This produces the equations (3.76).

In line with the work of Narayan and Shrivasta (2013), Umar and Singh (2014), Singh et al (2016) a range of values for the parameters ε' are used in studying the effect of small perturbation in the centrifugal force on the location around the triangular libration points. Specifically for the binary system Kruger - 60 $(\mu = 0.3937, c_d = 48002.33, q_1 = 0.99992, q_2 = 0.99996)$ and $RXJ\,0450,1-5856(\mu = 0.0967, cd = 299792458, q_1 = 0.9963, q_1 = 0.9965)$ with the aid of micro-soft Excel and Maple 18 Mathematical Software. The values obtained are given in the table below.

Table 4.2: Effects of ε' on $L_{4,5}$ and on the Jacobi Constant, C associated with the ZVCs that contain those point for kruger - 60.

ε'	x_p	x_p^*	$\pm y_p$	$\pm y_p^*$	C_{L_4}
-0.50	0.10129	0.10128	1.04979	1.04594	1.17436
-0.10	0.10129	0.10129	0.89583	0.89506	1.36673
-0.05	0.10129	0.10129	0.87659	0.87620	1.38593
0.00	0.10129	0.10129	0.85734	0.85734	1.40428
0.05	0.10129	0.10129	0.83810	0.83848	1.42185
0.10	0.10129	0.10129	0.81885	0.82488	1.43872
0.50	0.10129	0.10129	0.66489	0.66874	1.55635

$x_c = 0.1063$ $y_c = \pm 0.866025$ $C_c = 1.380065$ (subscript c indicates that the coordinate evaluation for the classical case)

Table 4.3: Effects of ε' on $L_{4,5}$ and on the Jacobi Constant, C associated with the ZVCs that contain those point for $RXJ\,0450,1-5856$.

ε'	x_p	x_p^*	$\pm y_p$	$\pm y_p^*$	C_{L_4}
-0.50	0.39718	0.39681	1.04904	1.04510	1.21060
-0.10	0.39718	0.39711	0.89508	0.89429	1.43331
-0.05	0.39718	0.39715	0.87583	0.87544	1.45631
0.00	0.39718	0.39718	0.85659	0.85659	1.47846
0.05	0.39718	0.39722	0.83734	0.83773	1.49985
0.10	0.39718	0.39726	0.81810	0.81888	1.52052
0.50	0.39718	0.39756	0.66414	0.67129	1.66886

$x_c = 0.4033$ $y = \pm 0.866025$ $C_c = 1.456325$ It is observed from the tables 2 and 3 that there is a significant change in the value of x_c and y_c (coordinate of the classical case of the system) due to the presence of all the perturbing factors. It is also seen that as ε' is increasing the values of the x coordinate is not affected by it, but when x is extended to accommodate more of ε' up to the first order product of ε' and other small quantities $(w_1, w_2, A_1, A_2, W_1, W_2)$, there is a significant increase in the value of x as ε' is increasing. On the other-hand the values of y is seen to be decreasing as ε' is increasing at the same rate. This can be seen in the figures below.

Figure 3: Location of Triangular Libration points for Kruger-60

Figure 4: Location of Triangular Libration points for RXJ0450, 1-5856

The effect of small perturbation in the centrifugal force on the x and y coordinate of L_4 can also be seen in the graph of ε' plotted against the coordinates of the triangular points below.

Figure 5: Effect of ε' on the coordinate of Kruger-60 Binary system

Figure 6: Effect of ε' the coordinate of RXJ0450, 1-5856 Binary system

The equations in (3.77) are the constant coefficients a, b, c and d of the characteristics equation (3.76) with eigenvalue λ, corresponding to variational equation of motion (3.75). The values of these coefficients are obtained in equation (3.85) and seen to be dependent on the the perturbing parameters.

The Hurwitz's determinants, D were obtained for the characteristics equation (3.76) and the values of the $D's_i (i = 1,2,3,4)$ have been computed for the binary systems: Kruger-60 and RXJ0450,1-5856 in table 4 and table 5. below.

Table 4.4: Effects of ε and ε' on the $D's_i$ for kruger-60 binary system

ε'	ε	D_1	D_2	D_3	D_4
-0.50	1.00	$4.01556E-09$	$4.75343E-08$	$-2.52302E-16$	$3.7479E-17$
-0.10	0.20	$4.01556E-09$	$1.70161E-08$	$-1.14183E-16$	$-1.62925E-16$
-0.05	0.10	$4.01556E-09$	$1.32013E-08$	$-9.69183E-17$	$-1.57376E-16$
0.00	0.00	$4.01556E-09$	$9.38654E-09$	$-7.96534E-17$	$-1.45027E-16$
0.05	-0.10	$4.0156E-09$	$5.57176E-09$	$-6.23886E-17$	$-1.25879E-16$
0.10	-0.20	$4.01556E-09$	$1.75699E-09$	$-4.51237E-17$	$-9.99303E-17$
0.50	-1.00	$4.01556E-09$	$-2.87612E-08$	$9.29952E-17$	$3.52452E-16$

Table 4.5: Effects of ε and ε' on the $D'S_i$ for RXJ0450,1-5856 binary system

ε'	ε	D_1	D_2	D_3	D_4
-0.50	1.00	$3.37839E-11$	$3.91579E-10$	$-1.42073E-20$	$7.6498E-22$
-0.10	0.20	$3.37839E-11$	$1.34821E-10$	$-5.50928E-21$	$-2.87948E-21$
-0.05	0.10	$3.37839E-11$	$1.02726E-10$	$-4.42203E-21$	$-2.62989E-21$
0.00	0.00	$3.37839E-11$	$7.06317E-11$	$-3.33478E-21$	$-2.22359E-21$
0.05	-0.10	$3.37839E-11$	$3.85369E-11$	$-2.24753E-21$	$-1.66059E-21$
0.10	-0.20	$3.37839E-11$	$6.44221E-12$	$-1.16028E-21$	$-9.40889E-22$
0.50	-1.00	$3.37839E-11$	$-2.50316E-10$	$7.53771E-21$	$1.04579E-20$

It is observed that D_1 is always positive for the two binary system. Due to the presence of the parameters of the small perturbations in the coriolis (ε) and centrifugal (ε') forces in the coefficients b and d of the characteristic equation of motion (3.78), it is expected that the nature of the roots of the equation (3.78) would be influenced by a change in the value of the perturbation.

However, it is seen that an increase in the value of ε' brings about changes in the value of D_2, that is, from positive to negative, D_3 from negative to positive and D_4 from positive to negative and again to positive. Since the values of D_3 is always fluctuating no matter the strength of $\varepsilon', |\varepsilon'| = 1$ then the D_i's cannot be all positive in the chosen range which implies that the real part of the roots of the characteristics equation cannot be all negative. Therefore according to Routh and Hurwitz's criteria for stability, the proposed system remains unstable.

3 Effects of Perturbations on the Periodic Orbit of the Generalized Restricted Three-Body Problem.

The discriminant, Δ of the perturbed, generalized, photo-gravitational RTBP with PR-drag force was obtained in equation (3.87) and was found to be dependent on the parameters $\varepsilon, \varepsilon'$, A_1, A_2, w_1, w_2, W_1 and W_2 and when $\mu = \frac{1}{2}$, the discriminant, $\Delta > 0$ and it implies that the solution of the characteristics equation (3.78) would consist of both real and complex conjugate roots. But when $\mu = 0$ the discriminant vanishes. That is

$$\Delta = 0$$

and since the critical mass value, μ_c is expected to exist when $\Delta = 0$, then $\mu_c = \mu = 0$ which implies that the critical mass value, μ_c does not exist in the interval $0 \le \mu \le \frac{1}{2}$ for this particular system.

The roots, $\lambda_i, i = 1,2,3,4$ obtained in equation (3.94) are seen to be affected by the small perturbations in the coriolis and centrifugal forces, oblateness, radiation pressure and the PR-drag force due to the presence of their parameters. The tables below shows the computation of the discriminant and roots of Kruger-60 and RXJ 0450, 1-5856 binary systems used as models to see the effects of small perturbations in the coriolis, ε and centrifugal, ε' forces on the RTBP under the influence of oblateness and radiation pressure force with PR-drag.

Table 4.6: Effects of ε and ε' on Δ and λ_i, $i = 1,2,3,4$ for kruger - 60

ε'	ε	Δ	$\lambda_{1,3}$	$\lambda_{2,4}$
-0.5	1	$-2.92373E+04$	± 0.1188458174	$-2.259476994E-09 \pm 3.243040908I$

-0.1	0.2	$1.69031E+02$	$2.387068724E-$[illegible]	[illegible]$\pm 1.508765248I$
-0.05	0.1	$1.86641E+02$	$0.3858037786\pm$[illegible]	[illegible]$\pm 1.060867363I$
0	0	$1.14764E+03$	$0.6510565577\pm$[illegible]	[illegible]$\pm .9620133291I$
0.05	-0.1	$2.10125E+03$	$0.8348199175\pm$[illegible]	[illegible]$\pm 0.8505993757I$
0.1	-0.2	$2.29849E+03$	$0.9840108877\pm$[illegible]	[illegible]$\pm 0.7210217957I$
0.5	-1	$1.97267E+05$	$1.088952547\pm$[illegible]	[illegible]$\pm 0.8723400436I$

Table 4.7: Effects of ε and ε' on Δ and λ_i, $i=1,2,3,4$ for RXJ 0450, 1-5856

ε'	ε	Δ	$\lambda_{1,3}$	$\lambda_{2,4}$
-0.5	1	$-1.05611E+04$	± 0.07154875190	$-1.861523864E-11\pm 3.243026245I$
-0.1	0.2	$3.41427E+02$	$8.491802263E-$[illegible]	[illegible]$\pm 1.649224500I$
-0.05	0.1	$2.05885E+01$	$2.024011448E-$[illegible]	[illegible]$\pm 1.259946411I$
0	0	$2.87922E+01$	$0.3940317902\pm$[illegible]	[illegible]$\pm 0.8132103367I$
0.05	-0.1	$1.02985+02$	$0.6436279371\pm$[illegible]	[illegible]$\pm 0.6673132109I$
0.1	-0.2	$7.82166E+01$	$0.8198949364\pm$[illegible]	[illegible]$\pm 0.4777841634I$
0.5	-1	$9.81527E+04$	± 0.4083305543	± 2.884643160

The equation of the period of oscillation obtained in equation (3.95) is a function of the mass parameter, μ, the coefficients of the characteristics equation of motion, a and c given in equation (3.85) and not dependent on the parameters of perturbations in the coriolis (ε) and centrifugal (ε') forces, mass reduction factors, (w_1, w_2) and the PR-drag force, (W_1, W_2) in the linear sense.

The orientation of the orbit, (α) and the semi-axes, (a', b') of the elliptic orbit obtained in equations (3.107) and (3.109) respectively are seen to be influenced by all the aforementioned perturbing factors due to the presence of their parameters.

5 Summary, Conclusion and Recommendation

1 Brief Introduction

In Space Dynamics, the study of Classical RTBP and its many generalizations have been of great interest to researchers over the years. This is due to the rise in the need for accuracy in determining astrometric positions, revealing peculiarities of components of motion and to draw conclusions on the stability of space vehicles to be launched. This has led to the necessity to take into account all possible physical properties (non-sphericity of the bodies, phase angle, surface area light, perturbing and drag forces) that affect the motion of particles in space.

2 Summary and Conclusion

The effects of small perturbations in the coriolis and centrifugal forces on the stability of the libration points (precisely the triangular points) of the RTBP was considered when the primaries are taken to be both oblate spheroids, radiating with PR-drag effect.

The equations of motion and the coordinates of the triangular libration points were obtained and their stability at these points was determined. The results obtained are given thus:

- The PR-drag force were seen to affect the equations of motion and libration points of the oblate photo-gravitational RTBP due to the presence of its parameters W_1, W_2 in the equations. Also due to the nature of the value of the coefficient, c in equation (3.48) which contradicts the condition necessary for stability by Murray (1994) shows that the motion of=1the infinitesimal body in the RTBP become unstable due to the Poynting-Robertson Drag effect from the primaries.

- The equations of motion obtained were perturbed further by introducing the parameters ϕ $(\phi = 1+\varepsilon)$ to the coriolis force and ψ $(\psi = 1+\varepsilon')$ to the centrifugal force. Its libration points were found to be influenced by the centrifugal force only. The coefficients of the characteristics equation corresponding to the variational equations of motion were seen to depend on the perturbing parameters, therefore the roots are affected by them and hence the stability of the system.

- It was discovered that the critical mass value μ_c does not exist in the interval $0 < \mu < 1/2$ for this particular system. The roots of the characteristics equation were determined and used to obtain the equation of the period of oscillation (T) which were not affected by the small perturbations in the coriolis and centrifugal forces and oblateness but on the PR-drag force. Furthermore, the orientation or direction of the orbit (α) was seen to be elliptic in nature and the semi-axes, (a', b') are found to be influenced by all the perturbing factors.

- The results obtained were verified by computing for the Kruger-60 and $RXJ\,0450{,}1-5856$ binary system. It was observed that the x coordinate is not affected by the change in value ε' of the centrifugal force while the values of the y coordinate decreases with increase in ε' thereby affecting the isosceles triangle obtained from other generalization. These can be seen in the figures given.

The values of the coefficients of the characteristics equation were computed and used to determine the Hurwitz's determinants, $D_{i'}s$. According to Routh and Hurwitz criteria for stability, it was observed that the system remains unstable.

Therefore in line with existing research, results of various generalizations involving small perturbations in the centrifugal force, radiation pressure forces, oblateness of primaries, Poynting-Robertson drag and even with the stabilizing nature of the coriolis force, it has been shown that the aforementioned are destabilizing forces and that this work is a generalization of the classical case and the work of others.

3 Contribution to Knowledge

This research work has answered the question on the stability of a small particle to be launched in the vicinity of oblate and radiating bodies, putting into consideration the Poynting-Robertson drag force and small perturbations in the coriolis and centrifugal forces.

Furthermore, this work would serve as form of reference to achieving more interesting and vital results in the subject area, Space Dynamics and would be of great and added value to researchers in space science and aerospace agencies.

4 Recommendation

Astrophysical evidence has revealed that the perturbing forces: oblateness, radiation pressure forces, PR-drag force, coriolis and centrifugal forces are all natural activities in our solar, extrasolar and stellar systems. A satellite (natural or artificial) is expected to navigate in the vicinity of the planets in our solar system in their stable orbits under the influence of these forces. Hence, our result provides information for Space/Astronomical Engineers to take into consideration, the destabilizing effects of all these small but significant perturbing forces when designing spacecraft that will navigate in the vicinity of the planets and binary stars. This work as a generalization of the classical case and the work of others is therefore recommended to serve as a form of reference to achieving more interesting and vital results in Space Dynamics and also an added value to researchers in space science and aerospace agencies.

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