

Multiple Perturbed Collocation Tau-method for Solving High Order Linear and Non-Linear Boundary Value Problems

Adebisi, A. F.

Department of Mathematical Physical Sciences,
Faculty of Basic and Applied Sciences,
Osun State University,
Osun State, Nigeria.

Taiwo, O. A.

Department of Mathematics,
Faculty of Physical Sciences,
University of Ilorin,
Ilorin, Nigeria

Adewumi, A. O.

Department of Mathematics,
Faculty of Sciences,
Obafemi Awolowo University, Ile-Ife,
Osun State, Nigeria

Bello, K. A.

Department of Mathematics,
Faculty of Physical Sciences,
University of Ilorin, Ilorin, Nigeria

Abstract

This paper is concerned with the numerical solution of high order linear and nonlinear boundary value problems of ordinary differential equations by Multiple Perturbed Collocation Tau Method (MPCTM). We assumed a perturbed approximate solution in terms of Chebyshev Polynomial basis function which is substituted into the special class of the problem considered. Thus, resulting into n -folds integration. After evaluation of n -fold integration, the resulting equation is then collocated at equally spaced interior points and the unknown constants in the approximate solution are then obtained by Gaussian elimination method which are then substituted back into the approximate solution. The proposed method is tested on several numerical examples, the approximate solution is in agreement with the exact solution. The approximate results obtained by the proposed method confirm the convergence of numerical solutions and are compared favorably with the existing methods available in literature.

Keywords: Accuracy, Boundary Value Problem, Errors, Chebyshev, Collocation, Multiple, Perturbed, Ordinary Differential equations

Introduction

In recent years, the studies of differential equations have been developed very rapidly. Differential equations are classified into two types namely: Ordinary differential equation (ODE) and Partial differential equation (PDE).

Problems involving ODE arise frequently in many applied areas including engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics e. t. c. Rasheed (2004), Kodalbajoo and Sharma

$$y_N(x) = \int \int \dots \int [f(x, y_N, y'_N, y''_N, y'''_N, y^{(4)}_N, y^{(n-1)}_N)] dx dx \dots dx \\ + \int \int \dots \int \left[\sum_{r=1}^n \tau_r T_{N-r+1}(x) \right] dx dx \dots dx + \sum_{i=1}^{r=0} \frac{1}{i!} c_i x^i \quad (14)$$

Thus, collocating equation (14) at point $x=x_i$, where

$$x_i = a + \frac{(b-a)i}{N+n+2}; i = 1, 2, 3, \dots, N+n+1 \quad (15)$$

to get

$$y_N(x_i) = \int \int \dots \int [f(x, y_N, y'_N, y''_N, y'''_N, y^{(4)}_N, y^{(n-1)}_N)] dx dx \dots dx \\ + \int \int \dots \int \left[\sum_{r=1}^n \tau_r T_{N-r+1}(x) \right] dx dx \dots dx + \sum_{i=1}^{r=0} \frac{1}{i!} c_i x^i \quad (16)$$

Equation (16) gives rise to $(N+n+1)$ linear algebraic systems of equations in $(N+2n+1)$ unknown constants, n extra equations are obtained from the boundary conditions. Altogether, we have $(N+2n+1)$ linear algebraic system of equations in $(N+2n+1)$ unknown constants which are then solved by Gaussian Elimination Method. Thereafter, the values of the unknown constants obtained are then substituted into the assumed solution given in equation (8) to give the approximate solution.

Discussion of the multiple perturbed collocation tau method on non-linear boundary value problem

In this section, we employed Newton-Raphson-Kantorovich linearization process derived from the Taylor series expansion to linearize equation (6) since the proposed method could not handle equation (6) directly [See Adewumi, 2014].

Let

$$\frac{d^n y_N}{dx^n} = f(x, y, y', y'', y''', \dots, y^{(n-1)}) y, a \leq x \leq b \quad (17)$$

be a given non-linear general equation where f is generally a non-linear.

Let

$$G(y, y', y'', y''', \dots, y^{(n)}) y = f(x) \quad (18)$$

$$\frac{\partial G}{\partial y_k} = f_y y + y = f_{y_k} y_k + f$$

$$\frac{\partial G}{\partial y'_k} = f_{y'} y = f_{y'_k} y_k$$

$$\begin{aligned}\frac{\partial G}{\partial y_k} &= f_{y_k} y = f_{y_k} y_k \\ \frac{\partial G}{\partial y_k''} &= f_{y_k''} y = f_{y_k''} y_k \\ &\vdots\end{aligned}\quad (19)$$

$$\begin{aligned}\frac{\partial G}{\partial y_k^{n-1}} &= f_{y_k^{n-1}} y = f_{y_k^{n-1}} y_k \\ \frac{\partial G}{\partial y_k^n} &= 1\end{aligned}$$

The linearization process derived from the Taylor series expansion in G is

$$G + \Delta y_k \frac{\partial G}{\partial y_k} + \Delta y_k' \frac{\partial G}{\partial y_k'} + \Delta y_k'' \frac{\partial G}{\partial y_k''} + \Delta y_k''' \frac{\partial G}{\partial y_k'''} + \Delta y_k^{iv} \frac{\partial G}{\partial y_k^{iv}} + \dots + \Delta y_k^n \frac{\partial G}{\partial y_k^n} \equiv 0 \quad (20)$$

where

$$\Delta y_{N,k} = y_{N,k+1}^j - y_{N,k}^j, \quad j = 0, 1, 2, \dots, n \quad (21)$$

Substituting equations (19) and (21) into equation (20), we obtained

$$\begin{aligned}& \frac{d^n y_N}{dx^n} + \dots - f(x, y_k, y_k', y_k'', y_k''', \dots, y_k^{n-1}) y_k + (y_{k+1} - y_k)(f_{y_k} y_n + f) \\ & + (y_{k+1}' - y_k')(f_{y_k'} y_n) + (y_{k+1}'' - y_k'')(f_{y_k''} y_n) + \dots + (y_{k+1}^{n-1} - y_k^{n-1})(f_{y_k^{n-1}} y_n) \\ & + (y_{k+1}^n - y_k^n)(1) - f(x, y_k, y_k', y_k'', y_k''', \dots, y_k^{n-1}) y_k + y_k^n \\ & \equiv 0\end{aligned} \quad (22)$$

Expansion of equation (22) gives

$$\begin{aligned}& y_{k+1}^n + y_{k+1}^{n-1} f_{y_k^{n-1}} + \dots + y_{k+1}' f_{y_k'} y_n + y_{k+1} (f_{y_k} y_n + f) - \\ & \left[\left(y_k^{n-1} f_{y_k^{n-1}} y_n + \dots + y_k'' f_{y_k''} y_n + y_k' f_{y_k'} y_n + y_k f \right) + y_k f \right] \\ & - f(x, y_k, y_k', y_k'', y_k''', \dots, y_k^{n-1}) y_k + y_k^n \equiv 0\end{aligned} \quad (23)$$

where N is the order of the problem considered i.e. $4, 5, 6, \dots$, $k = 0, 1, 2, 3, \dots$ is the number of iterations. Equation (23) is the n th general linearized equation of the non-linear problems considered.

Initial approximation scheme

The initial approximation $y_k(x)$ are assumed to be the simplest polynomials and it depends on the order of the problems i. e.

$$\text{Order2problem} = 1 + x$$

$$\text{Order3problem} = 1 + x + \frac{Ax^2}{2}$$

$$\text{Order4problem} = 1 + x + \frac{Ax^2}{2} + \frac{Bx^3}{3!}$$

$$\text{Order5problem} = 1 + x + \frac{Ax^2}{2} + \frac{Bx^3}{3!} + \frac{Cx^4}{4!}$$

⋮

$$\text{Ordernproblem} = \frac{x^0}{0!} + \frac{x^{n-(n-1)}}{1!} + \frac{Ax^{n-(n-2)}}{2!} + \dots + \frac{Zx^{n-1}}{n-1!}$$

The unknown parameters $A, B, \dots, Z$ are obtained using the end boundary conditions

$$y(0) = a$$

$$y(1) = b$$

Numerical examples

Numerical Example 1

Consider the following linear fourth order boundary value problem (Syed et al., (2006))

$$y^{iv}(x) = y(x) + y''(x) + e^x(x-3)$$

with boundary conditions

$$y(0) = 1, y'(0) = 0, y(1) = 0, y'(1) = -e$$

The exact solution is given as $y(x) = (1-x)e^x$

Numerical Example 2

Consider the following linear fifth order boundary value problem (Noor \& Mohyud-Din, (2007))

$$y^v(x) = y(x) - 15e^x - 10xe^x, 0 < x < 1$$

subject to the boundary conditions

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y(1) = 0, y'(1) = -e$$

The exact solution is given as $y(x) = x(1-x)e^x$

Numerical Example 3

Consider the following sixth order linear boundary value problem

$$y^{vi}(x) = y(x) - 6e^x, 0 \leq x \leq 1$$

subject to the boundary conditions

$$y(0) = 1, y''(0) = -1, y^{iv}(0) = -3, y(1) = 0, y''(1) = -2e, y^{iv}(1) = -4e$$

The exact solution is given as $y(x) = (1-x)e^x$

Numerical Example 4

Consider the following non-linear fourth order boundary value problem (Olagunju, (2012))

$$y^{iv}(x) + y(x) + y(x)y''(x) - (y'(x))^2 = e^x(1+x^2-4x) - 4x^2$$

with the boundary conditions

$$y(0) = y'(0) = 1, y(1) = 1 + e, y'(1) = 2 + e$$

The exact solution is given as $y(x) = x^2 + e^x$

Numerical Example 5

Consider the following non-linear fifth order boundary value problem (Ayyaz et al., (2013))

$$y^{(5)}(x) = e^{-x}y^2(x) - 15e^x - 10xe^x, 0 < x < 1$$

subject to the boundary conditions

$$y(0) = y'(0), y''(0) = 1, y(1) = y'(1) = e$$

The exact solution is given as $y(x) = e^x$

Numerical Example 6

Consider the following sixth order non-linear boundary value problem

$$y^{(6)}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, 0 \leq x \leq 1$$

subject to the boundary conditions

$$y(0) = 1, y'(0) = -1, y''(0) = 1, y(1) = \frac{1}{e}, y'(1) = -\frac{1}{e}, y''(1) = \frac{1}{e}$$

The exact solution is given as $y(x) = e^{-x}$

Remark: We have defined absolute error as

$$\text{Error} = |y(x) - y_N(x)|; a \leq x \leq b$$

6.0 NUMERICAL RESULTS AND ERRORS

Table 1: Numerical Results Obtained for Example 1 (CASE N=10)

x	Exact	Approximate	Error	Syed et al., (2006)
0	1.0000000000	0.9999999999	1.0E-10	0.000000000
0.1	0.9946538262	0.9946538261	1.0E-10	2.0000E-10
0.2	0.9771222064	0.9771222061	3.0E-10	7.0052E-10
0.3	0.9449011656	0.9449011651	5.0E-10	1.3501E-09
0.4	0.8950948188	0.8950948183	5.0E-10	2.0025E-09
0.5	0.8243606355	0.8243606350	5.0E-10	2.0025E-09
0.6	0.7288475200	0.7288475200	0.00000	2.5085E-09
0.7	0.6041258121	0.6041258122	1.0E-10	2.7220E-09
0.8	0.4451081856	0.4451081856	0.00000	5.0000E-10
0.9	0.2459603111	0.2459603113	2.0E-10	9.9476E-04
1.0	0.0000000000	0.0000000000	1.0E-10	0.000000000

$$y^{iv}(x) + y(x) + y(x)y''(x) - (y'(x))^2 = e^x(1+x^2-4x) - 4x^2$$

with the boundary conditions

$$y(0) = y'(0) = 1, y(1) = 1 + e, y'(1) = 2 + e$$

The exact solution is given as $y(x) = x^2 + e^x$

Numerical Example 5

Consider the following non-linear fifth order boundary value problem (Ayyaz et al., (2013))

$$y''(x) = e^{-x}y^2(x) - 15e^x - 10xe^x, 0 < x < 1$$

subject to the boundary conditions

$$y(0) = y'(0), y''(0) = 1, y(1) = y'(1) = e$$

The exact solution is given as $y(x) = e^x$

Numerical Example 6

Consider the following sixth order non-linear boundary value problem

$$y^{vi}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, 0 \leq x \leq 1$$

subject to the boundary conditions

$$y(0) = 1, y'(0) = -1, y''(0) = 1, y(1) = \frac{1}{e}, y'(1) = -\frac{1}{e}, y''(1) = \frac{1}{e}$$

The exact solution is given as $y(x) = e^{-x}$

Remark: We have defined absolute error as

$$\text{Error} = |y(x) - y_N(x)|; a \leq x \leq b$$

6.0 NUMERICAL RESULTS AND ERRORS

Table 1: Numerical Results Obtained for Example 1 (CASE N=10)

x	Exact	Approximate	Error	Syed et al., (2006)
0	1.0000000000	0.9999999999	1.0E-10	0.00000000
0.1	0.9946538262	0.9946538261	1.0E-10	2.0000E-10
0.2	0.9771222064	0.9771222061	3.0E-10	7.0052E-10
0.3	0.9449011656	0.9449011651	5.0E-10	1.3501E-09
0.4	0.8950948188	0.8950948183	5.0E-10	2.0025E-09
0.5	0.8243606355	0.8243606350	5.0E-10	2.0025E-09
0.6	0.7288475200	0.7288475200	0.00000	2.5085E-09
0.7	0.6041258121	0.6041258122	1.0E-10	2.7220E-09
0.8	0.4451081856	0.4451081856	0.00000	5.0000E-10
0.9	0.2459603111	0.2459603113	2.0E-10	9.9476E-04
1.0	0.0000000000	0.0000000000	1.0E-10	0.00000000

(Ayyaz et

1

Table 2: Numerical Results for Example 2 (CASE N=6)

x	Exact	Approximate	Error (N=6)	Noor & Mohyud-Din, (2007)
0	0.000000000000	5.9103911E-10	5.9103911E-10	0.000000
0.1	0.09946538262	0.09946538264	2.0000000E-11	4.00E-07
0.2	0.19542444130	0.19542444150	2.0000000E-10	3.10E-06
0.3	0.28347034970	0.28347034990	2.0000000E-10	8.80E-06
0.4	0.35803792750	0.35803792740	1.1000000E-10	1.60E-05
0.5	0.41218031780	0.41218031650	1.3000000E-09	2.30E-05
0.6	0.43730851200	0.43730851000	2.0000000E-09	2.70E-05
0.7	0.42288806850	0.42288806630	2.2000000E-09	2.60E-05
0.8	0.35608654850	0.35608654700	1.5000000E-09	1.80E-05
0.9	0.22136428000	0.22136427920	8.0000000E-10	6.90E-06
1.0	0.000000000000	4.4110039E-10	4.4110039E-10	0.000000

Table 3: Numerical Results for Example 2 (CASE N=10)

x	Exact	Approximate	Error (N=10)	Noor & Mohyud-Din, (2007)
0	0.000000000000	5.9103911E-12	5.9103911E-10	0.000000
0.1	0.09946538262	0.09946538264	2.0000000E-11	4.00E-07
0.2	0.19542444130	0.19542444150	2.0000000E-10	3.10E-06
0.3	0.28347034970	0.28347034990	2.0000000E-10	8.80E-06
0.4	0.35803792750	0.35803792740	1.1000000E-10	1.60E-05
0.5	0.41218031780	0.41218031650	1.3000000E-09	2.30E-05
0.6	0.43730851200	0.43730851000	2.0000000E-09	2.70E-05
0.7	0.42288806850	0.42288806630	2.2000000E-09	2.60E-05
0.8	0.35608654850	0.35608654700	1.5000000E-09	1.80E-05
0.9	0.22136428000	0.22136427920	8.0000000E-10	6.90E-06
1.0	0.000000000000	4.4110039E-10	4.4110039E-10	0.000000

N=10)

06)

Table 4: Numerical Results for Example 3 (CASE N=8)

x	Exact	Approximate	Errors		
			N=8	Mukesh et al., (2013)DTM	Mukesh et al., (2013) HPM
0	1.0000000000	0.9999999999	6.0000E-10	0.000000	0.00000000
0.1	0.9946538262	0.9946529448	8.8140E-07	6.57E-08	4.0933E-04
0.2	0.9771222064	0.9771207066	1.4998E-06	1.26E-08	7.7820E-04
0.3	0.9449011656	0.9448994851	1.6805E-06	1.74E-08	1.0705E-03
0.4	0.8950948188	0.8950934204	1.3984E-06	2.08E-07	1.2500E-03
0.5	0.8243606355	0.8243598450	7.9050E-07	2.21E-07	1.3220E-03
0.6	0.7288475200	0.7288474200	1.0000E-07	2.14E-07	1.2570E-03
0.7	0.6041258121	0.6041262287	4.1660E-07	1.85E-07	1.2579E-03
0.8	0.4451081856	0.4451087772	5.9160E-07	1.36E-07	7.7820E-05
0.9	0.2459603111	0.2459607186	4.0750E-07	7.23E-08	4.0933E-04
1.0	0.0000000000	1.600000E-09	1.6000E-09	*	*

* Denote results are not available at those points

Table 5: Numerical Results for Example 3 (CASE N=10)

x	Exact	Approximate	Errors		
			N=10	Mukesh et al., (2013)DTM	Mukesh et al., (2013) HPM
0	1.0000000000	0.9999999999	1.0E-10	0.000000	0.00000000
0.1	0.9946538262	0.9946538269	7.0E-10	6.57E-08	4.0933E-04
0.2	0.9771222064	0.9771222077	1.3E-09	1.26E-08	7.7820E-04
0.3	0.9449011656	0.9449011666	1.0E-09	1.74E-08	1.0705E-03
0.4	0.8950948188	0.8950948195	7.1E-10	2.08E-07	1.2500E-03
0.5	0.8243606355	0.8243606354	1.0E-10	2.21E-07	1.3220E-03
0.6	0.7288475200	0.7288475194	6.0E-10	2.14E-07	1.2570E-03
0.7	0.6041258121	0.6041258111	1.0E-09	1.85E-07	1.2579E-03
0.8	0.4451081856	0.4451081847	9.0E-10	1.36E-07	7.7820E-05
0.9	0.2459603111	0.2459603109	2.0E-10	7.23E-08	4.0933E-04
1.0	0.0000000000	4.000000E-10	4.0E-10	*	*

* Denote results are not available at those points.

Table 6: Numerical Results for Example 4 (CASE N=4)

x	Exact	Approximate			Error (k=2)	Olagunju CCAM (2012)
		(k=0)	(k=1)	(k=2)		
0	1.000000000	1.000000000	1.000000000	0.999999999	0.000000000	0.000000000
0.1	1.115170918	1.115266617	1.115294344	1.115281206	9.8577E-05	3.8165E-04
0.2	1.261402758	1.261673466	1.261761094	1.261719579	2.7980E-04	1.3736E-03
0.3	1.439858808	1.440268658	1.440419611	1.440348096	4.2552E-04	2.1772E-03
0.4	1.651824698	1.652283789	1.652480955	1.652387545	4.7959E-04	2.7372E-03
0.5	1.898721271	1.899133943	1.899347879	1.899246524	4.3488E-04	2.4812E-03
0.6	2.182118800	2.182417682	2.182616485	2.182521439	3.1935E-04	3.5080E-03
0.7	2.503752707	2.503917062	2.504068016	2.503996501	1.8003E-04	2.1937E-03
0.8	2.865540928	2.865597617	2.865685247	2.865643732	6.5785E-05	1.2109E-03
0.9	3.269603111	3.269602370	3.269636097	3.269622962	8.1380E-06	1.6087E-04
1.0	3.718281828	3.718281826	3.718281828	3.718281829	1.0000E-09	1.0000E-09

Table 7: Numerical Results for Example 4 (CASE N=6)

x	Exact	Approximate			Error (k=2)	Olagunju CCAM (2012)
		(k=0)	(k=1)	(k=2)		
0	1.000000000	1.000000000	1.000000001	1.000000000	0.000000000	0.000000000
0.1	1.115170918	1.115153721	1.115216038	1.1151713361	2.443E-06	3.8165E-04
0.2	1.261402758	1.261373014	1.261324083	1.261409284	6.526E-06	1.3736E-03
0.3	1.439858808	1.439844720	1.439422431	1.439867374	8.566E-06	2.1772E-03
0.4	1.651824698	1.651847877	1.650966621	1.651831487	6.789E-06	2.7372E-03
0.5	1.898721271	1.898783889	1.897591202	1.898722768	1.497E-06	2.4812E-03
0.6	2.182118800	2.182203879	2.180998544	2.182113650	1.150E-06	3.5080E-03
0.7	2.503752707	2.503833229	2.502914695	2.503742817	9.890E-06	2.1937E-03
0.8	2.865540928	2.865593313	2.865112276	2.865531129	9.799E-06	1.2109E-03
0.9	3.269603111	3.269620412	3.269500434	3.269598517	4.594E-06	1.6087E-04
1.0	3.718281828	3.718281830	3.718281828	3.718281827	1.000E-10	1.0000E-09

Table 8: Numerical Results for Example 5 (CASE N=6)

x	Exact	Approximate		Errors		
		(k=0)	(k=1)	(k=1)	Noor et al., (2009)	Ayyaz et al., (2013)
0	1.000000000	1.000000001	1.000000000	0.0000000	0.0000000	0.000000000
0.1	1.105170918	1.105170747	1.105170788	1.300E-07	2.30E-07	9.72896E-06
0.2	1.221402758	1.221401823	1.221402153	6.050E-07	1.60E-06	6.24961E-05
0.3	1.349858808	1.349856720	1.349857734	1.074E-06	4.60E-06	1.64145E-04
0.4	1.491824698	1.491821500	1.491823544	1.154E-06	8.90E-06	2.90616E-04
0.5	1.648721271	1.648717344	1.648720494	7.770E-07	1.30E-05	4.00814E-04
0.6	1.822118800	1.822114696	1.822118597	2.030E-07	1.60E-05	4.50825E-04
0.7	2.013752707	2.013749056	2.013752912	2.050E-07	1.60E-05	4.09640E-04
0.8	2.225540928	2.225538379	2.225541185	2.570E-07	1.20E-05	2.76514E-04
0.9	2.459603111	2.459602107	2.459603194	8.300E-08	5.10E-06	1.000168E-04
1.0	2.718281828	2.718281828	2.718281828	0.0000000	0.0000000	0.000000000

Table 9: Numerical Results for Example 5 (CASE N=10)

x	Exact	Approximate		Errors		
		(k=0)	(k=1)	(k=1)	Noor et al., (2009)	Ayyaz et al., (2013)
0	1.000000000	1.000000000	0.999999999	1.00E-10	0.0000000	0.000000000
0.1	1.105170918	1.105170526	1.105170936	1.80E-08	2.30E-07	9.72896E-06
0.2	1.221402758	1.221401657	1.221402818	6.00E-08	1.60E-06	6.24961E-05
0.3	1.349858808	1.349858381	1.349858810	4.30E-08	4.60E-06	1.64145E-04
0.4	1.491824698	1.491826521	1.491824627	7.10E-08	8.90E-06	2.90616E-04
0.5	1.648721271	1.648725457	1.648721053	2.18E-07	1.30E-05	4.00814E-04
0.6	1.822118800	1.822123888	1.822118514	2.86E-07	1.60E-05	4.50825E-04
0.7	2.013752707	2.013756841	2.013752494	2.13E-07	1.60E-05	4.09640E-04
0.8	2.225540928	2.225543158	2.225540873	5.50E-08	1.20E-05	2.76514E-04
0.9	2.459603111	2.459603746	2.459603144	3.30E-08	5.10E-06	1.000168E-04
1.0	2.718281828	2.718281828	2.718281828	0.0000000	0.0000000	0.000000000

Discu

usually
in clos
to obt
points

the bet
exists.

Table 10: Numerical Results for Example 6 (CASE N=8)

x	Exact	Approximate		Errors	
		(k=0)	(k=1)	k=1	Adewumi, EPBICM (2014)
0	1.0000000000	1.0000000000	1.0000000000	0.00000000	0.00000000
0.1	0.9048374180	0.9048373211	0.9048374230	5.000E-09	3.460E-07
0.2	0.8187307531	0.8187303670	0.8187307426	1.050E-08	2.010E-07
0.3	0.7408182207	0.7408171265	0.7408181106	1.101E-07	4.540E-06
0.4	0.6703200460	0.6703353620	0.6703197517	2.943E-07	6.530E-06
0.5	0.6065306597	0.6065291079	0.6065301790	4.807E-07	6.800E-06
0.6	0.5488116361	0.5488104133	0.5488110789	5.572E-07	5.200E-06
0.7	0.4965853038	0.4965845942	0.4965848414	4.624E-07	2.930E-06
0.8	0.4493289641	0.44932872028	0.4493287198	2.443E-07	1.050E-06
0.9	0.4065696597	0.4065696234	0.4065696091	5.060E-08	1.590E-07
1.0	0.3678794411	0.3678794411	0.3678794412	1.000E-10	8.280E-10

Table 11: Numerical Results for Example 6 (CASE N=10)

x	Exact	Approximate			Errors	
		(k=0)	(k=1)	(k=2)	k=2	Adewumi, EPBICM(2014)
0	1.0000000000	0.9999999999	1.0000000000	1.0000000000	0.00000000	0.00000000
0.1	0.9048374180	0.9048375309	0.9048374687	0.9048380400	5.070E-08	3.46E-07
0.2	0.8187307531	0.8187311492	0.8187309715	0.8187339230	2.184E-07	2.01E-07
0.3	0.7408182207	0.7408188774	0.7408186303	0.7408253368	4.096E-07	4.56E-06
0.4	0.6703200460	0.6703209449	0.6703205927	0.6703312833	5.460E-07	6.53E-06
0.5	0.6065306597	0.6065317998	0.6065312478	0.6065447342	5.880E-07	6.80E-06
0.6	0.5488116361	0.5488129500	0.5488121627	0.5488605220	5.266E-07	5.20E-06
0.7	0.4965853038	0.4965865814	0.4965856895	0.4965970388	3.857E-07	2.93E-06
0.8	0.4493289641	0.4493298437	0.4493291697	0.4493355963	2.056E-07	1.05E-06
0.9	0.4065696597	0.4065699109	0.4065697085	0.4065712356	4.880E-08	1.59E-07
1.0	0.3678794411	0.3678794410	0.3678794414	0.3678794411	0.00000000	8.28E-10

Discussion of results

Higher-order linear and non-linear ordinary differential equations are usually difficult to solve analytically i.e the exact solutions are difficult to obtain in closed form. For this reason, the proposed method serve as alternative method to obtain the exact solution for different values of N at equally spaced interior points without extra computation at cost.

The result obtained show that the higher the degree of approximant used the better the accuracy of the proposed method to the exact solution where such exists.

Conclusion

We have used Multiple Perturbed Collocation Tau Method to solve high order linear and nonlinear boundary value problems by employing Chebyshev basis functions as the approximate solution.

Six examples are considered with the method and we observed from the results obtained that as N increases, the results obtained by the proposed method are getting closer to the exact solutions for the case where the exact solutions are known in closed form.

References

- Abassy, T. A., El-Tawil, M. A and Zoheiry, H. E. (2007d): Solving Nonlinear Partial Differential Equations Using the Modified Variational Iteration Pade Technique. *Journal of Computational and Applied Mathematics*, 207, 73-91.
- Abassy, T. A., El-Tawil, M. A and Zoheiry, H. E. (2007c): Toward a Modified Variational iteration Method. *Journal of Computational and Applied Mathematics*, 207, 137-147.
- Abassy, T. A. (2010a): Improved Adomian Decomposition Method. *Computers and Mathematics with Applications*. 59,42-54.
- Abassy, T. A. (2010b): Modified Variational Iteration Method (Nonlinear Homogeneous Initial Value Problem). *Computer and Mathematics Applications*, 59, 912-918.
- Abbasbandy, S. Magyari, E, and Shivanian, E. (2009): The Homotopy Analysis Method for Multiple Solution of Nonlinear Boundary Value Problems, *Comm. Nonlinear Sci. Numer. Simul.*, 14, 3530-3536.
- Ayaz, F. (2013): On the Two-dimensional Differential Transform Method. *Journal of Computation and Applied Mathematics*, 143, 361-374.
- Adewumi, A. O. (2014): Tau Homotopy and Embedded Perturbed Integral Collocation Methods For Solving Boundary Value Problems. Ph. D Thesis, University of Ilorin. (Unpublished).
- Adibi, H. and Assani, P. (2010): Chebyshev Wavelets Method for Numerical Solution of Fredholm Integral Equations of the First Kind. *Maths Problem Eng.*
- Adomian, G. (1976): Nonlinear Stochastic Differential Equations. *Journal of Mathematics Analysis and Application*, 55, 441-452.
- Albasiny, E. I and Eskins, W. O (1969): Cubic Spline Solutions to Two Points Boundary Value Problems. *Computer Journal*, 151.
- Ali Shokri (2014): One and Two Step New Hybrid Methods for the Numerical Solution of First Order Value Problem. *Acta Universitatis Matthiae Belli, Series Mathematics*.
- Allan, F. M. (2007): Derivation of the Adomian Decomposition Method using the Homotopy Analysis Method. *Applied Mathematics and Computation*, 190, 6-14.
- Ariel, P. D., Hayat, T. and Asghar, S. (2006): Homotopy Perturbation Method and Axisymmetric Flow over a Stretching Sheet. *International Journal of Nonlinear Sci. and Num. Simul.*, 7, 399-406.

- Ariel, P. D. (2007a): Axisymmetric Flow Due to a Stretching Sheet with Partial Slip. *Comput. and Math. with Applications*, 54, 1169-1183.
- Ariel, P. D. (2009): Extended Homotopy Perturbation Method and Computation of flow Past a Stretching Sheet. *Comp. and Math. with Applications*, 58, 2402-2409.
- Ayyaz, A., Muhammed, A. I. and Syed, T. M. (2003): Hermite Wavelets Method for Boundary Value Problems. *International Journal of Modern Applied Physics*, 3(1), 38-47.
- Babolian, E. and Fahahzadeh, F. (2007): Numerical Solution of Differential Equation by Using Chebyshev Wavelet Operational Matrix of Integration. *Appl. Math. Comput.*, 188, 417-426.
- Bainov, D. D., Dimitrova, M. B. and Dishliev, A. D. (2000): Oscillation of the Bounded Solutions of impulsive Differential Difference Equations of Second Order. *Appl. Math. Comput.*, 114, 61-68.
- Catal, S. (2008): Solution of Free Vibration Equations of Beam on Elastic Soil by using Differential Transform Method. *Applied Mathematical Modeling*, 32, 1744-1757.
- Cao, J and Wang, J. (2004): Delay-Dependent Robust Stability of Uncertain Nonlinear Systems with Time Delay. *Appl. Math. Comput.*, 154, 289-297.
- Collatz, L. (1960): *The Numerical Treatment of Differential Equation*, Springer-Verlag, Berlin.
- Daries, A. R., Karageoghis, A. and Philips, T. N. (1988): Spectral Galerkin Methods for the Primary Two Points Boundary Value Problems in Modeling Viscoelastic Flow. *Int. J. Numer. Methods Eng.*, 26, 647-662.
- David, O. O. A. (1992): On Some Continuous Linear Multistep Method for IVPs, Phd Thesis (Unpublished), University of Ilorin, Ilorin, Nigeria.
- Dehghan, M. and Saadatmandi, A. (2008): Chebyshev Finite Difference Method for Fredholm Integro-Differential Equation. *Int. J. Comput. Math.*, 85, 123-129.
- Elmer, C. E. and Van Vleck, E. S. (2001): Travelling Wave solutions for Bistable Differential Difference Equations with Periodic Diffusion. *SIAM J. Appl. Math.*, 61, 1648-1679.
- Elmer, C. E. and Van Vleck, E. S. (2002): A Variant of Newtons Method for Solution of Travelling Wave Solutions of Bistable Differential.
- El-Kady and Khalil, M. (2011): Chebyshev Pseudo spectral Approximation for Solving Higher-order boundary Value Problems. *Applied Mathematics and Information Sciences*, 5, 342-357.
- Fazai-i-Haq, Arshed Ali and Iltaf Hussain (2011): Solution of Sixth-order Boundary Value Problems by Collocation Method Using Haar Wavelets. *International Journal of Physics Sciences*, 7(43), 5729-5735.
- Frazer, R. A., Jone, W. P. and Scan, S. W. (1937): *Approximation Functions and to the Solutions of Differential Equations*, Great Britain Air Ministry Aero. Res-Comm. Tech, 517-549.
- Fyte, D. H. (1969): The Use of Cubic Spline in the Solution of Two-Points Boundary Value Problems. *Computer Journal*, 12, 188.
- Fox and Parker (1968): *Chebyshev Polynomial in Numerical Analysis*, Oxford University Press, London
- Gamze, Y. and Mehmet, S. (2013): A Chebyshev Series Approximation for Linear

Second Order Partial Differential Equation with Complicated Condition. Gazi University Journal of Science (GUJ. Sci., 26(4), 515-525. al

Geng, F. (2010): A Modified Variational Iteration Method for Solving Riccati Differential Equations. Computers and Mathematics with Applications, 60, 1868-1872.

Grevat, B. S. (2005): Numerical Methods in Engineering and Science 7th Edition. Kanna Publisher Delhi.

Gulsu, M., Ozturk, Y. and Sezer, M. (2010): A New Collocation Method for Solution of the Mixed Linear Integro-Differential Difference Equations. Appl. Math. Comput., 216, 2183-2198.

Hashim, I. M., Chowdhury, S. H. and Mawa, S. (2008): On Multistage Homotopy Perturbation Method applied to Nonlinear Biochemical Reaction Models. Chaos Solitons and Fractals, 36, 823-827.

He, J. H. (1999a): Approximate Analytical Solution of Blasius Equations. Communications in Nonlinear Science and Numerical Simulations, 4(1), 75-78.

He, J. H. (1999b): Homotopy Perturbation Technique. Communications in Nonlinear Science and Numerical Simulations, 4(1), 75-78.

He, J. H. (1999c): Variational Iteration Method. A Kind of Nonlinear Analytical Techniques. Some Examples. International Journal of Nonlinear Mechanics, 34(4), 699-708.

He, J. H. (2000): Variational Iteration Method for Autonomous Ordinary Differential Systems. Appl. Math. Comput., 114, 115-123.

He, J. H. (2003): Homotopy Perturbation Method. A New Nonlinear Analytical Technique. Applied Mathematics and Computation, 135, 73-79.

Hermann Brunner (2004): Collocation Method for Volterra Integral and related Functional Differential Equation, Cambridge University Press, New York.

Hopkin, T. R. and Wait, R. (1978): A Comparison of Galerkin Collocation and the Method Lines for PDE's. International Journal Numer. Maths. Eng., 12(7), 1081-1107.

Hosseini, S. M. and Shahmorad, S. (2003): Numerical Solution of a Class of Integro-Differential Equations by the Tau Method with an Error Estimation. Appl. Math. Comput., 136, 559-570.

Kadalbajoo, M. K. and Sharma, K. K. (2002). Numerical Analysis of Boundary Value problems for Singularly Perturbed Differential Equations with Small Stiffs of Mixed Type. Journal Optim. Theory Appl., 115, 145-163.

Kadalbajoo, M. K. and Sharma, K. K. (2004). Numerical Analysis of Singularly Perturbed Delay Differential Equations with Layer Behavior. Appl. Math. Comput., 157, 11-28.

Karageophis, A., Phillips, T. N. and Davies, A. R. (1998): Special Collocation Method for the Primary Two-Points Boundary Value Problems in Modeling Viscoelastic Flows. Int. J. Numer. Math. Eng., 26, 805-813.

Lanzcos, C. (1956): Applied Analysis (New Jersey: Prentice Hall).

Lanzcos, C. (1957): Applied Analysis Prentice Hall, Endlewood Cliffs, New Jersey.

Liao (1992): The Proposed Homotopy Analysis Technique for the Solution of

condition.
15-525.al

cati
lications,

Edition.

ethod for
quations.

ultistage
chemical

uations.
ns, 4(1),

tions in

lytical
onlineair

Ordinary

lytical

related
w York.
and the
3., 12(7),

Class of
1 Error

oundary
uations
115, 145

larly
l. Math.

location
ems in
13.

Jersey.
n of

- Nonlinear Problems. PhD Thesis (Unpublished), Shangai Jiao Tong University.
- Liao, S. J. (2003b): Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman and Hall CRC Press.
- Liu, N. and Lin, E. B. (2009): Legendre Wavelet Method for Numerical Solution of Partial Differential Equations, 26(1), 496-505.
- Maleknajad, K. I. and Rabbaiu, M. (2007): Using Chebyshev Multiwavelet Basis for Solving Integral Equation. Int. J. Math., Manuscripts: 1, 104-110.
- Mason, J. C. and Handscomb, D. C. (2003): Chebyshev Polynomials Chapman & Hall-ERC, Boca Ratons London, New York, North Washington D. C.
- Muhammed, A. N., Syed, T. M. (2008): A New Approach to Fifth Order Boundary Value Problems. International Journal of Non-linear Sciences, 7(2), 143-148.
- Mukesh Groover and Arnn Kumar Tomer (2013): A New Approach to Evaluate Higher Order Differential equations by Differential Transform Method and Homotopy Perturbation Method Using Boundary Value Problems. Elair Appl. Math., 55, 13351-13355.
- Noor, M. A., Noor, K. I. and Mohyud-Din, S. T. (2009): Variational Iteration Method for Solving Sixth Order Boundary Value Problems. Comm. Non-linear Sci. Numer. Simul., 14, 2571-2580.
- Noor, M. A. and Mohyud-Din, S. T. (2008): Variational Iteration Method for Solving Higher- Order Boundary Value Problems. App. Math. Comput., 189, 1929-1942.
- Noor, M. A. and Mohyud-Din, S. T. (2007): An Efficient Algorithm for Solving Fifth Order Boundary Value Problems. Math. Comput. Modeling, 45, 954-964.
- Momani, S. and Odibat, Z. (2008): A Generalized Differential Transform Method for Linear Partial Differential Equations of Fractional Order. Applied Mathematics Letters, 21, 194-199.
- Odibat, Z. M. (2007): A New Modification of the Homotopy Perturbation Method for Linear and Nonlinear Operator. Applied Mathematical and Comput., 189, 746-753.
- Olagunju, A. S. (2012): Chebyshev Collocation Approximation Methods for Numerical Solution of Boundary Value Problems. Ph. D. Thesis (Unpublished). University of Ilorin, Ilorin, Nigeria.
- Okoro, F., Onumanyi, P. and Taiwo, O. A. (1990): A Collocation Approximation of Singularly Perturbed Second Order Differential Equations. Computational Mathematics, 40(3), 205-211.
- Ortiz, E. L. and Aliabadi, M. H. (1998): Numerical Treatment of Moving and Free Boundary Value problems with the Tau Method. Comp. Math. Appl., 35(8), 53-61.
- Raji, M. T. (2012): Collocation Approximation Methods for Solving Special Linea and Nonlinear Integro-Differential Equations. PhD Thesis (Unpublished), University of Ilorin, Ilorin, Nigeria.
- Rasheed, M. T. (2004): Numerical Solution of Functional Differential, Integral and Integro-differential Equations. Appl. Numer. Math., 156, 485-492.
- Rashidi, M. M. (2009): The Modified Differential Transform Method for Solving

- MHD Boundary Layer Equations. Computer Physics Communications, 180, 2210-2217.
- Syed, T. M. and Muhammed, A. N. (2006): Homotopy Perturbation Method for Solving Fourth Order Boundary Value Problems. Mathematical Problems Engineering, 1-14.
- Synader, M. A. (1966): Chebyshev Methods in Numerical Approximation. Prentice Hall, Inc. London.
- Taiwo, O. A. and Adebisi, A. F. (2014): Multiple Perturbed Collocation Tau Method for Special Class of Higher Order Linear Fredholm and Volterra Integro-Differential Equations. Projournal of Physical Science Research (PPSR). 2(3), 13-22.
- Taiwo, O. A. (1988): A Comparison of Three Global Collocation Methods for Boundary Value Problems with Some Error Estimation. AMSE Rev. AMSE Press, 7(1), 53-63.
- Taiwo, O. A. and Evans, D. J. (1997): On the Singularly Perturbed BVP by a Chebyshev Method. Inter. Journal Computer Maths., 71, 521-529.
- Taiwo, O. A. (2005): Comparison of Collocation Method's for the Solution of Second Order Nonlinear Boundary Value Problems. Int. Journal of Computer Mathematics, 82(11), 1389-1401.
- Taiwo, O. A. (1991): Collocation Methods for Singularly Perturbed Ordinary Differential Equations. University of Ilorin, Ilorin, Nigeria, Ph. D. Thesis (Unpublished).
- Taiwo, O. A. (1986): A Computational Method for ordinary Differential Equation and Error Estimation, University of Ilorin, Ilorin, Nigeria, M. Sc. Dissertation (Unpublished).
- Wazwaz, A. M. (2001a): Analytical Treatment for Variable Coefficient Fourth Order Parabolic Partial Differential Equations. Applied Mathematics and Computation, 23, 219-227.
- Wazwaz, A. M. The Numerical Solution of Fifth order Boundary Value problems.
- Zhao, J. and Corless, R. M. (2006): Compact Finite Difference Method for Integro-differential Equations. Appl. Math. Comput., 177, 271-288.

Tsal

Gaba

magai

bari ir

ma'an

kuma

zantuk

bayyar

al'umr

rayuw:

samu v

maganj

fusaha

wafoki

maganz

gargaji

tsakanin

iya sam

(2002, 2004), Bainov et al.(2000) and Cao and Wang (2004). The study of differential equations has great interest in contemporary research work. Several numerical methods such as the Successive Approximation, Adomian Decomposition, Chebyshev and Taylor Collocation, Haar Wavelets, Tau and Walsh series, Variational Iteration, Spectral Galerkin, Homotopy Perturbation, Legendre Wavelet e.t.c to mention a few [See Ortiz and Aliabadi (1988), Hosseini and Shahmorad (2003), Zhao and Corless (2006), Maleknejad (2007), Synader (1966), Gulsu et al. (2010), Noor and Muhy-din (2007, 2008), He (2000), Hopkin and Wait(1978), Mukesh and Arnn (2013), Liu and Lin (2009)] are used for their solutions.

Chebyshev Polynomials

The Chebyshev Polynomials of the first kind denoted by $T_N(x)$ is defined as

$$T_N(x) = \cos\{N\cos^{-1}x\}; \quad -1 \leq x \leq 1 \quad (1)$$

and the recurrence relation is given by

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x), \quad N \geq 1 \quad (2)$$

where N is the degree of the polynomial.

In general, the Chebyshev Polynomial valid in the interval $a \leq x \leq b$ is given as

$$T_N(x) = \cos\left[N\cos^{-1}\left(\frac{2x-b-a}{b-a}\right)\right] \quad (3)$$

and the recurrence relation is given as

$$T_{N+1}(x) = 2\left(\frac{2x-b-a}{b-a}\right)T_N(x) - T_{N-1}(x) \quad (4)$$

Few terms of the shifted Chebyshev Polynomials valid in the interval $[0,1]$ are given below:

$$T_0(x) = 1$$

$$T_1(x) = 2x - 1$$

$$T_2(x) = 8x^2 - 8x + 1$$

$$T_3(x) = 32x^3 - 48x^2 + 18x - 1$$

$$T_4(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

$$T_5(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

$$T_6(x) = 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$$

$$T_7(x) = 8192x^7 - 28672x^6 + 39424x^5 - 26990x^4 + 9408x^3 - 1568x^2 + 98x - 1$$

$$T_8(x) = 32768x^8 - 131072x^7 + 212992x^6 - 40224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1$$

Probl
For th
and nc

and

which

where

and f_1
the res

Methc
We dis
section
equatic

Discus
bound

(MPCT
stated i

Here, ϵ
to be d
Equatic

Problem Considered

For the purpose of our discussion, we deal with the following high order linear and non-linear boundary value problems of the type

$$\frac{d^n y}{dx^n} + \dots = f_1(x, y, y', y'', y''', y^{iv}, \dots, y^{n-1}) \quad (5)$$

and

$$\frac{d^n y}{dx^n} + \dots = f_2(x, y, y', y'', y''', y^{iv}, \dots, y^{n-1})y \quad (6)$$

which holds in some intervals $a \leq x \leq b$ together with the boundary conditions

$$\left. \begin{array}{ll} y(a) = A_0 & y(b) = B_0 \\ y'(a) = A_1 & y'(b) = B_1 \\ y''(a) = A_2 & y''(b) = B_2 \\ y'''(a) = A_3 & y'''(b) = B_3 \\ y^{iv}(a) = A_4 & y^{iv}(b) = B_4 \\ \vdots & \vdots \\ y^{n-1}(a) = A_{n-1} & y^{n-1}(b) = B_{n-1} \end{array} \right\} \quad (7)$$

where $A_0, A_1, A_2, A_3, \dots, A_{n-1}$ and $B_0, B_1, B_2, B_3, \dots, B_{n-1}$ are real finite constants and f_1 and f_2 are linear and non-linear functions respectively, n is the order of the respective differential equations.

Methodology

We discussed the method used to solve the classes of problems considered in section 3.0 given in equations (5) and (6) together with the conditions stated in equation (7)

Discussion of the multiple perturbed collocation tau method on linear boundary value problem

In this section, the Multiple Perturbed Collocation Tau Method (MPCTM2) is used to solve problems (5) and (6) together with the conditions stated in equation (7). Here, we assumed a trial approximate solution of the form

$$y(x) \approx y_N(x) = \sum_{r=0}^N a_r T_r(x) + \sum_{i=1}^n \tau_i T_{N-i+1}(x) \quad (8)$$

Here, $a_r (r \geq 0)$ are constants to be determined, $\tau_i (i \geq 1)$ are free Tau-parameters to be determined, $T_N(x)$ are Chebyshev Polynomials defined in section (2.0). Equation (5) is differentiated n -times to get