

SOME RESULTS OF OPIAL-TYPE INTEGRAL INEQUALITIES

By

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Certification

This is to certify that this study was carried out by ANTHONIO, Yisa Oluwatoyin at the Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria.

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Dedication

To "a friend in need is a friend indeed" Alhaji Ahmed Olajide Kafidipe (Jaeed).

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Abstract

Inequalities are essential branches of Mathematics and are useful tools in some theory of analyses. Some inequalities such as Wirtinger's, Holder's, Cauchy's, Minkowski's, Hardy's and Opial's inequalities are more currently used by researchers and these are of particular interest in this study. Opial inequality and its generalisations have various applications in the theories of differential and difference equations. Opial inequality on time-scale (an arbitrary non-empty closed subset of real numbers) was introduced by Hilger in order to unify discrete and continuous analyses. Little work has been done on the determination of best possible constant for Opial-type inequalities on time-scales, multi-function and higher order delta derivatives. This study aimed at generalising Opial-type inequalities with best possible constant. The objectives of the study were to: (i) obtain some deductions from Opial-type inequalities of Shum, Pachpatte, Olech, Calvert, Fabelurin, Oguntuase, Beesack, Yang, and Maroni; (ii) investigate the nature of the deduced Opial-type inequalities; (iii) obtain sharp bound of the new Opial-type inequalities; and (iv) generalise Opial-type inequalities on time-scales. The methodology adopted was based on the definite and indefinite integrals with modified Jensen's inequality. The theorem relating to this inequality states that:

Let $\psi \in C((\gamma, \omega), \mathbb{R})$ be convex. Then, for each $t \in (\gamma, \omega)$, there exist $B_t \in \mathbb{R}$ such that $\psi(\wp) - \psi(t) \geq \beta_t(\wp - t)$, for all $\wp \in (\gamma, \omega)$. Then

$$\left(\frac{\int_0^t \psi(s) d\lambda(s)}{\int_0^t d\lambda(s)} \right)^{\frac{\zeta}{\zeta-1}} \leq \left(\frac{\int_0^t \phi(\psi(s))^{\frac{1}{\zeta-1}} d\lambda(s)}{\int_0^t d\lambda(s)} \right).$$

implies

$$\left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \phi(\psi(s))^{\frac{1}{\zeta-1}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s) \right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta}.$$

The findings of the study were that:

(i) deductions from Shum(1974), Pachpatte(1986), Olech (1962), Calvert(1967), Fabelurin(2010), Oguntuase(2009), Beesack(1962), Yang(1983), and Maroni(1967) results were generalised;

(ii) the properties of Opial-type inequalities involving integral of functions and their derivatives were verified;

(iii) sharp bounds for the new Opial-type inequalities were obtained; and

(iv) Opial-type inequalities were generalised on time-scales.

The study concluded that modified Jensen's inequality was useful in refining and extending Opial-type inequalities in order to generalise some results on time-scales. The results recommended that refined Jensen's inequality is sufficient in obtaining Opial-type inequalities.

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1 GENERAL INTRODUCTION

1 Introduction

Inequality has been in existence since the beginning of the world, disparity in sizes, ages, heights and way of thinking of human being confirms the existence of inequality in nature.

The differences in mass, volume, length and area of objects give illustration of inequalities in non-living things.

Economically, variance in wages and salaries explains the concept of inequalities. This lead to income and wealth inequality. Another concept that explains inequality is the divergence found in the economic well being of individuals, group of people and countries at large.

In mathematics, an inequality can be expressed as a relation that exists between two values when they are not same. The notation $\alpha \neq \beta$ means that α is not equal to β . It does not say that one is greater than the other, or even that they can be compared in size. If the values in question are elements of an ordered set.

The concept of inequality is of great importance in mathematics and the study has a history dated back to the early eighteen century with a lot of applications in the theory of difference equations.

Some centuries ago, some inequalities were in their reigns, these include Wirtinger's

inequality, Holder's inequality, Cauchy's inequality, Miskwoki's inequality. Hardy's inequality

was discovered by G. Hardy in 1922. A few decades after this, a Polish Mathematician called Zdzisław Opial established an inequality involving integral of a function and its derivatives

which was named after him as Opial's inequality.

Since its discovery in early 1960 Opial inequality has proved to be one of the most useful inequality in analysis.

Its application in proving the existence and uniqueness solutions to initial and boundary value problems for differential equations has been noticed in the literature. The importance of

Opial's result, however, with the establishment of the best possible constant just after its

publication has lead to many simplifications of the original proof. In respect of this, many generalization and refinement has been considered by researchers in the literature.

Today, the study of Opial inequality has developed into a substantial field, with numerous significance applications.

Time scales was introduced by Hilger (1988) in order to unify discrete and continuous analysis. Hence, a time scale is an arbitrary nonempty closed subset of real numbers. Thus,

$$\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$$

that is the real numbers, integers, natural numbers and non negative integers are example of time scales, also are $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$ and the cantor set, while

$$\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{C}, (0,1),$$

that is the rational number, the irrational numbers, complex numbers, and the open interval between 0 and 1, are not time scales. We shall denotes time scale by the symbol T .

Time-scale T being a closed subset of $\mathbb{R}$, topological structure of T , especially from the openness point of view has various features. Obviously any subset A of T which is open in $\mathbb{R}$, is also open in T . The reverse is generally not true, though, as the simple example $T := \mathbb{Z}$ shows, where any subset in the induced topology is open in T but not open in $\mathbb{R}$. This is taken care of by distinguishing between $\mathbb{R}$ -openness and T -openness. In order to investigate the details of the notion of openness in time-scales, we must define the concept of neighborhood in this set-up. We give two different versions of neighborhood definitions, distinguishing between the concepts of $\mathbb{R}$ -neighborhood and T -neighborhood, giving way to the distinction between $\mathbb{R}$ -openness and T -openness.

Given a time-scale $T, t \in T$ and an $\varepsilon > 0$, we denote by

$$R_\varepsilon(t) := \{x \in \mathbb{R} : t - \varepsilon < x < t + \varepsilon\}$$

$$T_\varepsilon(t) := \{x \in T : t - \varepsilon < x < t + \varepsilon\}$$

the ε -neighborhoods of $t \in \mathbb{R}$ and T , respectively.

An interval, in the time-scale context, is always understood as the intersection of a real interval with a given time-scale.

2 Definition of Terms

The following definitions are used in this research work.

2.1 Convex Function

Let $f(x)$ be a real valued function defined on the interval $I = [a, b]$. f is said to be convex if for every $x_1, x_2 \in [a, b]$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

A function is said to be strictly convex if the equality is strict for $x_1 \neq x_2$.

$f(x)$ is said to be concave (strictly concave) if $-f(x)$ is convex (strictly convex).

2.2 Integrable or Summable

Let f be measurable and consider the two integral $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ where f^+ and f^- and $f^+ = \max(f, 0)$, $f^- = \min(f, 0)$. Then if at least one of the above is finite. we define

$$\int_E f d\lambda = \int_E f^+ d\lambda - \int_E f^- d\lambda.$$

If both integral are finite, we say f is integrable (or summable) on E in the Lebesgue sense.

2.3 Differentiable Function

Suppose that $f:(a,b) \rightarrow \mathbb{R}$ and $a < c < b$. Then f is differentiable at x with derivative $f'(x)$ if

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of points $x \in (a,b)$ for which this limit exists. If the limit exists for every $x \in (a,b)$ then we say f is differentiable on (a,b) .

2.4 Continuously Differentiable Function

Let $I \subset \mathbb{R}$ be an open interval. Then f is continuously differentiable on I if and only if f is differentiable on I and its derivative is continuous on I .

2.5 Absolutely Continuous Functions

A real-valued function f defined on $[a,b]$ is said to be absolutely continuously on $[a,b]$ if given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sum_{i=1}^n |f(x_i) - f(x_j)| < \varepsilon$$

for every finite collection (x_i, x_j) of non overlapping interval with

$$\sum_{i=1}^n |x_i - x_j| < \delta$$

Note that the sum and difference of two absolutely continuous functions is absolutely continuous. In fact, every indefinite integral is a absolutely continuous.

2.6 Some Adaptations of Jensen's Inequality

Let φ be continuous and convex and let $h(s,t)$ be non-negative, $s \geq 0, t \geq 0$ and λ be non-decreasing. Let $-\infty \leq \xi(t) \leq \eta(t) < \infty$ and suppose φ has a continuous inverse $(\varphi)^{-1}$ (which is necessarily concave).

Then,

$$\varphi^{-1} \left[\frac{\int_{\xi(t)}^{\eta(t)} h(t,s) d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)} \right] \geq \left[\frac{\int_{\xi(\xi)}^{\eta(t)} (\varphi)^{-1} h(t,s) d\lambda(s)}{\int_{\xi(t)}^{\eta(t)} d\lambda(s)} \right] \quad (1)$$

with the inequality sign reversed if φ is concave. The inequality (1.1) above is known as

Jensen's inequality for convex function. Setting $\varphi(u) = u^l$, $\xi(t) = \xi$ and $\eta(t) = \gamma$, then as a

consequence of (1.1), we have for $l \geq 1$

$$\left[\frac{\int_{\xi}^{\gamma} h(t, s) d\lambda(s)}{\int_{\xi}^{\gamma} d\lambda(s)} \right]^{\frac{1}{l}} \geq \left[\frac{\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{l}} d\lambda(s)}{\int_{\xi}^{\gamma} d\lambda(s)} \right] \quad (2)$$

which we write as

$$\left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{l}} d\lambda(s) \right]^l \leq \left[\int_{\xi}^{\gamma} d\lambda(s) \right]^{l-1} \left[\int_{\xi}^{\gamma} h(t, s) d\lambda(s) \right] \quad (3)$$

and for $0 < l < 1$, as

$$\int_{\xi}^{\gamma} h(t, s) d\lambda(s) \leq \left[\int_{\xi}^{\gamma} d\lambda(s) \right]^{1-l} \left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{l}} d\lambda(s) \right]^l \quad (4)$$

as the reverse of (1.4).

Furthermore, if $1 \leq l \leq p$, it follows from (1.3) that

$$\left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{lp}} d\lambda(s) \right]^l \leq \left[\int_{\xi}^{\gamma} d\lambda(s) \right]^{l-1} \left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{p}} d\lambda(s) \right] \quad (5)$$

The next definition is the modified Jensen's inequality on time scale.

Let $\psi, \varphi \in C([t, \varepsilon], \mathbb{R})$. Suppose φ is convex, ψ non-negative and s non-decreasing. Then,

$$\left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s) \right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta}. \quad (6)$$

2.7 Cauchy inequality

If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are sequences of numbers, then

$$\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n b_k^2 \right)^{\frac{1}{2}} \quad (7)$$

with equality if and only if the sequence $\{a\}_{n=1}^{\infty}$ and $\{b\}_{n=1}^{\infty}$ are proportional. Inequality (1.7) is called Cauchy inequality.

2.8 Hölder's inequality

Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ for $k = 1, \dots, n$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$

then

$$\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q \right)^{\frac{1}{q}} \quad (8)$$

with equality holding if and only if $\alpha a_k^p = \beta b_k^q$ and $k = 1, \dots, n$ where α and β are real non-negative constants with $\alpha^2 + \beta^2 > 0$. This inequality is called Hölder's inequality.

2.9 Minkowski inequality

If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ for $k = 1, \dots, n$ and $p > 0$, then

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |b_k|^p \right)^{\frac{1}{p}} \quad (9)$$

with equality holding if and only if the sequence $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are proportional (1.9) is called Minkowski inequality.

2.10 Geometric inequality

If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two real sequence such that $a_1 \leq \dots \leq a_n$ and $b_1 \leq \dots \leq b_n$ or $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_n$ then, the following inequality is true.

$$\left(\frac{1}{n} \sum_{k=1}^n a_k \right) \left(\frac{1}{n} \sum_{k=1}^n b_k \right) \leq \frac{1}{n} \left(\sum_{k=1}^n a_k b_k \right) \quad (10)$$

it is called chebysev s inequality.

2.11 Young inequality

Let f be a real-valued, continuous and strictly increasing function on $[0, c]$, $c > 0$.
If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \quad (11)$$

where f^{-1} is the inverse function of f equality holds in (1.11) if and only if $b = f(a)$.

Then, since this result is due to W. H. Young (1.11) is called Young's inequality.

2.12 Monotone

A sequence $\{a_n\}_{n=1}^{\infty}$, of the number is called:

- (a) monotone increasing if $a_{n+1} \geq a_n, \forall n \in \mathbb{N}$.
- (b) strictly monotone increasing if $a_{n+1} > a_n, \forall n \in \mathbb{N}$.
- (c) monotone decreasing if $a_{n+1} \leq a_n, \forall n \in \mathbb{N}$.
- (d) strictly monotone decreasing if $a_{n+1} < a_n, \forall n \in \mathbb{N}$.

2.13 Some Basic Definitions on Times Scales

We shall write the delta derivatives f^Δ for a function f defined on T as follows:

- (i) $f^\Delta = f'$ is the usual derivative if $T = \mathbb{R}$. and
- (ii) $f^\Delta = \Delta f$ is the usual forward difference operator if $T = \mathbb{Z}$.

Let T be time for, $t \in T$, we define the forward jump operator $\sigma: T \rightarrow T$ by

$$\sigma(t) := \inf \{s \in T : s > t\}$$

while the backward jump operator $\rho: T \rightarrow T$ by

$$\rho(t) := \sup \{s \in T : s < t\}$$

In this definition we put $\inf \emptyset = \sup T$ (i.e, $\sigma(t) = t$ if T has a maximum t) and $\sup \emptyset = \inf T$ (i.e, $\rho(t) = t$. If T has a minimum t where $\emptyset$ denotes the empty set.

If $\sigma(t) > t$, we say that t is right-scattered and while $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. Point that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu: T \rightarrow (0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t$$

If T has a left-scattered maximum m , the $T^k = T - m$. Otherwise $T^k = T$. In summary

$$T^k = \begin{cases} T \setminus ((\rho \sup T), \sup T) & \sup T < \infty \\ T, & \sup T = \infty \end{cases} \quad (12)$$

Like $T^k = T$ is defined as the $T_k = T \setminus (\inf T, \sigma(\inf T))$. If $|\inf T| < \infty$ and $T_k = T$ if $\inf T = -\infty$

For a function $f: T \rightarrow \mathbb{R}$, we define the derivative f^Δ as follows. Let $t \in T$. If there exist a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exist a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad (13)$$

for all $s \in U$, then f is said to be differentiable at t , and we call α the delta derivative of f at t and denote it by f^Δ .

If $T = \mathbb{R}$, then

$$f^\Delta(t) = f'(t) = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \text{ for all } t \in T \quad (14)$$

If $T = \mathbb{R}$, then $f^\Delta(t) = f(t+1) - f(t)$ for all $t \in T$. For a function $f: T \rightarrow \mathbb{R}$ the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \quad (15)$$

If f is continuous at t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{t \rightarrow \infty} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \rightarrow \infty} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

A useful formular is

$$f^\sigma = f + \mu f^\Delta \quad \text{where} \quad f^\sigma := f \circ \sigma$$

A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be right dense continuous (rd -continuous) if it is right continuous at each right dense point and there exist a finite left limit at all left dense points, and f is said to be differentiable if its derivatives exist. The space of rd -continuous functions denoted by $C_r(T, \mathbb{R})$. A time scale T is said to be regular if the following two conditions are satisfied simultaneously:

(i) For all $t \in T, \sigma(\rho(t)) = t$

(ii) For all $t \in T, \rho(\sigma(t)) = t$

Example 1.1 Taking $T = \mathbb{R}$

(i) If $T = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf \{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

We can also put $\rho(t) = t$.

Every point $t \in \mathbb{R}$ is dense. The graininess function μ is

$$\mu(t) \equiv 0 \quad \forall t \in T$$

Example 1.2 Taking $T = \mathbb{Z}$

(ii) If $T = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf \{s \in \mathbb{R} : s > t\} = \inf \{t+1, t+2, t+3, \dots\} = t+1$$

We observe that $\rho(t) = t-1$. Hence every point $t \in \mathbb{Z}$ isolated. The graininess function μ here is

$$\mu(t) \equiv 1 \quad \forall t \in T$$

Remark 3.1 If T is regular time, then both operator are invertible with $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.

The following theorem given the product and quotient rules for the derivatives of the product fg and the quotient $\frac{f}{g}$ (where $gg^\sigma \neq 0$) two derivatives functions f and g .

Theorem 1.1

Assume $f, g: T \rightarrow \mathbb{R}$ are delta differentiable at $t \in T$. Then

$$\begin{aligned}
(fg)^\Delta &= f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \\
(fg)^{\Delta\Delta} &= (f^\Delta g + f^\sigma g^\Delta)^\Delta \\
(fg)^{\Delta\Delta} &= f^{\Delta\Delta} g + f^{\Delta^\sigma} g^\Delta + f^{\sigma\Delta} g^\Delta + f^{\sigma\sigma} g^{\Delta\Delta} \\
(fg)^{\Delta\Delta} &= f^{\Delta\Delta} g + (f^{\Delta^\sigma} + f^{\sigma\Delta}) g^\Delta + f^{\sigma\sigma} g^{\Delta\Delta} \\
\left(\frac{f}{g}\right)^\Delta &= \frac{f^\Delta g - fg^\Delta}{gg^\sigma}
\end{aligned}$$

where we wrote f^{Δ^σ} for f^{Δ^σ} e.t.c

By using the product rule, we see that the derivatives of $f(t) = (t - \alpha)^m$ for $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ can be calculated as

$$f^\Delta(t) = ((t - \alpha)^m)^\Delta = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^\nu (t - \alpha)^{m-\nu-1}$$

As a special case when $\alpha = 0$. we see that the derivative of $f(t) = t^m$ for $m \in \mathbb{N}$ can be calculated as

$$(t^m)^\Delta = \sum_{\gamma=0}^{m-1} \sigma^\gamma t^{m-\gamma-1}$$

We see that when

(i) $T = \mathbb{R}$, we have $\sigma(t) = t, \mu(t) = 0, f^\Delta(t) = f'(t)$

We see that when

(ii) $T = \mathbb{Z}$, we have $\sigma(t) = t + 1, \mu(t) = 1, f^\Delta(t) = \Delta f(t)$

(iii) We note that when $T = h\mathbb{Z}, h > 0$ we have $\sigma(t) = t + h, \mu(t) = h$,

$$f^\Delta = \Delta_h f(t) = \frac{(f(t+h) - f(t))}{h}$$

(iv) We observe that when $T = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, we have $\sigma(t) = qt, \mu(t) = (q-1)t$

$$f^\Delta(t) = \Delta_q f(t) = \frac{(f(qt) - f(t))}{(q-1)h}$$

(v) We see that when $T = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2, \mu(t) = 1 + 2\sqrt{t}$,

$$f^\Delta(t) = \Delta_0 f(t) = \frac{(f(\sqrt{t} + 1)^2 - f(t))}{(1 + 2\sqrt{t})}$$

(vi) We see that when $T = T_k = \{t_n : n \in \mathbb{N}\}$, where (t_n) are the harmonic number that are defined by $t_0 = 0$ and $t_n = \sum_{k=1}^n \frac{1}{k}, n \in \mathbb{N}_0$ and we have $\sigma(t_n) = t_{n+1}, \mu(t_n) = \frac{1}{n+1}$,

$$f^\Delta(t) = \Delta_1 f(t) = (n+1) f(t_n)$$

(vii) We note that when $T_2 = \{\sqrt{n} : n \in \mathbb{N}\}$, we have $\sigma(t) = \sqrt{t^2 + 1}, \mu(t) = \sqrt{t^2 + 1} - 1$

$$f^\Delta(t) = \Delta_2 f(t) = \frac{(\sqrt{t^2 + 1}) - f(t)}{(\sqrt{t^2 + 1} - t)}$$

(viii) We see that when $T_3 = \{\sqrt[3]{n} : n \in \mathbb{N}\}$, we have $\sigma(t) = \sqrt[3]{t^3 + 1}, \mu(t) = \sqrt[3]{t^3 + 1} - t$

$$f^\Delta(t) = \Delta_3 f(t) = \frac{(\sqrt[3]{t^3+1}) - f(t)}{(\sqrt[3]{t^3+1} - t)}$$

For $a, b \in T$ and delta differentiable function f , the cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t) \Delta t = F(b) - F(a).$$

Theorem 1.2

Let $f, g \in C_{rd}([a, b], \mathbb{R})$ be rd-continuous functions $a, b, c \in \mathbb{R}$. Then, the following are valid

1. $\int_a^b (\alpha f(t) + \beta g(t)) \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t.$
2. $\int_a^b f(t) \Delta t = -\int_b^a f(t) \Delta t.$
3. $\int_a^b f(t) \Delta t = \int_c^b f(t) \Delta t + \int_a^c f(t) \Delta t.$
4. $\left| \int_a^b \alpha f(t) \Delta t \right| \leq \alpha \int_a^b |f(t)| \Delta t.$
5. $\int_a^b f(t) g^\Delta(t) \Delta t = f(t) g(t) \Big|_a^b \Delta t - \int_a^b f^\Delta(t) g^\sigma(t) \Delta t$
6. $\int_a^b f^\sigma(t) g^\Delta(t) \Delta t = f(t) g(t) \Big|_a^b \Delta t - \int_a^b f^\Delta(t) g(t) \Delta t$

An integration by parts can be expressed as:

$$\int_a^b f(t) g^\Delta(t) \Delta t = f(t) g^\Delta(t) \Big|_a^b - \int_a^b f^\Delta(t) g^\sigma(t) \Delta t$$

and infinity integrals are defined as

$$\int_a^\infty f(t) \Delta t \leq \lim_{b \rightarrow \infty} \int_a^b f(t) \Delta t.$$

We see that when $T = \mathbb{R}$, we have

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt$$

We see that when $T = \mathbb{Z}$, we have

$$\int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t)$$

We see that when $T = h\mathbb{Z}$, we have

$$\int_a^b f(t) \Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h$$

3 Statement of the Problem

In the last few decades, a lot of research has been carried out on Opial-type inequality. Some of the notable generalizations are those of Pachpatte, Yang and Wong. The research problem in this work is to provide new generalizations of some variations of Opial-type inequalities on time scales and their applications to differential, integral and integro-differential equations.

4 Aim and objectives of the Study

This study aimed at generalising Opial-type inequalities with possible values. The objectives of the study were to: (i) obtain some deductions from Opial's inequality, especially those of Shum, Pachpatte, Olech, Calvert, Fabelurin, Oguntuase, Beesack, Yang, and Maroni inequalities; (ii) investigate the properties of inequality involved integration of functions and their derivatives; (iii) obtain sharp bound of Shum, Pachpatte, Olech, Calvert, Fabelurin, Oguntuase, Beesack, Yang, and Maroni; and (iv) generalise opial-type inequalities on time scales.

5 Significance of the Study

The study will extend the frontier of the knowledge in providing better understanding of Opial-type inequalities on time scales. To improve results of some earlier researchers is envisaged. The results obtained will provide bound to differential and difference equations.

6 Research Methodology

The methodology adopted was based on the modified Jensen's inequality which stated as follows:

Let $\psi \in C((\gamma, \omega), \mathbb{R})$ be convex. Then, for each $t \in (\gamma, \omega)$, there exist $B_t \in \mathbb{R}$ such that $\psi(\wp) - \psi(t) \geq \beta_t(\wp - t)$, for all $\wp \in (\gamma, \omega)$. Then

$$\left(\frac{\int_0^t \psi(s) d\lambda(s)}{\int_0^t d\lambda(s)} \right)^{\frac{1}{\zeta}} \leq \left(\frac{\int_0^t \phi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)}{\int_0^t d\lambda(s)} \right).$$

implies

$$\left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \phi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s) \right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta}.$$

2 LITERATURE REVIEW

1 Some Classical Theorems

In 1960, Opial established the following interesting integral result.

Theorem 2.1

Let $\eta(\gamma) \in C'[0, \rho]$ be such that $\eta(0) = \eta(\rho) = 0$ and $\eta(\gamma) > 0$ in $(0, \rho)$. Then, the following inequality holds.

$$\int_0^{\rho} |\eta(\gamma) \eta'(\gamma)| d\gamma \leq \frac{\rho}{4} \int_0^{\rho} (\eta'(\gamma))^2 d\gamma. \quad (16)$$

In (2.1), the constant $\frac{\rho}{4}$ is the best possible.

Immediately after Opial, many authors came out with different methods and approaches to extend and generalize (2.1).

In 1962, Olech gave a simplified proof of the original inequality, he noted that in Theorem 2.1, the positive requirement of $\eta(\gamma)$ is unnecessary and that inequality (2.1) holds absolutely continuous in $[0, b]$. He however demonstrated this, in the theorem below.

Theorem 2.2

Let $\eta(\gamma)$ be absolutely continuous in $[0, \rho]$ such that $\eta(0) = \eta(\rho) = 0$ and $\eta(\gamma) > 0$ in $(0, \rho)$ with $\eta(\gamma) \in C'[0, \rho]$. Then following inequality holds.

$$\int_0^\rho \eta(\gamma) \eta'(\gamma) d\gamma \leq \frac{\rho}{2} \int_0^\rho (\eta'(\gamma))^2 d\gamma. \quad (17)$$

Beesack (1962) established and proved Theorem 2.3 which is an improvement on Theorem 2.2. This paved way for further results on Opial-type inequality. His results are stated in the theorem below.

Theorem 2.3

Let $-\infty \leq \alpha < \beta \leq \infty$, and $p(t), r(t)$ be functions which are positive and continuous on (α, β) . Let $(p(t)r(t))'$ be continuous on (α, β) . If $\alpha < \alpha' < \tau < \beta$ and $\Delta(t)$ is absolutely continuous on $[\alpha, \alpha']$, with $\Delta(t) = \int_a^t \Delta'(s) ds$ $t \in [\alpha, \tau]$.

We have

$$\int_\alpha^\tau |\Delta(t) \Delta'(t)| dt \leq \frac{1}{2} \int_\alpha^\tau \frac{dt}{p(t)} \int_\alpha^\tau p(t) ((\Delta'(t))^2 dt \quad (18)$$

In the same trend, Godunova and Levin (1967) stated and proved the following theorem.

Theorem 2.4

Let f be real-valued absolutely continuous function defined on $[0, b]$ with $f(0) = 0$. Let F be a convex, increasing function on $[0, \infty)$ with $F(0) = 0$. Then, the integral inequality

$$\int_0^b F'(t) |f(t)| |f'(t)| dt \leq F(t) \int_0^b |f'(t)| dt \quad (19)$$

Using Hölder's inequality with indices μ and ν Maroni (1967) obtained a generalized

form of Opial's inequality as follows.

Theorem 2.5

Let $p(t)$ be positive and continuous on $[\tau, \alpha]$ with $\int_{\alpha}^{\tau} p^{1-\mu}(t) dt < \infty$, where $\mu > 1$, $x(t)$ be absolutely function on $[\alpha, \tau]$ and $x(0) = 0$. Then, the following inequality holds.

$$\int_{\alpha}^{\tau} |x(t)x'(t)| dt \leq \frac{1}{2} \left(\int_{\alpha}^{\tau} p^{1-\mu}(t) dt \right)^{\frac{2}{\mu}} \left(\int_{\alpha}^{\tau} p(t) |x'(t)|^{\nu} dt \right)^{\frac{2}{\nu}} \quad (20)$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Equality holds in (2.5) if and only if $c \int_{\alpha}^{\tau} p^{1-\mu}(s) ds$.

This inequality serves as a generalization to all Opial-type inequalities existing before it in the literature.

As a further extension on the work of Maroni (1967), Calvert in 1967 obtained a convex-like structure of Opial-type inequality which he stated and proved in the theorem below.

Theorem 2.6

Assuming that

- (i) $\eta(t)$ is absolutely continuous in $[\alpha, \tau]$, and $\eta(\alpha) = 0$
- (ii) $\eta(t)$ is continuous, complex-valued, defined in the range of $\eta(t)$ and for all real value t of the form $t(s) = \int_{\alpha}^s |\eta(u)| du, \eta(|t|)$ for all t and $\eta(t)$ is real for $t > 0$ and is increasing there and;

(iii) $p(t)$ is positive, continuous and $\int_{\alpha}^{\tau} p^{1-\mu}(t) dt < \infty$, where, $\frac{1}{\mu} + \frac{1}{\nu} = 1$ Then, the following inequality holds.

$$\int_{\alpha}^{\tau} |f(\eta(t))\eta'(t)| dt \leq F \left(\left(\int_{\alpha}^{\tau} p^{1-\mu}(t) dt \right)^{\frac{1}{\mu}} \left(\int_{\alpha}^{\tau} p(t) |\eta'(t)|^{\nu} dt \right)^{\frac{1}{\nu}} \right). \quad (21)$$

where

$$F(t) = \int_0^t \eta(s) ds,$$

$t > 0$, equality holds only if $\eta(t) = \int_{\alpha}^t p^{1-\mu}(s) ds$.

It was Willet (1968) that first obtained an Opial-type inequality (2.1) that involved n^{th} derivative of the given function $\eta(\psi)$. His celebrated result is stated and proved in the theorem below:

Theorem 2.7

Let $\varpi(t) \in C^{(n)}[0, b]$ such that $\Delta^{(i)}(0) = 0, 0 \leq i \leq (n-1) (n \geq 1)$. Then, the following inequality holds.

$$\int_0^b |\varpi(\psi) \varpi^{(n)}(\psi)| d\psi \leq \frac{b^n}{2} \int_0^b (|\varpi^{(n)}(\psi)|)^2 d\psi \quad (22)$$

Beesack and Das (1968) established generalization of some results embodied in Theorem 2.8 as follows:

Theorem 2.8

Let ς, σ be real such that $\varsigma\sigma > 0$ and either $\varsigma+1 > 1$ or $\varsigma+\sigma < 0$. Let $p(t)$ and $q(t)$ be non-negative measurable functions on (α, τ) such that

$$\int_{\alpha}^{\tau} p(t)^{\frac{1}{\varsigma+\varsigma-1}} dt < \infty \quad (23)$$

and

$$K_1(\tau, \varsigma, \varsigma) = \left(\frac{\varsigma}{\sigma + \varsigma} \right)^{\frac{\sigma}{(\varsigma+\varsigma)}} \times \left(\int_{\alpha}^{\tau} q(t)^{\frac{(\varsigma+\sigma)}{\varsigma}} p(t)^{\frac{-\varsigma}{\varsigma}} \left(\int_{\alpha}^t p(s)^{\frac{-1}{(\varsigma+\sigma-1)}} ds \right)^{(\varsigma+\sigma-1)} dt \right)^{\frac{1}{\varsigma+\sigma}} < \infty$$

If $\zeta(t)$ is absolutely continuous on $[\alpha, \tau]$, $\zeta(\alpha) = 0$ and $\zeta'(t)$ do no change sign in (α, τ) . Then, the following inequality holds:

$$\int_{\alpha}^{\tau} q(t) |\zeta(t)|^{\varsigma} |\zeta'(t)|^{\sigma} dt \leq K(\tau, \varsigma, \sigma) \int_{\alpha}^{\tau} p(t) |\zeta'(t)|^{\varsigma+\sigma} dt \quad (24)$$

Remark 1.2:

For $\varsigma = \sigma = 1$, $p(t) = q(t) = 1$, $K(\tau, \varsigma, \sigma) = \frac{b}{2}$. Then, (2.9) reduces to (2.2). Similarly,

for $\varsigma = \sigma = 1$, $p(t), q(t)$, $K(\tau, \varsigma, \sigma) = \frac{b}{4}$ Then, inequality (2.9) reduces to (2.1)

In the same year, Das obtained an improvement on inequality (2.7). His result which is stated in Theorem 2.9 below paved way for many subsequent results on Opial-type inequality.

Theorem 2.9

Let $\Delta(t) \in C^{(n-1)}[0, b]$ be such that $\Delta^{(i)}(0) = 0, 0 \leq i \leq (n-1) (n \geq 1)$. Further more let $x^{(n-1)}(t)$ be absolutely continuous and $\int_0^b |\Delta(t)| dt < \infty$. The following inequality holds

$$\int_0^b |x(t)x^{(n)}(t)| dt \leq c_n a^n \int_0^b ((x^{(n)}(t))^2 dt \quad (25)$$

where $c_n = \frac{1}{2n!} \left(\frac{n}{2n-1} \right)^{\frac{1}{2}}$.

In 1969, Boyd and Wong introduced a new trend into the generalization of Opial-type inequality by adopting the method of boundary valued problems for ordinary differential equation. This innovation was considered a great improvement in the literature on generalization of Opial-type inequalities and his well known result is summarized in Theorem 2.10 below.

Theorem 2.10

Let $p(t)$ and $q(t)$ be non-negative continuously differential functions on $[0, b]$ such that boundary value problems

$$(p(t)(y'(t)^l))' = \lambda q(t)' y(t)^l \quad (26)$$

$$y(0) = 0, p(a)(y'(a))^l = \lambda q(a)y'(a), y'(t) > 0, t \in [0, a] \quad (27)$$

has a solution for some $\lambda > 0$. Let $\Delta(t)$ be absolutely continuous on $[0, b]$ and $\Delta(0) = 0$. Then for any $l > 0$ the following inequality holds:

$$\int_0^b q(t) |\Delta'(t) \Delta'(t)| dt \leq \frac{1}{\lambda_o(l+1)} \int_0^b p(t) |\Delta'(t)|^{l+1} dt \quad (28)$$

where (λ_o) is the only positive eigenvalue of (2.11), (2.12) and (2.13), equality holds if and only if $\Delta(t)$ is the eigenvalue of (2.11) and (2.13) corresponding to λ_o .

The argument that Opial's inequality can be easily generalized to abstract was established by Vranceanu (1973) established in the theorem below.

Theorem 2.11

Let H be a Hilbert space with the inner-product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let the function $\Delta: [0, b] \rightarrow H$ have a continuous derivative and $\Delta(0) = 0$. Then, the following inequality holds.

$$\int_0^b |\langle \Delta(t), \Delta'(t) \rangle| dt \leq \frac{b}{2} \int_0^b \|\Delta'(t)\|^2 dt \quad (29)$$

In 1974, Shum added his own contribution to knowledge. He used differential inequality for a result that looked more complex in the expression than other generalizations.

Theorem 2.12

Let $\Delta(t)$ be absolutely continuous on $[0, b]$ and $\Delta(0) = 0$. If $l > 0$ and $\int_0^b |(\Delta'(t))|^{l+1} dt < \infty$, then

$$\int_0^b |\Delta(t)|^l |\Delta'(t)| dt + \frac{lb^l}{l+1} \int_0^b \frac{g(t)}{t^{l+1}} dt \leq \frac{b^l}{l+1} \int_0^b |(\Delta'(t))|^{l+1} dt \quad (30)$$

where

$$g(t) = (l+1) \int_0^b |\Delta(s)|^l |\Delta'(s)| ds - |\Delta(t)|^{l+1} \geq 0. \quad (31)$$

If either $l < -1$ and both $\int_0^b |\Delta(t)|^l |\Delta'(t)| dt < \infty$ and $\int_0^b |(\Delta'(t))|^{l+1} dt < \infty$; or $-1 < l < 0$ and $\int_0^b |\Delta(t)|^l |\Delta'(t)| dt < \infty$, the reverse inequality holds. Further, for $l > 0$ or $-1 < l < 0$, equality holds in (2.15) if and only if $\Delta(t) = ct$, whereas for $l < -1$, equality never holds.

Remark 2.3

If $g(t) = 0$, (2.15) reduces to

$$\int_0^b |\Delta(t)|^l |\Delta'(t)| dt \leq \frac{b^l}{l+1} \int_0^b |(\Delta'(t))|^{l+1} dt \quad (32)$$

which is a special case of Shum's generalization.

In 1977, Pec̃ariĉ generalized Opial-type inequalities for the case of many independent variables. The result is as follows:

Theorem 2.12

Let $m \geq 2$ and let $f_i, D^1 f_i, \dots, D^m f_i, i = 1, \dots, n$ be real valued continuous function on τ with $f_i(t)|_{t_i} = a_i = 0, i = 1, \dots, n, j = 1, \dots, m$. or

$$f_i(t)|_{t_1} = a_1 = D_i(t)|_{t_2} = a_2 = \dots = D_i(t)|_{t_m} = a_m = 0 \quad i = 1, \dots, n$$

Let F be a non-negative and differentiable function on $[0, \infty]$ with $F(0, \dots, 0) = 0$ such that $D_i F, i = 1, \dots, n$ are non-negative continuous and non-decreasing on $[0, \infty]^n$. Then the integral inequality

$$\int_{\tau} \sum_{i=1}^n D_i F |f_1(t)|, \dots, |f_n(t)| \|D^m f_i(t)\| dt \leq F \left[\int_{\tau} |D^m f_1(t)| dt, \dots, \int_{\tau} |D^m f_n(t)| dt \right] \quad (33)$$

Theorem 2.13

Using the Holder's inequality, Beesack (1979) gave a refinement of Opial-type inequality

by establishing the conditions under which the weighted Opial-type inequality (2.19) holds

$$\left(\int_a^b |l(x)|^q |l'(x)|^s v(x) dx \right)^{\frac{1}{s+q}} \leq C \left(\int_a^b |l'(t)|^p u(t) dt \right)^{\frac{1}{p}} \quad (34)$$

holds for all l with $l(a) = 0$, where $p > 1, q > 0$ and $0 < s < p$. The case of negative values of p, q and s was also considered.

(2.20) has as special case (2.22)

$$\int_0^b |l(x)| |l'(x)|^s v(x) dx \leq \frac{b}{2} \int_0^b |l'(t)|^2 u(t) dt \quad (35)$$

and the weighted Hardy inequality

$$\left(\int_a^b \left| \int_a^x f(t) dt \right|^q v(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b |f(t)|^p u(t) dt \right)^{\frac{1}{p}} \quad (36)$$

Remark 2.4

For, if $q = s = 1, u(t) = v(x) = 1$ and $p = 2$, (2.19) is equivalent to (2.20) and if $s = 0$, (2.19) is equivalent to (2.21).

In 1980, Wong extended Opial's inequality in the following ways.

Theorem 2.14

Let $\Delta(t)$ be absolutely continuous on $[a, b]$ and $\Delta(0) = 0$. Let $l \geq 0, 0 \leq \alpha < \beta \leq b$. Then, the following holds.

$$\int_{\alpha}^{\beta} |\Delta^l(t) \Delta'(t)| dt \leq \frac{\beta^l}{l+1} \int_{\alpha}^{\beta} |(\Delta'(t))^{l+1}| dt \quad (37)$$

Cheng-Shyong(1980) established the following sum s inequality.

Theorem 2.15 (Shum Inequality)

Assume that $a, b \in \mathbb{R}, a < b, p > 0$, and $s \in C([a, b], (0, \infty))$ is non increasing. If $f \in C'([a, b], \mathbb{R})$ satisfies $f(a) = 0$, then

$$\int_a^b s(t) |f(t)|^p |f'(t)| dt + \frac{(b-a)^p}{p+1} \int_a^b \frac{p g(t)}{(t-a)^{p+1}} dt \leq \frac{(b-a)^p}{p+1} \int_a^b s(t) |f'(t)|^{p+1} dt \quad (38)$$

where

$$g(t) = (p+1) \int_0^t s(\tau) |f(\tau)|^p |f'(\tau)| d\tau - s(t) |f(t)|^{p+1} \geq 0 \quad (a \leq t \leq b)$$

We note that Theorem 2.15 implies Theorem 2.2 by choosing $a = 0, b = h, p = 1$ and $s(t) = 1$ for all $t \in [a, b]$.

In 1985 Fagbohun and Imoru generalized Opial-type inequality.

Theorem 2.16

Let $\lambda(t)$ be absolutely continuous and non-negative function on $[0, b]$, with $\lambda(0) = 0$ and $\lambda(t) > 0, t \in [0, b]$. The function $\Delta(t)$ is absolutely continuous on $[0, b]$ and that

$$\tilde{\Delta}(t) = \int_0^t |\Delta'(s) \oplus p(s)| d\lambda(s), t \in [0, b] \quad (39)$$

where $p(s)$ is a non-negative function on $[0, b]$, and $\oplus$ is a binary operation defined on $\mathbb{R} = (-\infty, +\infty)$, satisfying $|u \oplus v| \leq |u| \oplus |v|, u, v \in \mathbb{R}$. Let $f(u)$ be non-negative and non-decreasing function on $\mathbb{R}$ such that $\xi > 0, f(u)^{\frac{1}{\xi}}$ is convex and

$$f(u \oplus v) = f(u) f(v) \quad u, v \geq 0 \quad (40)$$

Let $\lambda(t), x(t), f(u)$ be as defined above and let $\xi > 0, \eta, l, m$ be real numbers such that either $l > 0, m > 0, l > \xi m$, or $l < 0, m < 0, l < \xi m$. Let $p(t), q(t)$ be non-negative and measurable functions on $[0, b]$ such that

$$K = \left(\frac{m}{l+m} \right)^{\frac{\xi m}{l}} \left[\int_0^b (q(t))^{\frac{l}{(l-\xi m)}} (\lambda(t))^{\frac{l(\eta-\xi)}{(l-\xi m)}} \times f(p(t))^{\frac{-m}{(l-\xi m)}} d\lambda(t) \right]^{\frac{(l-\xi m)}{l}} < \infty \quad (41)$$

Then, the following inequality holds

$$\int_0^b q(t) (\lambda(t))^n f\left(\frac{\tilde{\Delta}(t)}{\lambda(t)}\right) [f(|\Delta'(t)|)]^{\frac{m}{l}} d\lambda(t) \leq K \left[\int_0^b [f(|\Delta'(t)| \oplus p(t))]^{\frac{1}{\xi}} d\lambda(t) \right]^{\frac{\xi(l+m)}{l}} \quad (42)$$

The concept of generalizing Opial-type inequalities involving inequalities involving two functions and their derivatives was introduced by Pachpatte in 1986. This was established in Theorem 2.16 below.

Theorem 2.17

Let $p(t)$ and $q(t)$ be positive and continuous with $q(t)$ non-increasing on (α, τ) and let $x_1(t)$ and $x_2(t)$ be absolutely continuous on $[\alpha, \tau]$ and $x_1(t) = x_2(t) = x_2(t) = x(0) = 0$. Then, the following inequality holds:

$$\int_{\alpha}^{\tau} q(t) (|x_1'(t)x_2(t)| + |x_1(t)x_2'(t)|) dt \leq \frac{1}{2} \int_{\alpha}^{\tau} \frac{dt}{p(t)} \int_{\alpha}^{\tau} p(t) q(t) [|x_1'(t)|^2 + |x_2'(t)|^2] dt \quad (43)$$

In 1980, Milovanovic et. al established the following results:

Theorem 2.18

Let $p(t)$ be positive and continuous on (a, b) with $\int_a^b p(t) dt = p < \infty$. Let $x(t)$ be absolutely continuous on $[\alpha, \tau]$, $\Delta(\alpha) = 0$ and $\int_a^b r(t) (\Delta'(t))^2 dt < \infty$ where $r(t) = \frac{1}{p(t)}$. Then, the following inequality holds

$$\int_a^b \left(\int_0^t p(s) ds \right) |\Delta(t) \Delta'(t)| dt \leq \frac{p^2}{2A^2} \int_0^b r(t) |\Delta'(t)|^2 dt, \quad (44)$$

where A is the positive solution of the equation $\coth t = t$ ($A \equiv 1.996786$) holds if and only if $\Delta(t) = c \sinh \left(\frac{A}{p} \int_{\alpha}^t p(s) ds \right)$

In 1991, Sinnamon established an integral inequality of the type of Hardy and Opial mainly by Hölder's inequality in the theorem below

Theorem 2.19

Suppose that $0 < l < \infty, 0 < m < k < \infty, k > 1$ and $P(t), q(t) : (\alpha, \tau) \rightarrow [0, \infty]$ are measurable functions. Further, suppose that for all non-negative measurable function $y(t)$ on

(α, τ) then, the following Hardy's result holds:

$$\left(\int_{\alpha}^{\tau} (q^k(t) p^{-m}(t))^{\frac{1}{k-m}} \left(\int_{\alpha}^t y(s) ds \right)^{\frac{kl}{k-m}} dt \right)^{\frac{k-m}{kl}} \leq L \left(\int_{\alpha}^{\tau} P(t) |y(t)|^k dt \right)^{\frac{1}{k}}. \quad (45)$$

where L is a positive constant. Then, for $x(t) \in AC_o(\alpha, \tau)$ the following inequality holds

$$\int_{\alpha}^{\tau} q(t) |x(t)|^l |x'(t)|^m dt \leq L^l \left[\int_{\alpha}^{\tau} p(t) |x'(t)|^k dt \right]^{\frac{l+m}{k}}. \quad (46)$$

In 2006, Wu Shan He established the following inequality.

Theorem 2.20

Let $f(x), g(x), e(x)$ be the integrable functions defined on $[a, b]$ and $f(x) \geq 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$ for all $x, y \in [a, b]$ and let $p \geq q > 0, \frac{1}{p} + \frac{1}{q} \leq 1$. Then,

$$\begin{aligned} & \int_a^b f(x) g(x) dx \\ & \leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b g^q(x) dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b g^q(x) dx \int_a^b f^p(x) dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b g^p(x) e(x) dx \int_a^b f^p(x) dx - \int_a^b g^q(x) dx \int_a^b f^p(x) e(x) dx \right)^2 \right]^{\frac{1}{2p}} \end{aligned} \quad (47)$$

. In 2006, Imoru and Adeagbo-Sheikh established an integral inequality using the combination

of Jensen's inequality for convex functions and result of Minkowski to obtain the following

result:

Theorem 2.21

Let $g(x)$ is continuous, non-decreasing on $[a, b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0, 1 \leq p \leq q < \infty$ and $f(x)$ is non-negative Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $\frac{-p}{q} < \delta < 0$, then

$$\left[\int_a^b g^{\frac{\delta q}{p}}(x) \left(\int_a^x f(t) dg(t) \right)^q dg(x) \right]^{\frac{1}{q}} \leq C(a, b, p, q, \delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x) \right]^{\frac{1}{p}} \quad (48)$$

. where

$$C(a, b, p, q, \delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{p + \delta q} \right)^{\frac{1}{p}} g(b)^{p+\delta q} (g(b)^{-\delta} - g(a)^{-\delta})^{\frac{q(p-1)}{p}} > 0.$$

In a recent time, Oguntuase present a transparent proof of an integral inequality of Hardy type

by using Hölder's inequality to obtain an integral inequality which has as special cases Opial's

and Hardy's inequalities. However, we observed that the constant of (2.33) at the right hand

side is wrong. In his work, he refined the result of Adeagbo-Sheikh using standard Hölder's

inequality and obtained a better constant. This can be observed in the theorem stated below:

Theorem 2.22

Let g be continuous and non-decreasing on $[a, b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Let $q \geq p \geq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}} \left(\int_a^x f(t) dg(t) \right)^q dg(x) \right]^{\frac{1}{q}} \leq C(a, b, p, q, \delta) \left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x) \right]^{\frac{1}{p}} \quad (49)$$

where

$$C(a, b, p, q, \delta) = (-\delta)^{\frac{q(1-p)}{p}} \left(\frac{p}{p + \delta q} \right)^{\frac{p}{q}} g(b)^{p + \delta q} \left(g(b)^{-\delta} - g(a)^{-\delta} \right)^{\frac{q}{p}(p-1)} > 0$$

Fabelurin et. al(2010) got a result which is special case of Shum-Opial s inequality. This occur

when the main functions does not change in the interval $[a, b]$. In their work they took a class of functions that satisfies a condition and obtain generalization of the special case using

adaptation of Jensen's inequality for convex function. They obtained the following result:

Theorem 2.23

Let $f(t)$ be continuous functions and non-decreasing on $[a, b]$, and $0 \leq a \leq b < \infty$ with $f(t) > 0$ for $t > 0$. Suppose that $p \geq l \geq 1, q > 0, 0 < l + q \leq p$ and $\delta > 0$. Then,

$$\begin{aligned}
 & \left| \int_a^b t^{\delta-1} f^q(t) \left[\int_t^b f(s) ds \right]^l dt \right| \\
 & \leq [\delta^{-1}]^{(lp-(l+q)+p)/p} \left[\frac{p}{l+q} \right] \left[\left[\int_a^b (f^p(s)) s^{lp(1+\delta)/(l+q)-1} ds \right] \right]^{(l+q)/p} \\
 & + [\delta^{-1}]^{(lp-(l+q)+p)/p} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\left[\int_a^b f^p(s) s^{lp(1+\delta)/(l+q)-(1+\delta)} ds \right] \right]^{\frac{(l+q)}{p}}.
 \end{aligned} \tag{50}$$

Adeagbo-Sheikh and Fabelurin (2011), follows a trend in using Jensen's inequality in

conjunction with a form of Minkowski's integral inequality to establish a result that generalizes

Theorem 2.21. He established an integral inequality of Hardy-type mainly by Jensen's

inequality.

Theorem 2.24

Let g be a continuous function which is non-decreasing on $[a, b], 0 \leq a \leq b < \infty$, with $g(x) > 0$ for $x > 0$. Suppose that $p \geq q \geq 1, 0 < q + s \leq p, \delta > 0$ and $f(x)$ is non-negative, non-decreasing and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$, Then

$$\left[\int_a^b \left[\int_x^b f(t) dg(t) \right]^q f(x)^s v_1(x) dg(x) + B(p, q, s, \delta, g(a)) \right]^{\frac{1}{q+s}} \leq C(p, q, s, \delta)^{\frac{1}{q+s}} \left[\int_a^b f(t)^p u_1(t) dg(t) \right]^{\frac{1}{p}} \quad (51)$$

where

$$v_1(x) = g(x)^{\frac{\delta(q+s)}{p}-1} \left[g(x)^{-\delta} - g(b)^{-\delta} \right]^{\frac{(q+s)-pq}{p}}$$

$$u_1(t) = g(t)^{\frac{pq(1+\delta)}{q+s}-1}$$

$$C(p, q, s, \delta) = [\delta^{-1}]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s} \right]$$

and

$$B(p, q, s, \delta, g(a)) = C(p, q, s, \delta) g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t) \right]^{\frac{q+s}{p}}$$

In 1979, Beesack used different measure and obtained that if $x(t)$ is a positive and continuous function with $\int_a^\beta \frac{1}{x(s)} ds < \infty$ satisfies the inequality

$$\int_a^\beta |x'(t)x(t)| dt \leq \frac{1}{2} \left(\int_a^\beta \frac{dt}{p(t)} \right) \left(\int_a^\beta p(t)(x'(t))^2 dt \right) \quad (52)$$

Yang (1983) in his own write up replaced $p(t)$ by $p(t)q(t)$, chosen $x(t)$ as same and $q(t)$ is a positive, bounded and non increasing functions on $[\alpha, \beta]$

$$\int_a^\beta q(t) |x'(t)x(t)| dt \leq \frac{1}{2} \left(\int_a^\beta \frac{dt}{p(t)} \right) \left(\int_a^\beta p(t)q(t)(x'(t))^2 dt \right) \quad (53)$$

He also extended the result to a more generalized form. we observe that the following result of Yang have (2.36) and (3.37) and that

$$\int_a^\beta q(t) |x'(t)x(t)| dt \leq \frac{m}{l+m} (\beta - \alpha)^l \left(\int_a^\beta \frac{dt}{p(t)} \right) \left(\int_a^\beta p(t)q(t)(x'(t))^2 dt \right) \quad (54)$$

In 2001, Bohner and Kaymakçalan extended Theorem 2.2 to an arbitrary time scale on dynamic Opial inequality and proved the following:

Theorem 2.25(Dynamic Opial Inequality)

Let T be a time scale with $0 \in T$. If $f \in C'_{rd}([0, h]_T, \mathbb{R})$ with $h > 0$ and $h \in T$ satisfies $f(0) = 0$, then

$$\int_0^h |(f^\Delta)^2(t)| \Delta t \leq h \int_0^h (f(t)^\Delta)^2 \Delta t. \quad (55)$$

Theory can be applied to any nonempty closed set $T \subset \mathbb{R}$, the so-called underlying time scale. We note that $T = \mathbb{R}$ in observed that (2.25) results in (2.2). The following generalization of Theorem 2.25.

Theorem 2.26(Dynamic Beesack Inequality)

Let T be a time scale with $0 \in T$. If $f \in C'_{rd}([0, h]_T, \mathbb{R})$ with $h > 0$ and $h \in T$ satisfies $f(0) = 0$, then

$$\int_0^h |(f^\Delta)^2(t)| \Delta t + h \int_{\sigma(0)}^h \frac{g(t)}{t \sigma(t)} \Delta t \leq h \int_0^h (f(t)^\Delta)^2 \Delta t \quad (56)$$

where

$$g(t) = \int_0^t |(f^\Delta)^2(\tau)| \Delta \tau - (f(t))^\Delta \geq 0$$

We note that Theorem 2.26 implies Theorem 2.25. We also note that Theorem 2.26 implies

$$\int_0^h |f(t) f'(t)| dt + \frac{h}{2} \int_0^h \frac{g(t)}{t^2} dt \leq \frac{h}{2} \int_0^h (f'(t))^2 dt \quad (57)$$

where

$$g(t) = \int_0^t |(f^\Delta)^2(\tau)| \Delta \tau - (f(\tau)^\Delta)^2 \Delta \tau \geq 0$$

Theorem 2.2 since, for $T = \mathbb{R}$, we have $\sigma(t) = t$ for all $t \in T$.

The extension of Opial-type inequalities on the theory of time scales was initiated by Agarwal et al (2001) when they extended Theorem 2.1 on a time scale and obtained the following:

Theorem 2.26

Suppose that $x: [0, h] \rightarrow \mathbb{R}$ is delta differentiable with $x(0) = 0$, we have

$$\int_0^h |(x + x^\sigma)^\Delta(t)| \Delta t \leq h \int_0^h |x^\Delta|^2(t) \Delta t. \quad (58)$$

Bohner and Kaymakçalan (2001) obtained Opial type inequality on Opial dynamic inequality time scale with two different weight functions.

Theorem 2.27

Let r and q are positive rd -continuous functions on $[0, h]_T$, $\int_0^h \frac{\Delta t}{r(t)} < \infty$ and q is non increasing and $x: [0, h]_T \rightarrow \mathbb{R}$ is the delta differentiable with $x(0) = 0$ then

$$\int_0^h q^\sigma(t) |x(t) + x^\sigma(t)| x^\Delta(t) \Delta t \leq \left(\int_0^h \frac{\Delta t}{r(t)} \right) \left(\int_0^h r(t) q(t) |x^\Delta(t)|^2 \Delta t \right) \quad (59)$$

Srivastava et. al (2010) established some general weighted Opial-type inequalities on time scale which improved, generalized and extended some very recent results. They obtained the

following inequality.

Theorem 2.28

Let each of the functions $f(t)$ and $g(t)$ given by

$$f, g : [a, b] \cap \mathbb{T} \rightarrow \mathbb{R} \quad (b > a \geq 0)$$

be delta differentiation n times ($n \in \mathbb{N}$) on $[a, b] \cap \mathbb{T}$. Suppose also that

$$p \geq 0, \quad q \geq 1 \quad \text{and} \quad h(t) \in C_{rd}([a, b], [1, \infty)).$$

if the $f(t) = f^\Delta(a) = \dots f^{\Delta^{n-1}}(a) = 0$ and $g(t) = g^\Delta(a) = \dots g^{\Delta^{n-1}}(a) = 0$ ($n \in \mathbb{N}$) then,

$$\begin{aligned} & \int_a^b h(x) \left\{ |g(x)|^p |f^{\Delta^n}(x)|^q + |g(x)|^p |f^{\Delta^n}|^q \right\} \Delta x \\ & \leq \frac{2q}{p+q} \left[\left(\frac{b-a}{2} \right)^p \right]^n \int_a^b h(x) \left\{ |f(x)|^{p+q} |g^{\Delta^n}(x)|^{p+q} \right\} \Delta x \end{aligned} \quad (60)$$

George (2010) presented an Opial-type inequality on time scale.

He assume that $\mathbb{T}^k = \mathbb{T}$ and presented a time scales Opial type inequality. He also gave the generalized analogs of all these inequalities involving high order delta derivatives of functions on time scales.

Theorem 2.29

Let $f \in C_{rd}^m(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}, a \leq b$; $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Assume $f^{\Delta^k}(a) = 0, k = 0, 1, 2, 3, \dots, n-1$. and that f^{Δ^n} is increasing on $[a, b] \cap \mathbb{T}$. Here σ is continuous and $h_{n-1}(t, s)$ jointly continuous. Then,

$$\int_a^b |f(t)| |f(t)|^q \Delta(t) \leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta(\tau) \right)^{\frac{q}{p}} \Delta(t) \right) \left(\int_a^b |f^{\Delta^n}(\tau)|^{2q} \Delta(\tau) \right)^{\frac{1}{q}} \quad (61)$$

Theorem 2.30

Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and $y \in C_{rd}^{(n)}([a, b] \cap \mathbb{T})$. If $y^{\Delta^i}(a) = 0$, for $i = 1, 2, 3, \dots, n-1$, then

$$\int_a^b |y(t)| |y^{\Delta^n}(t)| \Delta(t) \leq \sqrt{\frac{1}{2}} \left(\int_a^b \left(\int_a^t |H_{n-1}(t, \sigma(s))|^2 \Delta(s) \right) \Delta(t) \right)^{\frac{1}{2}} \times \int_a^b |y^{\Delta^n}(t)|^2 \Delta(t) \quad (62)$$

Theorem 2.31

Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and l, m be positive real numbers such that $l+m > 1$ and $y \in C_{rd}^{(n)}([a, b] \cap \mathbb{T})$. If $y^{\Delta^i}(a) = 0$, for $i = 1, 2, 3, \dots, n-1$, then

$$\int_a^b |y(t)|^l |y^{\Delta^n}(t)|^m \delta(t) \leq \left(\frac{m}{l+m} \right)^{\frac{m}{l+m}} \left(\int_a^b H^{(l+m-1)}(t, s) \Delta(t) \right)^{\frac{1}{l+m}} \times \int_a^b |y^{\Delta^n}(t)|^{l+m} \Delta(t) \quad (63)$$

where

$$H(t, s) = \int_a^t H_{n-1}(t, \sigma(s))^{\frac{l+m}{l+m-1}} \Delta s$$

Theorem 2.32

Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and l, m be positive real numbers such that

$l+m>1$ and $y \in C_{rd}^{(n)}([a,b] \cap T)$. If $y^{\Delta^i}(a) = 0$, for $i = 1, 2, 3, \dots, n-1$, then

$$\int_a^b |y(t)|^l |y^{\Delta^n}(t)|^m \delta(t) \leq m^m \left(\int_a^b h_{(n-1)}(t, a) \Delta(t) \right)^l \int_a^b |y^{\Delta^n}(t)| \Delta(t) \quad (64)$$

In 2013, Qiao-Luan Li and Wing-Sum Cheung considered the Opial-type inequality which involves higher-order delta derivatives of two functions on time scales. Their results yield some of the recent results on Opial's inequality and provide some new estimates on such types of inequalities.

Theorem 2.33

Let $0 \leq r \leq s < t, s > 0, t > 1$ be real numbers, and let m, k , be integers with $0 \leq k \leq m-1$. Let $p > 0$ and $q \geq 0$ be measurable functions on $Y = [a, b] \cap T$. Further, let $f, g \in C_{rd}^{m-1}(Y)$ with $f^{\Delta^i}(a) = g^{\Delta^i}(a) = 0$, $i = 1, 2, \dots, m-1$ and let $f^{\Delta^{m-1}}, g^{\Delta^{m-1}}$ be absolutely continuous on Y such that the integrals $\int_a^b P(t) |f^{\Delta^i}(x)|^t \Delta x$ and $\int_a^b P(t) |g^{\Delta^i}(x)|^t \Delta x$ exist. Then we have,

$$\begin{aligned} & \int_a^b q(x) \left[\left| f^{\Delta^k}(x) \right|^r \left| g^{\Delta^m}(x) \right|^s + \left| g^{\Delta^k}(x) \right|^r \left| f^{\Delta^m}(x) \right|^s \right] \Delta x \\ & \leq 2^{1-\alpha} \left(\int_a^b h(x)^{\frac{t}{t-s}} \Delta x \right)^{1-\alpha} \times [\beta k^{1-\beta} F(b)G(b) + (1-\beta)k^\beta (F(b) + G(b))]^\alpha \end{aligned} \quad (65)$$

Agarwal et.al (2017) gave refinements of Opial inequalities on time scales that reduce in the continuous case to classical inequalities named after Beesack and Shum. The following inequality was established.

Theorem 2.34

Let T be a time scale. Assume that $a, b \in T, a < b, \sigma > 1$, and $s \in C'_{rd}([a, b]_T, (0, \infty))$ is non increasing. suppose

$$\kappa(t) = (t-a)^{\alpha-1} \text{ for all } t \in [a, b]_T.$$

If $f \in C'_{rd}([a, b]_T, \mathbb{R})$ satisfies $f(a) = 0$, then

$$\int_a^b s(t) |(f^\sigma)^\Delta(t)| \Delta t + \kappa(b) \int_{\sigma(a)}^b \frac{\kappa(t)g(t)}{\kappa(t)\kappa(\sigma(t))} \Delta t \leq \kappa(b) \int_a^b |f(t)^\Delta|^\sigma \Delta t \quad (66)$$

where

$$g(t) = \int_a^t s(\tau) |(f^\sigma)^\Delta(\tau)| \Delta \tau - |f(t)^\Delta|^\sigma \geq 0$$

We observed that, with $p = \sigma - 1$, Theorem 2.34 implies Theorem 2.15 since, for $T = \mathbb{R}$, we have

$$\frac{\kappa^\Delta(t)}{\kappa(t)\kappa(\sigma(t))} = \frac{p(t-a)^{p-1}}{(t-a)^{2p}} = \frac{p}{(t-a)^{p+1}}$$

We also seen that Theorem 2.34 implies Theorem 2.26 by choosing

$$a = 0, b = h, \sigma = 2, \text{ and } s(t) = 1 \text{ for all } t \in [a, b]_T$$

We concluded this chapter with consequences of above literature in which different methods

and approaches were used to obtain generalizations, extensions and special cases of Opial type inequality. Little work has been done in Opial type inequality on time scales. In the next

two chapters, we will use modified Jensen's inequality to obtain our results.

3 PRRLIMINARY RESULTS

1 Introduction

We shall discuss some results in this section, the generalizations of some results in the literature which of course, are inequalities involving integral of functions and its derivatives. However, adaptation of Jensen inequality plays a fundamental role in achieving our results for

the case of convex functions. Hence, we shall have our results by employing adapted Jensen's

inequality.

2 Generalization of Shum's Inequality of Opial-type

We obtain a special case of Shum's generalization of Opial's inequality. We take a

class of functions which does not change sign on $[a, b]$ and the result will direct extension of (2.17).

We have the following Theorem.

Theorem 3.1

Let $f(t)$ be absolutely continuous function and non-decreasing on $[a, b]$, and $0 < a \leq b < \infty$ with $f(t) > 0$ for $t > 0$. Suppose that $p \geq l \geq 1, q > 0, 0 < l + q \leq p$ and $\delta > 0$. Then,

$$\begin{aligned} & \int_a^b f^q(t) t^{\delta-1} \left[\int_t^b f(s) ds \right]^l dt \\ & \leq [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)-(1+\delta)}} ds \right]^{\frac{l+q}{p}} \\ & + [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)}-1} ds \right]^{\frac{l+q}{p}} \end{aligned} \quad (67)$$

Proof: By using the standard Jensen's inequality:

$$\left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{p}} d\lambda(s) \right]^l \leq \left[\int_{\xi}^{\gamma} d\lambda(s) \right]^{l-1} \left[\int_{\xi}^{\gamma} h(t, s)^{\frac{1}{p}} d\lambda(s) \right] \quad (68)$$

With

$$\left. \begin{aligned} h(s, t) &= f^{lp}(s) f^{pq}(t) t^{\delta(l+q)} s^{lp(1+\delta)}, \\ d\lambda(s) &= s^{-(1+\delta)} ds, \\ \gamma &= b \text{ and } \xi = t \end{aligned} \right\}$$

$$\begin{aligned} & \left[\int_t^b \left(f^{lp}(s) f^{pq}(t) t^{\delta(l+q)} s^{lp(1+\delta)} \right)^{\frac{1}{p}} s^{-(1+\delta)} ds \right]^l \\ & \leq \left[\int_t^b s^{-(1+\delta)} ds \right]^{l-1} \left[\int_t^b \left(f^{lp}(s) f^{pq}(t) t^{\delta(l+q)} s^{lp(1+\delta)} \right)^{\frac{1}{p}} s^{-(1+\delta)} ds \right] \end{aligned}$$

Then, we have

$$\begin{aligned} & f^q(t) t^{\frac{\delta(l+q)}{p}} \left[\int_t^b f(s) ds \right]^l \\ & \leq \left[\frac{s^{-(1+\delta)+1}}{-(1+\delta)+1} \right]_b^t \left[t^{\frac{\delta(l+q)}{p}} f^q(t) \int_t^b f^l(s) s^{(l-1)(1+\delta)} ds \right]^{l-1} \\ & = [t^{-\delta} - b^{-\delta}]^{l-1} [\delta^{-1}]^{l-1} t^{\frac{\delta(l+q)}{p}} f^q(t) \int_t^b f^l(s) s^{(l-1)(1+\delta)} ds \end{aligned} \quad (69)$$

Since $q > 0$, we have $f^q(t) \leq f^q(s) \forall s \in [t, b]$. Consequently,

$$f^q(t) \int_t^b f^l(s) s^{(l-1)(1+\delta)} ds \leq \int_t^b f^{l+q}(s) s^{(l-1)(1+\delta)} ds \quad (70)$$

Combining (3.4) and (3.5), yields,

$$\begin{aligned} & f^q(t) t^{\frac{\delta(l+q)}{p}} \left[\int_t^b f(s) ds \right]^l \leq [t^{-\delta} - b^{-\delta}]^{l-1} [(\delta)^{-1}]^{l-1} t^{\frac{\delta(l+q)}{p}} \int_t^b f^{l+q}(s) s^{(l-1)(1+\delta)} ds \\ & = [t^{-\delta} - b^{-\delta}]^{l-1} [\delta^{-1}]^{l-1} \int_t^b t^{\frac{\delta(l+q)}{p}} f^{l+q}(s) s^{(l-1)(1+\delta)} ds \end{aligned} \quad (71)$$

let $\zeta = \frac{l+q}{p}$ and since $0 < \frac{l+q}{p} \leq 1$, then ζ satisfies the condition in (3.3). Consequently, putting

$$\left. \begin{aligned} h(s, t) &= f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)} \\ d\lambda(s) &= s^{-(1+\delta)} ds \end{aligned} \right\} \quad (72)$$

We have,

$$\begin{aligned}
& \int_t^b f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)-(1+\delta)} ds \left[\int_t^b \left[f^{l+q}(s) t^{\frac{\delta(l+q)}{p}} s^{l(1+\delta)} \right]^{\frac{p}{l+q}} s^{-(1+\delta)} ds \right]^{\frac{l+q}{p}} \\
&= [\delta^{-1}]^{1-\frac{l+q}{p}} [t^{-\delta} - b^{-\delta}]^{1-\frac{l+q}{p}} t^{\frac{\delta(l+q)}{p}} \int_t^b \left[f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \right]^{\frac{l+q}{p}} ds
\end{aligned} \quad (73)$$

Combining (3.5) and (3.7), we have

$$f^q(t) t^{\frac{\delta(l+q)}{p}} \left[\int_t^b f(s) ds \right]^l \leq [\delta^{-1}]^{1-\frac{l+q}{p}} [t^{-\delta} - b^{-\delta}]^{1-\frac{l+q}{p}} t^{\frac{\delta(l+q)}{p}} \int_t^b \left[f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \right]^{\frac{l+q}{p}} ds \quad (74)$$

On simplifying and arranging last inequality, it becomes

$$f^q(t) t^{\frac{\delta(l+q)}{p}} \left[\int_t^b f(s) ds \right]^l [t^{-\delta} - b^{-\delta}]^{\frac{(l+q)-lp}{p}} \leq [\delta^{-1}]^{\frac{(l+q)-lp}{p}} t^{\frac{\delta(l+q)}{p}} \int_t^b \left[f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \right]^{\frac{l+q}{p}} ds \quad (75)$$

Let

$$\delta > 0 \text{ and } \left(\frac{(l+q)-lp}{p} \right) \leq 0.$$

Thus

$$[t^{-\delta}]^{\frac{(l+q)-lp}{p}} \leq [t^{-\delta} - b^{-\delta}]^{\frac{(l+q)-lp}{p}} \quad \forall t \in [a, b] \quad (76)$$

Putting inequality (3.9) into (3.10), we have

$$t^{\delta} f^q(t) \left[\int_t^b f(s) ds \right]^l \leq [\delta^{-1}]^{\frac{lp-(l+q)}{p}} t^{\frac{\delta(l+q)}{p}} \left[\int_t^b \left[f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \right]^{\frac{l+q}{p}} ds \right]^{\frac{l+q}{p}} \quad (77)$$

Multiplying both sides of (3.11) with t^{-1} then, integrate over $[a, b]$ with respect to t , we obtain

$$\int_a^b f^q(t) t^{\delta-1} f^q(t) \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b \left[f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} \right]^{\frac{l+q}{p}} ds \right]^{\frac{l+q}{p}} dt \quad (78)$$

Take the right hand side (3.12) and then integrate by parts to obtain

$$\begin{aligned}
& [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} dt \\
&= (-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} ds \\
&+ [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b f^p(t) t^{\frac{\delta(l+q)}{p}-1} t^{\frac{lp(1+\delta)}{l+q}-1} \times \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}-1} dt.
\end{aligned} \tag{79}$$

Simplifying (3.13), we have

$$\begin{aligned}
& [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} dt \\
&= (-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} ds \\
&+ [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b f^p(t) t^{\frac{\delta(l+q)}{p}-1} \times \left[t^\delta \int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}-1} dt.
\end{aligned} \tag{80}$$

using the fact that

$$\delta > 0, t^\delta \leq s^\delta \quad \forall [t, b]$$

we have

$$\begin{aligned}
& [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} dt \\
&= (-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} ds \\
&+ [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b f^p(t) t^{\frac{\delta(l+q)}{p}-1} \times \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} s^\delta ds \right]^{\frac{l+q}{p}-1} dt.
\end{aligned} \tag{81}$$

Simplifying (3.15) further, we obtain,

$$\begin{aligned}
& [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} dt \\
& = (-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} ds \\
& + [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b f^p(t) t^{\frac{lp(1+\delta)}{l+q}-1} \times \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-1} ds \right]^{\frac{l+q}{p}-1} dt.
\end{aligned} \tag{82}$$

Evaluating the last integral on the right hand side of (3.16), we have

$$\begin{aligned}
& [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \int_a^b t^{\frac{\delta(l+q)}{p}-1} \left[\int_t^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} dt \\
& = (-)[\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-(1+\delta)} ds \right]^{\frac{l+q}{p}} ds \\
& + [\delta^{-1}]^{\frac{lp-(l+q)}{p}} \left[\frac{p}{l+q} \right] \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{l+q}-1} ds \right]^{\frac{l+q}{p}} dt.
\end{aligned} \tag{83}$$

We combine (3.12) and (3.17) to obtain

$$\begin{aligned}
& \int_a^b f^q(t) t^{\delta-1} \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)}-(1+\delta)} ds \right]^{\frac{l+q}{p}} \\
& + [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)}-1} ds \right]^{\frac{l+q}{p}}.
\end{aligned} \tag{84}$$

This completes the proof of the Theorem.

Lemma 3.1

Let all assumptions of Theorem 3.1 hold. Then,

$$\int_a^b x'(t) x^l(t) dt \leq \frac{b^l}{l+1} \int_a^b (x'(t))^{l+1} dt + \frac{b^l}{l+1} a^{(l+1)\frac{1}{l}} \int_a^b (x'(t))^{l+1} dt. \tag{85}$$

Proof :

Suppose $q = 1$ and $p = l+1$ in inequality (3.18) above, then

$$\begin{aligned}
& \int_a^b t^{\delta-1} f(t) \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^l a^\delta \left[\int_a^b f^{l+1}(s) s^{l(1+\delta)-(1+\delta)} ds \right] \\
& + [\delta^{-1}]^l \left[\int_a^b f^{l+1}(s) s^{l(1+\delta)-1} ds \right]
\end{aligned} \tag{86}$$

Since $f(t) > 0, \forall t \in [0, b]$ we can write (3.21) as

$$\begin{aligned} \int_a^b t^{\delta-1} f(t) \left[\int_t^b f(s) ds \right]^l dt &\leq [\delta^{-1}]^l a^\delta \left[\int_a^b f^{l+1}(s) s^{(l-1)(1+\delta)} ds \right] \\ &+ [\delta^{-1}]^l \left[\int_a^b f^{l+1}(s) s^{l(1+\delta)-1} ds \right] \end{aligned} \quad (87)$$

Rearranging and factoring $t^{\delta-1}$ gives

$$0 \leq \int_a^b t^{\delta-1} \left[[\delta^{-1}]^l (f^{l+1}(t)) t^l + [\delta^{-1}]^l a^\delta f^{l+1}(t) t^{l-\delta} - f(t) \left[\int_t^b f(s) ds \right]^l \right] dt \quad (88)$$

Furthermore, since $t \geq 0$ on $[a, b]$, it then follows from (3.22) that

$$0 \leq \int_a^b \left[[\delta^{-1}]^l (f^{l+1}(t)) t^l + [\delta^{-1}]^l a^\delta f^{l+1}(t) t^{l-\delta} - f(t) \left[\int_t^b f(s) ds \right]^l \right] dt. \quad (89)$$

Rearranging (3.23) gives

$$\int_a^b f(t) \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^l \int_a^b (f^{l+1}(t)) t^l dt + [\delta^{-1}]^l a^\delta \int_a^b f^{l+1}(t) t^{l-\delta} dt. \quad (90)$$

Use the fact that $t \in [a, b]$ and $l > 0$, the (3.24) above yields

$$\int_a^b f(t) \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^l b^l \int_a^b (f^{l+1}(t)) dt + [\delta^{-1}]^l a^\delta b^l \int_a^b f^{l+1}(t) t^{-\delta} dt. \quad (91)$$

Note that $f(t) = (-f(t))$. Then, from (3.25).

$$\int_a^b (-f(t)) \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^l b^l \int_a^b (-f(t))^{l+1} dt + [\delta^{-1}]^l a^\delta b^l \int_a^b (-f(t))^{l+1} t^{-\delta} dt. \quad (92)$$

Now put $\delta = (1+l)^{\frac{1}{l}}$ in the (3.26) to obtain

$$\int_a^b (-f(t)) \left[\int_t^b f(s) ds \right]^l dt \leq \frac{b^l}{l+1} \int_a^b (-f(t))^{l+1} dt + \frac{b^l}{l+1} a^{(l+1)^{\frac{1}{l}}} \int_a^b (-f(t))^{l+1} t^{-\delta} dt \quad (93)$$

If we put $x(t) = \int_t^b f(s) ds$ and noting that $x'(t) = -f(t)$, then, (3.27) becomes (3.19).

Remark 3.1

If we let $a \rightarrow 0$ then, (3.27) becomes a special case of (2.17).

We have succeeded in obtaining a special generalization of the inequality of Shum, using Jensen's inequality for convex functions.

3 Generalization of Beesack's Inequality

In this section, we will obtain a generalization of Beesack's inequality from Jensen's

inequality for convex function is stated as follows.

$$(f(t))^{\frac{1}{\mu}} = \left(f \left(\frac{\int_0^\gamma \psi(t) d\lambda(t)}{\int_0^\gamma d\lambda(t)} \right) \right)^{\frac{1}{\mu}} \leq \left(\frac{\int_0^\gamma f(\psi(t))^{\frac{1}{\mu}} d\lambda(t)}{\int_0^\gamma d\lambda(t)} \right). \quad (94)$$

The result is the following:

Theorem 3.2

Let $x(t)$ and $\lambda(t)$ be absolutely continuous functions $[a, b]$ with $\lambda(t) > 0, t \in [a, b]$. Let $Q(t)$ and $R(t)$ be non-negative and measurable function on $[a, b]$ such that either $k > 0, l > 0, m > 0, l > \mu m$ or $l < 0$ and $l < \mu m$. then, the following inequality holds:

$$\int_a^b Q(t) \lambda'(t) x'(t)^m \left(\int_0^t x'(t)^l R'(t) d\lambda(t) \right)^l \leq \int_a^b Q(t) (\lambda'(t))^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-m)}{k}} R^{-m}(t) y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} \quad (95)$$

Proof: By using the Jensen's inequality:

$$\left(f \left(\frac{\int_0^\gamma \psi(t) d\lambda(t)}{\int_0^\gamma d\lambda(t)} \right) \right)^\mu \leq \left(\frac{\int_0^\gamma f(\psi(t))^\mu d\lambda(t)}{\int_0^\gamma d\lambda(t)} \right) \quad (96)$$

With $\psi(s) = x'(s)R(s)$ Using the fact that $f(u) = u^l$

$$\frac{\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l}{\lambda^l(t)} \leq \left(\frac{(x'(t)^l R(t)^l)^\mu d\lambda(t)}{\lambda(t)} \right)^\mu \quad (97)$$

$$\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda^{l-\mu}(t) \left(\int_0^t x'(t)^l R(t)^l)^\mu d\lambda(t) \right)^\mu \quad (98)$$

Setting $\mu = \frac{l}{k}$ then, inequality (3.32) becomes

$$\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda^{\frac{kl-l}{k}}(t) \left(\int_0^t (x'(t)^l R(t)^l)^{\frac{k}{l}} d\lambda(t) \right)^{\frac{l}{k}}$$

Putting $y(t) = \int_0^t x'(t)^k R(t)^k d\lambda(t)$ to obtain the following inequality

$$\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda(t)^{\frac{l(k-1)}{k}} y(t)^{\frac{l}{k}} \quad (99)$$

Now that

$$y'(t) = x'(t)^k R(t)^k \lambda'(t)$$

$$y'(t)^{\frac{m}{k}} = x'(t)^m R(t)^m \lambda'(t)^{\frac{m}{k}}$$

$$x'(t)^m = R(t)^{-m} \lambda'(t)^{\frac{m}{k}} y'(t)^{\frac{m}{k}}. \quad (100)$$

Then, multiplying both sides of (2.34) with $\lambda'(t)$ to have

$$\lambda'(t)x'(t)^m = \lambda'(t)^{\frac{k-m}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}}. \quad (101)$$

Combining both (3.33) and (2.35) to obtain

$$\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} \times y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} \quad (102)$$

Multiplying both sides of (3.36) with $Q(t)$ then integration over $[a, b]$ with respect t to obtain

$$\int_a^b Q(t) \lambda'(t) x'(t)^m \left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l dt \leq \int_a^b Q(t) \lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} dt \quad (103)$$

Next, by applying the Hölder's inequality with indices $\frac{k}{m}$ and $\frac{k}{k-m}$ to the right hand side

of (3.37), we have

$$\begin{aligned} & \int_a^b Q(t) \lambda'(t) |x'(t)|^m \left(\int_0^t x'(t) R(t) d\lambda(t) \right)^l dt \\ & \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t) \lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \left[\int_a^b y'(t) y(t)^{\frac{l}{m}} dt \right]^{\frac{m}{k}} \end{aligned} \quad (104)$$

Hence,

$$\begin{aligned} & \int_a^b Q(t) \lambda'(t) |x'(t)|^m \left(\int_0^t x'(t) R(t) d\lambda(t) \right)^l dt \\ & \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t) \lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \\ & \quad \left[\int_a^b \left(\int_0^t x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{l}{m}} \left(\int_0^t x'(t)^k R(t)^k d\lambda(t) \right) dt \right]^{\frac{m}{k}} \\ & = \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t) \lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \left[\int_a^b \left(\int_0^t x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{l+m}{m}} dt \right]^{\frac{m}{k}} \end{aligned} \quad (105)$$

and the proof is complete.

Lemma 3.2 Under the assumptions of Theorem 2.2 then the following inequality

$$\int_a^b q(t) x'(t)^m x(t)^l dt \leq L(l, m, k) \left[\int_a^b P(t) x'(t)^k dt \right]^{\frac{l+m}{k}} \quad (106)$$

holds .

Proof:

In Theorem 2.2, setting $\lambda'(t) = P(t)^{\frac{1}{k-1}}$, $Q(t) = q(t)P(t)^{\frac{1}{k-1}}$, $R(t) = P(t)^{\frac{1}{k-1}}$ and $\lambda(t) = \int_0^t P(t)^{\frac{1}{k-1}} dt$, then (3.40) yields

$$\begin{aligned}
& \int_a^b q(t) P(t)^{\frac{1}{k-1}} P(t)^{-\frac{1}{k-1}} x(t)^m \left(\int_0^t x'(t) dt \right)^l dt \\
& \leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} \right]^{\frac{k-m}{k}} \\
& \quad \left[\left(\int_a^b x'(t)^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt \right)^{\frac{l+m}{m}} \right]^{\frac{m}{k}} \\
& \leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} \right]^{\frac{k-m}{k}} \\
& \quad \left[\left(\int_a^b x'(t)^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt \right)^{\frac{l+m}{m}} \right]^{\frac{m}{k}}.
\end{aligned} \tag{107}$$

Further simplification yields,

$$\begin{aligned}
& \int_a^b q(t) x'(t)^m x(t)^l dt \\
& \leq \left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} dt \right]^{\frac{k-m}{k}} \left[\int_a^b P(t) x'(t)^k dt \right]^{\frac{l+m}{k}}
\end{aligned} \tag{108}$$

By setting

$$\tilde{L}(l, m, k) = \left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} dt \right]^{\frac{k-m}{k}}$$

then, (3.42) becomes

$$\int_a^b q(t) x'(t)^m x(t)^l dt \leq L(l, m, k) \left[\int_a^b P(t) x'(t)^k dt \right]^{\frac{l+m}{k}}, \tag{109}$$

where,

$$L(l, m, k) = \left(\frac{m}{l+m} \right)^{\frac{m}{k}} \tilde{L}(l, m, k) \tag{110}$$

The proof is complete.

Remark 3.2 In (3.43), setting $q(t) = P(t) = m = l = 1, k = 2$ and taking $x(t)$ to be absolutely continuous, then we have the following inequality

$$\int_a^b x'(t)x(t)dt \leq \left[\int_a^b t dt \right] \left[\int_a^b x'(t)^2 dt \right] \quad (111)$$

$$\int_a^b x'(t)x(t)dt \leq \frac{(b-a)^3}{2\sqrt{2}} \quad (112)$$

4 Generalizations of Sinnamon s result

In this section, Sinnamon (1991) also used Holder's inequality to obtain an integral

inequality which has as special case Hardy's inequality.

We obtain integral inequalities which are Opial-type inequalities, mainly by using Jensen

s inequality for the case of convex function.

We shall give our result using the well known Jensen's inequality for convex function.

Firstly, we need the following:

Theorem 3.3

Let $x(t)$ and $\lambda(t)$ be absolutely continuous functions $[a, b]$ with $\lambda(t) > 0, t \in [a, b]$. Let $Q(t)$ and $R(t)$ be non-negative and measurable function on $[a, b]$ such that either $k > 0, l > 0, m > 0, l > \mu m$ or $l < 0, m > 0$ and $l < \mu m$ where $l < |m - k|$, then, the following inequality holds:

$$\begin{aligned} & \int_a^b \left[Q(t)^{\frac{kl}{k-m}} \lambda'(t)^l R^{\frac{-klm}{k-m}} \lambda(t)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \left[\left(\int_0^t x'(t) R(t) d\lambda(t) \right)^l dt \right]^{\frac{1}{\omega}} dt \\ & \leq \int_a^b \left[y'(t)^{\frac{-m}{k}} y(t)^{\frac{l}{k}} dt \right]^{\frac{1}{k(l-m)}} \left[\left(Q(t) \lambda'(t) \lambda(t)^{\frac{l(k-m)}{k}} |x'(t)|^m dt \right)^{\frac{1}{\omega}} dt \right] \end{aligned} \quad (113)$$

Proof: By using Jensen's inequality:

$$\int_0^t d\lambda(s) \left[f \left(\frac{\int_0^t \psi(s) d\lambda(s)}{\int_0^t d\lambda(s)} \right) \right]^{\frac{1}{\mu}} \leq \int_0^t f(\psi(s))^{\frac{1}{\mu}} d\lambda(s) \quad (114)$$

with $\psi(s) = x'(t)R(t)$, $\xi = 0$ $\gamma = t$ and rearrange we have,

$$f \left(\frac{\int_0^t x'(t)R(t) d\lambda(t)}{\int_0^t d\lambda(t)} \right)^{\frac{1}{\mu}} \leq \frac{\int_0^t f(x'(t)R(t))^{\frac{1}{\mu}} d\lambda(t)}{\int_0^t d\lambda(t)} \quad (115)$$

Using the fact that $f(u) = u^l$, implying

$$\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda^{l-\mu}(t) \left(\int_0^t (x'(t)^l R(t)^l)^{\frac{1}{\mu}} d\lambda(t) \right)^{\mu} \quad (116)$$

Setting $\mu = \frac{l}{k}$ then, inequality (3.50) becomes

$$\begin{aligned} \left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l &\leq \lambda^{\frac{kl-l}{k}}(t) \left(\int_0^t (x'(t)^l R(t)^l)^{\frac{k}{l}} d\lambda(t) \right)^{\frac{l}{k}} \\ &\leq \lambda^{\frac{l(k-1)}{k}}(t) \left(\int_0^t x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{l}{k}} \end{aligned}$$

Putting $y(t) = \int_0^t x'(t)^k R(t)^k d\lambda(t)$ to obtain

$$\left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq \lambda(t)^{\frac{l(k-1)}{k}} y(t)^{\frac{l}{k}} \quad (117)$$

where

$$y'(t) = x'(t)^k R(t)^k \lambda'(t)$$

$$y'(t)^{\frac{m}{k}} = x'(t)^m R(t)^m \lambda'(t)^{\frac{m}{k}}$$

$$R(t)^{-m} \lambda'(t)^{\frac{-m}{k}} = x'(t)^m y'(t)^{\frac{-m}{k}} \quad (118)$$

Then, multiplying both sides of (3.52) with $Q(t) \lambda'(t)$ to have

$$Q(t) \lambda'(t)^{\frac{k-m}{k}} R(t)^{-m} = Q(t) \lambda'(t) |x'(t)|^m y'(t)^{\frac{-m}{k}} \quad (119)$$

Combining both (3.51) and (3.53) to obtain

$$Q(t) R(t)^{-m} \lambda'(t)^{\frac{k-m}{k}} \left(\int_0^t x'(t)R(t) d\lambda(t) \right)^l \leq Q(t) \lambda'(t) \lambda^{\frac{l(k-1)}{k}}(t) y'(t)^{\frac{-m}{k}} y(t)^{\frac{l}{k}} x'(t)^m \quad (120)$$

By integration both sides of (3.54) over $[a, b]$ with respect to t and by using Hölder's

inequality with indices $\frac{kl}{k-m}$ and $\frac{1}{\omega}$ on left hand side with $(l-m)$ and $\frac{1}{\omega}$ on the right hand side to obtain:

$$\begin{aligned} & \int_a^b \left[Q(t)^{\frac{kl}{k-m}} \lambda'(t)^l R^{\frac{-klm}{k-m}} \lambda(t)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \times \left[\left(\int_0^t x'(t) R(t) d\lambda(t) \right)^l dt \right]^{\frac{1}{\omega}} dt \\ & \leq \int_a^b \left[y'(t)^{\frac{-m}{k}} y(t)^{\frac{l}{k}} dt \right]^{\frac{1}{(l-m)}} \times \left[\left(Q(t) \lambda'(t) \lambda(t)^{\frac{l(k-m)}{k}} x'(t)^m dt \right)^{\frac{1}{\omega}} \right]^{\omega} \end{aligned} \quad (121)$$

This completes the proof of the Theorem 2.3

Remark 3.3

If we put $R(t) = P(t)^{\frac{1}{l(k-1)}}$, $Q(t) = q(t)^{\frac{1}{l}} P(t)^{\frac{1}{l(k-1)}}$, $\lambda'(t) = P(t)^{\frac{-1}{l(k-1)}}$ in (3.55), we obtain another useful inequality:

$$\begin{aligned} & \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)} \frac{1}{k-m} \frac{km}{(k-1)(k-m)}} \left(\int_0^t P(t)^{\frac{1}{l(k-1)}} dt \right)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \\ & \left[\left(\int_a^b \int_0^t x'(t) dt \right)^{\frac{l}{\omega}} \right]^{\omega} \\ & \leq \left[\left(\int_a^b x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{-m}{k}} \left(\int_a^b x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{l}{k}} dt \right]^{\frac{1}{(l-m)}} \\ & \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{\frac{k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega} \end{aligned} \quad (122)$$

implies

$$\begin{aligned} & \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k-(k-m)-km}{(k-1)(k-m)}} \left(\int_0^t P(t)^{\frac{1}{l(k-1)}} dt \right)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \left[\left(\int_a^b x(t) dt \right)^{\frac{l}{\omega}} \right]^{\omega} \\ & \leq \left[\left(\int_a^b x'(t)^k R(t)^k d\lambda(t) dt \right)^{\frac{l-m}{k}} \right]^{\frac{1}{(l-m)}} \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{\frac{k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega} \end{aligned} \quad (123)$$

Simplify further yields

$$\begin{aligned}
& \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{-m}{(k-m)}} \left(\int_0^t P(t)^{\frac{-1}{l(k-1)}} dt \right)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \left[\left(\int_a^b x(t) dt \right)^{\frac{l}{\omega}} \right]^{\omega} dt \\
& \leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t) \right) dt \right]^{\frac{1}{k}} \times \left[\left(\int_a^b q(t)^{\frac{1}{l}} |x'(t)|^m \int_0^t P(t)^{\frac{-k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega}
\end{aligned} \quad (124)$$

$$\begin{aligned}
& \left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{\frac{-1}{l(k-1)}} dt \right)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \left[\left(\int_0^t x'(t) dt \right)^{\frac{l}{\omega}} \right]^{\omega} \\
& \leq \left[\left(\int_a^b |x'(t)|^k R(t)^k d\lambda(t) \right) dt \right]^{\frac{1}{k}} \left[\left(\int_a^b q(t)^{\frac{1}{l}} |x'(t)|^m \int_0^t P(t)^{\frac{-k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega} \\
& \leq \left[\left(\int_a^b x'(t)^k P(t)^{\frac{1}{l}} dt \right) \right]^{\frac{1}{k}} \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{\frac{-k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega}
\end{aligned} \quad (125)$$

Remark 3.4

If $r(t) = P(t)^{\frac{-1}{l(k-1)}} = x'(t)$, $P(t) = P(t)^{\frac{1}{l}}$, $\omega = 0$ and

$$L = \left[\left(\int_a^b q(t)^{\frac{1}{l}} x'(t)^m \int_0^t P(t)^{\frac{-k-m}{k-1}} dt \right)^{\frac{1}{\omega}} \right]^{\omega} \text{ in (3.59) to obtain}$$

$$\left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_a^t r(s) ds \right)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \leq L \left[\int_a^b P(t) r(t)^k dt \right]^{\frac{1}{k}} \quad (126)$$

Theorem 3.4 Let takes all assumptions of Theorem 3.3. Then, the following inequality also holds:

$$\int_a^b Q(t) \lambda'(t) x'(t)^m \left(\int_0^t x'(t)^l R'(t) d\lambda(t) \right)^l \leq \int_a^b Q(t) (\lambda'(t))^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-m)}{k}} R^{-m}(t) y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} \quad (127)$$

Proof:

Rearrange equation (3.52) and simplify to obtain the following

$$x'(t)^m = R(t)^{-m} \lambda'(t)^{\frac{m}{k}} y'(t)^{\frac{m}{k}}. \quad (128)$$

Then, multiplying both sides of (3.62) with $\lambda'(t)$ to have

$$\lambda'(t)x'(t)^m = \lambda'(t)^{\frac{k-m}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}} \quad (129)$$

Combining both (3.51) and (3.63) to obtain:

$$\lambda'(t)x'(t)^m \times \left(\int_0^t x'(t)R(t)d\lambda(t) \right)^l \leq \lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} \times y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} \quad (130)$$

Multiplying both sides of (3.64) with $Q(t)$ then integration over $[a, b]$ with respect to t to obtain:

$$\int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t) \right)^l dt \leq \int_a^b Q(t)\lambda'(t)^{\frac{k-m}{k}} \lambda(t)^{\frac{l(k-1)}{k}} R(t)^{-m} y'(t)^{\frac{m}{k}} y(t)^{\frac{l}{k}} dt \quad (131)$$

Next, by applying the Hölder's inequality with conjugate exponents $\frac{k}{m}$ and $\frac{k}{k-m}$ to the right hand side of (3.65), we have:

$$\begin{aligned} & \int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t) \right)^l dt \\ & \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \left[\int_a^b y'(t)y(t)^{\frac{l}{m}} dt \right]^{\frac{m}{k}} \end{aligned} \quad (132)$$

Hence,

$$\begin{aligned} & \int_a^b Q(t)\lambda'(t)x'(t)^m \left(\int_0^t x'(t)R(t)d\lambda(t) \right)^l dt \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \\ & \left[\int_a^b \left(\int_0^t |x'(t)|^k |R(t)|^k d\lambda(t) \right)^{\frac{l}{m}} \left(\int_0^t |x'(t)|^k |R(t)|^k d\lambda(t) \right) dt \right]^{\frac{m}{k}} \\ & \leq \left[\int_a^b Q(t)^{\frac{k}{k-m}} \lambda'(t)\lambda(t)^{\frac{l(k-1)}{k-m}} R(t)^{\frac{-km}{k-m}} dt \right]^{\frac{k-m}{k}} \left[\int_a^b \left(\int_0^t x'(t)^k R(t)^k d\lambda(t) \right)^{\frac{l+m}{m}} dt \right]^{\frac{m}{k}} \end{aligned} \quad (133)$$

and the proof is complete.

Lemma 3.3:

Under the assumptions of Theorem 3.4, then the following inequality holds:

$$\int_a^b q(t)x'(t)^m x(t)^l dt \leq L^l \left[\int_a^b P(t)x'(t)^k dt \right]^{\frac{l+m}{k}} \quad (134)$$

Proof:

In Theorem 3.4, setting $\lambda'(t) = P(t)^{-\frac{1}{k-1}}$, $Q(t) = q(t)P(t)^{\frac{1}{k-1}}$, $R(t) = P(t)^{\frac{1}{k-1}}$ and $\lambda(t) = \int_0^t P(t)^{-\frac{1}{k-1}} dt$, then (3.68) yields

$$\begin{aligned}
& \int_a^b q(t) P(t)^{\frac{1}{k-1}} P(t)^{-\frac{1}{k-1}} x(t)^m \left(\int_0^t x'(t) dt \right)^l dt \\
& \leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_a^b P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} \right]^{\frac{k-m}{k}} \\
& \left[\left(\int_0^t |x'(t)|^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt \right)^{\frac{l+m}{m}} \right]^{\frac{m}{k}}
\end{aligned} \tag{135}$$

That is

$$\begin{aligned}
& \int_a^b q(t) x(t)^m \left(\int_0^t x'(t) dt \right)^l dt \\
& \leq \left[\int_a^b q(t)^{\frac{k}{k-m}} P(t)^{\frac{k}{(k-1)(k-m)}} P(t)^{-\frac{1}{k-1}} P(t)^{-\frac{km}{(k-1)(k-m)}} \left(\int_a^b P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} \right]^{\frac{k-m}{k}} \\
& \left[\left(\int_a^b |x'(t)|^k P(t)^{\frac{k}{k-1}} P(t)^{-\frac{k}{k-1}} dt \right)^{\frac{l+m}{m}} \right]^{\frac{m}{k}}
\end{aligned} \tag{136}$$

further simplification yields,

$$\begin{aligned}
& \int_a^b q(t) |x'(t)|^m x(t)^l dt \\
& \leq \left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} dt \right]^{\frac{k-m}{k}} \\
& \left[\int_a^b P(t) |x'(t)|^k dt \right]^{\frac{l+m}{k}}
\end{aligned} \tag{137}$$

By setting

$$L^l = \left[\int_a^b \left(q(t)^k P(t)^{-m} \right)^{\frac{1}{k-m}} \left(\int_0^t P(t)^{-\frac{1}{k-1}} dt \right)^{\frac{l(k-1)}{k-m}} dt \right]^{\frac{k-m}{k}}$$

then, (3.71) becomes

$$\int_a^b q(t) x'(t)^m x(t)^l dt \leq L^l \left[\int_a^b P(t) x'(t)^k dt \right]^{\frac{l+m}{k}} \tag{138}$$

Taking the $(l+m)^{th}$ root of both sides of (3.71) to obtain

$$\left[\int_a^b q(t) x(t)^l x'(t)^m dt \right]^{\frac{1}{l+m}} \leq L^{\frac{1}{l+m}} \left[\int_a^b P(t) x'(t)^k dt \right]^{\frac{1}{k}} \quad (139)$$

Remark 3.5

For, if $q = m$, $L^{\frac{1}{l+m}} = C$, $s = l$, $l'(x) = x'(t)$, $l(x) = x(t)$, $u(t) = P(t)$ $q(t) = v(t)$ and $p = k$, (3.73) coincides to (2.19).

5 Multifunctional Opial-type Integral Inequalities

In this section, we obtain a generalization of Opial-type inequalities which involving

functions and its derivatives from which we deduce an extension and improved versions of

several recent results.

We begin with the following theorems.

Theorem 3.5

Let $\Delta(v)$ be absolutely continuous, $\lambda(v)$ non decreasing functions on $[\eta, \Theta]$ for $0 \leq \eta \leq \Theta < \infty$ with $v > 0$ and convex function $\varphi(\lambda(v)) = \lambda(v)^\varsigma$. Let α, β, k and ς be real numbers such that $\delta \geq 0$, $\varsigma \geq 0$ and suppose $P(v)$ and $R(v)$ are non negative and measurable function on $[\eta, \Theta]$ such that

$$\Delta'(v) \times \varphi \left(\int_0^v \Delta'(v) P(v)^{\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv \right) \leq \lambda(v)^{\varsigma-\zeta} y(v)^\zeta \times P(v)^{\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \nabla'(v). \quad (140)$$

Then, the following inequality holds:

$$\int_\eta^\Theta \Delta'(v) \times \varphi \left(\int_0^v \Delta'(v) dv \right) \leq \int_\eta^\Theta \nabla(v)^\zeta d\nabla'(v). \quad (141)$$

Proof: By Jensen's inequality:

$$\frac{\varphi\left(\int_0^v \Delta'(v)R(v)d\lambda(v)\right)}{\lambda(v)^\zeta} \leq \frac{\left(\int_0^v \varphi(\Delta'(v)R(v))^{\frac{1}{\zeta}} d\lambda(v)\right)^\zeta}{\lambda(v)^\zeta}.$$

With

$h(s,t) = \Delta'(v)R(v)$, we have

$$\varphi\left(\int_0^v \Delta'(v)R(v)d\lambda(v)\right) \leq \lambda(v)^{\zeta-\zeta} \left(\int_0^v (\Delta'(v)R(v))^{\frac{1}{\zeta}} d\lambda(v)\right)^\zeta = \lambda(v)^{\zeta-\zeta} \nabla(v)^\zeta. \quad (142)$$

but,

$$\nabla(v) = \int_0^v \varphi(\Delta'(v)R(v))^{\frac{1}{\zeta}} \lambda'(v)$$

$$\nabla'(t)^\zeta = \Delta'(v)^\zeta R(v)^\zeta \lambda'(v)^\zeta. \quad (143)$$

Hence,

$$\Delta'(t) = R(t)^{-1} \lambda'(t)^{-1} \nabla'(t), \quad (144)$$

Combining both (3.76) and (3.78), yields (3.74) and the proof follows as:

Let $R(v) = P(v)^{-\frac{1}{k-1}}$, $\lambda'(v) = P(v)^{\frac{1}{k-1}}$, $\zeta = \zeta$ in Theorem 2.4 yields,

$$\Delta'(v) \times \varphi\left(\int_0^v \Delta'(v)P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv\right) \leq \lambda(v)^{\zeta-\zeta} y(v)^\zeta \times P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \nabla'(v).$$

Integrating both sides of the last inequality over $[\eta, \Theta]$ with the respect to t , which implies

$$\int_\eta^\Theta \Delta'(v) \times \varphi\left(\int_0^v \Delta'(v)dv\right) \leq \int_\eta^\Theta \nabla(v)^\zeta \nabla'(v)dv. \quad (145)$$

That is,

$$\int_\eta^\Theta \Delta'(v) \times \varphi\left(\int_0^v \Delta'(v)dv\right) \leq \frac{\nabla(v)^{\zeta+1}}{\zeta+1}$$

By using Hölders inequality with α and β we obtain

$$\begin{aligned} \frac{1}{\zeta+1} \nabla(\Theta)^{\zeta+1} &= \frac{1}{\zeta+1} \left(\int_\eta^\Theta \Delta'(v)dv\right)^{\zeta+1} = \frac{1}{\zeta+1} \left(\int_\eta^\Theta R^{-\frac{1}{\beta}}(v) R^{\frac{1}{\beta}} \Delta'(v)(v)d\tau\right)^{\zeta+1} \\ &\leq \frac{1}{\zeta+1} \left(\int_\eta^\Theta R^{1-\alpha}(v)dv\right)^{\frac{\zeta+1}{\alpha}} \left(\int_\eta^\Theta R(v) \Delta'(v)^\beta dv\right)^{\frac{\zeta+1}{\beta}} \end{aligned}$$

Therefore,

$$\int_\eta^\Theta \Delta'(v) \varphi(\Delta(v))dv \leq \frac{1}{\zeta+1} \left(\int_\eta^\Theta R^{1-\alpha}(v)dv\right)^{\frac{\zeta+1}{\alpha}} \left(\int_\eta^\Theta R(v) \Delta'(v)^\beta dv\right)^{\frac{\zeta+1}{\beta}} \quad (146)$$

which gives,

$$\int_\eta^\Theta \Delta'(v) \Delta(v)^\zeta dv \leq \frac{1}{\zeta+1} \left(\int_\eta^\Theta R^{1-\alpha}(v)dv\right)^{\frac{\zeta+1}{\alpha}} \left(\int_\eta^\Theta R(v) \Delta'(v)^\beta dv\right)^{\frac{\zeta+1}{\beta}} \quad (147)$$

Remark 3.7

(1) If $\zeta = 0$ in (3.81) yields,

$$\int_{\eta}^{\Theta} \Delta'(v) \Delta(v)^{\zeta} dv \leq \left(\int_{\eta}^{\Theta} R^{1-\alpha}(v) dv \right)^{\frac{1}{\alpha}} \left(\int_{\eta}^{\Theta} R(v) \Delta'(v)^{\beta} dv \right)^{\frac{1}{\beta}} \quad (148)$$

that is,

$$\int_{\eta}^{\Theta} \Delta'(v) \varphi(\Delta(v)) dv \leq \left(\int_{\eta}^{\Theta} R^{1-\alpha}(t) dt \right)^{\frac{1}{\alpha}} \left(\int_{\eta}^{\Theta} R(v) \Delta'(v)^{\beta} dv \right)^{\frac{1}{\beta}} \quad (149)$$

Also $\zeta = \zeta = 1$, and $\delta = 0$ in (2.82) reduces to

$$\int_{\eta}^{\Theta} \Delta'(v) \Delta(v) dv \leq \frac{1}{2} \left(\int_{\eta}^{\Theta} R^{1-\alpha}(v) dv \right)^{\frac{2}{\alpha}} \left(\int_{\eta}^{\Theta} R(v) \Delta'(v)^{\beta} dv \right)^{\frac{2}{\beta}}$$

(2) If $\zeta = \zeta = \alpha = R(v) = v = 1$ and $\beta = 2$ in (3.81) yields,

$$\int_0^{\Theta} \Delta'(v) \Delta(v) dv \leq \frac{\Theta}{2} \int_0^{\Theta} \Delta'(v)^2 dv \quad (150)$$

which is Opial-type inequality of (2.16).

If we set $1 - \delta = \zeta$ in (3.79) yields

$$\int_{\eta}^{\Theta} \Delta'(v) \times \varphi \left(\int_{\eta}^{\Theta} \Delta'(v) dv \right) dv \leq \int_{\eta}^{\Theta} \nabla(v)^{1-\delta} \nabla'(v) dv \leq \int_{\eta}^{\Theta} \varphi(\nabla(v)) d\nabla'(v). \quad (151)$$

By similarly argument with conjugate exponents α and β , we have

$$\int_{\eta}^{\Theta} \Delta'(v) dv = \int_{\eta}^{\Theta} R^{-\frac{1}{\beta}}(v) R^{\frac{1}{\beta}} \Delta'(v) dv \leq \left(\int_{\eta}^{\Theta} R^{1-\alpha}(v) dv \right)^{\frac{1}{\alpha}} \left(\int_{\eta}^{\Theta} R(v)^{\beta} \Delta'(v) dv \right)^{\frac{1}{\beta}}$$

Hence

$$\int_{\eta}^{\Theta} \Delta'(v) \times \varphi \left(\int_0^v \Delta'(v) dv \right) dv \leq \varphi \left[\left(\int_{\eta}^{\Theta} R^{1-\alpha}(v) dv \right)^{\frac{1}{\alpha}} \left(\int_{\eta}^{\Theta} R(v)^{\beta} \Delta'(v) dv \right)^{\frac{1}{\beta}} \right] \quad (152)$$

that is

$$\int_{\eta}^{\Theta} \Delta'(v) \varphi(\Delta(v)) dv \leq \varphi \left[\left(\int_{\eta}^{\Theta} R^{1-\alpha}(v) d \right)^{\frac{1}{\alpha}} \left(\int_{\eta}^{\Theta} R(v)^{\beta} \Delta'(v) dv \right)^{\frac{1}{\beta}} \right] \quad (153)$$

which is an Opial-type inequality of Calvert.

Similarly, we need the following theorems to obtain a new Opial-type inequalities by modified Jensen's inequality for the case of convex functions. **Theorem 3.6**

Let $\Delta(v), \lambda(v), f(u), R(v), \zeta$ and k be defined as in Theorem 2.4 and $\pi \geq 0, \rho \geq 0$ such that:

$$\Delta'(v) \varphi \left(\int_0^v \Delta'(v) R(v) d\lambda(v) \right) \leq \nabla'(v) R(v)^{-1} \lambda'(v)^{-1} \nabla(v)^{\delta-1} \quad (154)$$

Then, the following inequality holds:

$$\Delta'(v) \varphi \left(\int_0^v \Delta'(v) dv \right) \leq \nabla(v)^{\delta-1} \nabla'(v). \quad (155)$$

Proof: By using (1.4) and if $\zeta = \delta - 1$, (1.4) becomes

$$\left(\varphi \left(\frac{\int_0^v \Delta'(v) R(v) d\lambda(v)}{\int_0^v d\lambda(v)} \right) \right)^{\frac{1}{(\delta-1)}} \leq \left(\frac{\int_0^v \varphi(\Delta'(v) R(v))^{\frac{1}{\delta-1}} d\lambda(v)}{\int_0^v d\lambda(t)} \right),$$

that is

$$\varphi \left(\int_0^v \Delta'(v) R(v) d\lambda(v) \right) \leq \left(\int_0^v \varphi(\Delta'(v) R(v))^{\frac{1}{\delta-1}} d\lambda(v) \right)^{\delta-1},$$

but

$$\nabla'(v) R(v)^{-1} \lambda'(v)^{-1} = \Delta'(v) \quad (156)$$

then,

$$\Delta'(t) \varphi \left(\int_0^v \Delta'(v) R(v) d\lambda(v) \right) \leq \nabla'(v) R(v)^{-1} \lambda'(v)^{-1} \nabla(v)^{\delta-1}$$

This completes the proof of the Theorem.

Remark 3.7

Consider all conditions of Remark 2.5 then, we have

(1)

$$\Delta'(v) \times \varphi \left(\int_0^v \Delta'(v) P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \right) \leq \nabla'(v) \nabla(v)^{\delta-1} P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}}$$

Hence,

$$\Delta'(v) \varphi \left(\int_0^v \Delta'(v) dv \right) \leq \nabla(v)^{\delta-1} \nabla'(v)$$

(2) Also, integrate both side of the last inequality above over $[\eta, \Theta]$ with respect to v , yields

$$\begin{aligned} \int_{\eta}^{\Theta} \Delta'(v) \Delta(v)^{\delta-1} dv &\leq \int_{\eta}^{\Theta} \nabla(v)^{\delta-1} \nabla'(v) \\ &= \frac{1}{\delta} \nabla(\Theta)^{\delta} \leq \frac{1}{\delta} \left(\int_{\eta}^{\Theta} \Delta'(v) P(v)^{-\frac{1}{\rho}} P(v)^{\frac{1}{\rho}} dv \right)^{\delta} \leq \frac{1}{\delta} \left(\int_{\eta}^{\Theta} P(v)^{1-\pi} dv \right)^{\frac{\delta}{\pi}} \left(\int_{\eta}^{\Theta} \Delta'(v)^{\rho} P(v) dv \right)^{\frac{\delta}{\rho}}. \end{aligned}$$

$$\int_{\eta}^{\Theta} \Delta'(v) \Delta(v)^{\delta} dv \leq \frac{1}{\delta} \left(\int_{\eta}^{\Theta} P(v)^{1-\pi} dv \right)^{\frac{\delta}{\pi}} \left(\int_{\eta}^{\Theta} \Delta'(v)^{\rho} P(v) dv \right)^{\frac{\delta}{\rho}}.$$

that is,

$$\int_{\eta}^{\Theta} \Delta'(v) \varphi(\Delta(v)) dv \leq \frac{1}{\delta} \left(\int_{\eta}^{\Theta} P(v)^{1-\pi} dv \right)^{\frac{\delta}{\pi}} \left(\int_{\eta}^{\Theta} \Delta'(v)^{\rho} P(v) dv \right)^{\frac{\delta}{\rho}}$$

which also implies,

$$\int_{\eta}^{\Theta} \Delta'(v) \Delta(v)^{\delta} dv \leq \frac{\chi}{\delta} \left(\int_{\eta}^{\Theta} \Delta'(v)^{\rho} P(v) dv \right)^{\frac{\delta}{\rho}} \quad (157)$$

where,

$$\chi = \left(\int_{\eta}^{\Theta} P(\nu)^{1-\pi} d\nu \right)^{\frac{\delta}{\pi}}.$$

We shall have another result here, using the modified Jensen's inequality to proof a new Opial-type inequalities for multifunction.

Theorem 3.7

Let $\Delta_1(\nu)$ and $\Delta_2(\nu)$ be absolutely continuous, $\lambda(\nu)$ non decreasing functions on $[\eta, \Theta]$ for $0 \leq \eta \leq \Theta < \infty$ with $\nu > 0$, and convex function $\Delta_1(\nu) \leq \phi \left(\int_0^{\nu} \Delta_2'(\nu) d\nu \right), n = 1, 2$. Let l, k and ζ be real numbers such that $\zeta \geq 0$ and also let $R(\nu)$ be non negative and measurable function on $[\Theta, \eta]$ such that:

$$\Delta_2'(\nu) \times \phi \left(\int_0^{\nu} \Delta_2'(\nu) R(\nu) d\lambda(\nu) \right) \leq \nabla_2'(\nu)^{\zeta-\zeta} \nabla_1(t)^{\zeta} \times R(\nu)^{-1} \lambda'(\nu)^{-1} \quad (158)$$

Then, the following inequality holds:

$$\int_{\eta}^{\Theta} \Delta_2'(\nu) \times \phi \left(\int_0^{\nu} \Delta_2'(\nu) d\nu \right) \leq \int_{\eta}^{\Theta} \nabla_1(\nu) \nabla_2'(\nu) d\nu \quad (159)$$

Proof: Assume $h(s,t) = \Delta_2'(v)R(v)$, in Jensen's inequality then we have,

$$\frac{\phi\left(\int_0^v \Delta_2'(v)R(v)d\lambda(v)\right)}{\lambda(v)^\zeta} \leq \frac{\left(\int_0^v \phi(\Delta_2'(v)R(v))^{\frac{1}{\zeta}} d\lambda(v)\right)^\zeta}{\lambda(v)^\zeta}$$

Further simplification yields,

$$\phi\left(\int_0^v \Delta_2'(v)R(v)d\lambda(v)\right) \leq \lambda(v)^{\zeta-\zeta} \left(\int_0^v \phi(\Delta_2'(v)R(v))^{\frac{1}{\zeta}} d\lambda(v)\right)^\zeta \leq \nabla_2'(v)^{\zeta-\zeta} \nabla_1(v)^\zeta \quad (160)$$

Suppose,

$$\nabla_2'(v)^{\zeta-\zeta} = \lambda(v)^{\zeta-\zeta} \quad \text{and} \quad \nabla_1(v) = \int_0^v \phi(\Delta_2'(v)R(v))^{\frac{1}{\zeta}} \lambda'(v)$$

Hence,

$$\Delta_2'(v) = R(v)^{-1} \lambda'(v)^{-1} \nabla_1'(v) \quad (161)$$

Then,

$$\Delta_2'(v) \times \phi\left(\int_0^v \Delta_2'(v)R(v)d\lambda(v)\right) \leq \nabla_2'(v)^{\zeta-\zeta} \nabla_1(t)^\zeta \times \nabla_1'(v)R(v)^{-1} \lambda'(t)^{-1} \quad (162)$$

and the proof of the Theorem is complete.

Corollary 3.1

By setting $R(v) = P(v)^{-\frac{1}{k-1}}$, $\lambda'(v) = p(v)^{\frac{1}{k-1}}$, $\nabla_1'(v) = 1$, $\zeta = 1$ and $\varsigma = 2$ in (3.97) then,

$$\Delta_2'(v) \times \phi\left(\int_0^v \Delta_2'(v)P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv\right) \leq \nabla_2'(v)^{2-1} \nabla_1(v) \times P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}}$$

Proof: This implies that,

$$\int_\eta^\Theta \Delta_2'(v) \times \phi\left(\int_0^v \Delta_2'(v)dv\right) dv \leq \int_\eta^\Theta \nabla_1(v) \nabla_2'(v) dv$$

$$\int_\eta^\Theta \Delta_2'(v) \Delta_1(v) dv \leq \int_\eta^\Theta \nabla_1(v) \nabla_2'(v) dv \quad (163)$$

Similarly, replacing $\Delta_2(v)$ with $\Delta_1(v)$ in (3.97) yields,

$$\frac{\phi\left(\int_0^v \Delta_1'(v)R(v)d\lambda(v)\right)}{\lambda(v)^\zeta} \leq \frac{\left(\int_0^t \phi(\Delta_1'(t)R(v))^{\frac{1}{\zeta}} d\lambda(t)\right)^\zeta}{\lambda(t)^\zeta} \quad (164)$$

Hence,

$$\phi\left(\int_0^v \Delta_1'(v)R(v)d\lambda(v)\right) \leq \nabla_1'(v)^{\zeta-\zeta} \left(\int_0^v \phi(\Delta_1'(v)R(v))^{\frac{1}{\zeta}} d\lambda(v)\right)^\zeta = \nabla_1'(v)^{\zeta-\zeta} \nabla_2(v)^\zeta. \quad (165)$$

Let,

$$\nabla_1'(v)^{\zeta-\zeta} = \lambda(v)^{\zeta-\zeta} \quad \text{and} \quad \nabla_2(v) = \int_0^v \phi(\Delta_1'(v)R(v))^{\frac{1}{\zeta}} \lambda'(v) dv$$

Hence,

$$\Delta_1'(v) = \nabla_2'(v)R(v)^{-1} \lambda'(v)^{-1} \quad (166)$$

from (3.99) and (3.100), we have

$$\Delta_1'(v) \times \phi \left(\int_0^v \Delta_1'(v) R(v) d\lambda(v) \right) \leq \nabla_1'(v)^{\zeta-\zeta} \nabla_2(v)^\zeta \times \nabla_2'(v) R(v)^{-1} \lambda'(v)^{-1} \quad (167)$$

and the proof is complete.

Corollary 3.2

By setting $R(v) = P(v)^{-\frac{1}{k-1}}$, $\lambda'(v) = p(v)^{\frac{1}{k-1}}$, $\nabla_2'(v) = 1$, $\zeta = 1$ and $\zeta = 2$ in (3.101) yields,

$$\Delta_1'(v) \times \phi \left(\int_0^v \Delta_1'(v) P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv \right) \leq \nabla_1'(t)^{2-1} \nabla_2(v) \times P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \quad (168)$$

Proof:

Integrating both sides of (3.102) over $[\Theta, \eta]$ with the respect to v ,

$$\int_\eta^\Theta \Delta_1'(v) \times \phi \left(\int_0^v \Delta_1'(v) dv \right) dv \leq \int_\eta^\Theta \nabla_2(v) \nabla_1'(v) dv \quad (169)$$

If $\Delta_2(v) \leq \phi \left(\int_0^v \Delta_1'(v) dv \right)$ in (3.103) becomes

$$\int_\eta^\Theta \Delta_1'(v) \Delta_2(v) dv \leq \int_\eta^\Theta \nabla_2(v) \nabla_1'(v) dv \quad (170)$$

Adding both sides of (3.97) and (3.104) together to yields

$$\int_\eta^\Theta \Delta_2'(v) \Delta_1(v) dv + \Delta_1'(v) \Delta_2(v) dv \leq \int_\eta^\Theta \nabla(v)_1 \nabla_2'(v) dv + \nabla_2(v) \nabla_1'(v) dv \leq \nabla_1(\Theta) \nabla_2(\Theta).$$

Since $\nabla_1(\eta) = 0$ then,

$$\nabla_1(\Theta) = \int_\eta^\Theta \Delta_1'(v) dv = \int_\eta^\Theta \Delta_1'(v) P_1(v)^{-1} P_1(v) dv \leq \left(\int_\eta^\Theta P_1(v)^{-2} dv \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_1(v)^2 \Delta_1'(v)^2 dv \right)^{\frac{1}{2}}$$

Since $\nabla_2(a) = 0$ then,

$$\nabla_2(\Theta) = \int_\eta^\Theta \Delta_2'(v) dv = \int_\eta^\Theta \Delta_2'(v) P_1(v)^{-1} P_1(v) dv \leq \left(\int_\eta^\Theta P_2(v)^{-2} dv \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_2(v)^{-2} \Delta_2'(v)^2 dt \right)^{\frac{1}{2}}$$

Hence,

$$\begin{aligned} & \int_\eta^\Theta \Delta_1(v) \Delta_2'(v) dv + \Delta_1'(v) \Delta_2(v) dv \\ & \leq \left(\int_\eta^\Theta P_1(v)^{-2} dv \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_1(t)^2 \Delta_1'(v)^2 dv \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_2(v)^{-2} dv \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_2(v)^{-2} \Delta_2'(v)^2 dv \right)^{\frac{1}{2}} \end{aligned} \quad (171)$$

which is the desired inequality.

Remark 3.8

(a) In (3.104) with the fact that $f(t) = t^\zeta$ it becomes,

$$\int_\eta^\Theta \Delta_1'(v) \times \left(\int_0^v \Delta_1'(v) dv \right)^\zeta \leq \int_\eta^\Theta \nabla_2(v) \nabla_1'(v) dv.$$

that is,

$$\int_\eta^\Theta \Delta_1'(v) \Delta_1(v)^\zeta dv \leq \int_\eta^\Theta \nabla_2(v) \nabla_1'(v) dv \quad (172)$$

(b) We observe that $\nabla_1(v) = \nabla_2(v) = \nabla(v)$ and $\nabla(\eta) = 0$ in (2.106), we obtain

$$\int_{\eta}^{\Theta} \Delta'(v) \Delta(v)^{\varsigma} dv \leq \int_{\eta}^{\Theta} \nabla(v) \nabla'(v) dv \leq \frac{1}{2} \nabla(\Theta)^2 = \frac{\Theta}{2} \int_{\eta}^{\Theta} \nabla'(v)^2 dv \quad (173)$$

that is, If in (3.106) $\varsigma = 1$, it becomes the result in Olech.

Theorem 3.10

Let $\Delta_{\gamma}(v)$ and $\Delta_{\gamma+1}(v)$ be absolutely continuous, $\lambda(v)$ non decreasing functions on $[\eta, \Theta]$ for $0 \leq \eta \leq \Theta < \infty$ with $v > 0, i = \gamma, \gamma+1$. Let ς, k and ζ be real numbers such that $\zeta \geq 0$ and also let $R(v)$ be non negative and measurable function on $[\eta, \Theta]$ such that:

$$\Delta_{\gamma+1}'(v) \times \phi \left(\int_0^v \Delta_{\gamma+1}'(v) R(v) d\lambda(v) \right) \leq \nabla'_{\gamma+1}(v)^{\varsigma-\zeta} \nabla_{\gamma}(v)^{\zeta} \times R(v)^{-1} \lambda'(v)^{-1} \quad (174)$$

Then, the following inequality holds:

$$\int_{\eta}^{\Theta} \Delta'_{\gamma+1}(v) \times \phi \left(\int_0^v \Delta_{\gamma+1}'(v) dv \right) \leq \int_{\eta}^{\Theta} \nabla_{\gamma}(v) \nabla'_{\gamma+1}(v) dv \quad (175)$$

Proof: The proof of Theorem 3.8 could be sourced from the proof of Theorem 3.5.

Theorem 3.9

Assume all assumptions in Theorem 3.5 holds, we have

$$\Delta_{\gamma}'(v) \times \phi \left(\int_0^t \Delta_{\gamma}'(v) R(v) d\lambda(v) \right) \leq \nabla'_{\gamma}(v)^{\varsigma-\zeta} \nabla_{\gamma+1}(v)^{\zeta} \times R(v)^{-1} \lambda'(v)^{-1} \quad (176)$$

Then, the following inequality holds:

$$\int_{\eta}^{\Theta} \Delta'_{\gamma}(v) \times \kappa \left(\int_0^v \Delta_{\gamma}'(v) dv \right) \leq \int_{\eta}^{\Theta} \nabla_{\gamma+1}(v) \nabla'_{\gamma}(v) dv \quad (177)$$

Proof: The proof of Theorem 3.9 is similar to the proof of Theorem 2.8.

Corollary 3.3

By setting $\kappa(v) = v^{\varsigma}, R(t) = P(v)^{-\frac{1}{k-1}}, \lambda'(v) = p(v)^{\frac{1}{k-1}}, \zeta = 1$ and $\varsigma = 2$ in Theorem 2.4 yields,

$$\Delta_{\gamma+1}'(v) \times \phi \left(\int_0^v \Delta_{\gamma+1}'(v) P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv \right) \leq \nabla'_{\gamma+1}(v)^{2-1} \nabla_{\gamma}(v) P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \quad (178)$$

Proof: Integrating both sides of (3.112) over $[\eta, \Theta]$ with the respect to v ,

$$\int_{\eta}^{\Theta} \Delta'_{\gamma+1}(v) \times \phi \left(\int_0^v \Delta_{\gamma+1}'(v) dv \right) \leq \int_{\eta}^{\Theta} \nabla_{\gamma}(v) \nabla'_{\gamma+1}(v) dv \quad (179)$$

If $\Delta_{\gamma}(v) \leq \phi \left(\int_0^v \Delta_{\gamma+1}'(v) dv \right)$ in (2.113) becomes,

$$\int_{\eta}^{\Theta} \Delta_{\gamma+1}'(v) \Delta_{\gamma}(v) dv \leq \int_{\eta}^{\Theta} \nabla_{\gamma}(v) \nabla'_{\gamma+1}(v) dv \quad (180)$$

Corollary 3.4

By setting $R(v) = P(v)^{-\frac{1}{k-1}}, \lambda'(v) = p(v)^{\frac{1}{k-1}}, \zeta = 1$ and $\varsigma = 2$ in Theorem 3.5 yields,

$$\Delta_{\gamma}'(v) \times \phi \left(\int_0^v \Delta_{\gamma}'(v) P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} dv \right) \leq \nabla'_{\gamma}(t)^{2-1} \nabla_{\gamma+1}(v) \times P(v)^{-\frac{1}{k-1}} P(v)^{\frac{1}{k-1}} \quad (181)$$

Proof: Integrating both sides of (3.115) over $[\eta, \Theta]$ with the respect to γ , to get

$$\int_{\eta}^{\Theta} \Delta'_{\gamma}(v) \times \phi \left(\int_0^v \Delta_{\gamma}'(v) dv \right) \leq \int_{\eta}^{\Theta} \nabla_{\gamma+1}(v) \nabla'_{\gamma}(v) dv \quad (182)$$

If $\Delta_{\gamma+1}(\nu) \leq \varphi\left(\int_0^\nu \Delta_\gamma'(v)dv\right)$ in (2.116) becomes,

$$\int_\eta^\Theta \Delta_\gamma'(\nu) \Delta_{\gamma+1}(\nu) d\nu \leq \int_\eta^\Theta \nabla_{\gamma+1}(\nu) \nabla_\gamma'(\nu) d\nu \quad (183)$$

Adding both sides of inequality (3.114) and (3.117) together to yields,

$$\begin{aligned} & \int_\eta^\Theta \Delta_{\gamma+1}'(\nu) \Delta_\gamma(\nu) d\nu + \Delta_\gamma'(\nu) \Delta_{\gamma+1}(\nu) d\nu \\ & \leq \int_\eta^\Theta \nabla_\gamma(\nu) \nabla_{\gamma+1}'(\nu) d\nu + \nabla_{\gamma+1}(\nu) \nabla_\gamma'(\nu) d\nu = \int_\eta^\Theta (\nabla_\gamma(\nu) \nabla_{\gamma+1}(\nu)) d\nu \end{aligned}$$

$$\int_\eta^\Theta \Delta_\gamma(\nu) \Delta_{\gamma+1}'(\nu) d\nu + \Delta_\gamma'(\nu) \Delta_{\gamma+1}(\nu) d\nu \leq \nabla_\gamma(\Theta) \nabla_{\gamma+1}(\Theta) \quad (184)$$

Since $\nabla_\gamma(\eta) = 0$ then,

$$\begin{aligned} \nabla_\gamma(\Theta) &= \int_\eta^\Theta \Delta_\gamma'(\nu) d\nu = \int_\eta^\Theta \Delta_\gamma'(\nu) P_1(\nu)^{-1} P_1(\nu) d\nu \\ &\leq \left(\int_\eta^\Theta P_1(\nu)^{-2} d\nu \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_1(\nu)^2 \Delta_\gamma'(\nu)^2 d\nu \right)^{\frac{1}{2}} \end{aligned}$$

Also, $\nabla_{\gamma+1}(\eta) = 0$ then, $\nabla_{\gamma+1}(\Theta) = \int_\eta^\Theta \Delta_{\gamma+1}'(\nu) d\nu$

$$= \int_\eta^\Theta \Delta_{\gamma+1}'(\nu) P_1(\nu)^{-2} P_1^2(\nu) d\nu \leq \left(\int_\eta^\Theta P_2(t)^{-2} dt \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_2(\nu)^{-2} \Delta_{\gamma+1}'(\nu)^2 d\nu \right)^{\frac{1}{2}}$$

Thus,

$$\begin{aligned} & \int_\eta^\Theta \Delta_\gamma(\nu) \Delta_{\gamma+1}'(\nu) + \Delta_\gamma'(\nu) \Delta_{\gamma+1}(\nu) d\nu \leq \left(\int_\eta^\Theta P_1(\nu)^{-2} d\nu \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_1(\nu)^2 \Delta_\gamma'(\nu)^2 d\nu \right)^{\frac{1}{2}} \\ & \left(\int_\eta^\Theta P_2(\nu)^{-2} d\nu \right)^{\frac{1}{2}} \left(\int_\eta^\Theta P_2(\nu)^2 \Delta_{\gamma+1}'(\nu)^2 d\nu \right)^{\frac{1}{2}} \end{aligned} \quad (185)$$

which can also be written as:

$$\begin{aligned} & \int_\eta^\Theta [(\Delta_1(\nu) \Delta_2(\nu)) + (\Delta_3(\nu) \Delta_4(\nu)) + \dots + (\Delta_\gamma(\nu) \Delta_{\gamma+1}(\nu)) + (\Delta_{\gamma+1}(\nu) \Delta_{\gamma+2}(\nu))] d\nu \\ & \leq \int_\eta^\Theta [(\nabla_1(\nu) \nabla_2(\nu)) + (\nabla_2(\nu) \nabla_3(\nu)) + \dots + (\nabla_\gamma(\nu) \nabla_{\gamma+1}(\nu))] d\nu \end{aligned}$$

that is,

$$\int_\eta^\Theta \left(\sum_{i=1}^{\gamma} \Delta_i(\nu) \Delta_{i+1}(\nu) \right) d\nu \leq \int_\eta^\Theta \left(\sum_{i=1}^{\gamma} \nabla_i(\nu) \nabla_{i+1}(\nu) \right) d\nu \quad (186)$$

which indeed is an Opial-type inequality.

6 On Some Results of Hardy's and Opial's Inequality

In this section, we shall use inequality (1.3) to obtain both Abdulaziz Gbadebo Adeagbo-Sheikh and Olanrewaju Olabiyi Fabulurin, and James Oguntuase results.

The modified Minkowski integral inequalities (3.121) and (3.122) is stated in Opic and Kufner (1990) on page 21 and 22 as follows :

Lemma 3.9

Let $K(x, y)$ be a non negative measurable function on $(a, b) \times (c, d)$ and let $r \geq 1$. Then,

$$\left[\int_a^b \left[\int_c^d K(x, y) dy \right]^r dx \right]^{\frac{1}{r}} \leq \int_c^d \left[\int_a^b K^r(x, y) dy \right]^{\frac{1}{r}} \quad (187)$$

If $K(x, y) = H(x)G(y)$ where $x \in (a, b)$ and $y \in (c, d)$ then (2.122) reduces to

$$\left[\int_a^b G(x) \left[\int_a^b H(y) dy \right]^r dx \right]^{\frac{1}{r}} \leq \int_a^b H(y) \left[\int_y^b G(x) dx \right]^{\frac{1}{r}} \quad (188)$$

Our result is the following:

Theorem 3.10

Let g be continuous and non-decreasing on $[a, b]$, and $0 < a \leq b < \infty$ with $g(t) > 0$ for $t > 0$. Suppose that $l \geq p \geq 1, \nu > 0, 0 < \nu \leq l$ and $\delta > 0$. Let $f(x)$ be absolutely, non-decreasing function and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Then,

$$\begin{aligned} & \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^{\nu} dg(x) \\ & + \left[\delta \right]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\ & \leq \left[\delta \right]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] \left[\int_a^b f^l(t) g(x)^{l(1+\delta)-1} dg(t) \right]^{\frac{\nu}{l}}. \end{aligned} \quad (189)$$

Proof:

In Jensen's inequality (1.3) with

$$\left. \begin{aligned} h(x,t) &= f^l(t) f^p(x) g(x)^\delta g(t)^{l(1+\delta)} \\ d\lambda(s) &= g(t)^{-(1+\delta)} dg(t) \end{aligned} \right\} \quad (190)$$

Then, we have

$$\begin{aligned} & \left[\left(\int_a^x f^l(t) g(x)^\delta g(t)^{l(1+\delta)} f^p(t) \right)^{\frac{1}{l}} g^{-(1+\delta)} dg(t) \right]^l \\ & \leq \left[\int_a^x g(t)^{-(1+\delta)} dg(t) \right]^{l-1} \left[\int_a^x f^l(t) f^p(x) g(x)^\delta g(t)^{l(1+\delta)} g(t)^{-(1+\delta)} dg(t) \right] \end{aligned}$$

which implies

$$\begin{aligned} & \left[\int_a^x f(x)^{\frac{p}{l}} g(x)^{\frac{\delta}{l}} g(t)^{(1+\delta)} f(t) g^{-(1+\delta)} dg(t) \right]^l \\ & \leq [g(x)^{-\delta} - g(a)^{-\delta}]^{l-1} [\delta^{-1}]^{l-1} \left[\int_a^x f(t)^l g(x)^\delta f^p(x) g(t)^{(l-1)(1+\delta)} dg(t) \right] \end{aligned}$$

Therefore, we have

$$\begin{aligned} & f(x)^p g^\delta(t) \left[\int_a^x f(t) dg(t) \right]^l \\ & \leq [g(x)^{-\delta} - g(a)^{-\delta}]^{l-1} [\delta^{-1}]^{l-1} g(x)^\delta f^p(x) \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right] \end{aligned}$$

which gives

$$\begin{aligned} & f^p(x) g(x)^\delta [g(x)^{-\delta} - g(a)^{-\delta}]^{l-1} \left[\int_a^x f(t) dg(t) \right]^l \\ & \leq [\delta^{-1}]^{l-1} g(x)^\delta f^p(x) \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right] \end{aligned} \quad (191)$$

factor out $f(x)^p$, take l^{th} root of both sides, multiplying the powers with ν and simplifying the above yields

$$\begin{aligned} & g(x)^{\frac{\delta\nu}{l}} [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(l-1)}{l}} \left[\int_a^x f(t) dg(t) \right]^\nu \\ & \leq [\delta^{-1}]^{\frac{\nu(l-1)}{l}} g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \end{aligned} \quad (192)$$

Integrating both sides of (3.126) over $[a, b]$ with respect to $g(x)^{-1} dg(x)$, we have

$$\begin{aligned} & [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(l-1)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^\nu dg(x) \\ & \leq [\delta^{-1}]^{\frac{\nu(l-1)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} dg(x) \end{aligned} \quad (193)$$

Using integration by parts to simplifying the right hand side of above inequality

$$\begin{aligned}
& \left[\delta^{-1} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} dg(x) \\
& = (-) [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\
& + [\delta]^{\frac{\nu(1-l)-l}{l}} \int_a^b g(x)^{l(1+\delta)-1} f^l(t) \left[g(x)^{\delta} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}-1} dg(x).
\end{aligned} \tag{194}$$

Therefore, choose that $\delta > 0$, $g(x)^{\delta} \leq g(t)^{\delta}$ integral inequality above becomes

$$\begin{aligned}
& \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^{\nu} dg(x) \\
& \leq (-) [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\
& + [\delta]^{\frac{\nu(1-l)-l}{l}} \int_a^b g(x)^{l(1+\delta)-1} f^l(x) \left[\int_a^x g(x)^{l(1+\delta)-1} dg(t) \right]^{\frac{\nu}{l}-1} dg(x).
\end{aligned} \tag{195}$$

Evaluating the last integral on the right hand side of the inequality above, we have

$$\begin{aligned}
& \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^{\nu} dg(x) \\
& \leq (-) [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\
& + [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] \left[\int_a^b f^l(t) g(x)^{l(1+\delta)-1} dg(t) \right]^{\frac{\nu}{l}}.
\end{aligned} \tag{196}$$

That is

$$\begin{aligned}
& \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^{\nu} dg(x) \\
& + [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\
& \leq [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] \left[\int_a^b f^l(t) g(x)^{l(1+\delta)-1} dg(t) \right]^{\frac{\nu}{l}}
\end{aligned} \tag{197}$$

This completes the proof of the theorem.

Let

$$\begin{cases} v_2 = [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(1-l)}{l}} g(x)^{\frac{\delta\nu}{l}-1} \\ C_2(l, \nu, \delta) = [\delta]^{\frac{\nu(1-l)-l}{l}} \left[\frac{l}{\nu} \right] \\ B_2(l, \nu, \delta, g(a)) = C_2(l, \nu, \delta) g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\ u_2(x) = g(x)^{l(1+\delta)-1} \end{cases} \quad (198)$$

then, substitute (3.132) in (3.131), we obtain

$$\int_a^b \left[\int_a^x f(t) dg(t) \right]^\nu v_2(x) dg(x) + B_2(l, \nu, \delta, g(a)) \leq C_2(l, \nu, \delta) \left[\int_a^b f^l(t) u_2(x) dg(t) \right]^{\frac{\nu}{l}}. \quad (199)$$

Finding the ν^{th} root of both sides we have the following

$$\left[\int_a^b \left[\int_a^x f(t) dg(t) \right]^\nu v_2(x) dg(x) \right]^{\frac{1}{\nu}} + B_2(l, \nu, \delta, g(a))^{\frac{1}{\nu}} \leq C_2(l, \nu, \delta)^{\frac{1}{\nu}} \left[\int_a^b f^l(t) u_2(x) dg(t) \right]^{\frac{1}{l}}. \quad (200)$$

which generalizes Fabelurin result.

Remark 3.9

(i) We see this by letting $f(x) = |l'(x)|, l(x) = \int_a^x f(t) dg(t), v_2(x) = v(x), u_2(x) = u(x), g(x) = t$ and $B_2(l, \nu, \delta, g(a)) = 0$ and $C = C_2(l, \nu, \delta)^{\frac{1}{\nu}}$ in (3.134) reduces to the (2.21).

Furthermore, we shall need (3.126) to generalize Oguntuase result. To have this result, (3.126) becomes

$$g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f(t) dt \right]^\nu \leq [\delta^{-1}]^{\frac{\nu(1-l)}{l}} [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(1-l)}{l}} g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dt \right]^{\frac{\nu}{l}} \quad (201)$$

We integrate both sides of (3.135) with respect to $g(x)$ from a to b and then raising both sides to power $\frac{l}{\nu}$ to obtain

$$\begin{aligned} & \left(\int_a^b g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f(t) dt \right]^\nu dg(x) \right)^{\frac{l}{\nu}} \\ & \leq \left([\delta^{-1}]^{\frac{\nu(1-l)}{l}} \int_a^b [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(1-l)}{l}} g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f^l(t) g(t)^{(l-1)(1+\delta)} dt \right]^{\frac{\nu}{l}} dg(x) \right)^{\frac{l}{\nu}} \end{aligned} \quad (202)$$

We shall need Minkowski integral inequalities of (3.121) and (3.122) to the right hand side of (3.136) yields.

$$\begin{aligned}
& \left(\int_a^b g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f(t) dt \right]^\nu dg(x) \right)^{\frac{l}{\nu}} \\
& \leq [\delta^{-1}]^{l-1} \int_a^b g(t)^{(l-1)(1+\delta)} f^l(t) \left(\int_t^a [g(x)^{-\delta} - g(a)^{-\delta}]^{\frac{\nu(1-l)}{l}} g(x)^{\frac{\delta\nu}{l}} dg(x) \right)^{\frac{l}{\nu}} dg(t) \\
& \leq [\delta^{-1}]^{l-1} [g(x)^{-\delta} - g(a)^{-\delta}]^{l-1} \int_a^b g(t)^{(l-1)(1+\delta)} f^l(t) \left(\int_t^b g(x)^{\frac{\delta\nu}{l}} dg(x) \right)^{\frac{l}{\nu}} dg(t) \\
& = [\delta^{-1}]^{l-1} \left(\frac{l}{\delta\nu+l} \right)^{\frac{l}{\nu}} [g(x)^{-\delta} - g(a)^{-\delta}]^{(l-1)} \\
& \quad \int_a^b g(t)^{(l-1)(1+\delta)} f^l(t) \left[g(b)^{\frac{\delta\nu+l}{l}} - g(t)^{\frac{\delta\nu+l}{l}} \right]^{\frac{l}{\nu}} dg(t) \\
& \leq [\delta^{-1}]^{l-1} \left(\frac{l}{\delta\nu+l} \right)^{\frac{l}{\nu}} [g(x)^{-\delta} - g(a)^{-\delta}]^{(l-1)} \left[g(b)^{\frac{\delta\nu+l}{l}} - g(t)^{\frac{\delta\nu+l}{l}} \right]^{\frac{l}{\nu}} \\
& \quad \int_a^b g(t)^{(l-1)(1+\delta)} f^l(t) dg(t)
\end{aligned} \tag{203}$$

Therefore, (3.137) becomes

$$\begin{aligned}
& \left(\int_a^b g(x)^{\frac{\delta\nu}{l}} \left[\int_a^x f(t) dt \right]^\nu dg(x) \right)^{\frac{1}{\nu}} \\
& \leq [\delta^{-1}]^{l-1} \left(\frac{l}{\delta\nu+l} \right)^{\frac{l}{\nu}} [g(x)^{-\delta} - g(a)^{-\delta}]^{(l-1)} \left[g(b)^{\frac{\delta\nu+l}{l}} - g(t)^{\frac{\delta\nu+l}{l}} \right]^{\frac{l}{\nu}} \\
& \quad \left(\int_a^b g(t)^{(l-1)(1+\delta)} f^l(t) dg(t) \right)^{\frac{1}{l}}
\end{aligned} \tag{204}$$

which completes the proof of the Theorem .

We also see that (1.138) coincides with (2.34), if $q = \nu$, and $l = p$.

Remark 3.10:

(i) For the case $g(x) = x$, and $\nu = l = 1$ (3.138) becomes

$$\int_a^b x^\delta \left(\int_a^x f(t) dg(t) \right) dx \leq \frac{(g(b)^{\delta+1} - g(a)^{\delta+1})}{\delta+1} \left(\int_a^b f(t) dg(t) \right)$$

(ii) We also see that if $l = 1$ and $g(t) = 0$ (3.138) then, we get

$$\int_a^b g(x)^{\delta\nu} \left(\int_a^x f(t) dg(t) \right)^{\frac{1}{\nu}} dg(x) \leq \left(\frac{1}{\delta\nu+1} \right)^{\frac{1}{\nu}} g(b)^{\frac{\delta\nu+1}{\nu}} \int_a^b f(t) dg(x)$$

(iii) We observe that if

$$C = \left[\delta^{-1} \right]^{l-1} \left(\frac{l}{\delta\nu+l} \right)^{\frac{l}{\nu}} g(a)^{\frac{\delta\nu+l}{l}} \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{(l-1)} \left[g(b)^{\frac{\delta\nu+l}{l}} - g(a)^{\frac{\delta\nu+l}{l}} \right]^{\frac{l}{\nu}},$$

$g(x) = x, \nu = q,$ and $l = p,$ in (3.138) it reduces to weighted Hardy's (2.34).

4 MAIN RESULTS

The purpose of this section is to obtain some Opial-type inequalities for convex

functions, through the modified Jensen's inequality, which extends and generalizes some

recent results on time scale.

1 Jensen's inequalities on Time Scale

The original Jensen's Inequality proved by Jensen states that if $g \in C([a, b], (c, d))$

and $\phi \in C([a, b], \mathbb{R})$ is convex, then

$$\phi\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{1}{b-a} \int_a^b \phi(g(s))ds. \quad (1)$$

If f is strictly convex, then the inequality sign $\geq$ in (4.1) should be replaced by $>$.

The following Lemma was adopted from Agarwal and Bohner (2010)

Lemma 4.1

Let $\phi \in C((c, d), \mathbb{R})$ be convex. Then, for each $t \in (c, d)$, there exist $B_t \in \mathbb{R}$ such that

$$\phi(x) - \phi(t) \geq B_t(x - t), \text{ for all } x \in (c, d). \quad (2)$$

In this section, the extension of the results above shall be showed on time scales.

We shall extend our result to Opial's inequality on times scale.

Theorem 4.1

Let $[0, t] \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $\Delta \in C_{rd}([0, t], (c, d))$ and $R \in ([0, t]_{\mathbb{T}}, \mathbb{R})$ with

$$\int_0^t \Delta'(s) d\lambda(s) > 0$$

If ϕ is convex (4.1) could be written as follows:

$$\phi\left(\frac{\int_0^t \Delta'(t) R(t) d\lambda(t)}{\int_0^t d\lambda(t)}\right) \leq \left(\frac{\int_0^t \phi(\Delta'(t) R(t)) d\lambda(t)}{\int_0^t d\lambda(t)}\right) \quad (3)$$

if ϕ is strictly convex, then the inequality $\leq$ can be replaced by $<$.

Proof:

Since ϕ is convex it follows from Lemma 4.1 that for each $t \in (c, d)$ there exist $\beta_t \in \mathbb{R}$ such that (4.3) holds. Let

$$t = \frac{\int_0^t \Delta'(s) R(s) d\lambda(s)}{\int_0^t d\lambda(s)} \quad (4)$$

Thus

$$\begin{aligned} & \left(\int_0^t d\lambda(s)\right) \phi(t) \leq \int_0^t \phi(\Delta'(s) R(s)) d\lambda(s) \\ & \int_0^t \phi(\Delta'(s) R(s)) d\lambda(s) - \phi(t) \left(\int_0^t d\lambda(s)\right) \\ & = \int_0^t \phi(\Delta'(s) R(s) - \phi(t)) d\lambda(s) \geq \beta_t \int_0^t (\Delta'(s) R(s) - t) d\lambda(s) \\ & = \beta_t \left(\int_0^t R(s) \Delta'(s) d\lambda(s) - t \int_0^t d\lambda(s)\right) \\ & = \beta_t \left(\int_0^t R(s) \Delta'(s) d\lambda(s) - \frac{\int_0^t \Delta'(s) R(s) d\lambda(s)}{\int_0^t d\lambda(s)} \left(\int_0^t d\lambda(s)\right)\right) = 0 \end{aligned}$$

2 Opial-type inequalities on time scales

We present the results as follows:

Theorem 4.2

Let $\mathbb{T}$ be a time scale with $t \in \mathbb{T}$. Let ς, ζ be real numbers, let $h, q \in C_{rd}([\alpha, \beta]_{\mathbb{T}}, \mathbb{R})$ where h and q are positive rd-continuous functions on $[\alpha, \beta]_{\mathbb{T}}$ such that $\int_{[\alpha, t]} r(t) \Delta(s) < \infty$. We define ϕ as convex function and if $x: [\alpha, \beta]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\alpha(0) = 0$, then we have

$$\begin{aligned}
& \int_{[\alpha,\beta]} (\sqrt{q(t)})^{\zeta+1} x^\Delta(t) x^\zeta(t) \Delta(t) \\
& \leq \frac{1}{1+\zeta} (\beta-\alpha)^{\zeta-\zeta} \left(\int_{[\alpha,\beta]} \frac{\Delta(t)}{(\sqrt{h(t)})^{1+\zeta}} \right) \int_{[\alpha,\beta]} (\sqrt{h(t)q(t)})^{1+\zeta} x^\Delta(t)^{1+\zeta} \Delta(t)
\end{aligned}$$

Proof: Modified Jensen's inequality

$$\left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s) \right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{-\zeta}.$$

With $\psi(s) = \sqrt{q(s)}$, $d\lambda(s) = x^{\Delta}(s) \beta = t$ and $\alpha = \varepsilon$
we have,

$$\begin{aligned} (\beta - \alpha)^{-\zeta} &\leq \left(\int_{[\alpha, \beta]} \varphi(\psi(s))^{\frac{1}{\zeta}} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \psi(s) x^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha, \beta]} \varphi(\sqrt{q(s)})^{\frac{1}{\zeta}} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha, \beta]} (\sqrt{q(s)})^{\frac{\zeta}{\zeta}} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \end{aligned}$$

Hence,

$$\left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \leq \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{\zeta} (\beta - \alpha)^{\zeta - \zeta} \quad (5)$$

Now,

$$\gamma(t) = \int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s)$$

$$\gamma^{\Delta}(t) = \sqrt{q(t)} x^{\Delta}(t) \quad (6)$$

In view of (4.5) and (4.6), we obtain

$$x^{\Delta}(t) \left(\int_{[\alpha, \beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \frac{\gamma^{\Delta}(t)}{\sqrt{q(t)}} \gamma(t)^{\zeta}$$

which implies

$$(\sqrt{q(t)})^{\zeta + 1} x^{\Delta}(t) \left(\int_{[\alpha, \beta]} x^{\Delta}(s) \Delta(s) \right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \gamma^{\Delta}(t) \gamma(t)^{\zeta}$$

Integrating both side of the latter inequality with respect to delta derivative, we have the following inequality:

$$\begin{aligned} \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta + 1} x^{\Delta}(t) x^{\zeta}(t) \Delta(t) &\leq (\beta - \alpha)^{\zeta - \zeta} \int_{[\alpha, \beta]} \gamma^{\Delta}(t) \gamma(t)^{\zeta} \Delta(t) \leq \frac{1}{1 + \zeta} (\beta - \alpha)^{\zeta - \zeta} \gamma(\beta)^{1 + \zeta} \\ &= \frac{1}{1 + \zeta} (\beta - \alpha)^{\zeta - \zeta} \left(\int_{[\alpha, \beta]} \frac{1}{\sqrt{h(t)}} \int_{[\alpha, \beta]} \sqrt{h(t) q(t)} x^{\Delta}(t) \Delta(t) \right)^{1 + \zeta} \end{aligned}$$

Which implies,

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+1} x^\Delta(t) x^\zeta(t) \Delta(t) \\
& \leq \frac{1}{1+\zeta} (\beta - \alpha)^{\zeta-\zeta} \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{(\sqrt{h(t)})^{1+\zeta}} \right) \int_{[\alpha, \beta]} (\sqrt{h(t)q(t)})^{1+\zeta} x^\Delta(t)^{1+\zeta} \Delta(t)
\end{aligned} \tag{7}$$

The above (4.7) becomes (2.41) when $T = \mathbb{R}$ and $\zeta = \zeta = q(t) = 1$ is

$$\int_{[\alpha, \beta]} x'(t) x(t) \Delta(t) \leq \frac{1}{2} \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{h(t)} \right) \int_{[\alpha, \beta]} h(t) x'(t)^2 \Delta(t), \tag{8}$$

The case when $T = \mathbb{R}$ and $\zeta = \zeta = h(s) = 1$ in (4.7) coincides with the following inequality

$$\int_{[\alpha, \beta]} q(t) x'(t) x(t) \Delta(t) \leq \frac{(\beta - \alpha)}{2} \int_{[\alpha, \beta]} q(t) x'(t)^2 \Delta(t), \tag{9}$$

we also observe that inequality (4.7) reduces to (2.43) when $T = \mathbb{R}$ and $\zeta = \zeta = 1$.

$$\int_{[\alpha, \beta]} q(t) x^\Delta(t) x(t) \Delta(t) \leq \frac{1}{2} \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{h(t)} \right) \int_{[\alpha, \beta]} h(t) q(t) x^\Delta(t)^2 \Delta(t), \tag{10}$$

The next Theorem generalizes Opial type on time scale.

Theorem 4.3 Let T be a time scale with $t \in T$. Let ζ, ζ and σ be real numbers, let $h, q \in C_{rd}([\alpha, \beta]_T, \mathbb{R})$ and positive rd-continuous functions on $[\alpha, \beta]_T$ such that $\int_{[0, t]} r(t) \Delta(s) < \infty$ and $q(t)$ is non increasing on $[0, b]$. We define φ as convex function and if $x: [\alpha, \beta] \rightarrow \mathbb{R}$ is delta differentiable with $\alpha(0) = 0$, then we have

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \\
& \leq \frac{\sigma}{\zeta + \sigma} (\beta - \alpha)^\zeta \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{(\sqrt{h(t)})^{\zeta+\sigma}} \right) \int_{[\alpha, \beta]} (\sqrt{q(t)h(t)} x^\Delta(t))^{\zeta+\sigma} \Delta(t)
\end{aligned}$$

Proof: Modified Jensen's inequality

$$\left(\int_{[\alpha,\beta]} d\lambda(s)\right)^{-\zeta} \leq \left(\int_{[\alpha,\beta]} \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} \left(\int_{[\alpha,\beta]} \psi(t) d\lambda(s)\right)^{-\zeta} \left(\int_{[\alpha,\beta]} d\lambda(s)\right)^{\zeta}.$$

With

$$\begin{cases} \psi(s) = \sqrt{q(s)}, \quad \zeta = \frac{\zeta}{\sigma}, \quad \varphi(u) = u^{\zeta} \\ \text{and} \\ d\lambda(s) = x^{\Delta}(s) \end{cases}$$

We have,

$$\begin{aligned} (\beta - \alpha)^{-\zeta} &\leq \left(\int_{[\alpha,\beta]} \varphi(\psi(s))^{\frac{1}{\zeta}} x^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha,\beta]} \psi(s) x^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha,\beta]} \left(\sqrt{q(s)}\right)^{\frac{\zeta}{\sigma}} x^{\Delta}(s) \Delta(s)\right)^{\frac{\zeta}{\sigma}} \left(\int_{[\alpha,\beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\frac{\zeta}{\sigma}} \\ &\leq \left(\int_{[\alpha,\beta]} \sqrt{q(s)}^{\sigma} x^{\Delta}(s) \Delta(s)\right)^{\frac{\zeta}{\sigma}} \left(\int_{[\alpha,\beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\frac{\zeta}{\sigma}} \end{aligned}$$

Which is,

$$\left(\int_{[\alpha,\beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq \left(\int_{[\alpha,\beta]} \sqrt{q(s)}^{\sigma} x^{\Delta}(s)^{\sigma} \Delta(s)\right)^{\frac{\zeta}{\sigma}} (\beta - \alpha)^{\frac{\zeta(\sigma-1)}{\sigma}} \quad (11)$$

Now,

$$\gamma(t) = \int_{[\alpha,\beta]} \sqrt{q(s)}^{\sigma} x^{\Delta}(s)^{\sigma} \Delta(s)$$

$$\gamma^{\Delta}(t) = \sqrt{q(t)}^{\sigma} x^{\Delta}(t)^{\sigma} \quad (12)$$

In view of (4.11) and (4.12), we obtain

$$x^{\Delta}(t)^{\sigma} \left(\int_{[\alpha,\beta]} \sqrt{q(s)} x^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq (\beta - \alpha)^{\frac{\sigma(\zeta-1)}{\sigma}} \frac{\gamma^{\Delta}(t)}{\sqrt{q(t)}^{\sigma}} \gamma(t)^{\frac{\zeta}{\sigma}}$$

$$(\sqrt{q(t)})^{\zeta+\sigma} x^{\Delta}(t)^{\sigma} \left(\int_{[\alpha,\beta]} x^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq (\beta - \alpha)^{\frac{\sigma(\zeta-1)}{\sigma}} \gamma^{\Delta}(t) \gamma(t)^{\frac{\zeta}{\sigma}}$$

Which implies that,

$$\begin{aligned} \int_{[\alpha,\beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^{\Delta}(t)^{\sigma} \left(\int_{[\alpha,\beta]} x^{\Delta}(s) \Delta(s)\right)^{\zeta} \Delta(t) &\leq (\beta - \alpha)^{\frac{\sigma(\zeta-1)}{\sigma}} \int_{[\alpha,\beta]} \gamma^{\Delta}(s) \gamma(s)^{\frac{\zeta}{\sigma}} \Delta(t) \\ &\leq \frac{\sigma}{\zeta + \sigma} (\beta - \alpha)^{\frac{\sigma(\zeta-1)}{\sigma}} \gamma(t)^{\frac{\zeta+\sigma}{\sigma}} \end{aligned}$$

By Hölder's inequality

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{\frac{\zeta(\sigma-1)}{\sigma}} \left(\left[\int_{[\alpha, \beta]} \gamma(t)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right]^{\frac{\sigma}{\zeta+\sigma}} \left[\int_{[\alpha, \beta]} \Delta(t) \right]^{\frac{\zeta}{\zeta+\sigma}} \right)^{\frac{\zeta+\sigma}{\sigma}} \\
& = \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^\zeta \left(\left[\int_{[\alpha, \beta]} \gamma(t)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right] \right) \\
& = \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^\zeta \left(\left[\int_{[\alpha, \beta]} \left(\sqrt{q(t)}^\sigma x^\Delta(t)^\sigma \right)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right] \right) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^\zeta \int_{[\alpha, \beta]} (\sqrt{q(s)} x^\Delta(t))^{\zeta+\sigma} \Delta(t)
\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^\zeta \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{(\sqrt{h(t)})^{\zeta+\sigma}} \right) \int_{[\alpha, \beta]} (\sqrt{q(t)h(t)} x^\Delta(t))^{\zeta+\sigma} \Delta(t)
\end{aligned} \tag{14}$$

which coincides with (2.50), if $l = m = \sigma = \zeta = 1$.

By replacing the exponent components of (4.13) with $\frac{\zeta+\sigma}{\sigma}$ and $\frac{\zeta+\sigma}{(\sigma-\zeta)(\sigma-1)}$ we have

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{\frac{\zeta(\sigma-1)}{\sigma}} \left(\left[\int_{[\alpha, \beta]} \gamma(t)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right]^{\frac{\sigma}{\zeta+\sigma}} \left[\int_{[\alpha, \beta]} \Delta(t) \right]^{\frac{(\sigma-\zeta)(\sigma-1)}{\zeta+\sigma}} \right)^{\frac{\zeta+\sigma}{\sigma}}
\end{aligned}$$

By simplification,

$$\int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{(\zeta-1)} \left(\left[\int_{[\alpha, \beta]} \gamma(t)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right] \right)$$

Therefore,

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{(\zeta-1)} \left(\left[\int_{[\alpha, \beta]} \left(\sqrt{q(t)}^\sigma x^\Delta(t)^\sigma \right)^{\frac{\zeta+\sigma}{\sigma}} \Delta(t) \right] \right) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{\zeta-1} \int_{[\alpha, \beta]} (\sqrt{q(t)} x^\Delta(t))^{\zeta+\sigma} \Delta(t)
\end{aligned}$$

Then,

$$\begin{aligned}
& \int_{[\alpha, \beta]} (\sqrt{q(t)})^{\zeta+\sigma} x^\Delta(t)^\sigma x(t)^\zeta \Delta(t) \\
& \leq \frac{\sigma}{\zeta+\sigma} (\beta-\alpha)^{\zeta-1} \left(\int_{[\alpha, \beta]} \frac{\Delta(t)}{(\sqrt{h(t)})^{\zeta+\sigma}} \right) \int_{[\alpha, \beta]} (\sqrt{q(t)h(t)} x^\Delta(t))^{\zeta+\sigma} \Delta(t)
\end{aligned} \tag{15}$$

In (4.15) reduces to (2.43), if $\zeta = \sigma = 1$.

We need the following Theorem to prove the next result.

Theorem 4.4 Let T be a time scale and $\alpha, \beta \in T$ and let $\varphi: [\alpha, \beta]_T \rightarrow \mathbb{R}^+$ be a absolutely continuous convex function and $C_{rd}([\alpha, \beta]_T, \mathbb{R}^+)$ be such that $\xi(t)$ is non-increasing on $[\alpha, \beta]$ and Suppose that $x: [\alpha, \beta]_T \rightarrow \mathbb{R}^+$ is delta differentiable with $x(\alpha) = 0$. Then,

$$\int_{[\alpha, \beta]} \xi(t) x^\Delta(t)^\eta x^\zeta(t) \Delta(t) \leq \frac{\eta}{\eta+\zeta} (\beta-\alpha)^\eta \int_{[\alpha, \beta]} \left(\xi^{\frac{\eta}{\eta+\zeta}}(t) x^\Delta(t)^\eta \right)^{\frac{\eta+\zeta}{\eta}} \Delta(t) \tag{16}$$

Proof : Modified Jensen 's inequality

$$\left(\int_{\varepsilon}^t d\lambda(s)\right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s)\right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s)\right)^{-\zeta}.$$

With

$$\begin{cases} \psi(s) = \xi^{\frac{1}{\eta+\zeta}}(s), \quad \zeta = \frac{\zeta}{\eta} \\ \text{and} \\ d\lambda(s) = x^{\Delta}(s) \end{cases}$$

we have,

$$\begin{aligned} & (\beta - \alpha)^{-\zeta} \\ & \leq \left[\int_{[\alpha, \beta]} \varphi\left(\xi^{\frac{1}{\eta+\zeta}}(s)\right)^{\frac{1}{\eta}} x^{\Delta}(s) \Delta(s) \right]^{\frac{\zeta}{\eta}} \left[\int_{[\alpha, \beta]} \xi^{\frac{1}{\eta+\zeta}}(s) x^{\Delta}(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\frac{\zeta}{\eta}} \end{aligned}$$

by convexity, we have

$$\begin{aligned} & (\beta - \alpha)^{-\zeta} \\ & \leq \left[\int_{[\alpha, \beta]} \left(\xi^{\frac{\zeta}{\eta+\zeta}}(s)\right)^{\frac{1}{\eta}} x^{\Delta}(s) \Delta(s) \right]^{\frac{\zeta}{\eta}} \left[\int_{[\alpha, \beta]} \xi^{\frac{1}{\eta+\zeta}}(s) x^{\Delta}(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\frac{\zeta}{\eta}} \\ & \leq \left[\int_{[\alpha, \beta]} \xi^{\frac{\eta}{\eta+\zeta}}(s) x^{\Delta}(s) \Delta(s) \right]^{\frac{\zeta}{\eta}} \left[\int_{[\alpha, \beta]} \xi^{\frac{1}{\eta+\zeta}}(s) x^{\Delta}(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\frac{\zeta}{\eta}} \end{aligned}$$

we see that

$$\nu(t) = \int_{[\alpha, \beta]} \xi^{\frac{\eta}{\eta+\zeta}}(s) x^{\Delta}(s)^{\eta} \Delta(s)$$

$$\nu^{\Delta}(t) = \xi^{\frac{\eta}{\eta+\zeta}}(t) x^{\Delta}(t)^{\eta} \quad (17)$$

Further simplification yields,

$$\begin{aligned}
& \left(\int_{[\alpha, \beta]} \xi^{\frac{1}{\eta+\varsigma}} x^\Delta(s) \Delta(s) \right)^\varsigma \leq \left[\int_{[\alpha, \beta]} \xi^{\frac{\eta}{\eta+\varsigma}} x^\Delta(s) \Delta(s) \right]^\frac{\varsigma}{\eta} (\beta - \alpha)^{-\frac{\varsigma}{\eta}} \\
& \leq \left[\int_{[\alpha, \beta]} \xi^{\frac{\eta}{\eta+\varsigma}} x^\Delta(s) \Delta s \right]^\frac{\varsigma}{\eta} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} = v(t)^\frac{\varsigma}{\eta} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}}
\end{aligned} \tag{18}$$

In view of (4.17) and (4.18),

$$\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^\Delta(t) \left(\int_{[\alpha, \beta]} \xi^{\frac{1}{\eta+\varsigma}}(s) x^\Delta(s) \Delta(s) \right)^\varsigma \leq (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} v^\Delta(t) v(t)^\frac{\varsigma}{\eta}$$

Which implies,

$$\xi(t) x^\Delta(t) \left(\int_{[\alpha, \beta]} x^\Delta(s) \Delta(s) \right)^\varsigma \leq (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} v^\Delta(t) v(t)^\frac{\varsigma}{\eta}$$

Also,

$$\begin{aligned}
& \int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x^\varsigma(t) \Delta(t) \leq (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} \int_{[\alpha, \beta]} v^\Delta(t) v(t)^\frac{\varsigma}{\eta} \Delta(t) \\
& \leq (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} \frac{\eta}{\varsigma + \eta} v(\beta)^{\frac{\varsigma+\eta}{\eta}}
\end{aligned}$$

Applying Hölder's with conjugate exponents $\frac{\zeta+\eta}{\zeta}$ and $\frac{\eta}{\zeta+\eta}$

$$\begin{aligned}
& \int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x^\varsigma(t) \Delta(t) \\
& \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} \left[\left(\int_{[\alpha, \beta]} \left(\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^\Delta(t)^\eta \right)^{\frac{\eta+\varsigma}{\eta}} \Delta(t) \right)^{\frac{\eta}{\eta+\varsigma}} \left(\int_{[\alpha, \beta]} \Delta(t) \right)^{\frac{\varsigma}{\eta+\varsigma}} \right]^{\frac{\varsigma+\eta}{\eta}}
\end{aligned} \quad (19)$$

Therefore, we have

$$\begin{aligned}
& \int_{[\alpha, \beta]} \xi(t) x^\Delta(t)^\eta x^\varsigma(t) \Delta(t) \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} (\beta - \alpha)^{\frac{\varsigma}{\eta}} \int_{[\alpha, \beta]} \left(\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^\Delta(t)^\eta \right)^{\frac{\eta+\varsigma}{\eta}} \Delta(t) \\
& = \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\varsigma} \int_{[\alpha, \beta]} \xi(t) x^\Delta(t)^{\eta+\varsigma} \Delta(t)
\end{aligned} \quad (20)$$

When $T = \mathbb{R}$, $\xi(s) = 1$ and x is continuous function on $[\alpha, \beta]$, then (4.20) yields

$$\int_{[\alpha, \beta]} x'(s)^\eta x(s)^\varsigma ds \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^\varsigma \int_{[\alpha, \beta]} x'(s)^{\eta+\varsigma} ds.$$

If the conjugate exponents in (4.19) are replaced with $\frac{\varsigma + \eta}{\eta}$ and $\frac{\varsigma + \eta}{\eta^2 \varsigma + \varsigma - \eta \varsigma}$ we have

$$\begin{aligned}
& \int_{[\alpha, \beta]} \xi(t) x^\Delta(t)^\eta x^\varsigma(t) \Delta(t) \\
& \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} \left[\left(\int_{[\alpha, \beta]} \left(\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^\Delta(t)^\eta \right)^{\frac{\eta+\varsigma}{\eta}} \Delta(t) \right)^{\frac{\eta}{\eta+\varsigma}} \left(\int_{[\alpha, \beta]} \Delta(t) \right)^{\frac{\eta^2 \varsigma + \varsigma - \eta \varsigma}{\varsigma + \eta}} \right]^{\frac{\varsigma+\eta}{\eta}} \\
& \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\frac{\varsigma(\eta-1)}{\eta}} (\beta - \alpha)^{\frac{\eta^2 \varsigma + \varsigma - \eta \varsigma}{\eta}} \int_{[\alpha, \beta]} \left(\xi^{\frac{\eta}{\eta+\varsigma}}(t) x^\Delta(t)^\eta \right)^{\frac{\eta+\varsigma}{\eta}} \Delta(t) \\
& \leq \frac{\eta}{\eta + \varsigma} (\beta - \alpha)^{\varsigma} \int_{[\alpha, \beta]} \xi(t) x^\Delta(t)^{\eta+\varsigma} \Delta(t)
\end{aligned} \quad (21)$$

Finally, if $T = \mathbb{R}, \eta = \xi(t) = 1 = \varsigma$ and $\alpha = 0$, we note that (4.21) becomes the celebrated Olech of (2.16) on time scale. **Theorem 4.5** Let $\alpha, \beta \in T$ and let $\varphi : [\alpha, \beta]_T \rightarrow \mathbb{R}^+$ be a absolutely continuous convex function and $\nu \in C_{rd}([\alpha, \beta]_T, \mathbb{R}^+)$ with $\int_{[\alpha, \beta]} \nu(s)^{1-\varepsilon} \Delta(s) < \infty$ be such that $\xi(t)$ is non- increasing on $[\alpha, \beta]$ and Suppose that $x : [\alpha, \beta]_T \rightarrow \mathbb{R}^+$ is delta differentiable with $\alpha(0) = 0$ and $\eta > 0$. Then, the following inequality holds:

$$\int_{[\alpha,\beta]} \xi(t) x^\Delta(t) x(t)^\varsigma \Delta(t) \leq (\beta - \alpha)^{\varsigma - \zeta} \frac{\chi}{1 + \zeta} \left(\int_{[\alpha,\beta]} \xi^{\frac{\gamma}{1+\varsigma}}(t) \nu(t) x^\Delta(t)^\gamma \Delta(t) \right)^{\frac{1+\zeta}{\gamma}} \quad (22)$$

where

$$\chi = \left(\int_{[\alpha,\beta]} \nu(t)^{1-\varepsilon} \Delta(t) \right)^{\frac{1+\zeta}{\varepsilon}}$$

proof: Modified Jensen's inequality

$$\left(\int_{[\alpha,\beta]} d\lambda(s)\right)^{-\zeta} \leq \left(\int_{[\alpha,\beta]} \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} \left(\int_{[\alpha,\beta]} \psi(t) d\lambda(s)\right)^{-\zeta} \left(\int_{[\alpha,\beta]} d\lambda(s)\right)^{\zeta}.$$

With

$$\begin{cases} \psi(s) = \xi^{\frac{1}{1+\zeta}}(s) \\ \text{and } d\lambda(s) = x^\Delta(s) \end{cases}$$

We have,

$$\begin{aligned} & (\beta - \alpha)^{-\zeta} \\ & \leq \left[\int_{[\alpha,\beta]} \varphi\left(\xi^{\frac{1}{1+\zeta}}(s)\right)^{\frac{1}{\zeta}} x^\Delta(s) \Delta(s) \right]^{\zeta} \left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\zeta} \end{aligned}$$

By convexity,

$$\begin{aligned} & (\beta - \alpha)^{-\zeta} \\ & \leq \left[\int_{[\alpha,\beta]} \left(\xi^{\frac{\zeta}{1+\zeta}}(s) \right)^{\frac{1}{\zeta}} x^\Delta(s) \Delta(s) \right]^{\zeta} \left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\zeta} \\ & \leq \left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{\zeta} \left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{-\zeta} (\beta - \alpha)^{-\zeta} \end{aligned}$$

$$\begin{aligned} \varpi(t) &= \int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \\ \varpi^\Delta(t) &= \xi^{\frac{1}{1+\zeta}}(t) x^\Delta(t) \end{aligned} \tag{23}$$

Which gives,

$$\left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \left[\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right]^{\zeta} \tag{24}$$

Combining (4.23) and (4.24), then

$$\xi^{\frac{1}{1+\zeta}}(t) x^\Delta(t) \left(\int_{[\alpha,\beta]} \xi^{\frac{1}{1+\zeta}}(s) x^\Delta(s) \Delta(s) \right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \varpi(t)^{\zeta} \varpi^\Delta(t) \tag{25}$$

That is,

$$\xi^{\frac{1}{1+\zeta}}(t) \xi^{\frac{\zeta}{1+\zeta}}(t) x^\Delta(t) \left(\int_{[\alpha,\beta]} x^\Delta(t) \Delta(t) \right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \varpi(t)^{\zeta} \varpi^\Delta(t)$$

Which yields,

$$\int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x(t)^\zeta \Delta(t) \leq (\beta - \alpha)^{\zeta - \zeta} \int_{[\alpha, \beta]} \varpi(t)^\zeta \varpi^\Delta(t) \Delta(t) = (\beta - \alpha)^{\zeta - \zeta} \frac{1}{1 + \zeta} \varpi(t)^{1 + \zeta}$$

Further simplification yields,

$$\begin{aligned} \int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x(t)^\zeta \Delta(t) &\leq (\beta - \alpha)^{\zeta - \zeta} \frac{1}{1 + \zeta} \left(\int_{[\alpha, \beta]} \xi^{\frac{1}{1 + \zeta}}(t) x^\Delta(t) \Delta(t) \right)^{1 + \zeta} \\ &\leq (\beta - \alpha)^{\zeta - \zeta} \frac{1}{1 + \zeta} \left(\int_{[\alpha, \beta]} v(t)^{\frac{1}{\gamma}} \xi^{\frac{1}{1 + \zeta}}(s) x^\Delta(t) v(t)^{\frac{1}{\gamma}} \Delta(t) \right)^{1 + \zeta} \end{aligned}$$

Which implies,

$$\begin{aligned} \int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x(t)^\zeta \Delta(t) &\leq (\beta - \alpha)^{\zeta - \zeta} \frac{1}{1 + \zeta} \left(\int_{[\alpha, \beta]} v(t)^{\frac{\varepsilon}{\gamma}} \Delta(t) \right)^{\frac{1 + \zeta}{\varepsilon}} \left(\int_{[\alpha, \beta]} \xi^{\frac{\gamma}{1 + \zeta}}(s) v(s) x^\Delta(t)^\gamma \Delta(t) \right)^{\frac{1 + \zeta}{\gamma}} \\ &\leq (\beta - \alpha)^{\zeta - \zeta} \frac{1}{1 + \zeta} \left(\int_{[\alpha, \beta]} v(t)^{1 - \varepsilon} \Delta(t) \right)^{\frac{1 + \zeta}{\varepsilon}} \left(\int_{[\alpha, \beta]} \xi^{\frac{\gamma}{1 + \zeta}}(t) v(t) x^\Delta(s)^\gamma \Delta(t) \right)^{\frac{1 + \zeta}{\gamma}} \end{aligned}$$

We have,

$$\int_{[\alpha, \beta]} \xi(t) x^\Delta(t) x(t)^\zeta \Delta(t) \leq (\beta - \alpha)^{\zeta - \zeta} \frac{\chi}{1 + \zeta} \left(\int_{[\alpha, \beta]} \xi^{\frac{\gamma}{1 + \zeta}}(t) v(t) x^\Delta(t)^\gamma \Delta(t) \right)^{\frac{1 + \zeta}{\gamma}} \quad (26)$$

If $T = \mathbb{R}$ and $\xi(t) = \xi^{\frac{\gamma}{1 + \zeta}}(s) = 1 = \zeta = \zeta$ in (4.26) yields

$$\int_{[\alpha, \beta]} x'(s) x(s) ds \leq \frac{\chi}{2} \left(\int_{[\alpha, \beta]} v(s) x'(s)^\gamma ds \right)^{\frac{2}{\gamma}}$$

which is equivalent to Maroni of (2.5).

We note that if $T = \mathbb{R}$, $\xi(t) = \xi^{\frac{\gamma}{1 + \zeta}}(t) = 1 = \zeta = (\beta - \alpha)$, $\zeta = 0$ and $\gamma = 2$ in (4.26), we have

$$\int_{[\alpha, \beta]} x'(t) x(t) dt \leq \left(\int_{[\alpha, \beta]} v(t)^{1 - \varepsilon} dt \right)^{\frac{1}{\varepsilon}} \left(\int_{[\alpha, \beta]} v(t) x'(t)^\gamma dt \right)^{\frac{1}{\gamma}}$$

which is a special case of Opial-type inequalities, since φ is non-decreasing. The last inequality is equivalent to the result of Anthonio (2014).

3 On Opial-Type Inequalities Related to Pachpatte's Result

We present the results as follows:

Theorem 4.6 Let T be a time scale with $s, h \in T$. Let ς, ζ be real numbers, let $\aleph, \chi \in C_{rd}([\alpha, \beta]_T, \mathbb{R})$ where h and χ are positive rd-continuous functions on $[\alpha, \beta]_T$ such that $\int_{[0, t]} r(t) \Delta(s) < \infty$. We define φ as convex function and if $\wp_{\gamma_1}, \wp_{\gamma_2} : [\alpha, \beta]_T \rightarrow \mathbb{R}$ is delta differentiable with $\alpha(0) = 0$, then we have

$$\begin{aligned} & \int_{[\alpha, \beta]} (\sqrt{\chi(h)})^{\varsigma+1} \wp_{\gamma_1}^{\Delta}(h) \wp_{\gamma_2}^{\varsigma}(h) \Delta(h) \\ & \leq \frac{1}{1+\zeta} (\beta - \alpha)^{\varsigma-\zeta} \left(\int_{[\alpha, \beta]} \frac{\Delta(h)}{(\sqrt{\aleph(h)})^{1+\zeta}} \right) \int_{[\alpha, \beta]} (\sqrt{\aleph(h)\chi(h)})^{1+\zeta} \wp_{\gamma_1}^{\Delta}(h)^{1+\zeta} \Delta(h) \end{aligned} \quad (27)$$

Proof: Modified Jensen's inequality

$$\left(\int_{\varepsilon}^t d\lambda(s)\right)^{-\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta} \left(\int_{\varepsilon}^t \psi(t) d\lambda(s)\right)^{-\zeta} \left(\int_{\varepsilon}^t d\lambda(s)\right)^{-\zeta}.$$

With

$$\begin{cases} \psi(s) = \sqrt{\chi(s)} \\ \text{and } d\lambda(s) = \wp_{\gamma_1}^{\Delta} \Delta(s) \end{cases}$$

We have,

$$\begin{aligned} (\beta - \alpha)^{-\zeta} &\leq \left(\int_{[\alpha, \beta]} \varphi(\psi(s))^{\frac{1}{\zeta}} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha, \beta]} \psi(s) \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha, \beta]} \varphi(\sqrt{\chi(s)})^{\frac{1}{\zeta}} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &= \left(\int_{[\alpha, \beta]} (\sqrt{\chi(s)})^{\frac{\zeta}{\zeta}} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\ &\leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \end{aligned}$$

which is

$$\begin{aligned} &(\beta - \alpha)^{-\zeta} \\ &\leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{-\zeta} (\beta - \alpha)^{-\zeta} \end{aligned}$$

Hence,

$$\left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} (\beta - \alpha)^{\zeta - \zeta} \quad (28)$$

Now,

$$\gamma_1(\hbar) = \int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)$$

$$\gamma_1^{\Delta}(\hbar) = \sqrt{\chi(\hbar)} \wp_{\gamma_1}^{\Delta}(\hbar) \quad (29)$$

In view of (4.28) and (4.29), we obtain

$$\wp_{\gamma_1}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \frac{\gamma_1^{\Delta}(\hbar)}{\sqrt{\chi(\hbar)}} \gamma_1(\hbar)^{\zeta}$$

$$(\sqrt{\chi(\hbar)})^{\zeta + 1} \wp_{\gamma_1}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \wp_{\gamma_1}^{\Delta}(s) \Delta(s)\right)^{\zeta} \leq (\beta - \alpha)^{\zeta - \zeta} \gamma_1^{\Delta}(\hbar) \gamma_1(\hbar)^{\zeta} \quad (30)$$

Furthermore, the proof of Theorem 4.6, with substitution of $\psi(s) \rightarrow \sqrt{\chi(s)}$ and

$d\lambda(s) \rightarrow \phi_{\gamma_2}^\Delta(s)$ in above modified Jensen's inequality, then we have

$$\begin{aligned}
(\beta - \alpha)^{-\zeta} &\leq \left(\int_{[\alpha, \beta]} \varphi(\psi(s))^{\frac{2}{\zeta}} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \psi(s) \wp_{\gamma_1}^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\
&\leq \left(\int_{[\alpha, \beta]} \varphi(\sqrt{\chi(s)})^{\frac{1}{\zeta}} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\
&\leq \left(\int_{[\alpha, \beta]} (\sqrt{\chi(s)})^{\frac{\zeta}{\zeta}} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_1}^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta} \\
&\leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta}
\end{aligned}$$

which is

$$\begin{aligned}
&(\beta - \alpha)^{-\zeta} \\
&\leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{-\zeta} (\beta - \alpha)^{-\zeta}
\end{aligned}$$

Hence,

$$\left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \leq \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} (\beta - \alpha)^{-\zeta} \quad (31)$$

Now,

$$\gamma_2(t) = \int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s)$$

$$\gamma_2^{\Delta}(\hbar) = \sqrt{\chi(\hbar)} \wp_{\gamma_2}^{\Delta}(\hbar) \quad (32)$$

In view of (4.31) and (4.32), we obtain

$$\begin{aligned}
\wp_{\gamma_2}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \sqrt{\chi(s)} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} &\leq (\beta - \alpha)^{\zeta - \zeta} \frac{\gamma_2^{\Delta}(\hbar)}{\sqrt{\chi(\hbar)}} \gamma_2(\hbar)^{\zeta} \\
(\sqrt{\chi(\hbar)})^{\zeta + 1} \wp_{\gamma_2}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} &\leq (\beta - \alpha)^{\zeta - \zeta} \gamma_2^{\Delta}(\hbar) \gamma_2(\hbar)^{\zeta} \quad (33)
\end{aligned}$$

Adding both side of (4.30) and (4.33) yields

$$\begin{aligned}
&(\sqrt{\chi(\hbar)})^{\zeta + 1} \wp_{\gamma_1}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \wp_{\gamma_1}^{\Delta}(s) \Delta(s) \right)^{\zeta} + (\sqrt{\chi(\hbar)})^{\zeta + 1} \wp_{\gamma_2}^{\Delta}(\hbar) \left(\int_{[\alpha, \beta]} \wp_{\gamma_2}^{\Delta}(s) \Delta(s) \right)^{\zeta} \\
&\leq (\beta - \alpha)^{\zeta - \zeta} \gamma_1^{\Delta}(\hbar) \gamma_1(\hbar)^{\zeta} + (\beta - \alpha)^{\zeta - \zeta} \gamma_2^{\Delta}(\hbar) \gamma_2(\hbar)^{\zeta} \quad (34)
\end{aligned}$$

Integrating both side of the above (4.34) with respect to delta derivative, we have the following inequality:

$$\begin{aligned}
&\int_{[\alpha, \beta]} (\sqrt{\chi(\hbar)})^{\zeta + 1} \left(\wp_{\gamma_2}^{\Delta}(\hbar) \wp_{\gamma_1}^{\zeta}(\hbar) + \wp_{\gamma_1}^{\Delta}(\hbar) \wp_{\gamma_2}^{\zeta}(\hbar) \right) \Delta(\hbar) \\
&\leq \frac{1}{1 + \zeta} (\beta - \alpha)^{\zeta - \zeta} \left(\int_{[\alpha, \beta]} \frac{\Delta(\hbar)}{(\sqrt{\chi(\hbar)})^{1 + \zeta}} \right) \int_{[\alpha, \beta]} (\sqrt{\chi(\hbar)})^{1 + \zeta} \left(\wp_{\gamma_1}^{\Delta}(\hbar)^{1 + \zeta} + \wp_{\gamma_2}^{\Delta}(\hbar)^{1 + \zeta} \right) \Delta(\hbar) \quad (35)
\end{aligned}$$

Remark 4.1

We see that if $T = R$ and

$q(t) = \chi(\hbar), \zeta = \varsigma = 1, \wp_{\gamma_1}^{\varsigma}(\hbar) = x_1(t), x_1'(t) = \wp_{\gamma_1}^{\Delta}(\hbar), \wp_{\gamma_2}(\hbar) = x_2(t), x_2'(t) = \wp_{\gamma_2}^{\Delta}(\hbar), \aleph(\hbar) = p(t)$ and $\Delta(\hbar) = dt$ (4.35) reduces to (2.27).

We obtain generalization of several results in Opial-type inequalities on time scale theory.

Indeed, above result has showed Opial's inequality on time scales by using modified

Jensen.

4 Higher Order of Opial Integral Inequalities on Time Scales

We obtain integral inequalities which involving higher order derivatives of Opial-type inequalities, mainly by using Jensen's inequality for the case of convex function.

In this section, some various types of Opial's inequality involving several functions and their higher order derivatives are presented on a closed subset of T on $[0,1]$. The

well-known Jensen's would be adopted to extend and generalizes some results on time scale

on higher Order Opial inequality.

To proof this we need the following Theorem.

Theorem 4.7

Let $\Gamma(\Omega)$ be absolutely continuous function and T be a time scale with $\rho, \sigma \in T, \delta \in C_{rd}^{(n)}([\rho, \sigma], \cap T)$. If $\Gamma^{\Delta^n}(\varepsilon) = 0$ is δ -differentiable function such that $i = 1, 2, 3, 4, \dots, n-1$, then the following inequality holds:

$$\begin{aligned} & \int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^2 \delta(\omega)^{\zeta} \Delta(\omega) \\ & \leq \left(\frac{1}{2}\right)^{\frac{1}{\zeta+1}} \left(\int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{(\zeta-\zeta)(\zeta+1)} \Delta(\Omega) \right] \Delta(\omega) \right)^{\frac{1}{\zeta+1}} \times \int_{\varepsilon}^t \kappa^{\Delta^n}(\omega)^2 \Delta(\omega) \end{aligned} \quad (36)$$

Proof: Modified Jensen's inequality

$$\left(\int_{\varepsilon}^t \varphi(\psi(s)) d\lambda(s) \right)^{\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{\zeta-\zeta}.$$

With

$$\begin{cases} \varphi(\Omega) = \Omega^{\zeta}, \quad \psi(\Omega) = \Gamma^{\Delta^n}(\Omega) \\ \text{and} \\ d\lambda(\Omega) = \Delta(\Omega) \end{cases}$$

We have,

$$\left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \leq (t - \varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \varphi(\Gamma^{\Delta^n}(\Omega))^{\frac{1}{\zeta}} \Delta(\Omega) \right)^{\zeta}$$

which implies

$$\left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \leq (t - \varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \quad (37)$$

Let,

$$\delta(\omega) = \int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega)$$

$$\delta^{\Delta}(\omega) = \Gamma^{\Delta^n}(\omega) \quad (38)$$

combining (4.37) and (4.38) we have

$$\delta^{\Delta}(\omega) \delta(\omega)^{\zeta} \leq (t - \varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \Gamma^{\Delta^n}(\omega)$$

Now

$$\begin{aligned} & \delta^{\Delta}(\omega) \delta(\omega)^{\zeta} \\ & \leq (t - \varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta} \Delta(\Omega) \right) \Gamma^{\Delta^n}(\omega) \leq (t - \varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta+1} \Delta(\Omega) \right). \end{aligned}$$

The above inequality can be written as follows

$$\delta^{\Delta}(\omega) \delta(\omega)^{\zeta} \leq \left(\int_{\varepsilon}^t (t - \varepsilon)^{(\zeta-\zeta)(\zeta+1)} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \times \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta+1} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}}$$

Then, multiplying both sides with $\Gamma^{\Delta^n}(\Omega)$, yields

$$\begin{aligned} & \delta^{\Delta}(\omega)^2 \delta(\omega)^{\zeta} \\ & \leq \Gamma^{\Delta^n}(\Omega) \left(\int_{\varepsilon}^t (t - \varepsilon)^{(\zeta-\zeta)(\zeta+1)} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \times \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta+1} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \end{aligned}$$

Note that $\nu(\omega) = \int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta+1} \Delta(\Omega)$, $\nu^{\Delta}(\omega) = \Gamma^{\Delta^n}(\omega)^{\zeta+1}$

$$\delta^{\Delta}(\Omega)^2 \delta(\Omega)^{\zeta} \leq \left(\int_{\varepsilon}^t (t - \varepsilon)^{(\zeta-\zeta)(\zeta+1)} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \times \left[\nu(\omega) \nu^{\Delta}(\omega) \right]^{\frac{1}{\zeta+1}}$$

which implies

$$\begin{aligned}
& \int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^2 \delta(\omega)^{\varsigma} \Delta(\omega) \\
& \leq \int_{\sigma}^{\rho} \left(\int_{\varepsilon}^t (t-\varepsilon)^{(\varsigma-\zeta)(\zeta+1)} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \times \left[\int_{\sigma}^{\rho} \nu(\omega) \nu(\omega)^{\Delta} \Delta(\omega) \right]^{\frac{1}{\zeta+1}} \Delta(\omega) \\
& = \int_{\sigma}^{\rho} \left(\int_{\varepsilon}^t (t-\varepsilon)^{(\varsigma-\zeta)(\zeta+1)} \Delta(\Omega) \right)^{\frac{1}{\zeta+1}} \times \left[\frac{1}{2} \nu(\omega)^2 \right]^{\frac{1}{\zeta+1}} \Delta(\omega)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^2 \delta(\omega)^{\varsigma} \Delta(\omega) \\
& \leq \left(\frac{1}{2} \right)^{\frac{1}{\zeta+1}} \left(\int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{(\varsigma-\zeta)(\zeta+1)} \Delta(\Omega) \right] \Delta(\omega) \right)^{\frac{1}{\zeta+1}} \times \int_{\varepsilon}^t \Gamma^{\Delta^n}(\omega)^2 \Delta(\omega)
\end{aligned} \tag{39}$$

Remark 4.1

In (4.39), if $\zeta = 1$ and $\varsigma = 2$ we have,

$$\begin{aligned}
& \int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^2 \delta(\omega)^2 \Delta(\omega) \\
& \leq \left(\frac{1}{2} \right)^{\frac{1}{2}} \left(\int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^2 \Delta(\Omega) \right] \Delta(\omega) \right)^{\frac{1}{2}} \times \int_{\varepsilon}^t (\Gamma^{\Delta^n}(\omega))^2 \Delta(\omega)
\end{aligned}$$

Theorem 4.7

Let $\Gamma(\Omega)$ are absolutely continuous functions and T be a time scale with $\lambda, \mu \in T, \delta \in C_{rd}^{(n)}([\lambda, \mu], \cap T)$ with ς, ζ , and m which are numbers. If $\Gamma^{\Delta^n}(\varepsilon) = 0$ is δ -differentiable function such that $i = 1, 2, 3, 4, \dots, n-1$, then the following inequality holds:

$$\begin{aligned}
& \int_{\mu}^{\lambda} \delta^{\Delta}(\omega)^{2m} \delta(\omega)^{\varsigma} \Delta(\omega) \\
& \leq \left(\frac{m}{\zeta+m} \right)^{\frac{m}{\zeta+m}} \left(\int_{\mu}^{\lambda} \left(\int_{\varepsilon}^t (t-\varepsilon)^{\frac{(\varsigma-\zeta)(\zeta+m)}{\zeta+m-1}} \Delta(\omega) \right)^{\frac{\varsigma(\zeta+m-1)}{\zeta+m} \times \frac{\zeta+m}{\varsigma}} \Delta(\omega) \right)^{\frac{1}{\zeta+m}} \\
& \times \left[\int_{\mu}^{\lambda} \delta(\omega)^{\zeta+m} \Delta(\omega) \right]
\end{aligned} \tag{40}$$

proof: Modified Jensen's inequality

$$\left(\int_{\varepsilon}^t \varphi(\psi(s)) d\lambda(s) \right)^{\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{\zeta - \zeta}.$$

With

$$\left\{ \begin{array}{l} \varphi(\Omega) = \Omega^{\zeta}, \quad \psi(\Omega) = \Gamma^{\Delta^n}(\Omega) \\ \text{and} \\ d\lambda(\Omega) = \Delta(\Omega) \end{array} \right.$$

We have,

$$\left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \leq (t - \varepsilon)^{\zeta - \zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \quad (41)$$

$$\delta(\omega) = \int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega)$$

$$\delta^{\Delta}(\omega)^m = \Gamma^{\Delta^n}(\omega)^m \quad (42)$$

In view of (4. 41) and (4.42) we have,

$$\delta^{\Delta}(\omega)^m \delta(\omega)^{\zeta} \leq (t - \varepsilon)^{\zeta - \zeta} \delta(\omega)^{\zeta} \Gamma^{\Delta^n}(\omega)^m$$

that is

$$\delta^{\Delta}(\omega)^m \delta(\omega)^{\zeta} \leq (t - \varepsilon)^{\zeta - \zeta} \int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega)^{\zeta + m} \Delta(\Omega)$$

simplify further, multiplying both sides with $\Gamma^{\Delta^n}(\Omega)^m$ and applying Hölder's inequality yields

$$\begin{aligned}
& \delta^\Delta(\Omega)^{2m} \delta(\Omega)^\zeta \\
& \leq \Gamma^{\Delta^n}(\Omega)^m \left(\int_\varepsilon^t (t-\varepsilon)^{\frac{(\zeta-\zeta')(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\zeta(\zeta+m-1)}{\zeta+m}} \\
& \times \left(\int_\varepsilon^t \Gamma^{\Delta^n}(\Omega)^{\zeta+m} \Delta(\Omega) \right)^{\frac{1}{\zeta+m}}
\end{aligned}$$

Now, in above put $\nu(\omega) = \int_\varepsilon^t \Gamma^{\Delta^n}(\Omega)^{\zeta+m} \Delta(\Omega)$, $\nu^\Delta(\omega)^{\frac{m}{\zeta+m}} = \Gamma^{\Delta^n}(\omega)^m$, $\nu(\varepsilon) = 0$ to have the following inequality

$$\begin{aligned}
& \int_\mu^\lambda \delta^\Delta(\omega)^{2m} \delta(\omega)^\zeta \Delta(\omega) \\
& \leq \int_\mu^\lambda \left(\int_\varepsilon^t (t-\varepsilon)^{\frac{(\zeta-\zeta')(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\zeta(\zeta+m-1)}{\zeta+m}} \times \left(\nu^\Delta(\Omega)^{\frac{m}{\zeta+m}} \nu(\Omega) \Delta(\Omega) \right)^{\frac{1}{\zeta+m}} \Delta(\omega)
\end{aligned}$$

that is

$$\begin{aligned}
& \int_\mu^\lambda \delta^\Delta(\omega)^{2m} \delta(\omega)^\zeta \Delta(\omega) \\
& \leq \left(\int_\mu^\lambda \left(\int_\varepsilon^t (t-\varepsilon)^{\frac{(\zeta-\zeta')(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\zeta(\zeta+m-1)}{\zeta+m} \times \frac{\zeta+m}{\zeta}} \Delta(\Omega) \right)^{\frac{1}{\zeta+m}} \left(\int_\mu^\lambda \nu^\Delta(\Omega) \nu(\Omega)^{\frac{\zeta}{m}} \Delta(\Omega) \right)^{\frac{m}{\zeta+m}} \Delta(\omega)
\end{aligned}$$

which implies

$$\begin{aligned}
& \int_\mu^\lambda \delta^\Delta(\omega)^{2m} \delta(\omega)^\zeta \Delta(\omega) \\
& \leq \left(\frac{m}{\zeta+m} \right)^{\frac{m}{\zeta+m}} \left(\int_\mu^\lambda \left(\int_\varepsilon^t (t-\varepsilon)^{\frac{(\zeta-\zeta')(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\zeta(\zeta+m-1)}{\zeta+m} \times \frac{\zeta+m}{\zeta}} \Delta(\omega) \right)^{\frac{1}{\zeta+m}} \left(\nu(\lambda)^{\frac{\zeta+m}{m}} \right)^{\frac{m}{\zeta+m}}
\end{aligned}$$

therefore, we have

$$\begin{aligned}
& \int_{\mu}^{\lambda} \delta^{\Delta}(\omega)^{2m} \delta(t)^{\varsigma} \Delta(\omega) \\
& \leq \left(\frac{m}{\zeta+m} \right)^{\frac{m}{\zeta+m}} \left(\int_{\mu}^{\lambda} \left(\int_{\varepsilon}^t (t-\varepsilon)^{\frac{(\varsigma-\zeta)(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\varsigma(\zeta+m-1) \times \zeta+m}{\zeta+m} \frac{\zeta+m}{\varsigma}} \Delta(\omega) \right)^{\frac{1}{\zeta+m}} \nu(\lambda) \\
& \leq \left(\frac{m}{\zeta+m} \right)^{\frac{m}{\zeta+m}} \left(\int_{\mu}^{\lambda} \left(\int_{\varepsilon}^t (t-\varepsilon)^{\frac{(\varsigma-\zeta)(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\varsigma(\zeta+m-1) \times \zeta+m}{\zeta+m} \frac{\zeta+m}{\varsigma}} \Delta(\omega) \right)^{\frac{1}{\zeta+m}} \\
& \times \left(\int_{\mu}^{\lambda} \Gamma^{\Delta^n}(\Omega)^{\zeta+m} \Delta(\Omega) \right)
\end{aligned}$$

then, we have

$$\begin{aligned}
& \int_{\mu}^{\lambda} \delta^{\Delta}(\omega)^{2m} \delta(\omega)^{\varsigma} \Delta(\omega) \\
& \leq \left(\frac{m}{\zeta+m} \right)^{\frac{m}{\zeta+m}} \left(\int_{\mu}^{\lambda} \left(\int_{\varepsilon}^t (t-\varepsilon)^{\frac{(\varsigma-\zeta)(\zeta+m)}{\zeta+m-1}} \Delta(\Omega) \right)^{\frac{\varsigma(\zeta+m-1) \times \zeta+m}{\zeta+m} \frac{\zeta+m}{\varsigma}} \Delta(\omega) \right)^{\frac{1}{\zeta+m}} \\
& \times \left(\int_{\mu}^{\lambda} \delta(\omega)^{\zeta+m} \Delta(\omega) \right)
\end{aligned} \tag{43}$$

In (4.43), if $\zeta = m = 0$ and $\varsigma = 2$ its reduces to

$$\int_{\mu}^{\lambda} \delta(\omega) \Delta(\omega) \leq (t-\varepsilon)(\lambda-\mu)^2$$

Theorem 4.9

Let $\Gamma(\Omega)$ are absolutely continuous functions and T be a time scale with $\rho, \sigma \in T, \delta \in C_{rd}^{(n)}([\rho, \sigma], \cap T)$ with ς, ζ , and ν which are numbers. If $\Gamma^{\Delta^n}(\varepsilon) = 0$ is δ -differentiable function such that $i = 1, 2, 3, 4, \dots, n-1$, then the following inequality holds:

$$\int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\varsigma} \Delta(\omega) \leq \int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{\varsigma-\zeta} \Delta(\Omega) \right]^{\zeta} \Delta(\omega) \times \left[\int_{\sigma}^{\rho} \delta(s) \Delta(\omega) \right]$$

Proof: Modified Jensen's inequality

$$\left(\int_{\varepsilon}^t \varphi(\psi(s)) d\lambda(s) \right)^{\zeta} \leq \left(\int_{\varepsilon}^t \varphi(\psi(s))^{\frac{1}{\zeta}} d\lambda(s) \right)^{\zeta} \left(\int_{\varepsilon}^t d\lambda(s) \right)^{\zeta-\zeta}.$$

With

$$\begin{cases} \varphi(\Omega) = \Omega^{\zeta}, \quad \psi(\Omega) = \Gamma^{\Delta^n}(\Omega) \\ \text{and} \\ d\lambda(s) = \Delta(\Omega) \end{cases}$$

We have,

$$\left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta} \leq (t-\varepsilon)^{\zeta-\zeta} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\omega) \right)^{\zeta} \quad (44)$$

$$\delta(\omega) = \int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega)$$

$$\delta^{\Delta}(\omega)^{\nu} = \Gamma^{\Delta^n}(\omega)^{\nu}$$

$$\delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\zeta} \leq (t-\varepsilon)^{\zeta-\zeta} \delta^{\Delta}(\Omega)^{\nu} \delta(\omega)^{\zeta} \quad (45)$$

combining (4.44) and (4.45) to obtain the following inequality,

$$\delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\zeta} \leq (t-\varepsilon)^{\zeta-\zeta} \delta^{\Delta}(\omega)^{\nu} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta}$$

$$\delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\zeta} \leq (t-\varepsilon)^{\zeta-\zeta} \Gamma^{\Delta^n}(t)^{\nu} \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\Omega) \right)^{\zeta}$$

$$\begin{aligned} & \int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\zeta} \Delta(\omega) \\ & \leq \int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{\zeta-\zeta} \Delta(\Omega) \right]^{\zeta} \left[\left(\Gamma^{\Delta^n}(\Omega) \right) \left(\int_{\varepsilon}^t \Gamma^{\Delta^n}(\Omega) \Delta(\omega) \right)^{\frac{\zeta}{\nu}} \right]^{\nu} \Delta(\omega) \\ & \leq \int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{\zeta-\zeta} \Delta(\Omega) \right]^{\zeta} \times \left[\int_{\varepsilon}^t \delta^{\Delta}(\omega) \delta(\omega)^{\frac{\zeta}{\nu}} \Delta(\Omega) \right]^{\nu} \Delta(\omega) \end{aligned}$$

We note that

$$\int_{\sigma}^{\rho} \delta^{\Delta}(\omega) \delta(\omega)^{\frac{\zeta}{\nu}} \Delta(\omega) \leq \nu \int_{\sigma}^{\rho} \left(\delta(s)^{\frac{\zeta+\nu}{\nu}}(\Omega) \right)^{\Delta} \Delta(\Omega) = \nu \delta(\Omega)^{\frac{1}{\nu}} = \nu(\rho)^{\frac{1}{\nu}}. \quad (46)$$

See above Theorem 4.7, we have

$$\int_{\sigma}^{\rho} \delta^{\Delta}(\omega)^{\nu} \delta(\omega)^{\zeta} \Delta(\omega) \leq \int_{\sigma}^{\rho} \left[\int_{\varepsilon}^t (t-\varepsilon)^{\zeta-\zeta} \Delta(\Omega) \right]^{\zeta} \times \left[\int_{\varepsilon}^t \delta(\omega) \Delta(\omega) \right]^{\nu} \Delta(\omega).$$

5 GENERAL CONCLUSION

1 Summary and Conclusion

The result of this study is an extension of inequality of Pachpate, Shum, Olech, Calvert,

Fabelurin, Oguntuase, Beesack, Bohner, Bilil ü r and Yang s results. These authors used various

methods and approaches to actualized their results. We have succeeded in generalizing Pachpate, a special case of Shum, Olech, Calvert, Fabelurin, Oguntuase, Beesack, Bohner, Bilil ü r

and Yang's results and extensions.

We guarantee the existence of a primitive anti-derivative function for continuous functions. We introduced the concept of Jensen which is an essential tool throughout this work.

The modification of Jensen's inequality was also outlined to add value to knowledge.

However, the following has attested that the Jensen's inequality used in this work is in

order:

Theorem 5.1

Let g be continuous and non-decreasing on $[a, b]$, and $0 < a \leq b < \infty$ with $g(t) > 0$ for $t > 0$. Suppose that $l \geq p \geq 1, \nu > 0, 0 < \nu \leq l$ and $\delta > 0$. Let $f(x)$ be absolutely, non-decreasing function and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Then,

$$\begin{aligned} & \left[g(x)^{-\delta} - g(a)^{-\delta} \right]^{\frac{\nu(1-l)}{l}} \int_a^b g(x)^{\frac{\delta\nu}{l}-1} \left[\int_a^x f(t) dg(t) \right]^\nu dg(x) \\ & + [\delta]^{\frac{\nu(1-l)}{l}} \left[\frac{l}{\nu} \right] g(a)^{\frac{\delta\nu}{l}} \left[\int_a^b f^l(t) g(t)^{(l-1)(1+\delta)} dg(t) \right]^{\frac{\nu}{l}} \\ & \leq [\delta]^{\frac{\nu(1-l)}{l}} \left[\frac{l}{\nu} \right] \left[\int_a^b f^l(t) g(t)^{l(1+\delta)-1} dg(t) \right]^{\frac{\nu}{l}}. \end{aligned} \quad (47)$$

Theorem 5.2 Let $x(t)$ and $\lambda(t)$ be absolutely continuous functions $[a, b]$ with $\lambda(t) > 0, t \in [a, b]$. Let $Q(t)$ and $R(t)$ be non-negative and measurable function on $[a, b]$ such that either $k > 0, l > 0, m > 0, l > \mu m$ or $l < 0, m > 0$ and $l < \mu m$ where $l < |m - k|$, then, the following inequality holds:

$$\begin{aligned} & \int_a^b \left[Q(t)^{\frac{kl}{k-m}} \lambda'(t)^l R^{\frac{-klm}{k-m}} \lambda(t)^{\frac{kl}{k-m}} dt \right]^{\frac{k-m}{kl}} \times \left[\left(\int_0^t x'(t) R(t) d\lambda(t) \right)^l dt \right]^{\frac{1}{\omega}} dt \\ & \leq \int_a^b \left[y'(t)^{\frac{-m}{k}} y(t)^{\frac{l}{k}} dt \right]^{\frac{1}{k(l-m)}} \times \left[\left(Q(t) \lambda'(t) \lambda(t)^{\frac{l(k-m)}{k}} |x'(t)|^m dt \right)^{\frac{1}{\omega}} dt \right] \end{aligned} \quad (48)$$

Theorem 5.3 Let $f(t)$ be absolutely continuous function and non-decreasing on $[a, b]$, and $0 < a \leq b < \infty$ with $f(t) > 0$ for $t > 0$. Suppose that $p \geq l \geq 1, q > 0, 0 < l + q \leq p$ and $\delta > 0$. Then,

$$\begin{aligned} & \int_a^b f^q(t) t^{\delta-1} \left[\int_t^b f(s) ds \right]^l dt \leq [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] a^{\frac{\delta(l+q)}{p}} \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)-(1+\delta)}} ds \right]^{\frac{l+q}{p}} \\ & + [\delta^{-1}]^{\frac{lp-(l+q)+p}{p}} \left[\frac{p}{l+q} \right] \left[\int_a^b f^p(s) s^{\frac{lp(1+\delta)}{(l+q)-1}} ds \right]^{\frac{l+q}{p}} \end{aligned} \quad (49)$$

Theorem 5.4 Let T be a time scale with $s, h \in T$. Let ς, ζ be real numbers, let $\aleph, \chi \in C_{rd}([\alpha, \beta]_T, \mathbb{R})$ where h and χ are positive rd-continuous functions on $[\alpha, \beta]_T$ such that $\int_{[0,t]} r(t) \Delta(s) < \infty$. We define φ as convex function and if $\wp_{\gamma_1}, \wp_{\gamma_2} : [\alpha, \beta]_T \rightarrow \mathbb{R}$ is

delta differentiable with $\alpha(0) = 0$, then we have

$$\begin{aligned} & \int_{[\alpha, \beta]} (\sqrt{\chi(h)})^{\zeta+1} \wp_{\gamma_1}^{\Delta}(h) \wp_{\gamma_2}^{\zeta}(h) \Delta(h) \\ & \leq \frac{1}{1+\zeta} (\beta - \alpha)^{\zeta-\zeta} \left(\int_{[\alpha, \beta]} \frac{\Delta(h)}{(\sqrt{\aleph(h)})^{1+\zeta}} \right) \int_{[\alpha, \beta]} (\sqrt{\aleph(h)\chi(h)})^{1+\zeta} \wp_{\gamma_1}^{\Delta}(h)^{1+\zeta} \Delta(h) \end{aligned} \quad (50)$$

2 Recommendation

Generalizations of Opial-type inequalities obtained could be used to obtain better bounds for Opial-type inequalities and the results can be utilized in the solutions of differential, integral and integro-differential equations.

3 Suggestion for Further Studies

One suggestion from what we have done is to explore the possibility of using Bernoulli

inequality instead of modified Jensen's inequality in obtaining both Hardy-type and Opial-type

inequalities.

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