

ON THE RESPONSE OF PLATE ON WINKLER FOUNDATION TO A MOVING DISTRIBUTED LINE LOAD

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ABSTRACT

The dynamic response of a plate on Winkler foundation subjected to a moving distributed line load is presented.

The assumption of variable separation mode shapes method transforms the partial differential equation governing the motion of the plate, which in general is a function of x , y and t , into a function of one independent variable. The resulting equation is analysed by a finite difference method. The results for simply supported boundary conditions are used as illustrative example. It is found that distributed moving mass problem produces the upper bound deflection, when compared with a concentrated moving mass problem.

1. INTRODUCTION

The structures in which the dynamic effects of moving loads play the most important role are bridges, particularly the railway bridges whose behaviour under these effects have been studied and will continue to be studied because of the significance of safety in road and rail transportations. As a matter of fact, these structures have been of interest since the time of erection of the early railway bridges over a century ago.

Publications on moving load on plates have appeared in the literature [2-3]. A comprehensive treatment of the subject for the vibration of structure resulting from moving concentrated loads was given by Fryba [2].

Gbadayan, et al [4] analyzed distributed moving load problem of dynamic displacement of rectangular plate without foundation using assumed-mode method. Sun [5] considered the time-harmonic electrodynamic Green function of plates for line loads without foundation.

In this present study the work in [4] is extended by considering a plate resting on a Winkler foundation subjected to moving distributed line load.

2. FORMULATION OF THE PROBLEM

The classical differential equation of motion for the transverse displacement $W(x, y, t)$ of a plate in z direction is given by

$$D \nabla^4 W(x, y, t) + \rho \frac{\partial^2 W}{\partial t^2} = F(x, y, t) - q(x, y, t) \quad (1)$$

where D is the flexural rigidity and is defined by

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

E is Young's modulus, h is the plate thickness, μ is Poisson's ratio, ρ is the mass density per unit area of the plate and

$$\nabla^4 = \nabla^2 \nabla^2,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian operator.

The Winkler elastic foundation model, which performs well in many circumstances of rigid pavement design, is given by

$$q(x, y, t) = kW(x, y, t) \quad (2)$$

The substitution of equation (2) into equation (1) gives

$$D\nabla^2 \nabla^2 W(x, y, t) + kW(x, y, t) + \rho \frac{\partial^2}{\partial t^2} W(x, y, t) = F(x, y, t) \quad (3)$$

where the external excitation $F(x, y, t)$ is characterized by

$$F(x, y, t) = -\bar{M}g\left(1 + \frac{c}{g} \frac{d^2 W}{dt^2}\right) \left[\left(H\left(x - \left(vt - \frac{d}{2}\right)\right) - H\left(x - \left(vt + \frac{d}{2}\right)\right) \right) \delta(y - y_p) \right] \quad (4)$$

c is a constant, g is an acceleration due to gravity, $\bar{M}$ is the mass of the moving distributed loads. The acceleration of the moving mass can be written as

$$\ddot{W}(x, y, t) = v^2 \frac{\partial^2 W}{\partial x^2} + 2v \frac{\partial^2 W}{\partial x \partial t} + \frac{\partial^2 W}{\partial t^2} \quad (5)$$

where v is the constant velocity of the mass moving along straight line $y = y_p$.

The first term on the right side of equation (5) is the centripetal acceleration of moving mass, the second term is the Coriolis acceleration and the last term is the support plate acceleration at the point of contact with the moving mass.

To transform equation (3) into ordinary differential equation, we assumed variable separation mode shapes for external excitation and displacement function as follows respectively.

$$F(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} X_i(x) Y_j(y) T_{ij}(t) \quad (6)$$

and

$$W(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} X_i(x) Y_j(y) R_{ij}(t) \quad (7)$$

The substitution of equation (5) into equation (4) and making use of variable separable mode-shape (6) and (7) yields

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} X_i(x) Y_j(y) T_{ij}(t) = -\frac{\bar{M}g}{d} \left[1 + \frac{c}{g} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ v^2 X_i''(x) Y_j(y) R_{ij}(t) + 2v X_i'(x) Y_j(y) \dot{R}_{ij}(t) + X_i(x) Y_j(y) \ddot{R}_{ij}(t) \right\} \right] \left[H\left(x - \left(vt - \frac{d}{2}\right)\right) - H\left(x - \left(vt + \frac{d}{2}\right)\right) \right] \delta(y - y_p) \quad (8)$$

Multiplying equation (8) by $X_m(x) Y_n(y)$ and taking definite double integral with respect to x and y of both sides along the intervals $[0, L_x]$ and $[0, L_y]$ respectively we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} T_{ij}(t) \int_0^{L_x} X_i(x) X_m(x) dx \int_0^{L_y} Y_j(y) Y_n(y) dy = & -\frac{Mg}{d} \left[\int_0^{L_x} X_m(x) G(x-e) dx \int_0^{L_y} Y_n(y) \delta(y-y_p) dy \right. \\ & + \frac{c}{g} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \{ V^2 R_{ij}(t) \int_0^{L_x} X_i'(x) X_m(x) G(x-e) dx \int_0^{L_y} Y_j(y) Y_n(y) \delta(y-y_p) dy \\ & + 2V \dot{R}_{ij}(t) \int_0^{L_x} X_i'(x) X_m(x) G(x-e) dx \int_0^{L_y} Y_j(y) Y_n(y) \delta(y-y_p) dy \\ & \left. + \ddot{R}_{ij}(t) \int_0^{L_x} X_i(x) X_m(x) G(x-e) dx \int_0^{L_y} Y_j(y) Y_n(y) \delta(y-y_p) dy \right] \quad (9) \end{aligned}$$

where

$$G(x-e) = H(x-(vt-\frac{d}{2})) - H(x-(vt+\frac{d}{2}))$$

The integrals in the right hand side of equation (9) give zero except for $i=m$ and $j=n$ [3] and with identity

$$\int_0^{L_x} X(x) \delta(x-a) = X(a) \quad \text{for } 0 < a < L_x$$

equation (9) reduces to

$$\begin{aligned} \alpha T_{ij}(t) = & -\frac{Mg}{d} [Y_j(y_p) \int_{w-d/2}^{w+d/2} X_m(x) dx + \frac{c}{g} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Y_i(y_p) Y_n(y_p) \{ V^2 R_{ij}(t) \int_{w-d/2}^{w+d/2} X_i'(x) X_m(x) dx + \\ & 2V \dot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} X_i'(x) X_m(x) dx + \ddot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} X_i(x) X_m(x) dx \}] \quad (10) \end{aligned}$$

where w is the normalized constant.

Assuming free vibration, the displacement function takes the form

$$W(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} X_i(x) Y_j(y) e^{\pm i \omega_{ij} t} \quad (11)$$

where ω_{ij} is the circular frequency and γ is the complex number which has the property

$$\gamma^2 = -1$$

The substitution of equation (11) into the free vibration equation of motion yields

$$D(X_i''''(x) Y_j(y) + 2X_i''(x) Y_j''(y) + X_i(x) Y_j''''(y)) = (\rho \omega_{ij}^2 - K) X_i(x) Y_j(y) \quad (12)$$

By virtue of equation (8), (9), (10) and (12) equation (3) becomes

$$\begin{aligned} \rho \omega_{ij}^2 R_i(t) + \ddot{R}_{ij}(t) = & -\frac{\overline{Mg}}{d\alpha} [Y_i(y_p) \int_{w-d/2}^{w+d/2} X_m(x) dx \\ & + \frac{c}{g} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Y_i(y_p) Y_n(y_p) \{ V^2 R_{ij}(t) \int_{w-d/2}^{w+d/2} X_i'(x) X_m(x) dx \\ & + 2V \ddot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} X_i'(x) X_m(x) dx + \ddot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} X_i(x) X_m(x) dx \} \end{aligned} \quad (13)$$

3. SOLUTION PROCEDURE

Equation (13) could be solved for different boundary conditions of the plate. This is accomplished by substituting the displacement kernel for a set of particular boundary conditions in equation (13) and the solution is obtained for the function of time $R_{ij}(t)$. The displacement is obtained from equation (7).

For the fact that study of the response of simply supported plates gives a very useful insight into the dynamic behaviours of bridges, we consider the problem of a rectangular plate simply supported along all edges with the following boundary conditions

$$\begin{aligned} W(0, y, t) &= W(L_x, y, t) = 0 \\ \frac{\partial^2 W(0, y, t)}{\partial x^2} &= \frac{\partial^2 W(L_x, y, t)}{\partial x^2} = 0 \\ W(x, 0, t) &= W(x, L_y, t) = 0 \\ \frac{\partial^2 W(x, 0, t)}{\partial y^2} &= \frac{\partial^2 W(x, L_y, t)}{\partial y^2} = 0 \end{aligned} \quad (14)$$

The initial conditions can be written as

$$\begin{aligned} W(x, y, 0) &= 0 \\ \frac{\partial W(x, y, 0)}{\partial t} &= 0 \end{aligned} \quad (15)$$

The normalized displacement kernel for the simply supported rectangular plate is

$$X_i(x) Y_j(y) = -\frac{2\alpha}{\sqrt{L_x L_y}} \sin\left(\frac{i\pi x}{L_x}\right) \sin\left(\frac{j\pi y}{L_y}\right) \quad (16)$$

Application of equation (16) in equation (12) yields

$$\omega_{ij}^2 = \frac{1}{\rho} \left\{ \pi^4 D \left[\left(\frac{i}{L_x} \right)^2 + \left(\frac{j}{L_y} \right)^2 \right]^2 \right\} + K \quad (17)$$

For simply supported boundary conditions, equation (13) becomes

$$\begin{aligned} \rho \omega_{ij}^2 R_i(t) + \rho \ddot{R}_{ij}(t) &= -\frac{2\bar{M}g}{d\sqrt{L_x L_y}} \left[\alpha \sin\left(\frac{j\pi y_p}{L_y}\right) \int_{w-d/2}^{w+d/2} \sin\left(\frac{m\pi}{L_x} x\right) dx \right. \\ &- 2 \frac{c\alpha^2}{g\sqrt{L_x L_y}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin\left(\frac{i\pi y_p}{L_y}\right) \sin\left(\frac{n\pi y_p}{L_y}\right) \left\{ V^2 R_{ij}(t) \left(\frac{i\pi}{L_x}\right)^2 \int_{w-d/2}^{w+d/2} \sin\left(\frac{i\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_x} x\right) dx \right. \\ &+ 2V\dot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} \frac{i\pi}{L_x} \cos\left(\frac{i\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_x} x\right) dx + \ddot{R}_{ij}(t) \int_{w-d/2}^{w+d/2} \sin\left(\frac{i\pi}{L_x} x\right) \sin\left(\frac{m\pi}{L_x} x\right) dx \left. \right\} \end{aligned} \quad (18)$$

Evaluation of the integrals in equation (18) reduces the equation to

$$\begin{aligned} \ddot{R}_{ij}(t) + \omega_{ij}^2 R_i(t) &= -4 \frac{\bar{M}gL_x}{\rho d\sqrt{L_x L_y}} \left[\frac{1}{m} \sin\left(\frac{j\pi y_p}{L_y}\right) \sin\left(\frac{m\pi vt}{L_x}\right) \sin\left(\frac{m\pi d}{2L_x}\right) \right. \\ &- \frac{c\alpha}{g\sqrt{L_x L_y}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin\left(\frac{i\pi y_p}{L_y}\right) \sin\left(\frac{n\pi y_p}{L_y}\right) \left\{ \left(\frac{i\pi}{L_x}\right)^2 V^2 R_{ij}(t) \right. \\ &- \ddot{R}_{ij}(t) \left\{ \frac{1}{(i-m)} \cos\left(\frac{\pi vt(i-m)}{L_x}\right) \sin\left(\frac{\pi d(i-m)}{2L_x}\right) \right. \\ &- \frac{1}{(i+m)} \cos\left(\frac{\pi vt(i+m)}{L_x}\right) \sin\left(\frac{\pi d(i+m)}{2L_x}\right) \right\} \\ &- 2\left(\frac{i\pi}{L_x}\right) V\dot{R}_{ij}(t) \left\{ \frac{1}{(i+m)} \sin\left(\frac{\pi vt(i+m)}{L_x}\right) \sin\left(\frac{\pi d(i+m)}{2L_x}\right) \right. \\ &- \left. \left. \frac{1}{(i-m)} \sin\left(\frac{\pi vt(i-m)}{L_x}\right) \sin\left(\frac{\pi d(i-m)}{2L_x}\right) \right\} \right\}; i \neq m \end{aligned} \quad (19)$$

and

$$\begin{aligned} \ddot{R}_{ij}(t) + \omega_{ij}^2 R_i(t) &= -4 \frac{\bar{M}gL_x}{\rho d\pi\sqrt{L_x L_y}} \left[\frac{1}{m} \sin\left(\frac{j\pi y_p}{L_y}\right) \sin\left(\frac{m\pi vt}{L_x}\right) \sin\left(\frac{m\pi d}{2L_x}\right) \right. \\ &- \frac{c\alpha}{2g\sqrt{L_x L_y}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin\left(\frac{i\pi y_p}{L_y}\right) \sin\left(\frac{n\pi y_p}{L_y}\right) \left\{ \left(\frac{i\pi V}{L_x}\right)^2 R_{ij}(t) \right. \\ &- \ddot{R}_{ij}(t) \left\{ \frac{\pi d}{L_x} - \frac{1}{m} \cos\left(\frac{2m\pi y_p}{L_x}\right) \sin\left(\frac{md\pi}{L_x}\right) \right\} \\ &- \left. \left. \frac{V\pi}{L_x} \dot{R}_{ij}(t) \sin\left(\frac{2m\pi vt}{L_x}\right) \sin\left(\frac{m\pi d}{L_x}\right) \right\} \right\}; i = m \end{aligned} \quad (20)$$

The equivalence of equation (19) for concentrated moving problem are obtained by taking the limit of equation (19) as d tends to zero.

Thus we have,

$$\begin{aligned} \ddot{R}_{ij}(t) + \omega_{ij}^2 R_i(t) = & -4 \frac{MgL_x}{\rho\pi\sqrt{L_x L_y}} \left[\frac{\pi}{2L_x} \sin\left(\frac{j\pi y_p}{L_y}\right) \sin\left(\frac{m\pi vt}{L_x}\right) \right. \\ & - \frac{c\alpha}{g\sqrt{L_x L_y}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin\left(\frac{i\pi y_p}{L_y}\right) \sin\left(\frac{n\pi y_p}{L_y}\right) \left\{ \left(\frac{i\pi V}{L_x}\right)^2 R_{ij}(t) - \ddot{R}_{ij}(t) \right\} \left\{ \frac{\pi}{2L_x} \cos\left(\frac{\pi vt(i-m)}{L_x}\right) \right. \\ & \left. \left. - \frac{\pi}{2L_x} \cos\left(\frac{\pi vt(i+m)}{L_x}\right) \right\} \right\} - 2 \frac{iV\pi}{L_x} \dot{R}_{ij}(t) \left\{ \frac{\pi}{2L_x} \sin\left(\frac{\pi vt(i+m)}{L_x}\right) - \frac{\pi}{2L_x} \sin\left(\frac{\pi vt(i-m)}{L_x}\right) \right\} \right] \end{aligned} \quad (21)$$

by virtue of the identity

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Using the trigonometric identity, equation (21) may be written as

$$\begin{aligned} \ddot{R}_{ij}(t) + \omega_{ij}^2 R_i(t) = & - \frac{2Mg}{\rho\sqrt{L_x L_y}} \left[\sin\left(\frac{m\pi vt}{L_x}\right) \left[\sin\left(\frac{j\pi y_p}{L_y}\right) \right. \right. \\ & - \frac{c\alpha}{g\sqrt{L_x L_y}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sin\left(\frac{i\pi y_p}{L_y}\right) \sin\left(\frac{n\pi y_p}{L_y}\right) \left\{ \left(\frac{i\pi}{L_x}\right)^2 V^2 R_{ij}(t) - \ddot{R}_{ij}(t) \right\} \left\{ 2 \sin\left(\frac{i\pi vt}{L_x}\right) \right\} \right. \\ & \left. \left. - 2 \frac{i\pi}{L_x} V \dot{R}_{ij}(t) (2 \cos \frac{i\pi vt}{L_x}) \right\} \right] \end{aligned} \quad (22)$$

4. RESULTS

The results of a simply supported plate carrying a moving distributed line mass along the line $y = \frac{L_y}{2}$ were obtained. To ensure the stability and convergence of solution, sufficient small t-steps have been used in the program. Considering the first modal shape, equation (21) and (22) reduces to the form

$$\ddot{R}(t) + P(t)R(t) + Q(t)R(t) = Z(t) \quad (23)$$

Numerical method of central finite difference scheme was used to solve equation (23) together with their boundary values. The resulting tridiagonal system is

$$(2 - h\dot{P}(t_i))R_{i-1} + (-4 + 2h^2\Phi(t_i))R_i(t) + (2 + hP(t_i))R_{i+1} = 2h^2Z(t_i) \quad (24)$$

$i = 1, 2, 3, 4$

where $\Delta t = h$

The numerical solution of this system exist and unique if h is chosen such that the sub diagonal elements

$$2 - h\dot{P}(t_i) > 0 \quad (25)$$

The main diagonal elements

$$-4 + 2h^2\Phi(t_i) < 0 \quad (26)$$

the super diagonal elements

$$2 + hP(t_i) > 0 \quad (27)$$

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Since $h > 0$ and assuming $\Phi(t_i) \leq 0$, then inequalities (25), (26) and (27) combine as

$$h \leq \frac{2}{\max |p(t_i)|} \quad (28)$$

if $\Phi(t_i) > 0$, h is chosen such that it also satisfies

$$h^2 \leq \frac{2}{\max |\Phi(t_i)|}$$

Computer program was used to solve the system of equations. In addition, the matrix is not ill conditioned. The stability that may be induced in finite difference boundary value techniques is avoided by using a Gauss elimination process, in which the largest coefficient in the current column is used as pivot. With its use, we are certain of good results. [1]

For the purpose of providing numerical examples the first mode of vibration is presented using the following values $E = 2.07 \times 10^8 \text{ kg m}^{-2}$, $V = 5 \text{ m/s}$, $\frac{M}{PL_x L_y} = 0.2$

The moving force solution is obtained by chosen $c = 0$ while the moving mass solution is obtained when $c = 1$. The results have been displayed graphically in Figures 1 – 4 and the discussion are presented as follows.

Figure 1 shows the deflection of the plate's center at various times for distributed moving mass and force problems taking $d = 0.8 \text{ m}$. The contribution of inertia effect of the line load is significantly noticed since the moving mass problem produces the maximum deflection. To indicate the effect of the foundation on the dynamic behaviour of the plate, variation of K is shown in Figure 2. It shows that the greater the value of K , the less the maximum deflection of the plate.

Figure 3 compares the maximum deflection of distributed moving mass with the one of concentrated moving mass. The effect of varying the distribution of the mass on the plate was also investigated as shown in Figure 4. It shows a sharp increase of maximum deflection with increase of d for a very small value of d . It shows a decrease in maximum deflection with increase in $d \geq 0.2$.

5. CONCLUSION

The problem of assessing the dynamic response of a plate traversed by a distributed line mass has been investigated. The mathematical procedure employed for the solution is applicable to all boundary conditions with practical applications. The effects of moving distributed line mass were investigated. An important conclusion drawn from this study is that caution should be taken in using concentrated moving mass problem to predict the dynamic behaviours of plate since the study shows that maximum plate deflection is noticed with distributed moving mass problem (see Figure 4).

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Figure 1. Moving mass and moving force mid-plate deflection ($k=20$).

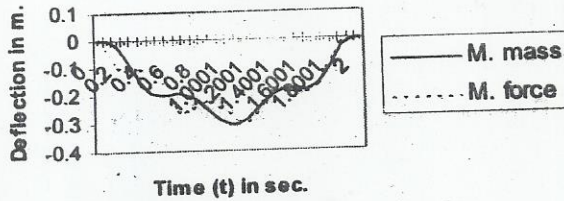


Figure 2. Variation of K

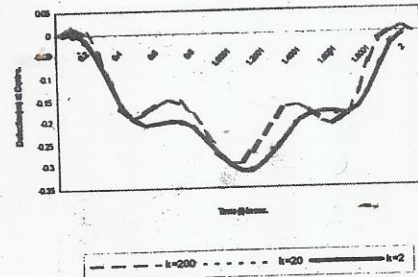


Figure 3. Mid-plate deflection for line mass and point mass ($k=20$)

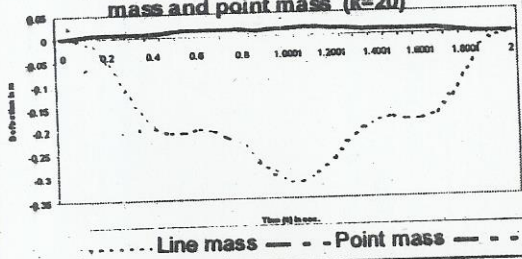


Figure 4. Maximum plate deflection due to variational distribution of a line mass.

