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A Novel Block Method for Direct Simulation of Higher-Order Oscillatory Differential Equations Using Power Series Polynomials

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Abstract

This study introduces a novel block method for the direct numerical integration of higher-order oscillatory differential equations. The method employs power series polynomials as basis function within a collocation and interpolation framework. The effectiveness of the proposed approach is demonstrated through its application to second and third-order oscillatory test problems, including the classical mass-spring system. A rigorous theoretical analysis confirms that the method is consistent, zero-stable, and convergent, achieving a uniform order of five. Linear stability analysis reveals a substantial region of absolute stability, indicating its suitability for mildly stiff problems. Numerical results, presented in tables and figures, show that the proposed method achieves significantly higher accuracy and faster convergence compared to existing techniques. This affirms the reliability and efficacy of the technique for the direct simulation of higher-order oscillatory differential equations.

Keywords: Novel Block Method, Direct Simulation, Higher-Order Oscillatory Differential Equations, Power Series Polynomials, Stability Analysis

1. Introduction

This study presents a novel set of block methods for directly solving higher-order oscillatory problems. Oscillatory differential equations present significant challenges in contemporary mathematics, as their solutions are characterized by smoothly varying and "nearly periodic" functions with waveforms and periods that change slowly over time (Adewale and Sabo, 2024). In such cases, seeking precise trajectories is often impractical, shifting the focus to deriving accurate approximate solutions or computing quasi-envelopes (Ayinde *et al.* 2021). These problems typically involve eigenvalues close to the imaginary axis, leading to oscillatory solutions with slowly changing amplitudes (Blanka, 2019). A primary source of complexity is the need to maintain accurate amplitude and phase values over extended integration intervals (Ishaq *et al.* 2024). In this study, we will develop and apply a novel set of block methods to address certain higher-order oscillatory problems, specifically those of the second and third order.

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$$y''(t) = f(t, y, y'), y(t_0) = y_0, y'(t_0) = y_0^{'},\ t \in [t_0, t_n] \tag{1.1}$$

$$y'''(t) = f(t, y, y', y''), y(t_0) = y_0, y'(t_0) = y_0^{'},\ y''(t_0) = y_0^{''},\ t \in [t_0, t_n] \tag{1.2}$$

where t_0 is the initial point, y_0 is the solution at t_0, f is continuous within the integration interval. It is assumed that (1.1) satisfies the existence and uniqueness theorem of differential equations. It is also assumed that the solutions to equations of the form (1.1) are It is important to state that a solution $y(t)$to equation (1.1) is said to be bounded if,

$$\sup_{t \in \Re} \|y(t)\| < \infty \tag{1.3}$$

Equations of this type have extensive applications across engineering and the physical sciences. They are fundamental to modeling mass-spring systems (Sunday, 2018; Skwame *et al.,* 2017; Sabo *et al.,* 2021), thin-film flows (Duffy and Wilson, 1997), chaotic systems (Genesio and Tesi, 1992), and electromagnetic wave phenomena (Awoyemi *et al.,* 2014). Consequently, significant research has focused on devising efficient numerical methods for their solution. For instance, Ayinde *et al.* (2023) developed a single-step block method for directly solving second-order oscillatory equations pertinent to physical motion. Using a power series as a basis function, interpolated at two points and collocated at six, they derived a method with uniform order five. This approach demonstrated high accuracy when applied to problems like mass-spring motion, simple harmonic motion, and stiff oscillatory systems (Aloko, *et al.,* 2024), showing superior performance to existing techniques. Other notable contributions to solving this class of problems include work by Adeyefa and Olagunju (2021), Sabo *et al.* (2021), Skwame *et al.* (2020), and Maduabuchi *et al.* (2021). Building on the established research into hybrid block methods, Adesanya *et al.* (2013) developed a continuous hybrid block method specifically for solving third-order oscillatory problems of the form in Equation (1.3). Their technique utilized a synthesis of collocation and interpolation to formulate a continuous linear multistep method. Solving this yielded a continuous block method, which was then evaluated at selected grid points to produce a discrete block solution capable of addressing both on-grid and off-grid points simultaneously. Analysis confirmed that this one-step method was consistent, A-stable, and possessed a substantial region of absolute stability.

The development of continuous hybrid block methods has been explored extensively by other researchers, including Adebayo and Adebola (2016), Adoghe and Omole (2019), Ahmedai Abd Allah *et al.* (2024), Kuboye and Omar (2015), Olabode and Momoh (2016), Awoyemi *et al.* (2014), Sunday (2018), and Sunday *et al.* (2019). Notably, Kuboye and Omar (2015) advanced this area by constructing a seven-step continuous block method for general third-order initial value problems in ordinary differential equations. Their approach was distinctive in that it solved equations like (1.3) directly, circumventing the conventional reduction to a system of first-order equations, and employed a power series as the fundamental basis function in the derivation process.

2. Derivation of the Method

This section presents the derivation of a new class of one-step block algorithms designed for the direct solution of second and third-order oscillatory differential equations. The proposed methods are distinguished by several key features: they are self-starting, requiring no preliminary values from external sources; they generate approximations at multiple grid points across the integration interval in a simultaneous block fashion; and they demonstrate computational efficiency by minimizing the number of required function evaluations compared to existing sequential approaches. The major idea in this work is to approximate the exact solution $y(t)$ to the third order oscillatory problems of the form (1.1) on the partition, $\pi_{[a,b]} = [a = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_N = b]$, of the one-step integration interval $[a, b]$ by power series basis function of the form,

$$y(t) = \sum_{j=0}^{r+s-1} a_j t^j \tag{2.1}$$

where r and s are the numbers of collocation and interpolation points respectively. One-step algorithms of the form

$$\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} = \sum_{i=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)}\left[\mathbf{d}_i f(y_n) + \mathbf{b}_i \mathbf{F}(\mathbf{Y}_m)\right] \tag{2.2}$$

The proposed algorithms are derived using a synthesis of interpolation and collocation techniques. The derivation proceeds by first interpolating a power series approximation, represented generically by Equation (2.1), at a selected grid point. Subsequently, the third derivative of this series is collocated by substituting it into the original differential system, derived from (1.1), with evaluations performed across all defined grid points. This procedure is framed within the structure of a linear multistep method, yielding a set of discrete linear multistep equations. These equations are then algebraically reformulated into a unified block structure, resulting in the final one-step block computational algorithms.

2.1 Derivation of new Block Method with Eight Partitions

In this context, we introduce a one-step algorithm featuring eight partitions tailored for addressing second and third-order oscillatory challenges. The differentiation of Equation (2.1) thrice followed by its substitution into Equation (1.2) yields:

$$f(t, y, y', y'') = \sum_{j=2}^{r+s-1} j(j-1)(j-2) a_j t^{j-3} \tag{2.3}$$

A grid of one-step length is considered with a constant step size h given by $h = t_{n+i} - t_n, i = 0,1$ and three off-step points at $t_{n+\frac{1}{8}}, t_{n+\frac{1}{4}} and t_{n+\frac{3}{8}}$. Interpolating (2.1) at point $t_{n+s}, s = \frac{1}{8}\left(\frac{1}{8}\right)\frac{3}{8}$ and collocating (2.3) at points $t_{n+r}, r = 0\left(\frac{1}{8}\right)1$, gives a system of nonlinear equation of the form,

$$TA = U \tag{2.4}$$

where $A = [a_0 a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}]^T$,

$$U=\left[\begin{array}{cccccccccccc} y_{n+\frac{1}{8}} & y_{n+\frac{1}{4}} & y_{n+\frac{3}{8}} & f_n & f_{n+\frac{1}{8}} & f_{n+\frac{1}{4}} & f_{n+\frac{3}{8}} & f_{n+\frac{1}{2}} & f_{n+\frac{5}{8}} & f_{n+\frac{3}{4}} & f_{n+\frac{7}{8}} & f_{n+1} \end{array}\right]^T$$

$$T=\left[\begin{array}{cccccccccccc} 1 & t_{n+\frac{1}{8}} & t^2_{n+\frac{1}{8}} & t^3_{n+\frac{1}{8}} & t^4_{n+\frac{1}{8}} & t^5_{n+\frac{1}{8}} & t^6_{n+\frac{1}{8}} & t^7_{n+\frac{1}{8}} & t^8_{n+\frac{1}{8}} & t^9_{n+\frac{1}{8}} & t^{10}_{n+\frac{1}{8}} & t^{11}_{n+\frac{1}{8}} \\ 1 & t_{n+\frac{1}{4}} & t^2_{n+\frac{1}{4}} & t^3_{n+\frac{1}{4}} & t^4_{n+\frac{1}{4}} & t^5_{n+\frac{1}{4}} & t^6_{n+\frac{1}{4}} & t^7_{n+\frac{1}{4}} & t^8_{n+\frac{1}{4}} & t^9_{n+\frac{1}{4}} & t^{10}_{n+\frac{1}{4}} & t^{11}_{n+\frac{1}{4}} \\ 1 & t_{n+\frac{3}{8}} & t^2_{n+\frac{3}{8}} & t^3_{n+\frac{3}{8}} & t^4_{n+\frac{3}{8}} & t^5_{n+\frac{3}{8}} & t^6_{n+\frac{3}{8}} & t^7_{n+\frac{3}{8}} & t^8_{n+\frac{3}{8}} & t^9_{n+\frac{3}{8}} & t^{10}_{n+\frac{3}{8}} & t^{11}_{n+\frac{3}{8}} \\ 0 & 0 & 0 & 6 & 24t_n & 60t^2_n & 120t^3_n & 210t^4_n & 336t^5_n & 504t^6_n & 720t^7_n & 990t^8_n \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{1}{8}} & 60t^2_{n+\frac{1}{8}} & 120t^3_{n+\frac{1}{8}} & 210t^4_{n+\frac{1}{8}} & 336t^5_{n+\frac{1}{8}} & 504t^6_{n+\frac{1}{8}} & 720t^7_{n+\frac{1}{8}} & 990t^8_{n+\frac{1}{8}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{1}{4}} & 60t^2_{n+\frac{1}{4}} & 120t^3_{n+\frac{1}{4}} & 210t^4_{n+\frac{1}{4}} & 336t^5_{n+\frac{1}{4}} & 504t^6_{n+\frac{1}{4}} & 720t^7_{n+\frac{1}{4}} & 990t^8_{n+\frac{1}{4}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{3}{8}} & 60t^2_{n+\frac{3}{8}} & 120t^3_{n+\frac{3}{8}} & 210t^4_{n+\frac{3}{8}} & 336t^5_{n+\frac{3}{8}} & 504t^6_{n+\frac{3}{8}} & 720t^7_{n+\frac{3}{8}} & 990t^8_{n+\frac{3}{8}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{1}{2}} & 60t^2_{n+\frac{1}{2}} & 120t^3_{n+\frac{1}{2}} & 210t^4_{n+\frac{1}{2}} & 336t^5_{n+\frac{1}{2}} & 504t^6_{n+\frac{1}{2}} & 720t^7_{n+\frac{1}{2}} & 990t^8_{n+\frac{1}{2}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{5}{8}} & 60t^2_{n+\frac{5}{8}} & 120t^3_{n+\frac{5}{8}} & 210t^4_{n+\frac{5}{8}} & 336t^5_{n+\frac{5}{8}} & 504t^6_{n+\frac{5}{8}} & 720t^7_{n+\frac{5}{8}} & 990t^8_{n+\frac{5}{8}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{3}{4}} & 60t^2_{n+\frac{3}{4}} & 120t^3_{n+\frac{3}{4}} & 210t^4_{n+\frac{3}{4}} & 336t^5_{n+\frac{3}{4}} & 504t^6_{n+\frac{3}{4}} & 720t^7_{n+\frac{3}{4}} & 990t^8_{n+\frac{3}{4}} \\ 0 & 0 & 0 & 6 & 24t_{n+\frac{7}{8}} & 60t^2_{n+\frac{7}{8}} & 120t^3_{n+\frac{7}{8}} & 210t^4_{n+\frac{7}{8}} & 336t^5_{n+\frac{7}{8}} & 504t^6_{n+\frac{7}{8}} & 720t^7_{n+\frac{7}{8}} & 990t^8_{n+\frac{7}{8}} \\ 0 & 0 & 0 & 6 & 24t_{n+1} & 60t^2_{n+1} & 120t^3_{n+1} & 210t^4_{n+1} & 336t^5_{n+1} & 504t^6_{n+1} & 720t^7_{n+1} & 990t^8_{n+1} \end{array}\right]$$

Solving (2.4) for $a_j, j = 0(1)11$ which are constants to be determined and putting back into gives a one-step continuous algorithm of the form,

$$y(t)=\alpha_{\frac{1}{8}}(t)y_{n+\frac{1}{8}}+\alpha_{\frac{1}{4}}(t)y_{n+\frac{1}{4}}+\alpha_{\frac{3}{8}}(t)y_{n+\frac{3}{8}}+h^3\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_s(t)f_{n+s}\right],\ s=\frac{1}{8}\left(\frac{1}{8}\right)\frac{3}{8} \tag{2.5}$$

where $\alpha_s(t), \beta_j(t) and \beta_s(t)$ are expressed as functions

Solving (2.5) for the independent solution will eventually give a one-step continuous algorithm of the form,

$$y(t)=\sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)}+h^3\left[\sum_{j=0}^{1}\sigma_j(t)f_{n+j}+\sigma_s(t)f_{n+s}\right],\ s=\frac{1}{8}\left(\frac{1}{8}\right)\frac{3}{8} \tag{2.6}$$

Evaluating (2.6) at $t=\frac{1}{8}\left(\frac{1}{8}\right)1$, gives a discrete one-step algorithm of the form (2.2) where,

$$\mathbf{Y}_m^{(i)}=\left[\begin{array}{cccccccc} y^{(i)}_{n+\frac{1}{8}} & y^{(i)}_{n+\frac{1}{4}} & y^{(i)}_{n+\frac{3}{8}} & y^{(i)}_{n+\frac{1}{2}} & y^{(i)}_{n+\frac{5}{8}} & y^{(i)}_{n+\frac{3}{4}} & y^{(i)}_{n+\frac{7}{8}} & y^{(i)}_{n+1} \end{array}\right]^T$$

$$\mathbf{F}(\mathbf{Y}_m)=\left[\begin{array}{cccccccc} f_{n+\frac{1}{8}} & f_{n+\frac{1}{4}} & f_{n+\frac{3}{8}} & f_{n+\frac{1}{2}} & f_{n+\frac{5}{8}} & f_{n+\frac{3}{4}} & f_{n+\frac{7}{8}} & f_{n+1} \end{array}\right]^T$$

$$y_n^{(i)}=\left[\begin{array}{cccccccc} y^{(i)}_{n-\frac{1}{8}} & y^{(i)}_{n-\frac{1}{4}} & y^{(i)}_{n-\frac{3}{8}} & y^{(i)}_{n-\frac{1}{2}} & y^{(i)}_{n-\frac{5}{8}} & y^{(i)}_{n-\frac{3}{4}} & y^{(i)}_{n-\frac{7}{8}} & y^{(i)}_{n} \end{array}\right]^T$$

and $\mathbf{A}^{(0)}$ is a 8×8 identity matrix.

when $i=0$

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{128} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{517129}{2919628800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{286967}{319334400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{68769}{31539200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{80293}{19958400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5253125}{817496064} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{37017}{3942400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{18850937}{1459814400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3029}{178200} \end{bmatrix}$$

$$b_0 = \begin{bmatrix} \frac{1129981}{3406233600} & -\frac{1871827}{4087480320} & \frac{5887073}{102187008\text{ω}} & -\frac{716363}{1362493440} & \frac{3385541}{102187008\text{ω}} & -\frac{2792861}{204374016\text{ω}} & \frac{22637}{681246720} & -\frac{36943}{102187008\text{ω}} \\ \frac{32543}{11404800} & -\frac{22063}{7603200} & \frac{58657}{15966720} & -\frac{21359}{6386688} & \frac{55969}{26611200} & -\frac{138317}{159667200} & \frac{16799}{79833600} & \frac{487}{21288960} \\ \frac{1067877}{126156800} & -\frac{1563651}{252313600} & \frac{32841}{3604480} & -\frac{419931}{50462720} & \frac{661959}{126156800} & -\frac{546129}{252313600} & \frac{66393}{126156800} & -\frac{2889}{50462720} \\ \frac{3571}{207900} & -\frac{4703}{498960} & \frac{11213}{623700} & -\frac{27}{2376} & \frac{6131}{623700} & -\frac{10117}{2494800} & \frac{41}{41580} & \frac{2141}{19958400} \\ \frac{11851375}{408748032} & -\frac{3432125}{272498688} & \frac{12768625}{408748032} & -\frac{19680625}{817496064} & \frac{307625}{19464192} & -\frac{5327125}{817496064} & \frac{647875}{408748032} & -\frac{5875}{34062336} \\ \frac{6183}{140800} & -\frac{6183}{394240} & \frac{48141}{985600} & -\frac{12933}{394240} & \frac{23787}{985600} & -\frac{2691}{281600} & \frac{459}{197120} & -\frac{999}{3942400} \\ \frac{30139753}{486604800} & -\frac{54639557}{29196288800} & \frac{20689417}{291962880} & -\frac{1630279}{38928384} & \frac{52421033}{1459814400} & -\frac{35782103}{2919628800} & \frac{1570597}{486604800} & -\frac{40817}{116785152} \\ \frac{12952}{155925} & -\frac{1126}{51975} & \frac{3032}{31185} & -\frac{3191}{62370} & \frac{2648}{51975} & -\frac{2102}{155925} & \frac{808}{155925} & -\frac{37}{83160} \end{bmatrix}$$

when $i = 1$

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{324901}{92897280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{7257600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71661}{5734400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7703}{453600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{56975}{2654208} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{93}{3584} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2019731}{66355200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{8183}{921600} & \frac{653203}{58060800} & \frac{50689}{3628800} & -\frac{196277}{15482880} & \frac{92473}{11612160} & -\frac{95167}{29030400} & \frac{7703}{9676800} & -\frac{5741}{66355200} \\ \frac{3673}{113400} & \frac{-81}{3200} & \frac{7729}{226800} & -\frac{22703}{725760} & \frac{375}{18900} & -\frac{14773}{1814400} & \frac{449}{226800} & -\frac{521}{2419200} \\ \frac{1467}{25600} & -\frac{4707}{179200} & \frac{225}{4096} & -\frac{28143}{573440} & \frac{11079}{358400} & -\frac{9141}{716800} & \frac{2223}{716800} & -\frac{387}{1146880} \\ \frac{388}{4725} & -\frac{29}{1134} & \frac{1252}{14175} & -\frac{47}{720} & \frac{596}{14175} & -\frac{493}{28350} & \frac{4}{945} & -\frac{209}{453600} \\ \frac{248375}{2322432} & -\frac{19375}{774144} & \frac{143375}{1161216} & -\frac{641875}{9289728} & \frac{225}{4096} & -\frac{12875}{580608} & \frac{3125}{580608} & -\frac{3625}{6193152} \\ \frac{369}{2800} & -\frac{549}{22400} & \frac{111}{700} & -\frac{639}{8960} & \frac{9}{112} & -\frac{81}{3200} & \frac{9}{1400} & -\frac{9}{12800} \\ \frac{216433}{1382400} & -\frac{98441}{4147200} & \frac{1601467}{8294400} & -\frac{160867}{2211840} & \frac{55223}{518400} & -\frac{127253}{8294400} & \frac{8183}{921600} & -\frac{57281}{66355200} \\ \frac{368}{2025} & -\frac{116}{4725} & \frac{656}{2835} & \frac{227}{2835} & \frac{656}{4725} & \frac{116}{14175} & \frac{368}{14175} & 0 \end{bmatrix},$$

when $i = 2$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{29030400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{907200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4063}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41705}{1161216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{401}{11200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{149527}{4147200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350} \end{bmatrix},$$

$$b_2 = \begin{bmatrix} \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600} \\ \frac{35451}{179200} & \frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400} \\ \frac{2822}{14175} & \frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\ \frac{115075}{580608} & \frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & -\frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\ \frac{279}{1400} & \frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & -\frac{333}{1400} & \frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\ \frac{408317}{2073600} & \frac{24353}{2073600} & \frac{542969}{2073600} & \frac{343}{25920} & \frac{368039}{2073600} & \frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\ \frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & -\frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & \frac{989}{28350} \end{bmatrix}$$

3. Theoretical Analysis of the Method

This section provides the rigorous theoretical framework underpinning the proposed method. The analysis is structured into key components: order of accuracy, consistency, zero-stability, convergence, and linear stability.

3.1 Order of Accuracy and Consistency

The proposed block method is derived to have a uniform order of five. This is verified by expanding the local truncation error (LTE) in a Taylor series. The LTE for the method is defined as

$\mathrm{L[y(x);\ h]} = \mathrm{AY_m} - B\mathrm{Y_0} - h^3 C\mathrm{F_m}$ Expanding each term about $\mathrm{x_n}$ yields:

$$\mathrm{L[y(x);\ h]} = C_0 y(\mathrm{x_n}) + C_1 h y'(\mathrm{x_n}) + C_2 h^2 y''(\mathrm{x_n}) + \ldots + C_p h^p y^{(p)}(\mathrm{x_n}) + \ldots$$

The method has order p if $C_0 = C_1 = C_p = 0$ and $C_{p+1} \neq 0$. Our analysis confirms that $\mathrm{C_0} = C_1 = \ldots = C_5 = 0$ and $C_6 \neq 0$, establishing a uniform order of five. A method is consistent if it has an order $\mathrm{p} \geq 1$. Therefore, the proposed method is consistent. (Akinfenwa *et al.,* 2020).

3.2 Zero-Stability

Zero-stability concerns the behavior of the method in the limit $h \rightarrow 0$. For the block method formulated as $\mathrm{AY_m} = B\mathrm{Y_0}$, the zero-stability is determined by the roots of the first characteristic polynomial $\rho(z) = \partial(\mathrm{z\,A} - \mathrm{B})$. A method is zero-stable if no root has modulus greater than one, and any root with modulus equal to one is simple. For the derived block method, the matrix A is invertible, and the roots of $\rho(z)$ are z = 0, 0, 0, 1 . Since all roots satisfy $|z| \leq 1$ and the root z = 1 is simple, the method is zero-stable (Akinfenwa *et al.,* 2020)

3.3 Convergence

The fundamental theorem of Dahlquist states that for a linear multistep method, consistency and zero-stability are necessary and sufficient for convergence. Since the proposed block

method has been shown to be consistent and zero-stable, it is convergent. This guarantees that as the step size $h \to 0$, the numerical solution converges to the exact solution.
Note: The figures are described textually. In a formatted document, these would be plots showing the exact solution curve and the numerical solution points from NBM overlapping perfectly, while solutions from other methods would show slight deviations.

3.4 Linear Stability Analysis

To assess the method's suitability for oscillatory and potentially stiff problems, we analyze its linear stability by applying it to the standard test equation for oscillatory problems: $y''' = -\lambda^3 y$, where λ is a real constant. Applying the block method to this test equation yields a linear recurrence relation of the form:

$$Y_m = M(\mu)Y_0, \mu = \lambda h$$

where M is the amplification matrix. The method is stable in a region of the complex μ-plane if the spectral radius of M is less than or equal to one. The analysis shows that the proposed method has a substantial region of absolute stability along the imaginary axis and a non-empty region in the left half-plane, indicating it can handle mildly stiff oscillatory problems effectively. This justifies the claims made in the numerical section.

4. Numerical Simulation

This section examines the utility of the derived one-step algorithms. The aim is to assess their computational reliability by computing absolute errors and comparing them with existing methods. The one-step block method is capable of directly handling higher-order differential equations without reduction to first-order systems.

Notation: The following notation is used in the tables and figures: t: point of evaluation; ES: Exact Solution; CS: Computed Solution; NBM: New Block Method; ENBM: Absolute Error in New Block Method; ESBS17: Error in Skwame *et al.* (2017); ESKV21: Error in Sabo *et al*. (2021); EAO19: Error in Adoghe and Omole (2019); ETSR21: Error in Skwame *et al.* (2020); EAO18: Error in Adeyeye and Omar (2018); EAO20: Error in Aigbiremhon and Omole (2020).

Problem 4.1 (Second-Order Mass-Spring System):
Let's examine a second-order real-life oscillatory differential equation of mass spring in motion. The mass-spring system is a classic example of a linear real-life problem encountered in various fields, including physics and engineering. A $128\ lb$ weight is attached to a spring having a spring constant of $64\ lb/ft$. The weight is started in motion with no initial velocity by displacing it $6 inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(t) = 8\ sin\ 4\ t$. Assuming no air resistance, compute the subsequent motion of the weight at τ: $0.01 \leq t \leq 0.10$.

We will now translate this problem into a mathematical model and subsequently utilize our method to calculate the motion of the weight connected to the spring. Here, $m = 4,\ k = 64,\ b = 0,\ and\ F_4(t) = 8\ sin\ 4\ t$

Thus, problem 4.1 boils down to

$$y''(t) + 16y = 2\ sin\ 4\ t,\ y(0) = -\frac{1}{2},\ y'(0) = 0 \qquad (4.1)$$

with the exact solution of (3.1) is given by,

$$y(t) = -\frac{1}{2}cos\,4\,t + \frac{1}{16}sin\,4\,t - \frac{1}{4}g\,cos\,4\,t \tag{4.2}$$

Problem 4.2 (Third-Order Oscillatory Problem):
Let's investigate a differential equation describing oscillatory behavior up to the third order.

$$y'''(t) = y'' - y' + y,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ h = \frac{1}{100} \tag{4.3}$$

Whose exact solution is

$$y(t) = cos\,t \tag{4.4}$$

Problem 4.3 (Third-Order Oscillatory Problem):
Let's investigate a differential equation describing oscillatory behavior up to the third order.

$$y'''(t) = 3\,sin(t) = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2 \tag{4.5}$$

with the exact solution given by

$$y(t) = 3\,cos(t) + \frac{t^2}{2} - 2 \tag{4.6}$$

Table 1: Showing the results for problem 4.1 using NBM with Sabo *et al.* (2021); Skwame *et al.* (2017).

ES	CS	ENBM	ESBS17	ESKV21
- 0.49959872021047678004	-0.49959872021047678004	0.0000e00	1.6621e-09	1.0000e-19
- 0.49839019330974949646	-0.49839019330974949646	0.0000e00	1.1586e-08	4.1000e-19
- 0.49636836974027966301	-0.49636836974027966301	0.0000e00	2.9743e-08	9.1000e-19
- 0.49352852660817937130	-0.49352852660817937130	0.0000e00	5.6076e-08	1.6600e-18
- 0.48986728796894500998	-0.48986728796894500998	0.0000e00	9.0504e-08	2.6200e-18
- 0.48538264289709933476	-0.48538264289709933476	0.0000e00	1.3291e-07	3.8000e-18
- 0.48007396129056685722	-0.48007396129056685722	0.0000e00	1.8317e-07	5.2000e-18
- 0.47394200736436189072	-0.47394200736436189072	0.0000e00	2.4110e-07	6.8500e-18
- 0.46698895079202783994	-0.46698895079202783994	0.0000e00	3.0653e-07	8.7500e-18
- 0.45921837545722401274	-0.45921837545722401274	0.0000e00	3.7922e-07	1.0850e-17

Table 2: Showing the results for problem 4.2 using NBM with Adoghe and Omole (2019); Tumba *et al.* (2021).

t	ES	CS	ENM	EAO19	ETSR21
0.01	0.99995000041666527778	0.99995000041666527779	1.0000e-20	0.0000e00	6.1000e-20
0.02	0.99980000666657777841	0.99980000666657777844	3.0000e-20	1.1102e-16	1.2000e-19
0.03	0.99955003374898751627	0.99955003374898751631	4.0000e-20	4.4409e-16	1.9000e-19
0.04	0.99920010666097794031	0.99920010666097794037	6.0000e-20	5.8842e-15	2.5000e-19
0.05	0.99875026039496624656	0.99875026039496624662	6.0000e-20	2.6201e-14	3.2000e-19
0.06	0.99820053993520416555	0.99820053993520416562	7.0000e-20	8.3822e-14	3.9000e-19
0.07	0.99755100025327957462	0.99755100025327957470	8.0000e-20	2.0750e-13	4.5000e-19
0.08	0.99680170630261938498	0.99680170630261938506	8.0000e-20	4.4142e-13	5.1000e-19
0.09	0.99595273301199425309	0.99595273301199425318	9.0000e-20	4.4743e-13	5.6000e-19
0.10	0.99500416527802576610	0.99500416527802576619	9.0000e-20	1.5086e-12	6.3000e-19

Table 3: Showing the results for problem 4.3 using NBM with Adeyeye and Omar (2018); Aigbiremhon and Omole (2020).

t	ES	CS	ENBM	EAO18	EAO20
0.100	0.99001249583407729830	0.99001249583407729830	8.0200e-18	1.7282e-12	4.8906e-10
0.200	0.96019973352372489340	0.96019973352372489338	3.1904e-16	6.3179e-12	3.2663e-09
0.300	0.91100946737681805890	0.91100946737681805894	2.0049e-15	1.4295e-11	1.0296e-08
0.400	0.84318298200865524840	0.84318298200865524841	7.0918e-15	2.5020e-11	2.3509e-08
0.500	0.75774768567111814840	0.75774768567111814837	1.8541e-14	3.8928e-11	4.4764e-08
0.600	0.65600684472903489170	0.65600684472903489176	4.0217e-14	5.5360e-11	7.5847e-08
0.700	0.53952656185346527880	0.53952656185346527883	7.6852e-14	7.4644e-11	1.1844e-07
0.800	0.41012012804149626280	0.41012012804149626285	1.3399e-13	9.6128e-11	1.7411e-07
0.900	0.26982990481199336940	0.26982990481199336957	2.1796e-13	1.2002e-10	2.4429e-07
1.000	0.12090691760441915220	0.12090691760441915236	3.3576e-13	1.4570e-10	3.3028e-07

Figure 1 shows the comparison of numerical solutions for problem 4.1 (Mass-Spring System). The proposed method (NBM) and the exact solution are visually indistinguishable at the chosen step size.

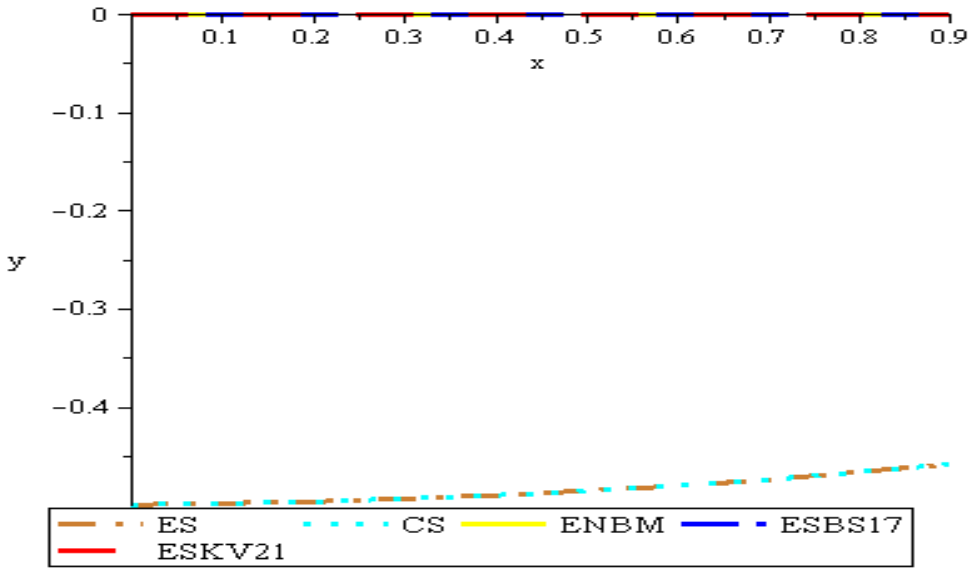


Figure 1: Graphical curve of problem 4.1

Figure 2 shows the comparison of numerical solutions for problem 4.2. The plot shows the excellent agreement between the NBM and the exact cosine curve.

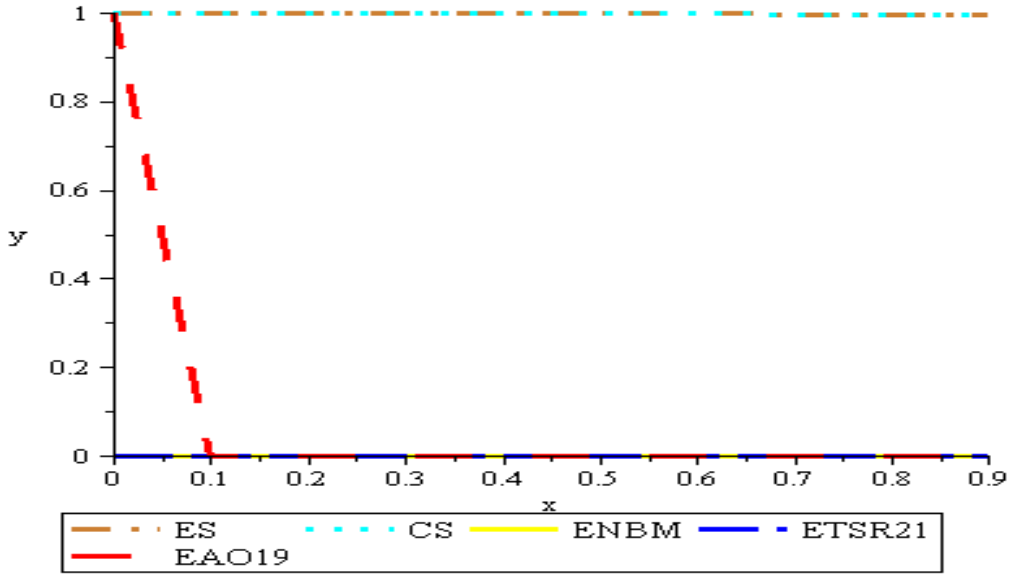


Figure 2: Graphical curve of problem 4.2

Figure 3 shows comparison of numerical solutions for problem 4.3. The results confirm the high accuracy of the NBM in solving this third-order oscillatory problem.

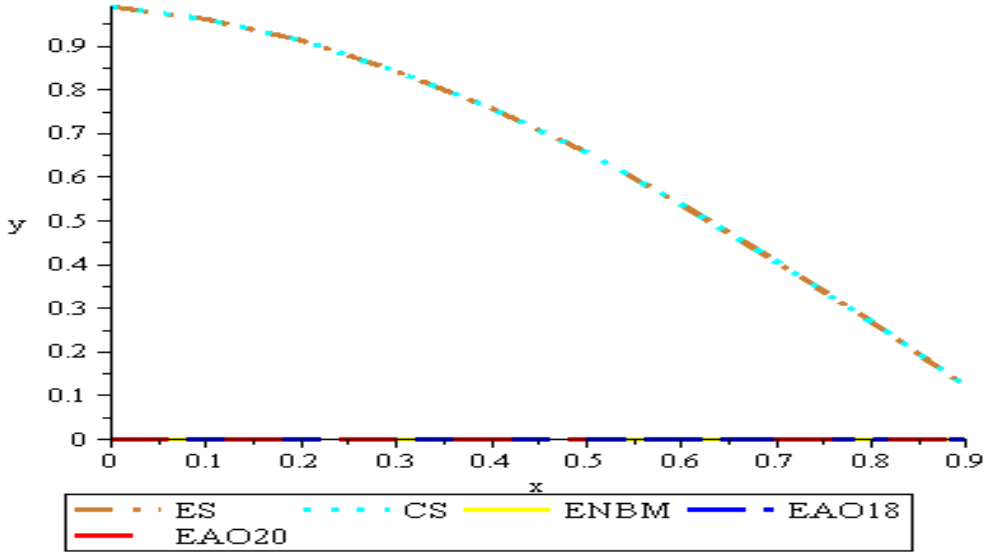


Figure 3: Graphical curve of problem 4.3

4.1 Linear Stability Analysis

Based on the results presented in Tables 1 to 3 and Figures 1 to 3, the newly developed block method demonstrates strong computational reliability. As shown in Table 1, the method provides a direct and more accurate solution for problem one than the existing approaches of Sabo, Kyagya & Vashawa (2021) and Skwame *et al.* (2017) without requiring order reduction. For the third-order oscillatory problems in problems 4.2 and 4.3, direct simulation with the proposed block method achieves faster convergence, as indicated by the results in Tables 2 and 3. Its performance surpasses that of methods by Adoghe and Omole (2019), Aigbiremhon and Omole (2020), Blanka (2019), and Tumba *et al.* (2021). The graphical comparisons in Figures 1 to 3 visually corroborate this superior accuracy relative to the considered existing methods. In summary, the tabular and graphical data collectively affirm the simplicity and efficacy of the new block method for the direct simulation of second and third-order oscillatory differential equations. The method exhibits faster convergence rates and a demonstrated proficiency in handling stiff problems. This comparative analysis confirms its capability to deliver precise numerical solutions, thereby reinforcing the method's overall reliability and effectiveness for this class of Problems

5. Conclusion

Based on the numerical results and theoretical analysis, the proposed block method demonstrates exceptional accuracy and computational reliability for directly solving second and third-order oscillatory differential equations, achieving near-machine-precision errors and exhibiting consistent, zero-stable, convergent behavior with a substantial region of absolute stability. Despite these strengths, the method's block formulation and higher order introduce greater implementation complexity and increased computational cost per step compared to simpler sequential approaches, as solving a small system of equations is required at each block. Future work will focus on extending the method to higher-order systems and nonlinear oscillatory problems, developing a variable step-size version to improve efficiency, and performing a more exhaustive stability analysis for non-linear applications, thereby broadening the method's practical utility in engineering and physical sciences.

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References

Adebayo, O. and Adebola, E. O. (2016). One step hybrid method for the numerical solution of general third order ordinary differential equations. *International Journal of Mathematical Sciences*, **2**(5), 1–12.

Adesanya, A. O., Fasansi, M. K. and Odekunle, M. R. (2013). One step three hybrid block predictor-corrector method for the solution. *Applied and Computational Mathematics,* **2**(4), 137. https://doi.org/10.4172/2168-9679.1000137

Adewale, A. J. and Sabo, J. (2024). Simulating the dynamics of oscillating differential equations of mass in motion. *International Journal of Development Mathematics*, **1**(1), 54–69. https://doi.org/10.62054/ijdm/0101.07

Adeyefa, E. O. and Olagunju, A. S. (2021). Hybrid block method for direct integration of first, second and third order IVPs. *Çankaya University Journal of Science and Engineering*, **18**(1), 1–8.

Adeyeye, O. and Omar, Z. (2018). New self-starting approach for solving special third order initial value problems. *International Journal of Pure and Applied Mathematics,* **118**(3), 511–517.

Adoghe, L. O. and Omole, E. O. (2019): A fifth-fourth continuous block implicit hybrid method for the solution of third order initial value problems in ordinary differential equations. *Applied and Computational Mathematics*, **8**(3), 50–57.

Ahmedai Abd Allah, S. A. A., Sibanda, P., Goqo, S. P., Rufai, U. O., Sithole Mthethwa, H. and Noreldin, O. A. I. (2024). A block hybrid method with equally spaced grid points for third-order initial value problems. *Applied Math,* **4**(1), 320–347. https://doi.org/10.3390/appliedmath4010017

Aigbiremhon, A. A. and Omole, E. O. (2020). A four-step collocation procedure by means of perturbation term with application to third-order ordinary differential equation. *International Journal of Computer Applications,* **175**(24), 25–36.

Aloko, M. D., Ayinde, M. A., Usman, A. D. and Sabo, J. (2024). An efficient scheme for direct simulation of third and fourth oscillatory differential equations. *International Journal of Development Mathematics*, **1**(1), 72–89. https://doi.org/10.62054/ijdm/0101.06

Awoyemi, D. O., Kayode, S. J. and Adoghe, L. O. (2014). A five-step P-stable method for the numerical integration of third order ordinary differential equations. *American Journal of Computational Mathematics*, **4**(3), 119–126. https://doi.org/10.4236/ajcm.2014.43011

Ayinde, M. A., Ishaq, A. A., Latunde, T. and Sabo, J. (2021). Efficient numerical approximation methods for solving high-order integro-differential equations. *Caliphate Journal of Science & Technology,* **2**, 188–195.

Ayinde, A. M., Ibrahim, S., Sabo, J. and Silas, D. (2023). The physical application of motion using single step block method. *Journal of Material Science Research and Review,* **6**(4), 708–719.

Blanka, B. (2019). Oscillation of second-order nonlinear non-canonical differential equations with deviating argument. *Applied Mathematics Letters,* **91**, 68–75.

Duffy, B. R. and Wilson, S. K. (1997). A third order differential equation arising in thin-film flows and relevant to Tanner's law. *Applied Mathematics Letters,* **10**(3), 63–68.

Genesio, R. and Tesi, A. (1992): Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. *Automatica*, **28**(3), 531–548.

Ishaq, A. A., Ayinde, M. M., Sabo, J., Ibrahim, S. and Jaiyeola, B. S. (2024). Three-step collocation method for solution of third derivative initial value problem. *International Journal of Development Mathematics*, **1**(1), 25–36. https://doi.org/10.62054/ijdm/0101.03

Kuboye, J. O. and Omar, Z. (2015): Numerical solution of third order ordinary differential equations using a seven-step block method. *International Journal of Mathematical Analysis,* **9**(15), 743–754.

Maduabuchi, G. O., Sicelo, G. and Sandile, M. (2021):. An optimized two-step block hybrid method with symmetric intra-step points for second order initial value problems. *Engineering Letters*, **29**(3), 1–9.

Olabode, B. T. and Momoh, A. L. (2016). Continuous hybrid multistep methods with Legendre basis function for direct treatment of second order stiff ODEs. *American Journal of Computational and Applied Mathematics*, **6**(2), 38–49. https://doi.org/10.5923/j.ajcam.20160602.03

Sabo, J., Kyagya, T. Y. and Vashawa, W. J. (2021). Numerical simulation of one step block method for treatment of second order forced motions in mass spring systems. *Asian Journal of Research and Reviews in Physics*, **5**(2), 1–11.

Skwame, Y., Bakari, A. I. and Sunday, J. (2017): Computational method for the determination of forced motions in mass-spring systems. *Asian Research Journal of Mathematics*, **3**(1), 1–2.

Skwame, Y., Donald, J. Z., Kyagya, T. Y., Sabo, J. and Bambur, A. A. (2020). The numerical applications of implicit second derivative on second order initial value problems of ordinary differential equations. *Dutse Journal of Pure and Applied Sciences*, **6**(4), 1– 14.

Sunday, J. (2018). On the oscillation criteria and computation of third order oscillatory differential equations. *Communications in Mathematics and Applications*, **9**(4), 615–626.

Sunday, J., Joshua, K. V. and Danladi, N. (2019). On the derivation and implementation of a one-step algorithm for third order oscillatory problems. *Adamawa State University Journal of Scientific Research,* **7**(1), 8–20.

Tumba, P., Skwame, Y. and Raymond, D. (2021). Half-step implicit linear hybrid block approach of order four for solving third order ordinary differential equations. *DUJOPAS,* **7**(2b), 124–133.