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# EARLY COEFFICIENTS OF CLOSE-TO-STAR FUNCTIONS OF TYPE $\alpha$

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ABSTRACT. We obtain sharp bounds on the early coefficients of certain close-to-star analytic functions of type $\alpha$ in the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$.

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## 1. INTRODUCTION

Let $A$ be the class of functions of the form:

$$f(z) = z + a_2z^2 + \cdots$$

which are analytic in the unit disk $E = \{z \in \mathbf{C}\colon |z| < 1\}$. A function $f \in A$ is said to be close-to-star if there exists a starlike functions $g(z) = z + b_2z^2 + \cdots$ such that:

$$\operatorname{Re} \frac{f(z)}{g(z)} > 0, \quad z \in E. \tag{1}$$

The concept of close-to-starlikeness was first introduced by Reade (see [5]). Close-to-star functions are not necessarily univalent in the open unit disk. However, they bear close relations to close-to-convex functions ($\operatorname{Re} f'(z)/g'(z) > 0$, $g$ is convex) similar to those which exist between the classes of starlike and convex functions. For instance, the well known Alexander theorem ($f$ is convex if and only if $zf'$ is starlike) hold in similar manner between close-to-convex and close-to-star functions, that is, $f$ is close-to-convex if and only if $zf'$ is close-to-star. Also by choosing $f = g$ (self-star or self-convex), it can be easily seen that every convex function is close-to-convex and similarly that every starlike function is close-to-star.

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Furthermore, by a simple application of a result of Miller and Mocanu [3], it can be shown that Re $f'(z)/g'(z) > 0$ implies Re $f(z)/g(z) > 0$ and since every convex function is starlike it follows that there exists a star map $g$ such that (1) is satisfied. It is thus easily verified that close-to-convex functions are close-to-star in the open unit disk.

In this work we use the notion of Bazilevic functions [6] to introduce the class of close-to-starlike maps of type $\alpha$ in $E$. We say: let $\alpha$ be a nonzero positive real number and suppose there exists a starlike map $g(z)$ such that:

$$\text{Re}\,\frac{f(z)^\alpha}{g(z)^\alpha} > 0, \quad z \in E \tag{2}$$

then $f(z)$ is said to be close-to-starlike of type $\alpha$ in $E$. Powers in (2) are meant as principal determinations only. The geometric condition (2) implies that $f(z)^\alpha/g(z)^\alpha$ belongs to the class $P$ of analytic functions:

$$p(z) = 1 + c_1 z + \cdots$$

which have positive real part in $E$.

We shall denote this class of functions by $C_\alpha^*$. As already noted, type $\alpha$ close-to-star functions are not necessarily univalent. Our concern in the present work is the determination of sharp bounds on the early coefficients of functions of the class $C_\alpha^*$. Our results are presented in Section 3. We state the needed lemmas in the next section.

## 2. PRELIMINARY

In our proof, we shall depend on the well known inequalities (namely the Caratheodory lemma and coefficient functionals for starlike functions):

$$|c_n| \le 2, \quad n \ge 1$$

$$|b_n| \le n, \quad n \ge 2$$

$$|b_3 - \lambda b_2^2| \le 3 - 4\lambda, \quad \lambda \le \frac{3}{4}$$

($\lambda$ real) and the following lemma.

**Lemma 1** [1]. *Let $p \in P$. Then*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| = \begin{cases} 2(1-\sigma) & \text{if } \sigma \le 0, \\ 2 & \text{if } 0 \le \sigma \le 2, \\ 2(\sigma - 1) & \text{if } \sigma \ge 2. \end{cases}$$

The above lemma is a consequence of the well known Caratheodory-Toeplitz inequality $|c_2 - \frac{1}{2}c_1^2| \leq 2 - \frac{1}{2}|c_1|^2$.

Next we state and prove the main result.

## 3. MAIN RESULTS

**Theorem 1:** Let $f \in C_\alpha^*$. Then

$$|a_2| \leq 2 + \frac{2}{\alpha}$$

$$|a_3| \leq \begin{cases} 3 + \frac{1}{\alpha^2}(2+4\alpha) & \text{if } 0 < \alpha \leq 1, \\ 3 + \frac{6}{\alpha} & \text{if } \alpha \geq 1. \end{cases}$$

and

$$|a_4| \leq \begin{cases} \frac{4}{3}(5 + 3\alpha + \frac{3}{\alpha} + \frac{1}{\alpha^2}) & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ 4(1 + \alpha + \frac{2}{\alpha}) & \text{if } \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

**Proof:** Since $f \in C_\alpha^*$ there exist a Caratheodory functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ such that $f(z)^\alpha / g(z)^\alpha = p(z)$ so that

$$\frac{f(z)^\alpha}{z^\alpha} = \frac{g(z)^\alpha}{z^\alpha} p(z).$$

Expanding both sides in series form we have

$$\begin{aligned}\frac{f(z)^\alpha}{z^\alpha} &= 1 + \alpha a_2 z + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2} a_2^2\right) z^2 \\ &\quad + \left(\alpha a_4 + 2\frac{\alpha(\alpha-1)}{2} a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} a_2^3\right) z^3 + \cdots\end{aligned}$$

and

$$\begin{aligned}\frac{g(z)^\alpha}{z^\alpha} p(z) &= 1 + (c_1 + \alpha b_2) z + \left(c_2 + \alpha(b_2 c_1 + b_3) + \frac{\alpha(\alpha-1)}{2} b_2^2\right) z^2 \\ &+ \left(c_3 + \alpha(b_4 + b_3 c_1 + b_2 c_2) + \frac{\alpha(\alpha-1)}{6}(6 b_2 b_3 + 3 b_2^2 c_1 + (\alpha-2) b_2^3)\right) z^3 \\ &+ \cdots\end{aligned}$$

Comparing coeficients we obtain

$$\alpha a_2 = c_1 + \alpha b_2 \tag{3}$$

$$\alpha a_3 = c_2 + b_2 c_1 + \alpha b_3 + \frac{1-\alpha}{\alpha}\frac{c_1^2}{2} \tag{4}$$

$$\alpha a_4 = c_3 + b_3 c_1 + b_2 c_2 + \alpha b_4 + \frac{1-\alpha}{2\alpha} b_2 c_1^2 + \frac{1-\alpha}{\alpha} c_1 c_2 + \frac{(1-\alpha)(1-2\alpha)}{6\alpha^2} c_1^3$$

Thus using the inequalities $|c_n| \le 2$ and $|b_n| \le n$ and the triangle inequality, we have the bound on $a_2$ as in the theorem.

As for $a_3$, suppose $\alpha$ lies in the half-open interval $(0, 1]$. Then by triangle inequality we have

$$\alpha|a_3| \le |c_2| + |b_2||c_1| + \alpha|b_3| + \frac{1-\alpha}{\alpha}\frac{|c_1^2|}{2}$$

which again by the inequalities $|c_n| \le 2$ and $|b_n| \le n$ yields the first bound for $a_3$ as stated. If $\alpha \ge 1$, then we write

$$\alpha|a_3| \le |b_2||c_1| + \alpha|b_3| + \left|c_2 - \frac{\alpha-1}{\alpha}\frac{c_1^2}{2}\right|$$

so that by Lemma 2 (taking $\sigma = \frac{\alpha-1}{\alpha}$) we obtain the second inequality for $a_3$.

Next to $a_4$. Suppose that $\alpha$ lies in the half-open interval $(0, \frac{1}{2}]$. Then applying the inequalities $|c_n| \le 2$ and $|b_n| \le n$ and the triangle inequality, we have the first part of the bounds on $a_4$ as in the theorem. For $\alpha$ in the closed interval $[\frac{1}{2}, 1]$, then we rewrite $a_4$ such that

$$\alpha|a_4| = |c_3| + |b_3||c_1| + |b_2||c_2| + \alpha|b_4| + \frac{1-\alpha}{2\alpha}|b_2||c_1|^2 + \frac{1-\alpha}{\alpha}|c_1|\left|c_2 - \frac{2\alpha-1}{3\alpha}\frac{c_1^2}{2}\right|.$$

Hence, applying the inequalities $|c_n| \le 2$ and $|b_n| \le n$ and Lemma 2 with $\sigma = \frac{2\alpha-1}{3\alpha}$, we obtain the second inequality for $a_4$ as required.

Next we establish a sharp bound on the Fekete-Szego functional $|a_3 - \lambda a_2^2|$ for the class $C_\alpha^*$.

**Theorem 2:** Let $\lambda \le \frac{1}{2}$ be a real number. Then for any $f \in C_\alpha^*$,

$$|a_3 - \lambda a_2^2| \le 9 - 12\lambda + \frac{2}{\alpha^2}(1 - \alpha - 2\lambda).$$

**Proof:** From (3) and (4) we find that

$$a_3 - \lambda a_2^2 = b_3 - \lambda b_2^2 + (1-2\lambda)b_2c_1 + c_2 + \frac{1-\alpha-2\lambda}{\alpha^2}\frac{c_1^2}{2}. \tag{5}$$

Now suppose $1 - \alpha - 2\lambda \ge 0$, that is $\lambda \le (1-\alpha)/2$. Then applying the triangle inequality and the inequalities $|c_n| \le 2$ and $|b_3 - \lambda b_2^2| \le$

$3-4\lambda$ for real $\lambda \leq \frac{3}{4}$ we obtain the bound as desired. Next we suppose $(1-\alpha)/2 \leq \lambda \leq 1/2$. Then from (5) we write

$$|a_3 - \lambda a_2^2| \leq |b_3 - \lambda b_2^2| + (1-2\lambda)|b_2||c_1| + \left| c_2 - \frac{2\lambda + \alpha - 1}{\alpha^2} \frac{c_1^2}{2} \right|$$

so that with Lemma 1 and the basic inequalities we again arrive at the desired bound for the functional.

## 4. CONCLUDING REMARKS

We note that bounds on higher coefficients of functions of the class $C_\alpha^*$ may not be tractable via the simple approach contained herein. Interested reader may wish to consider a technique due Nehari and Netanyahu [4] as employed in [2, 6].

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## REFERENCES

[1] K. O. Babalola, *On the coefficients of a certain class of analytic functions*, Advances in Inequalities for Series(Edited by S. S. Dragomir and A. Sofo), Nova Science Publishers, 1-13, 2008.

[2] K. O. Babalola, *Bounds on the coefficients of certain analytic and univalent functions*, Mathematica (Cluj), Tome **50** (73), No. 2, 139-148, 2008.

[3] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65**, 289-305, 1978.

[4] Z. Nehari and E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1), 15-23, 1957.

[5] Y. O. Park and S. Y. Lee, *On a class of strongly close-to-star functions*, Bull. Korean Math. Soc, **37** (4), 755–764, 2000.

[6] R. Singh, *On Bazilevic functions*, Proc. Amer. Math. Soc. **38**, 261–271, 1973.

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