MAGNETOHYDRODYNAMIC HEAT AND MASS TRANSFER OF NON-NEWTONIAN FLUIDS FLOW THROUGH VERTICAL PLATE IN THE PRESENCE OF THERMO-PHYSICAL PARAMETERS

FALODUN, Bidemi Olumide

(16/68 EW 004)

APRIL, 2021.

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B.Sc. (EKSU) 2011, M. Tech (FUTA) 2016

BEING A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, FACULTY OF PHYSICAL SCIENCES, UNIVERSITY OF ILORIN, ILORIN, NIGERIA,

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY (Ph.D.) DEGREE IN MATHEMATICS.

APRIL, 2021

CERTIFICATION

This is to certify that the research work reported in this thesis was done by FALODUN, Bidemi Olumide with Matriculation number 16/68EW004 in the Department of Mathematics, University of Ilorin, Ilorin, Nigeria, for the award of a Doctor of

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DEDICATION

This research work is dedicated to the Almighty God, the author and giver of wisdom.

DECLARATION

I, Bidemi Olumide, FALODUN, a Ph.D. student in the Department of Mathematics, University of Ilorin, Ilorin, hereby declare that this thesis entitled ”Magnetohydrodynamics heat and mass transfer of non-Newtonian fluids flow through vertical plate in the presence of thermo-physical parameters”, submitted by me is based on my actual and original work. Any materials obtained from other sources or work done by any other persons or institutions have been duly acknowledged. In addition, the research has been approved by the University of Ilorin Ethical Review Committee.

....

Bidemi Olumide, FALODUN Date

(16/68 EW 004)

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ABSTRACT

Fluid is a substance that deforms continuously when subjected to shear stress. Such fluid can either be Newtonian or non-Newtonian. The Newtonian fluid obeys the law of viscosity while non-Newtonian fluid does not obey the law of viscosity. The present study is concentrated on non-Newtonian fluids. The study of non-Newtonian fluids attracted the attention of numerous researchers in the field of fluid dynamics due to its rheological applications in mechanical and chemical engineering processes. Fluids of this type are generally complex and are considered under science of deformation and flow. Non-Newtonian fluids are applicable in the movement of biological liquids, food processing, liquid cosmetics, dyes, lubricants and puncturing sludge are few examples of rheological fluids. This study aimed at examining magnetohydromagnetic (MHD) heat and mass transfer of non-Newtonian fluids flow through vertical plate (both porous and non-porous) in the presence of thermo-physical parameters using spectral methods. The equations governing the study are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu_b(T)}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial u}{\partial y}\frac{\partial \mu_b(T)}{\partial T}\frac{\partial T}{\partial y}$$
$$-\frac{k_0}{\rho}\left(v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x \partial y^2} - 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)$$
$$+g\beta_t(T-T_\infty) + g\beta_c(C-C_\infty) - \frac{\sigma B_0^2}{\rho}u - \frac{\mu_b}{k\rho}\left(1+\frac{1}{\beta}\right)u \quad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho c_p}\frac{\partial T}{\partial y}\frac{\partial k(T)}{\partial y} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{\mu_b(T)}{\rho c_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2$$
$$+\frac{Q_0}{\rho c_p}(T-T_\infty) + \frac{Dk_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \quad (3)$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - k_l(C-C_\infty) - \frac{\partial}{\partial y}(V_T C) + \frac{Dk_T}{T_m}\frac{\partial^2 T}{\partial y^2} \quad (4)$$

together with the boundary conditions

$$u = Bx, v = -v(x), T = T_w, C = C_w \quad , at\ y = 0 \quad (5)$$

$$u \longrightarrow 0, T \longrightarrow T_\infty, C \longrightarrow C_\infty, as\ y \longrightarrow \infty \quad (6)$$

u and v represents the relations $u = {}^{\partial\psi}{}_{\partial y}$ and $v = -{}^{\partial\psi}{}_{\partial x}$. In the definition of u and v, $\psi(x,y)$ is the stream function which automatically satisfies the continuity equation

(1).

The objectives of the study are:

(i) to examine the physics of the problem of heat and mass transfer non-Newtonian fluids flow in a semi-infinite vertical plate and vertical porous plate;

(ii) to examine the influence of pertinent flow parameters such as magnetic parameter, radiation parameter, heat generation parameter and so on, on the flow of non-Newtonian fluids within the boundary layer regime;

(iii) to determine the physical quantities of engineering interest such as skin friction, Nusselt number and Sherwood number on all flow parameters; and

(iv) to test the accuracy and validity of the spectral relaxation method and spectral homotopy analysis method.

Scholars in the field of fluid dynamics now consider spectral methods as essential tools in solving highly coupled non-linear differential equations. Spectral methods involve approximating the unknown functions using truncated series of orthogonal functions or polynomials. The spectral relaxation method (SRM) employed the concept of Gauss-Seidel method to decouple system of differential equations. The spectral homotopy analysis method (SHAM) is based on a blend of the Chebyshev pseudospectral method with the homotopy analysis method. To apply SHAM on differential equations, the domain $[0,L]$ of the problem is first transformed to the domain $[-1,1]$. The partial differential equations which govern the model were simplified with the help of appropriate similarity variables and non-dimensional quantities. The transformed non-linear coupled ordinary differential equations along with the boundary conditions were solved numerically using spectral methods. The results obtained are as follows:

(i) It was discovered that as the value of the viscoelastic parameter increases, the velocity profile close to the plate gradually decrease while far away from the plate, it slightly increase;

(ii) The result revealed that variable viscosity and thermal conductivity greatly affects the fluids within the boundary layer as it enhances velocity and temperature

respectively;

(iii) It was found that the effects of Soret and Dufour on the temperature and concentration profiles are opposite;

(iv)It was found that the results obtained were useful in food processing, drilling operations and bioengineering. Also, Soret parameter on the fluid flow is significant in isotope separation;

(v) It was found that increase in the thermal Grashof number drastically increase the hydrodynamic boundary layer thickness; and

(vi) The numerical methods used in this study were found useful in solving highly non-linear differential equations in science and engineering.

In this study, it can be concluded that the momentum and thermal boundary layer thickness drastically increase with increase in Casson parameter and viscoelastic non-Newtonian fluid. Also, it can be concluded that Soret and Dufour parameter simultaneously affect the hydrodynamic boundary layer thickness. It is recommended that the findings of this investigation be used in plasma studies, geothermal energy extractions, generators and control of boundary layer in the field of aerodynamics. The method used in the investigation is recommended for solving highly non-linear differential equations in engineering.

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Nomenclature

English Alphabet

C-dimensional concentration D-mass diffusivity g-acceleration due to gravity (Unit: m/s^2) k-porosity term u, n-velocity components in x-direction (Unit:m/s) v, w-velocity components in y-direction

(Unit:m/s) $k(T)$-thermal conductivity (Unit:W/mK)

Greek alphabets

$alpha$-fluid thermal diffusivity μ-coefficient of
viscosity (Unit:Kg/ms) ρ-fluid density
(Unit:Kg/m^3) σ-electrical conductivity of the fluid
(Unit:$m\Omega/m$) ν-viscosity (Unit:m^2/s)

$\mu_b(T)$-dynamic viscosity with respect to temperature (Unit:Kg/ms)
Φ-inclined angle (Unit:degree) θ-fluid temperature (Unit:K) φ-
fluid concentration β_t-thermal expansion coefficient (Unit: K^{-1}) β_c-
concentration expansion coefficient φ_∞-free stream concentration
θ_∞-free stream temperature (Unit:K) ψ-stream function
(Unit:$m^{-2}s^{-1}$) **Subscripts** c_p-specific heat at constant pressure
(Unit:J/Kgk) k_T-thermal diffusion ratio

T_m-mean fluid temperature (Unit:K)

T_∞-free stream temperature (Unit:K)
C_∞-free stream concentration

B_0-externally imposed magnetic field strength in y-direction (Unit:Wb/m^2)
q_r-radiative heat flux (Unit:W/m^2) c_s-concentration susceptivity

U_0-the scale of free stream velocity

T_w-wall temperature (Unit:K)

C_w-concentration at the wall

K_0-viscoelastic term

D_B-Brownian coefficient (Unit:m^2/s) D_T-
thermophoresis coefficient (Unit:m^2/s) k_l-chemical
reaction parameter

V_T -thermophoretic velocity **Superscripts** u^0-
velocity components in x^0-direction (Unit:m/s)
v^0-velocity components in y^0-direction (Unit:m/s)
t^0-time (Unit: s) n^0- constant

k_r'-chemical reaction parameter

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CHAPTER ONE: GENERAL INTRODUCTION

1.1 Background information to the study

The concept of fluid is experienced in everyday life. The air we breath in and out, the rivers and seas in our environment, the water that flows in our tap at home, and so on are all phenomena of fluid. All this phenomena are capable of moving. Their study is called fluid mechanics. When fluid flow is under the action of forces such as buoyancy and drag, it continuously deforms as long this force is applied. The major fluid forces of interest in this study is the buoyancy and drag force. The drag force slows down the fluid motion by acting in a direction that is opposite to the relative flow velocity while buoyancy force describes how a body floats. The deformation of fluids is by shearing forces. In as much the elastic limit of a solid is not exceeded, the strain is a function of the applied stress. This strain is independent of time when force is applied and in as much as the elastic limit is not exceeded, deformation will disappear when force does not exist again. On the other hand, the rate of strain of a fluid is proportional to the applied stress. The flow of a fluid keeps moving in as much there is an applied force which does not recover the original shape the moment the applied force is removed (Douglas et al., 2005). The science of flow of gas or liquid is classified as hydraulics and hydrodynamics. Hydraulics is when water is at rest or motion. Fluids are divided into liquids and gases. A liquid substance is hard to compress and are regarded as incompressible under external pressure. The two major characteristics of a fluid from its study are its compressibility and viscosity. All fluids possess viscosity. Any fluid that does not have viscosity are classified to be an ideal or perfect fluids. In a real sense, such fluids does not exist. However, any fluid with low viscosities are considered to be ideal fluid under certain conditions.

Heat is transmitted from one object to another whenever a temperature difference exists in a medium or between media. When an object is hot, it possesses lots of energy but has less when its cold. Heat is transferred by the means of conduction, means of convection and means of radiation. Heat transfer that involves transportation of energy in-between parts of continuum by kinetic energy between particles groups at atomic level. Convection involves energy transfer by the movement of fluid and molecular diffusion. Thermal radiation involves the emission of energy into all bodies by the process of electromagnetic radiation (Anyakoha, 2010). If human beings is inside a room, the warm air buoys off the body to warm the room. Immediately the person decides to leave the room, some small buoyancy-driven flows of the air continues owing the walls can not be isothermal perfectly.

Heat gets to the earth from the sun by radiation because sunlight is part of thermal radiation that is generated by the hot plasma of the sun. In addition, when human beings are very close to fire, the heat reaches the body by radiation. Hence, radiation does not require any material medium but conduction and convection requires a material medium. Anyakoha (2010) explains that convection involves heat transfer by the actual movement of the molecules that are heated from the region that is hot to cooler region of fluid. Mohammed (2014) states that thermal radiation effect becomes important when the surface temperature and ambient temperature difference is very large. Roseland approximation define heat flux added to the energy equation when the fluid is optically thick. The radiation parameter is the reciprocal of Stephan number which measures the usefulness of thermal radiation to conduction heat transfer. Hence, larger value of radiation parameter indicates the amount of heat energy produced by radiative flux is large and it is injected into a system resulting to an enhancement in the fluid temperature.

In polymer industries, fluids used have a viscosity which varies rapidly with temperature. As a result of the Navier-Stokes equation coupled with the energy equation, viscous dissipation and thermal radiation plays a significant role in fluids with strong temperature dependence (Bird et al., 1960). A practical example is the behaviour of cooking oil and how it moves in a frying pan when heated by cooking stove gas. In the same vein, radiation have numerous usefulness in sciences and engineering especially in space technology and polymer processing industry. In view of these applications, Olanrewaju (2012) reported similarity solution for natural convection by taking effects of viscous dissipation and thermal radiation into consideration. Researchers have shown interest in the analysis that involves a varied conductivity alongside viscosity contribution on fluid flow. The analysis of Salawu and Dada (2016) heat transfer analysis of varied viscosity alongside thermal conductivity on inclined magnetism was solved numerically. The phenomena of viscous dissipation is an irreversible process in which the required energy to deform the behaviour of fluid flow is simplified into heat energy. It involves the transformation of mechanical energy into heat energy. According to Gireesha et al. (2012), increase in viscous dissipation gives rise to the rate of heat transfer.

In the last few decades, efforts have been devoted to the analysis of combined heat plus mass transfer effects on fluid flow due to their applications in geophysics, astrophysics, engineering and physics. Heat plus mass transfer process of chemical reaction is either heterogeneous or homogeneous. Chemical reaction effect on fluid flow is based on nature. According to Manglesh and Gorla (2012), a chemical reaction will be of first order in as much reaction rate is proportional to concentration. Naturally, existence of pure water or water is impossible but foreign mass could be mixed naturally with air or water. The study of heat and mass transfer is useful for many chemical technologies in food processing, production of polymer and in the manufacturing of ceramics or glassware. According to Srinivasacharya et al. (2018), the double diffusive steady flow is due to combined action of convection and molecular dispersion.

1.2 Statement of the problem

This research considered the analysis of MHD heat with mass transport. Pertinent flow parameters is presented using graphs and the engineering interest quantities are computed. The number of problems examined are:

(i) Effects of thermo-physical properties heat with mass transport of MHD a viscoelastic liquid past a half-infinite movable plate.
(ii) Contribution of varying viscosity and thermal conductivity of non-Newtonian liquids flow in a porous vertical plate with Soret-Dufour effects; and.

(iii) Thermophoresis, Soret-Dufour effects on convective motion of non-Newtonian nanofluids past an inclined plate situated in a penetrable medium.

All the problems above were solved numerically using spectral methods. This study test the accuracy and spectral relaxation and spectral homotopy analysis techniques were validated for solving model fluid flow problems in the field of fluid mechanics.

1.3 Justification of the study

This study shows that many areas have been neglected in previous works and the present study examined some of these areas. For instance, this study elucidate combined Soret-Dufour influence in heat with mass transfer processes. Soret and Dufour terms are studied as a second order phenomena because of its small order of magnitude as prescribed by Fick's and Fouriers' laws. Because of this, they have often been forgone in the problem of heat with mass transport. According to Eckert and Drake (1972), these effects are significant in a clear fluids. They are of great importance because of their vital role in science and engineering like Soret in the separation of isotopes. Furthermore, few works have been conducted on influence effects of parameters like thermophoresis, radiation, heat generation around the boundary layer region. Hence, this study aimed at examining the effects of all flow parameters. Also, many researchers in the past have assumed that viscosity and thermal conductivity is constant in their exploration. This is not realistic because the viscous and thermal conductivity of any fluid changes as it collides together during movement. It worth noting that higher temperature which will eventually lead to enhancement in the transport phenomenon by reducing the viscosity and thermal conductivity within the momentum and thermal boundary layer. Thus, the viscosity and thermal conductivity elucidated in this study as varied quantities.

1.4 Aim and objectives of the study

The aim of this study is to explore magnetohydrodynamics heat with mass transport of non-Newtonian liquids flow past a vertical plate with thermo-physical parameters. The objectives of the study are:

(i) to examine the physics of the problem of heat with mass transfer non-Newtonian fluids model in a half-infinite vertical plate and vertical porous plate;

(ii) to examine the effect of flow parameters such as magnetic parameter, radiation parameter, heat generation parameter, etc on the non-Newtonian fluids model within the boundary layer regime;

(iii) to elucidate the effects of flow parameters on quantities of engineering interest; and

(iv) to elucidate how accurate and valid of spectral relaxation and spectral homotopy analysis methods are.

1.5 Definition of terms

(i) Viscosity: Viscosity is a property that is very important in the existence of a fluid. It determines opposition to shearing stresses. It is used in determining the internal friction of the that resist fluid flow. Its existence in a fluid is as a result of cohesion and interaction between fluid particles. All fluids are viscous except where the viscosity effects are minimal. When the viscosity of a fluid is minimal, it is called an ideal fluid. Fluids are generally treated as a fluid that posses viscosity in order to examine loss of pressure as a result of a flow, drag acting on the flow where the body separates. The S.I. units of viscosity is Ns/m^2. Kinematic viscosity defines the ratio of dynamic viscosity to fluid density. It is denoted by ν and mathematically written as $\nu = {}^{dynamicviscosity}{}_{density} =$

$\frac{\mu}{\rho}$. Its S.I. units is $m\ ^2/s$. The Newton's law as regards viscosity states that shear stress (τ) on the fluid element layer is proportional to rate of shear strain, that is$\tau \propto \frac{du}{dy}$. Any fluid which obey Newtons law of viscosity is a Newtonian fluid (Gebhart, 1962).

(ii) Newtonian and Non-Newtonian fluids: Newtonian fluids are fluid with shearing stress τ proportional to the velocity gradient$\frac{du}{dy}$. For this kind of fluid, the dynamic viscosity (i.e the constant of proportionality called coefficient of viscosity) does not make any difference compare to rate of deformation. Examples are water, kerosene, oil or air. All the examples obeys the law guiding the concept of viscosity because they are not in agreement with the linear relationship shear stress rate of deformation. Such fluids are blood, mud flows liquid pulp and polymer solutions.

Non-Newtonian fluids does not conform with the law of viscosity. Such fluids are liquid pulp, mud flows, polymer solutions, blood etc. They are classified by rheological diagram. Their mechanical behaviour is in science of deformation and flow of substance due to complexity. It has numerous importance in industry and technology. Because they are complex, no constitutive equation which exibits such properties of non-Newtonian fluids. Power law and grade two or three models are simple fluid models because of their shortcomings whose fluid flow is not realistic. The Power-law model is applicable in modeling shear-dependent viscosity fluids but does not predict the effects of elasticity while grade two or three fluids gives details to the elasticity effects and has non shear dependent viscosity. Maxwell non-Newtonian model is used in predicting stress relaxation. Examples of Casson non-Newtonian fluid model are soup, tomato sauce, Jelly, honey, concentrated Juice etc.

The blood of human is also treated as Casson fluid because of the presence of fibrinogen, protein and globulin in an aqueous plasma base, red blood cell of human can form a structure known as aggregator rouleaux. Consider the aggregates or rouleaux acts in a plastic solid, manner and yield stress is present and identified yield stress that is constant in Casson fluid (Fung; 1984). Viscoelastic fluids are also non-Newtonian. Their constitutive equations usually result in higher-order derivative terms in the momentum equations. These make it difficult to solve in comparison with Newtonian fluids (Tonekaboni et al., 2012). These kinds of fluids are hereby solved using some analytical and numerical methods such as homotopy analysis method and spectral methods. Walters in 1962 proposed the Walters-B fluid to elucidate fluids with viscosity with short memory elastic effects and capable of stimulating accurately many complex polymeric, biotechnological and tribiological fluids (Frank, 1990).

(iii) Steady and Unsteady flow: Steady flow is the flow that all characteristics of fluid flow state such as density, pressure, velocity and so on do not change with time. However, unsteady flow is the flow that all characteristics of fluid flow state remain constant with time. A practical example is the water tap we use daily. When water is coming out of the tap when the handle is turned, it is an unsteady flow but when the opening is left constant, it is a steady flow (Robert and Murray, 2002).

(iv) Laminar and Turbulent flow: According to Reynolds, in 1883, what distinguish between laminar and turbulent flows is the reynold number. The Reynolds number is employed to characterize a flow and used to examine whether a flow is laminar or turbulent. An example is whenever a tap is open a little with a low velocity, there is a smooth flowing of water with surface in a laminar state. Increase in the tap velocity create an opaque with a rough surface flowing of water in a turbulent state of flow.

When the fluid particle in a stream is tampered with, at inertia there is a flow in a new direction and the forces produces due to surrounding fluid which helps it to conform the stream motion of the rest while it is accurate in turbulent flow. Effects of any deviation is eliminated by the viscous shear stress. The ratio of the inertial force to the force of viscous acting on the fluid particle

(Douglas et al.,2005).

(v) Incompressible and Compressible fluids: An incompressible fluid is liquid while compressible fluid is gas. Whenever a liquid is highly pressurized, compressibility is put into consideration in the case of liquid. A typical example is oil in a hydraulic machine. On the other hand, the compressibility in a gas could be disregarded when change in pressure is small (Frank, 1990).

(vi) Compressible and Incompressible flows: Compressible flows are those type of flow that has a non constant density (ρ) i.e ρ 6= *constant*. On the other hand, the type of flow that has density being constant is incompressible flow. Fluids that flows incompressibility are generally liquids (Robert and Murray, 2002).

(vii) Boundary layer: Prandtl divide the idea behind a viscous flow into region near the wall affected by friction and region outside the unaffected. The first layer is the one at the boundary and main region where the fluid flows is the second. However, boundary layer is the layer of a fluid where contribution of viscosity is important.

The boundary layer approximation explains that when Reynolds number is high, the surface is partitioned into outer region unaffected by viscosity, close region where viscosity is significant (Eckert and Drake, 1972).

(viii) Dimensional analysis: Dimensional analysis describes the relationship that is between physical quantities using their dimensions and their units of measurement. It is basically used in fluid dynamics to convert a unit from one form to another. It is applicable in real life problems in physics (Robert and Murray, 2002).

(ix) One, Two and Three-dimensional flow: A one-dimensional flow is the flow that all its factors (velocity, pressure etc) which describes the flow at any given instant varies only along the direction of flow. It varies with time the moment the flow is unsteady. An example is the flow through a pipe. A two-dimensional flow is when flow factors varies in the direction of flow. An example is a flow past a weir whose cross-section and infinite width is constant and perpendicular to the plane can be taken as two-dimensional (Frank, 1990).

(x) Radiation parameter: Thermal radiation plays an important role in the problem of heat plus mass transport most importantly in design of energy conversion system which works at a higher temperature. For instance, satellites, gas turbines, space vehicles etc. Therefore, effect of radiation is of great significance in a surrounding with high temperature. The thermal radiation within the system is as a result of hot walls emission and working fluids (Seigel and Howell, 1971). The thermal radiation generally indicates the relative contribution of conduction transfer of heat and thermal radiation transfer. It is necessary to say that, temperature profiles are more effected by thermal radiation while compared with concentration profiles. Increasing thermal radiation within the boundary layer means radiation heat loss enhances the ambient temperature and hereby augument cooling rate of fluid and reverses the flow in the boundary layer (Makinde and Olanrewaju, 2011). The presence of radiation term in the energy equation result to nonlinear equation and lead to difficulty in computation. Owing to these difficulties, radiation effect on convective flows has been studied with reasonable simplifications (Bird et al., 1960).

(xi) Magnetic parameter: According to Idowu and Falodun in 2018, induced magnetic field current in a movable conductive fluid. This induced currents hereby polarize the fluid and make changes to the magnetic field. Practically, in MHD heat plus mass transfer problem there is a migration of conductor into an electric current magnetic field which is induced and hereby creates its own magnetic field. This conductor is associated with fluids with complex

motions. The induced current by virtue of electrically conducting fluids moved as a result of magnetic field which gives rise to a resistive force (Lorentz) and retards its motion. Kumar and Sivaraj (2013) studied the problem of heat plus mass transfer with MHD. Their study established the fact that the magnetic
field decreases the fluid motion of an electrically conducting fluid within the boundary layer. The study of non-Newtonian fluids with magnetic field gained attention of many authors because of its significance in science and technology. Higher temperature plasma oil exploration, nuclear reactors, plasma studies, MHD generators and accelerators, liquid metals fluids and power generation systems are some important applications of MHD (Bird et al., 1960).

(xii) Soret and Dufour: In the combined investigation of heat plus mass transfer effect, Dufour or diffusion-thermal is thermal energy flux resulting from concentration gradients while Soret or thermo-diffusion effect is the contribution to the mass fluxes as a result of temperature gradient. Both Soret and Dufour effects are influencial whenever there is chemical reaction species at a surface in the domain of the fluid with lower density more than the surrounding fluid (Makinde and Olanrewaju, 2011). The Dufour term is found in the energy equations while Soret term is found in the specie equations. According to Idowu and Falodun (2018), when the Soret term is increases, there is higher thermal diffusion which brings increase to the fluid velocity. Positive Soret term stabilizes the flow. Idowu and Falodun (2018) explained that increase in temperature has every tendency of decreasing both density and mass fraction of solute concentration when the Soret parameter is greater than zero (i.e Soret number$\geq$ 0). This is refers to as thermal gradient with cooperative solutal meaning that all the solute diffuses to cold regions but for Soret to be less than zero (i.e Soret number$\leq$ 0), there is increase in the temperature and density of the fluid. This is refers to as thermal gradient with competitive solutal meaning that, all the solute diffuses to the warmer region. Both Soret and Dufour are significant when there is high concentration gradient and temperature gradient which cannot be neglected. Soret-Dufour are interesting physical phenomena while considering heat and mass transfer simultaneously. Soret-Dufour in heat transfer process are generally neglected because they posses smaller order of magnitude than the effects prescribed by the laws of Fouriers and Ficks. Soret effect is very useful in separation of isotope and when gases is mixed together at a high molecular and medium weight(Eckert and Drake, 1972). Eckert and Drake (1972) reported the usefulness of these effects in convective transport phenomenon.

(xiii) Viscous dissipation: The recent trend on the study of viscous dissipation is noticeable. This term is found in the energy equations. After dimensionalization or after the application of similarity variables, its results to a dimensionless Eckert number is capable of causing increase in the velocity profiles by adding more energy to the flow within the layer. Eckert number explains the relationship between kinetic energy of the flow enthalpy. According to Idowu and Falodun (2018), greater Eckert number gives rise to fluid velocity because heat energy is produced which is stored in the liquid fluid as a result of frictional heating (Gebhart, 1962).

Dissipation term is important when there is high the viscous in the fluid. For example, in polymer processing where temperature is high, dissipation term cannot be neglected. Viscous dissipation effect is very much important in geophysical flow and operations in the industries which are symbolized by Eckert number (Gebhart, 1962).

(xiv) Thermophoresis: Thermophoresis is a phenomenon that occurs when two or more moving particles are mixed together subject to the temperature gradient force and types of particles that react differently in the system. Thermophoresis is either positive or negative. It is positive when the particles moves from higher temperature to lower temperature and negative when it is the other way. Thus, the larger species experienced the positive thermophoresis while the lighter species experience negative thermophoresis. The phenomenon of thermophoretic has numerous practical applications as the particles move separately due to temperature gradient force. These particles are separated by the force after mixing them together and prevented from mixing it already separated. The phenomenon is very useful in most industries to bring separation to larger or smaller particles of polymer from their solvent.

(xv) Heat generation: Heat generation is a scientific method of generating heat in the system by either chemical or nuclear process. The example of this phenomenon include atmospheric motion heat is generated sunlight absorption. (Tasuka et al., 2016). The heat generation term is added to the energy equation. At times, temperature difference may exist in the surface and ambient fluid (Idowu and Falodun, 2018).

(xvi) Porous media: The flow through porous media finds practical application in design of filters, transpiration cooling, boundary layer control and gaseous diffusion. Problems on porous media can also be found in the manufacturing of wires, fibres and glass in polymer industries. Increase in the porous media is increase within the boundary layer allows passage of more fluid particles and leads to reduction of fluid velocity whereas the fluid temperature will be enhanced in the boundary layer (Eckert and Drake, 1972).

(xvii) Chemical reaction: The problem of heat plus mass transfer in the presence of chemical reaction plays a predominant role indrying, cooling of nuclear reactors, chemical vapour deposition on surfaces, petroleum industries, power and cooling industries. A phenomenon of this type frequently occurs in nature. The problem on chemical reaction is modeled as heterogeneous and homogeneous processes depending on how it occurs at the interface or as a volume reaction of single phase volume reaction. It could be reversible or nonreversible. A homogeneous reaction takes place in a phase while heterogeneous reaction occurs in an area that is restricted within boundary of a phase. At times, reaction rate may be subject to the species concentration itself (Alao et al., 2016).

When rate of reaction is proportional to concentration, the chemical reaction is of first order (Cussler, 1988). An example is emission of nitrogen dioxide to the atmosphere. The release of nitrogen dioxide to the atmosphere reacts chemically with hydrocarbons that are unburned aided by sunlight and produce peroxyacetylnitrate (Cussler, 1988).

(xviii) Skin friction coefficient: It occurs due to the drag of viscous in the boundary layer around object. The part of the layer is generally laminar and thin. However, the flow becomes turbulent towards the tail section of the object. Law of viscosity is explained by Isaac Newton that the shear stress (τ) that is between layers of a fluid is proportional to velocity gradient $\left(\frac{\partial u}{\partial y}\right)$in a perpendicular direction to the layer. Therefore, the viscous drag at the shell of body or wall is$\tau_w = \left(\mu\frac{\partial u}{\partial y}\right)$and the skin friction coefficient becomes $C_f = \frac{2\tau_w}{\rho u_\infty^2}$.

(xix) Nusselt number: It is the division of convective to conductive heat transfer in the boundary layer.

$$Nu = \frac{Convective\ heat\ transfer\ coefficient}{Conductive\ heat\ transfer\ coefficient} = \frac{h_w}{k}$$

(xx) Sherwood number: It is applicable in mass transfer problem. It is the division of convective to diffusive mass transfer.

$$Sherwoodnumber\ (Sh) = \frac{Convective\ mass\ transport}{Conductive\ masstransport} = \frac{h_w L}{D}$$

1.6 Outline of the thesis

While chapter one of this study is the general introduction, chapter two deals with literature review to shapen the aim and objectives of the study. This study has reviewed related published works following the present study and its goals. Chapter three comprises of the formulation, derivation, dimensionalization and method of approach to the problems. The results obtained and detailed discussions had been described in chapter four. The last chapter five contain the summary of the entire work followed by cited literature. This chapter also contains the conclusion of the study and its recommendation.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

The mathematical analysis of fluid dynamics arising from science and engineering in the form of differential equations which do not have analytical solutions has led many scholars to propose different numerical methods.

In this chapter, some of many contributions in the literature are reviewed. Their methods of approach and findings of different flow parameters are discussed extensively.

2.2 Review of existing works

The concept of MHD is to study the movement of a conducting fluid electrically because of an imposed magnetism. Magnetic fields are produced by electric current. MHD is of great importance in engineering mostly in power, metallurgy astrophysics and geophysics. Due to this importance, several authors presented the flow of magnetohydrodynamics. Manglesh and Gorla (2012) elucidate MHD thermal radiation, rotation and chemical reaction of unsteady viscoelastic slip flow.In their analysis, they obtained solutions of the problem analytically for momentum, energy and concentration equations. The Nusselt number and the frequency of oscillation was found to increase because of increase in Prandtl number. The work of Alam et al. (2014) was on heat plus mass transport boundary layer motion past an inclined penetrable plate by considering suction, Soret, and hall current effect. Their flow equations were simplified into first order total differential equations before applying Runge-Kutta fourth-fifth order with shooting technique. In their study, the thickness of the thermal boundary layer was found to increase as the Prandtl number rises which implies heat transfer at lower case. In 2014, Ibrahim discussed MHD, dissipative, radiative and chemical reaction effects through a penetrable channel and a non-Isothermal stretching sheet. In his investigation, similarity transformation were used to simplify the flow equations to total differential equations. Shooting method was used in solving the equations and he observed velocity and temperature fields decreases as a result of wall temperature parameter. Sreenivasulu et al. (2016) studied boundary layer slip flow over an exponential permeable stretching sheet by considering thermal radiation, MHD, Joule heating and viscous dissipation effects. Their flow equations were transformed to total differential equations with the help of similarity transformation and solved numerically. Pattnaik and Biswal (2015) presented solution of free convective flow through porous media with MHd and time dependent. Their flow equations were simplified to ordinary differential equations using super imposing a solution with steady time dependent transient part which resulted to set of ordinary differential equations and this set of ordinary differential equations were solved by Laplace transformation method. Also in their analysis, they concluded that a higher Prandtl fluid as well as a higher Schmidt fluid (heavier species) were counterproductive in accelerating fluid motion and further assisted by the presence of a porous medium. Recently, Ashish (2017) have studied transient free convective MHD flow past an exponentially accelerated vertical porous plate with variable temperature through a porous medium. He solved the problem of MHD flow analytically by the Laplace transform technique and his result revealed that skin friction increases with increase in magnetic parameter, suction parameter, radiation parameter and accelerating parameter.

Eswaramoorthi et al.(2015) have studied effect of radiation on MHD convective flow and heat transfer of a viscoelastic fluid over a stretching surface. In their analysis, a similarity transformation was used to reduce their governing nonlinear partial differential equations into ordinary differential equations. The transformed equations were then solved using homotopy analysis method. They found out that the momentum boundary layer thickness decreases as the unsteadiness parameter, viscoelastic parameter and Hartmann number increases. Natural convection heat and mass transfer in MHD fluid flow past a moving vertical plate with variable surface temperature and concentration in a porous medium was considered by Javaherdeh et al.(2015). In their analysis, implicit finite-difference method was used in solving their model and their results show that the higher the porosity parameter, the more sharply is the reduction in both Nusselt and sherwood number. Renuka et al.(2015) investigated radiation effect on MHD slip flow past a stretching sheet with variable viscosity and heat source/sink. Runge-kutta fourth-order technique along with shooting method were used in solving their model equations and they found out that the radiation parameter elevates the skin friction and reduces the heat transfer. Reddy et al.(2016) have studied unsteady MHD free convection flow characteristics of a viscoelastic fluid past a vertical porous plate. They solved their model using perturbation scheme and their result revealed that elasticity of the fluid and the Lorentz force reduces the velocity. Swapna et al.(2017) recently studied chemical reaction and thermal radiation effects on MHD convective oscillatory flow through a porous medium bounded by two vertical porous plates. Perturbation techniques were used to solve their model analytically and they found out that rate of heat transfer slightly increase for large values of heat source parameter.

Ahmed et al.(2010) have studied unsteady MHD free convective flow past a vertical porous plate immersed in a porous medium with hall current, model analytically and found out that when Soret number decreases, there is an increase in the main flow and cross flow velocities. Gundagani et al.(2012) considered finite element solution of thermal radiation effect on unsteady MHD flow past a vertical plate with variable suction. Finite element method were used to solve their nondimensional governing equations. They discovered that, increasing the radiation parameter causes an increase in the velocity and temperature within the boundary layer. Sarada and Shanker (2013) have studied the effect of chemical reaction on an unsteady MHD free convection flow past an infinite vertical porous plate with variable solution. In their analysis, finite difference method were used to solve the flow equations and they found out that an increase in the Prandtl number decreases the Nusselt number. Prakash et al. (2014) have studied effects of chemical reaction and radiation absorption on MHD flow of dusty viscoelastic fluid. They solved their modeled equations analytically using perturbation technique. Their result revealed that when heat source/sink parameter and Prandtl number increases, the temperature gradually decreases. Chandra et al.(2015) have studied thermal and solutal buoyancy effect on MHD boundary layer flow of a viscoelastic fluid past a porous plate with varying suction and heat source in the presence of thermal diffusion. In their analysis, they used multiple parameter perturbation on velocity and simple perturbation method on the temperature and concentration. They discovered that the presence of thermal diffusion increases fluid velocity while magnetic field reduces the fluid velocity. Sulochana et al.(2016) considered numerical investigation of chemically reacting MHD flow due to a rotating cone with thermophoresis and Brownian motion. In their analysis, Runge-kutta based shooting technique were used to solve their model and their results revealed that the thermophoresis and Brownian motion parameters control the heat and mass transfer rates. Vijayakumar and Keshava (2017) recently investigated MHD boundary layer flow of a visco-elastic fluid past a porous plate with varying suction and heat source or sink in the presence of thermal radiation and diffusion. They solved their flow equations numerically using finite difference method and found out that influence of heat source enhance the fluid temperature.

Islam et al.(2014) studied effect of conduction variation on MHD natural convection flow along a vertical flat plate with thermal conductivity. They solved their transformed non-linear equations using implicit finite difference method with Kellerbox technique. They found out that the temperature within the boundary layer increases when increasing magnetic parameter. Jain (2014) studied combined influence of hall current and Soret effect on chemically reacting magneto-micropolar fluid flow from radiative rotating vertical surface with variable suction in slip-flow regime. Perturbation techniques were used in solving their equations and they found out that micro-rotation profiles and couple stress coefficient decrease with an increase in suction parameter, rotation parameter and viscosity ratio. Haritha et al.(2015) studied effects of thermo-diffusion, thermal radiation, radiation absorption on convective flow past stretching sheet in a rotating fluid. Galerkin finite element analysis with three nodded line segment were used in their analysis. Jana and Das (2015) studied influence of variable fluid properties, thermal radiation and chemical reaction on MHD slip flow over a flat plate. In their analysis, they used similarity transformation on their governing equations to have non-linear ordinary differential equations and solved with the help of symbolic software MATHEMATICAL. Hemalatha and Bhaskar (2015) studied effects of thermal radiation and chemical reaction on MHD free convection flow past a moving vertical plate with heat source and convective surface boundary condition. They transformed their governing equation using similarity transformation and solved same using Runge-kutta method along with shooting technique. They found out that increase in the radiation parameter gives a little increase in the velocity.

Heat and Mass transfer (or double diffusion) in non-Newtonian fluids attracted the attention of researchers in fluid dynamics over decades due to its applications in engineering such as heat exchanger devices, nuclear waste disposals, chemical catalytic reactors and processes etc. The buoyancy due to temperature and concentration gradients drive double diffusive flow. Simultaneous occurrence of heat and mass transfer in a moving fluid makes the relations between the energy flux and the driving potentials to be more complicated. Dufour or diffusion-thermal effect is the energy flux caused by a composition gradient while temperature gradient create mass fluxes and this is called Soret or thermal-diffusion effect. The combined effects of Soret and Dufour are often neglected by authors in literature because of their smaller order of magnitude than the effects prescribed by Fick's laws. The effects of Soret and Dufour are of great importance like Soret in isotope separation. Ibrahim and Sunneth (2015) discussed chemical reaction and Soret effects on unsteady MHD flow of a viscoelastic fluid past an impulsively started infinite vertical plate with heat source/sink. Alao et al.(2016) presented effects of thermal radiation, Soret and

Dufour on an unsteady heat and mass transfer flow of a chemically reacting fluid past a semi-infinite vertical plate with viscous dissipation. In their study, governing equations representing the physical model are system of partial differential equations and were transformed into systems of coupled non-linear partial differential equation by introducing non-dimensional variables. The resulting equations were solved using the spectral relaxation method (SRM). Their results shows that an increase in Eckert number of the fluid actually increases the velocity and temperature profiles of the flow. Whereas an increase in thermal radiation parameter reduces the temperature distribution when the plate is being cooled. Adeniyan (2016) investigated Soret-Dufour and stress work effects on hydromagnetic free convection of a chemically reactive stagnation vertical surface with heat generation and variable thermal conductivity. Recently, Hayat et al. (2017) presented Soret and Dufour effects on MHD peristaltic transport of Jeffrey fluid in a curved channel with convective boundary conditions. In their analysis, it was found out that velocity is not symmetric about center line for curvature parameter and their results revealed that maximum velocity decreases with an increase in the strength of magnetic field. Among great authors who have presented the effects of Soret and Dufour discussed in this work are Gbadeyan et al. (2011); Subhakar et al. (2012); Mahbub et al. (2013); Jimoh et al. (2014) and Gangadhar and Suneetha (2015). Others are Vedavathi et al. (2015); Bilal et al. (2016); Prakash et al. (2016) and Sharma and Aich (2016). All the studies mentioned above used a numerical method in analyzing their results and all found out that the effects of both Soret and Dufour are in alternate to each other on the temperature and concentration profiles.

Thermal radiation plays a vital role in the design of advanced energy conversion system when at a very high temperature. The emission of hot walls and the working fluid indicate the presence of thermal radiation within the system, Seigel and Howell (1971). When the surrounding fluid is at high temperature, thermal radiation is of great importance and it has received researchers attention in recent time. Shateyi et al.(2010) studied the effects of thermal radiation, hall currents, Soret and

Dufour on MHD flow by mixed convection over a vertical surface in porous media. In their analysis, transient, non-linear and coupled governing equations were solved by adopting a perturbative series expansion about a small parameter. Radiation effect on an unsteady MHD free convective flow past a vertical porous plate in the presence of Soret have been studied by Anand Rao et al. (2012). Their problem was governed by systems of coupled non-linear partial differential equations whose exact solutions are difficult to obtain, if possible. So, they employed Galerkin finite element method for the solution. Their result revealed that an increase in the Soret number leads to increase in the velocity and temperature. Gundagani et al. (2012) investigated solution of thermal radiation effect on unsteady MHD flow past a vertical porous plate with variable suction. Non-dimensional governing equations are formed with the help of suitable dimensionless governing parameter in their analysis. The resultant coupled non dimensional governing equations are solved by a finite element method. In their result, it was found out that increase in the magnetic field leads to decrease in the velocity field and rise in the thermal boundary thickness. Devi et al. (2014) studied radiation and mass transfer effects on MHD boundary layer flow due to an exponentially stretching sheet with heat source. In their analysis, the initial governing boundary layer equations were transformed to systems of ordinary differential equations, and solved numerically by a fourth order Runge-Kutta method along with shooting technique. Their results show that the momentum boundary layer thickness decreases, while both thermal and concentration boundary layer thicknesses increases with an increase in the magnetic field intensity. Sreenivasa (2016) presented effect of thermal radiation, dissipation and chemical reaction on mixed convective heat and mass transfer flow past a stretching sheet with hall effects in slip flow regime. Similarity solutions were obtained using suitable transformations in their analysis. The similarity ordinary differential equations was then solved by MATLAB routine bvp4c. Idowu et al. (2016) studied finite element analysis on MHD Jeffry fluid flow with radiative heat transfer past a vertical porous plate moving through a binary mixture. Finite element method was used in their analysis. Sarma and Govardhan (2016) studied thermo-diffusion and diffusion-thermo effects on free convection heat and mass transfer of vertical surface in a porous medium with viscous dissipation and thermal radiation. In their study, the two-dimensional boundary-layer partial differential equations that govern the flow problem were transformed by a similarity transformation into a system of ordinary differential equations which were solved numerically. Their results revealed that as the viscous dissipation increases, the velocity profiles increases whereas concentration and temperature profiles decreases.

The interaction between two or more chemicals to produce new compounds is called chemical reaction. A chemical reaction can be classified as heterogeneous and homogeneous reaction. Heterogeneous reaction deals with reactants of different states of matter while homogeneous reaction deals with reactant of the same states of matter. Hady et al. (2008) reported influence of chemical reaction on mixed convection of Non-Newtonian fluids along non-Isothermal horizontal surface in porous media. In their study, numerical calculations were carried out for various values of dimensionless parameters and analysis of the results obtained show that the flow field were influenced appreciably by the chemical reaction, the buoyancy ratio, the viscosity index and the power-law of the wall temperature parameter. Sugunamma et al.(2013) discussed inclined magnetic field and chemical reaction effects on flow over a semi-infinite vertical porous plate through porous medium. Kishore et al. (2014) studied the effect of chemical reaction on MHD free convection flow of dissipative fluid past an exponentially accelerated vertical plate. Their flow equations are governed by coupled non-linear partial differential equations. The dimensionless equations of the problem were solved numerically by the unconditionally stable finite difference method of Dufort-Frankels type. Their result show that the rate of concentration transfer increases with increase values of magnetic parameter, Schmidt number, Prandtl number, and chemical reaction parameter while it decreases with an increase in thermal Grashof number, mass Grashof number and Eckert number. Ram et al. (2015) investigated stability of chemical reacting double diffusive free convective flow of Maxwell fluid through porous medium with internal heating under magnetic field. Their study explained that the effect of magnetic brings a very slow flow of fluid particles so that the Darcy model is taken into account in the momentum equation and the onset of stabilities at free-free boundaries is calculated analytically. Kala and Rawat (2015) reported effect of chemical reaction and oscillatory suction on MHD flow through porous media in the presence of pressure. Their study employed perturbation technique to find the solution for velocity field and concentration distribution. It is observed that with the increase in the value of Schmidt number, Chemical reaction parameter the concentration of the flow field shows decrease at all points and hence thickness of concentration boundary layer

decreases.

The type of liquid which disobey the Newtons law of viscosity are non-Newtonian. Liquids that behave in such manner are not common. Tooth pastes, mud flows, slurries, blood are examples of non-Newtonian fluids. They are mostly complex and are examined under flow (rheology) and science of deformation. Many researchers in recent time in fluid mechanics are exploring non-Newtonian liquid owing to the rheological applications in chemical along with mechanical engineering. In the examination of Ramana et al. (2017), it was finalized that mass transport motion rate in Casson liquid improved more than Maxwell liquid. The exploration of Walters-B liquid in boundary layer motion was elucidated by Tonekboni and Ramin (2012). They showed as elasticity is upsurge stress on the boundary enlarges in stagnation motion problem and decreases in Sakiadis as well as Blasius motion. The examination of Pushpalatha et al. (2017) indicate that, higher Casson liquid term leads to degeneration in the velocity plot while temperature along with concentration plots degenerates. The elucidation of Fagbade et al. (2018) found out that increase in the viscoelastic parameter lessens the velocity graphy. Animasaun et al. (2016) finalized that the contribution of Casson liquid parameter is explored by varying plastic dynamic viscosity which lessens both the temperature and velocity graph. The elucidation of Pramanik (2014) explained that higher Casson liquid term decrease the velocity graph but accelerate the temperature plot as blowing/suction term is negative or positive. Barik et al. (2017) elucidate laminar steady MHD motion of visco-elastic liquid through a penetrable pipe and embedded in a penetrable channel. The concluding remarks was that suction along with elasticity are counter productive when the skin friction becomes enlarged. Mukhpadhyay et al. (2013) elucidated Casson liquid motion past an unsteady stretchable surface. They detected that boundary layer thickness along with velocity are degenerating function of the Casson liquid term.

Examples of non-Newtonian fluid are micropolar fluid, viscoelatic fluid, powerlaw fluid etc. Non-Newtonian fluid are considered to be more important and appropriate in many technological applications compared to the Newtonian fluid. Rana et al. (2013) presented on the onset of thermosolutal instability in a layer of an elastico-viscous nanofluid in porous medium. In their analysis, they applied perturbation solutions in solving their transformed equations. Their results revealed that kinematic viscolasticity has no effect on the onset of stationary convection. Kumar and Singh (2007) studied instability of two-rotating viscoelastic (Walters'B) superposed fluids with suspended particles in porous medium. They employed perturbation techniques in solving their model equations and their result shows that the system is unstable or stable depending on the kinematic viscoelaticity whether it is greater than or smaller than the medium permeability divided by medium porosity. In another development, Prasad et al.(2011) reported unsteady free convection heat and mass transfer in a Walters'-B viscoelastic flow past a semi-infinite vertical plate: a numerical study. In their study, the finite difference scheme of the CrankNicolson type was employed to solve their dimensionless unsteady, coupled, and non-linear partial differential conversations equation. They observed that, when the viscoelaticity parameter increase, the velocity gradually increase close to the plate surface. Ahmad (2011) studied visco-elastic boundary layer flow past a stretching plate and heat transfer with variable thermal conductivity. The problem of boundary layer flow of viscoelaticity fluid (Walters'-B Liquid Model) was solved analytically and observed that as we move away the stretching plate with dynamic region, the temperature field increase gradually as viscoelatic parameter increases. Jimoh et al. (2015) presented numerical study of unsteady free convective heat transfer in Walters-B viscoelastic flow over an inclined stretching sheet with heat source and magnetic field. An implicit finite difference method of Crank-Nicolson type is employed in solving the dimensionless governing equations. They observed that when the heat source parameter increases, the velocity and temperature increases within the boundary layer.

The contribution of variable viscosity along with thermal conductivity on fluid motion has gained the attention of many scholars in recent time. The analysis of such finds applications in system of underground storage as well as extraction of geothermal energy. A varied viscosity in a liquid motion assist in assuming the type of motion and transport rate of heat. Also, the variation of thermal conduction on analysis of heat transport assist in getting accurate facts of the thermal transport. Choudhury and Hazarika in (2008) presented the contribution of variable viscosity along with thermal conductivity on MHD motion owing to a point sink. Their model equations of motion was tackled by utilizing the shooting approach. Hazarika and Konch (2016) elucidate variable thermal conductivity along with viscosity contributions on MHD free convection dusty liquid along a vertical penetrable plate with generation of heat. It was finalized in the exploration that increase in viscosity term degenerate the liquid velocity and dust phase. However, it brings enhancement at higher variable thermal conductivity term. The recent analysis of Hazarika and Phukon (2017) on a varied viscosity as well as thermal conductivity was tackled by utilizing the shooting techniques. Their contribution shows that higher viscosity resulted to degeneration in skin friction and enhancement in Nusselt number . Manjunatha and Gireesha (2016) explored the contribution of varying thermal conductivity and viscosity on MHD motion and heat transfer of a dusty liquid. They finalized that higher varied viscosity term lessens the dust phase along with liquid

velocity.

The effect of thermophoresis on mixed convection flow of MHD non-Newtonian nanofluid play an important role in space technology and in high temperature. The phenomenon of thermophoresis is used in industries to separate large or small polymer particles from their solvent. It plays a significant role in the mechanism of devices involving small micron sized particles and large temperature gradients in the fields. Mondal et al. (2018) considered thermophoresis on MHD mixed convection mass transfer in their study. Their transformed set of non-linear coupled ordinary differential equations were solved numerically. Their result revealed that increase in thermophoresis brings decrease to the concentration of the fluid. Jayachandra

Babu et al. (2017) considered thermophoresis and Brownian motion in their study. Their transformed governing equation were solved using Runge-Kutta and Newton's method. Their result revealed that thermal and solutal Grashof numbers regulate the temperature and concentration fields. It is noticed that thermophoresis and Brownian motion parameters is opposite to each other on the local Sherwood number. Muthuraj et al. (2016) studied influence of chemical reaction with heat and mass transfer. Their analysis was done using perturbing technique. Their result shows that chemical reaction greatly influence the mass transfer. Kataria and Patel (2016) studied effects of radiation and chemical reaction on MHD Casson fluid flow. Their flow equations were solved using Laplace transform technique. Their results were presented in close form and the result revealed that increase in chemical reaction parameter brings decrease to the concentration field. Mass along with heat transport on mixed convective motion of chemically reating nano liquid was examined by Mahanthesh et al. (2016). They utilized Laplace transform systematic approach to obtain a closed formed approach. It is detected in the study that higher chemical reaction parameter leads to rapid degeneration in the dimensionless concentration graphs.

An exploration of non-Newtonian liquids motion past a stretchable sheet was investigated by Ramana Reddy et al. (2018). Their motion equations were tackled by utilizing Runge-Kutta Fehlberg systematic approach. Their analysis indicates that casson liquid attains greater velocity in comparison to Maxwell liquid. Animasaun and Pop (2017) explored non-Newtonian Carreau liquid motion driven by catalytic reactions surface by utilizing shooting approach. They finalized that the temperature field in the motion of viscoelastic carreau liquid is higher than the Newtonian liquid. Gireesha et al. (2018) elucidated mass and heat transport of nano liquid Oldroyd-B past a stretchable sheet. Then simplified equations were tackled by utilizing RKF-45 approach. Their outcome reveals that nonlinear radiation becomes more effective than linear radiation. Mass and heat transport in MHD Casson liquid past an exponentially penetrable and stretchable surface have been considered by Raju et al. (2016). They used similarity transformation to reduce their flow equations into ordinary differential equations. These equations were solved by Matlab bvp4c package. It was concluded in the study that Casson fluid showed better performance of heat transfer when compared with Newtonian fluid. Srinivasacharya et al. (2018) studied double dispersion influence on micropolar fluid. Their transformed governing equations were solved using successive linearization method. The study concluded that dispersion coefficients have strong influence on heat and mass convective transfer. Raju et al. (2015) studied effects of radiation and Soret on MHD nanofluid flow. They used similarity transformation to reduce the governing equations and solve the resulting equations numerically. It was concluded in the study that the velocity, temperature and concentration boundary layer increases as a result of increase in nanoparticle volume fraction. Many researchers in the field of fluid dynamics now considered spectral methods as an essential tools in solving a highly coupled and nonlinear differential equations. Spectral relaxation method (SRM) is an iterative method which employed the Gauss-seidel approach. SRM is efficient, accurate, and it solves both ordinary and partial differential equations. Motsa and Makukula (2013) considered SRM for steadiness motion of von Karman of Reiner-Rivlin liquid with injection/suction, viscous dissipation and Joule heating. They suggested in their study that SRM can be extended to related problems in fluid mechanics applications. Haroun et al.(2015) studied unsteady natural convective boundary layer flow of MHD nanofluid over a stretching surfaces with chemical reaction using the spectral relaxation method. They found out that the values of skin friction increases when increasing the values of the nanoparticle volume fraction and magnetic parameter. Kameswaran et al.(2013) presented a spectral relaxation method for thermal dispersion and radiation effects in a nanofluid flow. Their analysis shows that the convergence rate of SRM is significantly improved when it is used with the succesive over-relaxation method. Magugula et al.(2016) recently considered a bivariate spectral relaxation method for unsteady magnetohydrodynamic flow in porous media. Their study revealed that the new approach is an improvement over the spectral relaxation method. Motsa et al.(2014) considered spectral relaxation method and spectral Quasilinearization method against Keller box method and they found out that the methods are efficient in terms of computational accuracy and speed compared to Keller box. Motsa et al.(2014) considered a spectral relaxation approach for unsteady boundary-layer flow and heat transfer of a nanofluid over a permeable stretching/shrinking sheet. In their analysis, it was found out that spectral quasilinearization method converges faster than spectral relaxation method but spectral relaxation method is more accurate than spectral quasilinearation method. A comparison between spectral purturbation and spectral relaxation approach for unsteady heat and mass transfer by MHD mixed convection flow over an impulsively stretched vertical surface with chemical reaction effect is investigated by Agbaje and Motsa (2015). Awad et al. (2015) studied the effect of thermophoresis on unsteady Oldroyd-B nanofluid flow over stretching surface. They used spectral relaxation method in their analysis and found out that Brownian motion on the rate of heat transfer are negligible.

Many of the aforementioned explorations targeted the analysis of non-Newtonian liquids at uniform physical attributes. The explorations on variable viscosity along with thermal conductivity are Newtonian liquid analysis. It worth noting that no exploration on contributions of thermophoresis on Dufour-Soret. The analysis of Mondal et al. (2017); Srinivasa et al. (2014); Kalyani et al. (2015); Vedavathin et al. (2015) just to mention a few explored Dufour-Soret contributions on Newtonian model with uniform physical attributes. Double diffusive motion of a non-Newtonian liquids find usefulness in many process in engineering such as food processing, biosystems, petroleum reservoirs, and industrial processes. Dufour or diffusion-thermal contribution portrays the energy flux owing to composition gradient. Temperature also create mass fluxes named Soret or thermal-diffusion contribution. These terms are seen in the concentration as well as temperature motion equations. The Dufour contribution is added to the temperature motion equation while Soret contribution is added to the concentration motion equation. This study considered unsteady, steady, laminar, two-dimensional, mixed and free convective flow of an incompressible non-Newtonian fluids through several medium. The plates has two coordinates (x,y).

While x−coordinate is taken along the plate in the upward direction, y−coordinate normal to the plate as shown in figure 3.1, 3.2 and 3.3 of the three problems considered in this study. The temperature and concentration at the wall (T_w) and (C_w) were kept constant. For a heated wall $T_w > T_\infty$, $C_w > C_\infty$ while $T_w < T\infty$, $C_w < C\infty$ is applicable to a cooled plate. The flow generated in the three problems is as a result of a stretched surface caused by simultaneous application of a magnetic field of uniform strength applied in y−direction. Soret and Dufour parameters are considered in this study because the level of species concentration is assumed to be high. The present study examined some of the areas which have been neglected in previous works.

To our best of knowledge, no explorations have been on effects of thermo-physical parameters on non-Newtonian fluids. This exploration is motivated owing to the past explorations and its application in industries such as the use of Soret in separation of isotope and in polymer industries. This research is aimed at examining the contributions of variable thermal conductivity and viscosity on non-Newtonian liquids flow through vertical penetrable plate with Dufour-Soret contribution. The varieties of Casson liquid explored in this research are concentrated fruit juice and tomatoe source while chromatography and polymethly methacrylate are the WaltersB liquid. These liquids finds usefulness in industrial engineering as well as polymer industries. Owing to this usefulness, this research becomes very essential to engineers and scientist. A robust numerical approach based on the use of homotopy analysis along with Chebyshev spectral collocation techniques. This techniques is efficient with a faster computational analysis more than the homotopy analysis approach. Key flow parameters are depicted graphically while computations of quantities of interest in engineering are depicted using table. The present outcomes were compared with existing works and were in conformity to each other.

CHAPTER THREE: Methodology

3.1 Introduction

In this chapter, non-dimensional quantities and similarity variables are introduced on the equations that governed the three problems considered in this study. The procedure of applying these quantities are shown in this chapter. The qualitative analysis of the three problems are also discussed. The qualitative analysis is done to examine if the transformed equations for the three problems have a solution, and if the solution exists, we want to know if, it is unique. It worths mentioning that, this study is only interested in the steady state to carry out the qualitative analysis.

The use of spectral method is in approximating the unknown functions with the help of truncating series of orthogonal functions or polynomials (Canuto et al., 1988). To apply the spectral method,the domain of the flow equations are defined in the close interval [−1,1] by considering the transformation $\eta = \frac{(b-a)(\tau+1)}{2}$to connect [$a$,$b$] to [−1,1]. Hence, the spectral method is implemented on the transformed closed interval. The spectral relaxation method (SRM), as proposed by Motsa (2012) employed the concept and the idea behinde Gauss-Seidel to decouple system of differential equations. The use of SRM also involves the application of differentiation matrix D in approximating the derivatives of all unknown variables at a specified collocation points. The spectral homotopy analysis method (SHAM) is a numerical method that combines the Chebyshev collocation method with homotopy analysis method (HAM). The SHAM as introduced by Motsa et al. (2010) employs the concept of HAM with the Chebyshev spectral collocation method. Spectral method are now a very useful numerical method in solving both linear and nonlinear differential equations in science and engineering because of high accuracy in getting solution to problems that have smooth functions (Canuto et al., 1988; Trefethen, 2000). L is the scaling parameter whose choice of values determines how congruent the result is at infinity, that is, the entire boundary layer.

3.2 Formulation of the research problem one

Flow of free convection viscoelastic liquid model past a half-infinite vertically upward plate along with oscillatory suction that is dependent of time with a transient magnetism is explored (see Idowu and Falodun; 2018). The plate is taken to be endless in x^0-direction, hence the x^0-axis is considered along the vertically endless plate and y^0-axis normal to it as depicted in figure 3.1. The plate moves in y^0-direction and at a point there is no continuity in the flow towards x^0-direction and thus neglected in the continuity equation.Hence, the flow equations such as energy, concentration and momentum is considered as a function of t and y alone.

The magnetism (B_0) of constant strength as imposed is transversely in opposite direction to the motion. At the initial motion at $t \leq 0$, both plate and fluid has constant temperature. Heat absorption or generation, Dufour along with Soret are explored in this analysis. Magnetic Reynolds number is considered to be little such that induced magnetism could be insignificant. Walters-B non-Newtonian type of liquid is explored in this research while its constitutive equation according to Choudhury and Das (2014) is given follows:

$$\sigma_{ik} = -pg_{ik} + \sigma_{ik0} \tag{3.1}$$

$$\sigma_{0ik} = a\eta_0 e_{ik} - 2k_0 e_{0ik} \tag{3.2}$$

Here η_0 means limiting kinematic viscosity with little shear rates, K_0 means elastic coefficient, σ^{ik} means stress tensor, p means isotropic pressure, g_{ik} means metric tensor with coordinate system x^i, v_i means velocity vector, σ_{ik} means Cauchy stress tensor, e^{ik} means deformation rate tensor. The contravarient way of e^{ik} is written

as:

$$e'^{ik} = \frac{\partial e^{ik}}{\partial t} + v^m e^{ik}_{,m} - V^k_{,m} e^{im} - v^i_{,m} e^{mk} \tag{3.3}$$

Where e^{ik} is the rate of deformation tensor convected derivative is given as

$$2e_{ik} = v_{i,k} + v_{k,i} \tag{3.4}$$

At little shear rate, (η_0) which is the limiting viscosity is defined as:

$$\eta_0 = \int_0^\infty N(\tau)d\tau \quad and \;\; k_0 = \int_0^\infty \tau N(\tau)d\tau \tag{3.5}$$

$N(\tau)$ means relaxation spectrum as explained in Walters (1962). The model explained above is correct for Walters-B approximation when relaxation time is considered while terms such as:

$$\int_0^\infty t^n N(\tau)d\tau \quad , n \geq 2 \tag{3.6}$$

is neglected while k_0 is considered to be significant

Considering all the stated assumptions above and Boussinesqs evaluation, the flow model and its boundary constraints are written as:

Continuity equation

$$\frac{\partial v'}{\partial y'} = 0, \tag{3.7}$$

Momentum equation

$$\frac{\partial u'}{\partial t'}+v'\frac{\partial u'}{\partial y'}=\nu\frac{\partial^2 u'}{\partial y'^2}+g\beta_t(T-T_\infty)+g\beta_c(C-C_\infty)-\frac{\sigma B_0^2 u}{\rho}-\frac{K_0}{\rho}\left(\frac{\partial^3 u'}{\partial t'\partial y'^2}+v'\frac{\partial^3 u'}{\partial y'^3}\right) \tag{3.8}$$

Energy equation

$$\frac{\partial T}{\partial t'}+v'\frac{\partial T}{\partial y'}=\alpha\frac{\partial^2 T}{\partial y'^2}-\frac{1}{\rho c_p}\frac{\partial q_r}{\partial y'}+\frac{\mu}{\rho c_p}\left(\frac{\partial u'}{\partial y'}\right)^2+\frac{DK_T\partial^2 C}{C_s c_p \partial y'^2}+\frac{\beta^* u}{\rho c_p}(T_\infty-T)+\frac{Q_0}{\rho c_p}(T-T_\infty) \tag{3.9}$$

Concentration equation

$$\frac{\partial C}{\partial t'}+v'\frac{\partial C}{\partial y'}=D\frac{\partial^2 C}{\partial y'^2}-K_r'^2(C-C_\infty)+\frac{DK_T\partial^2 T}{T_m\partial y'^2}. \tag{3.10}$$

subject to:

$u = U_0$, $T = T_w + \psi(T_w - T_\infty)e^{n_0 t_0}$, $C = C_w + \psi(C_w - C_\infty)e^{n_0 t_0}$ *at* $y^0 = 0$ (3.11)

$u \longrightarrow 0, T \longrightarrow T_\infty, C \longrightarrow C_\infty$ as $y^0 \longrightarrow \infty$ (3.12)

The continuity equation (3.7) is evaluated by utilizing integration approach to obtain v = *constant*. This implies that the suction velocity at the plate is a unform function. It is considered to be constant and dependent of time following Alao et al. (2016) in this research work as:

$$v' = -V_0(1+\epsilon A e^{n't'}) \tag{3.13}$$

In this research, the heat flux is such that $\frac{\partial q_r}{\partial y'} >> \frac{\partial q_r}{\partial x'}$ because the flow equations are functions of t^0 and y^0 respectively. Hence, the radiative flux in the x^0- direction $\frac{\partial q_r}{\partial x'}$ is neglected. Therefore, $\frac{\partial q_r}{\partial y'}$ means heat flux that dominate the flow. Now, assuming distinct temperature within the fluid layers flow regime is so little that T^4 is evaluated as a linear form in terms of temperature at ambient vicinity (T_∞).

Using Taylor series to expand T^4 about T_∞ while terms of higher order is forgone.

Let us examine the expansion in Taylor series of T^4 about T_∞

$$T^4 = T_\infty + (T-T_\infty)T'(T_\infty) + \frac{(T-T_\infty)^2}{2!}T''(T_\infty) + \dots + \frac{(T-T_\infty)^n}{n!}T^n(T_\infty) \tag{3.14}$$

The above series becomes:

$$T^4 = 4T_\infty^3 T - 3T_\infty^4 \tag{3.15}$$

Using Roseland approximation, the radiative heat flux is given by;

$$q_r = -\frac{4\sigma_0}{3k_e}\frac{\partial T^4}{\partial y'} \tag{3.16}$$

Here σ_0 means Stefan-Boltzman constant while ke means coefficient of mean absorption. By utilizing the approximation of Roseland, the present analysis is basically on optically thick liquid. Based on equations (3.15) and (3.16), (3.9) reduces to:

$$\frac{\partial T}{\partial t'} + v'\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y'^2} + \frac{16\sigma_0}{3\rho c_p k_e}T_\infty^3\frac{\partial^2 T}{\partial y'^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u'}{\partial y'}\right)^2 + \frac{D_m}{C_s}\frac{k_T}{c_p}\frac{\partial^2 C}{\partial y'^2}$$
$$+\frac{\beta^* u}{\rho c_p}(T_\infty - T) + \frac{Q_0}{\rho c_p}(T - T_\infty). \tag{3.17}$$

To simplify the flow equations (3.8)-(3.10) and the boundary conditions (3.11) and

(3.12) in a dimensionless form, non-dimensional quantities of the following forms are introduced:

$$u = \frac{u'}{u_0},\quad y = \frac{v_0^2 y'}{\upsilon},\quad t = \frac{v_0^2 t'}{\upsilon},\quad n = \frac{\upsilon n'}{v_0^2}, \tag{3.18}$$

$$\begin{aligned}
\vartheta &= \frac{T - T_\infty}{T_w - T_\infty},\quad \varphi = \frac{C - C_\infty}{C_w - C_\infty},\quad P_r = \frac{\upsilon\rho c_p}{k} = \frac{\upsilon}{\alpha},\\
Sc &= \frac{\upsilon}{D}, Gr = \frac{g\beta_t \upsilon(T_w - T_\infty)}{u_0 v_0^2}, Gm = \frac{g\beta_c \upsilon(C_w - C_\infty)}{u_0 v_0^2}\\
Ec &= \frac{u_0^2}{c_p(T_w - T_\infty)}, \alpha = \frac{K_0 V_0^3}{\rho\nu^2}, \Delta = \frac{\beta^* u_0 \nu}{\rho c_p v_0^2}, \delta = \frac{Q_0 \nu}{\rho c_p v_0^2}\\
k_r^2 &= \frac{k_r'^2 \upsilon}{v_0^2}, Rr = \frac{16\sigma_0 T_\infty^3}{3k_e k}, M = \frac{\sigma B_0^2 \upsilon}{\rho v_0^2}.
\end{aligned}$$

(3.19)

, (3.20)

(3.21)

(3.22)

3.2.1 Non-dimensionalization of momentum equation of research problem one

$\frac{\partial u'}{\partial t'} = \frac{\partial u'}{\partial u} \times \frac{\partial u}{\partial t'} = \frac{\partial u'}{\partial u} \times \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial t'} = u_0 \times \frac{\partial u}{\partial t} \times \frac{v_0^2}{\nu}$

$$\frac{\partial u'}{\partial t'} = \frac{u_0 v_0^2}{\nu}\frac{\partial u}{\partial t} \qquad (3.23)$$

$\frac{\partial u'}{\partial y'} = \frac{\partial u'}{\partial u} \times \frac{\partial u}{\partial y'} = \frac{\partial u'}{\partial u} \times \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial y'} = u_0 \times \frac{\partial u}{\partial y} \times \frac{v_0^2}{\nu}$

$$\frac{\partial u'}{\partial y'} = \frac{u_0 v_0^2}{\nu}\frac{\partial u}{\partial y}$$

$\frac{\partial^2 u'}{\partial y'^2} = \frac{\partial}{\partial y'}(\frac{\partial u'}{\partial y'}) = \frac{\partial}{\partial y'}(\frac{u_0 v_0^2}{\nu}\frac{\partial u}{\partial y}) = \frac{u_0 v_0^2}{\nu}\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})\frac{\partial y}{\partial y'} = \frac{v_0^2}{\nu}\frac{\partial^2 u}{\partial y^2}\frac{v_0^2}{\nu}$

$$\frac{\partial^2 u'}{\partial y'^2} = \frac{u_0 v_0^4}{\nu^2}\frac{\partial^2 u}{\partial y^2}$$

$$g\beta_t(T' - T_\infty) = g\beta_t\vartheta(T_w - T_\infty$$

$$g\beta_c(C' - C_\infty) = g\beta_c\varphi(C_w - C_\infty$$

$$\frac{\sigma B_0^2 u'}{\rho} = \frac{\sigma B_0^2 u u_0}{\rho}$$

(3.24)

(3.25)

) (3.26)

) (3.27)

(3.28)

$\frac{\partial^3 u'}{\partial t \partial y'^2} = \frac{\partial}{\partial t'}(\frac{\partial^2 u'}{\partial y'^2}) = \frac{\partial}{\partial t'}(\frac{u_0 v_0^4}{\nu^2}\frac{\partial^2 u}{\partial y^2})$

$\frac{\partial^3 u'}{\partial t \partial y'^2} = \frac{u_0 v_0^4}{\nu^2}\frac{\partial}{\partial t'}(\frac{\partial^2 u}{\partial y^2}) = \frac{u_0 v_0^3}{\nu^2}\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial y^2}) \times \frac{\partial t}{\partial t'}$

$$\frac{\partial^3 u'}{\partial t \partial y'^2} = \frac{u_0 v_0^5}{\nu^3} \frac{\partial^3 u}{\partial t \partial y^2} \quad (3.29)$$

$\frac{\partial^3 u'}{\partial y'^3} = \frac{\partial}{\partial y'}(\frac{\partial^2 u'}{\partial y'^2}) = \frac{\partial}{\partial y'}(\frac{u_0 v_0^4}{\nu^2}\frac{\partial^2 u}{\partial y^2}) = \frac{u_0 v_4^3}{\nu^2}\frac{\partial}{\partial y}(\frac{\partial^2 u}{\partial y^2}) \times \frac{\partial y}{\partial y'}$

$$\frac{\partial^3 u'}{\partial y'^3} = \frac{u_0 v_0^5}{\nu^3}\frac{\partial^3 u}{\partial y^3}\frac{v_0}{\nu} = \frac{u_0 v_0^5}{\nu^3}\frac{\partial^3 u}{\partial y^3}$$

(3.30)

substituting equations (3.23) – (3.30) into equation (3.8) we have;

$$\frac{u_0 v_0^2}{\nu}\frac{\partial u}{\partial t} - \frac{u_0 v_0^2}{\nu}(1+\epsilon e^{nt})\frac{\partial u}{\partial y} = \frac{u_0 v_0^4}{\nu}\frac{\partial^2 u}{\partial y^2} + g\beta_t \vartheta(T_w - T_\infty) + g\beta_c \varphi(C_w - C_\infty)$$

$$-\frac{\sigma B_0^2 v_0 u}{\rho} - \frac{K_0}{\rho}(\frac{u_0 v_0^5}{\nu^3}\frac{\partial^3 u}{\partial t \partial y^2} - \frac{u_0 v_0^5}{\nu^3}(1+\epsilon e^{nt})\frac{\partial^3 u}{\partial y^3}) \quad (3.31)$$

dividing all through by $\frac{v_0^3}{\nu}$ to have;

$$\frac{\partial u}{\partial t} - (1+\epsilon e^{nt})\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \frac{g\beta_t \vartheta(T_w - T_\infty)}{v_0^3} + \frac{g\beta_c \varphi(C_w - C_\infty)}{v_0^3} - \frac{\sigma B_0^2 \nu}{\rho v_0^2}u$$

$$-\frac{K_0 v_0^2}{\rho \nu^2}(\frac{\partial^3 u}{\partial t \partial y^2} - (1+\epsilon e^{nt})\frac{\partial^3 u}{\partial y^3}) \quad (3.32)$$

Simplifying further to obtain:

$$\frac{\partial u}{\partial t} - (1+\epsilon e^{nt})\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + Gr\vartheta + Gm\varphi - M^2 u - \alpha(\frac{\partial^3 u}{\partial t \partial y^2} - (1+\epsilon e^{nt})\frac{\partial^3 u}{\partial y^3}) \quad (3.33)$$

where;

$Gr = \frac{g\beta_t (T_w - T_\infty)}{v_0^3}, Gm = \frac{g\beta_c (C_w - C_\infty)}{v_0^3}, M^2 = \frac{\sigma B_0^2 \nu}{\rho v_0^2}, \alpha = \frac{K_0 v_0^2}{\rho \nu^2}$

are the Grashof, mass Grashof, magnetic parameter, and viscoelastic parameter.

3.2.2 Non-dimensionlization of the energy equation of research problem one

$$\frac{\partial T}{\partial t'} = \frac{\partial T}{\partial \vartheta} \times \frac{\partial \vartheta}{\partial t'} = \frac{\partial T}{\partial \vartheta} \times \frac{\partial \vartheta}{\partial t} \times \frac{\partial t}{\partial t'} = (T_w - T_\infty) \times \frac{\partial \vartheta}{\partial t} \times \frac{v_0^2}{\nu}$$

$$\frac{\partial T}{\partial t'} = \frac{v_0^2(T_w - T_\infty)}{\nu}\frac{\partial \vartheta}{\partial t} \quad (3.34)$$

$$\frac{\partial T}{\partial y'} = \frac{\partial T}{\partial \vartheta} \times \frac{\partial \vartheta}{\partial y'} = \frac{\partial T}{\partial \vartheta} \times \frac{\partial \vartheta}{\partial y} \times \frac{\partial y}{\partial y'} = (T_w - T_\infty) \times \frac{\partial \vartheta}{\partial y} \times \frac{v_0}{\nu}$$

$$\frac{\partial T}{\partial y'} = \frac{v_0(T_w - T_\infty)}{\nu}\frac{\partial \vartheta}{\partial y}$$

$$\frac{\partial^2 T}{\partial y'^2} = \frac{\partial}{\partial y'}(\frac{\partial T}{\partial y'}) = \frac{\partial}{\partial y'}(\frac{v_0(T_w - T_\infty)}{\nu}\frac{\partial \vartheta}{\partial y}) = \frac{v_0(T_w - T_\infty)}{\nu}\frac{\partial}{\partial y'}(\frac{\partial \vartheta}{\partial y})$$

$$\frac{\partial^2 T}{\partial y'^2} = \frac{v_0(T_w - T_\infty)}{\nu}\frac{\partial}{\partial y}(\frac{\partial \vartheta}{\partial y})\frac{\partial y}{\partial y'} = \frac{v_0(T_w - T_\infty)}{\nu}\frac{\partial^2 \vartheta}{\partial y^2} \times \frac{v_0}{\nu}$$

$$\frac{\partial^2 T}{\partial y'^2} = \frac{v_0^2(T_w - T_\infty)}{\nu^2}\frac{\partial^2 \vartheta}{\partial y^2}$$

$$\beta\frac{^*u'}{\rho c_p}(T_\infty - T) = \frac{\beta\ ^*uu_0}{\rho c_p}(T_\infty - T$$

$$\frac{Q_0}{\rho c_p}(T - T_\infty) = \frac{Q_0}{\rho c_p}\vartheta(T_w - T_\infty$$

(3.35)

(3.36)

) (3.37)

) (3.38)

$$-\frac{\partial q_r}{\partial y'} = \frac{16\sigma_0 T_\infty^3 v_0^2 (T_w - T_\infty)}{3k_e \rho c_p \nu^2}\frac{\partial^2 \vartheta}{\partial y^2} \tag{3.39}$$

$\frac{\partial^2 C}{\partial y'^2} = \frac{\partial}{\partial y'}(\frac{\partial C}{\partial y'}) = \frac{\partial}{\partial y'}(\frac{\partial C}{\partial \varphi}\times\frac{\partial \varphi}{\partial y}\times\frac{\partial y}{\partial y'}) = \frac{\partial}{\partial y'}((C_w - C_\infty)\times\frac{\partial \varphi}{\partial y}\times\frac{v_0}{\nu})\ \frac{\partial^2 C}{\partial y'^2} = \frac{v_0(C_w - C_\infty)}{\nu}\frac{\partial}{\partial y}(\frac{\partial \varphi}{\partial y})\frac{\partial y}{\partial y'} = \frac{v_0(C_w - C_\infty)}{\nu}\frac{\partial^2 \varphi}{\partial y^2}\frac{v_0}{\nu}$

$$\frac{\partial^2 C}{\partial y'^2} = \frac{v_0^2 (C_w - C_\infty)}{\nu^2}\frac{\partial^2 \varphi}{\partial y^2} \tag{3.40}$$

putting equations (3.34) – (3.40) into (3.9) yields:

$$\frac{v_0^2(T_w - T_\infty)}{\nu}\frac{\partial \vartheta}{\partial t} - \frac{v_0^2(T_w - T_\infty)}{\nu}(1+\epsilon e^{nt})\frac{\partial \vartheta}{\partial y} = \frac{\alpha v_0^2 (T_w - T_\infty)}{\nu^2}\frac{\partial^2 \vartheta}{\partial y^2} + \frac{16\sigma_0 T_\infty^3 v_0^2 (T_w - T_\infty)}{3k_e \rho c_p \nu^2}\frac{\partial^2 \vartheta}{\partial y^2}$$

$$+\frac{\mu v_0^4}{\rho c_p \nu^2}(\frac{\partial u}{\partial y})^2 + \frac{Dk_T v_0^2 (C_w - C_\infty)}{c_s c_p \nu^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\beta\ {}^* u_0 (T_\infty - T_w)}{\rho c_p}u\vartheta + \frac{Q_0 (T_w - T_\infty)}{\rho c_p}\vartheta \tag{3.41}$$

dividing all through by $\frac{v_0^2(T_w - T_\infty)}{\nu}$ and simplify to obtain:

$$\frac{\partial \vartheta}{\partial t} - (1+\epsilon A e^{nt})\frac{\partial \vartheta}{\partial y} = \frac{\alpha}{\nu}\frac{\partial^2 \vartheta}{\partial y^2} + \frac{16\sigma_0 T_\infty^3}{3k_e \rho c_p \nu}\frac{\partial^2 \vartheta}{\partial y^2} + \frac{v_0^2}{c_p (T_w - T_\infty)}(\frac{\partial u}{\partial y})^2$$

$$+\frac{Dk_T (C_w - C_\infty)}{c_s c_p \nu (T_w - T_\infty)}\frac{\partial^2 \varphi}{\partial y^2} - \frac{\beta\ {}^* u_0 \nu}{\rho c_p v_0^2}u\vartheta + \frac{Q_0 \nu}{\rho c_p v_0^2}\vartheta \tag{3.42}$$

Simplifying further to obtain:

$$\frac{\partial \vartheta}{\partial t} - (1+\epsilon A e^{nt})\frac{\partial \vartheta}{\partial y} = (\frac{1+Rr}{Pr})\frac{\partial^2 \vartheta}{\partial y^2} + Ec(\frac{\partial u}{\partial y})^2 + Du\frac{\partial^2 \varphi}{\partial y^2} - \Delta u\vartheta + \delta\vartheta \tag{3.43}$$

where;

$Pr = \frac{\nu}{\alpha}, Rr = \frac{16\sigma_0 T_\infty^3}{3k_0 k}, Ec = \frac{v_0^2}{c_p(T_w - T_\infty)}, \Delta = \frac{\beta\ {}^* u_0 \nu}{\rho c_p v_0^2}, \delta = \frac{Q_0 \nu}{\rho c_p v_0^2}, Du = \frac{Dk_T (C_w - C_\infty)}{c_s c_p \nu (T_w - T_\infty)}$

are the Prandtl, thermal radiation term, Eckert, heat generation/absorption term, heat source/sink parameter and Dufour number.

3.2.3 Non-dimensionalization of the concentration equation of the research problem one

$$\frac{\partial C}{\partial t'} = \frac{\partial C}{\partial \varphi} \times \frac{\partial \varphi}{\partial t'} = \frac{\partial C}{\partial \varphi} \times \frac{\partial \varphi}{\partial t} \times \frac{\partial t}{\partial t'} = (C_w - C_\infty) \times \frac{\partial \varphi}{\partial t} \times \frac{v_0^2}{\nu}$$

$$\frac{\partial C}{\partial t'} = \frac{v_0^2(C_w - C_\infty)}{\nu}\frac{\partial \varphi}{\partial t} \quad (3.44)$$

$$\frac{\partial C}{\partial y'} = \frac{\partial C}{\partial \varphi} \times \frac{\partial \varphi}{\partial y'} = \frac{\partial C}{\partial \varphi} \times \frac{\partial \varphi}{\partial y} \times \frac{\partial y}{\partial y'} = (C_w - C_\infty) \times \frac{\partial \varphi}{\partial y} \times \frac{v_0}{\nu}$$

$$\frac{\partial C}{\partial y'} = \frac{v_0(C_w - C_\infty)}{\nu}\frac{\partial \varphi}{\partial} \quad (3.45)$$

$$K_c'(C - C_\infty) = \frac{K_c v_0^2}{\nu}\varphi(C_w - C_\infty) \quad (3.46)$$

substituting equations (3.44),(3.45),(3.46),(3.40) and (3.36) into the concentration equation (3.10) to obtain:

$$\frac{v_0^2(C_w - C_\infty)}{\nu}\frac{\partial \varphi}{\partial t} - \frac{v_0^2(C_w - C_\infty)}{\nu}(1 + \epsilon e^{nt})\frac{\partial \varphi}{\partial y} = D\frac{v_0^2(C_w - C_\infty)}{\nu^2}\frac{\partial^2 \varphi}{\partial y^2}$$
$$-\frac{K_c v_0^2}{\nu}\varphi(C_w - C_\infty) + \frac{Dk_T v_0^2(T_w - T_\infty)}{T_m \nu^2}\frac{\partial^2 \vartheta}{\partial y^2} \quad (3.47)$$

dividing all through by $\frac{v_0^2(C_w - C_\infty)}{\nu}$ yields:

$$\frac{\partial \varphi}{\partial t} - (1 + \epsilon e^{nt})\frac{\partial \varphi}{\partial y} = \frac{D}{\nu}\frac{\partial^2 \varphi}{\partial y^2} - K_c\varphi + \frac{Dk_T(T_w - T_\infty)}{T_m \nu(C_w - C - \infty)}\frac{\partial^2 \vartheta}{\partial y^2} \quad (3.48)$$

simplifying (3.48) further yields:

$$\frac{\partial \varphi}{\partial t} - (1 + \epsilon e^{nt})\frac{\partial \varphi}{\partial y} = \frac{1}{Sc}\frac{\partial^2 \varphi}{\partial y^2} - K_c\varphi + Sr\frac{\partial^2 \vartheta}{\partial y^2} \quad (3.49)$$

where;

$Sc = \frac{\nu}{D}, Sr = \frac{Dk_T(T_w - T_\infty)}{T_m \nu(C_w - C_\infty)}, K_r^2 = \frac{k_r'^2 \nu}{v_0^2}$

are Chemical reaction term, Schmidt, and Soret number.

3.2.4 Transformation of the boundary conditions of research problem one

$u^0 = u_0$ *and* $u^0 =$
$uu_0 \, u_0 = uu_0$

$$u = 1 \quad (3.50)$$

$$T = T_\infty + \epsilon(T_w - T_\infty)e^{n't'} \quad and \quad T' = \vartheta(T_w - T_\infty) + T_\infty$$
$$\implies T_\infty + \epsilon(T_w - T_\infty)e^{\frac{nv_0^2}{\nu}\times\frac{t\nu}{v_0^2}} = \vartheta(T_w - T_\infty) + T_\infty$$

simplifying the above further yields:

$$\vartheta = 1 + \epsilon e^{nt} \quad (3.51)$$

Also,

$$C = C_\infty + \epsilon(C_w - C_\infty)e^{n't'} \quad and \quad C = \varphi(C_w - C_\infty) + C_\infty$$
$$\implies C_\infty + \epsilon(C_w - C_\infty)e^{\frac{nv_0^2}{\nu}\times\frac{t\nu}{v_0^2}} = \varphi(C_w - C_\infty) + C_\infty$$

simplifying the above further yields:

$$\varphi = 1 + \epsilon e^{nt} \quad (3.52)$$

Also,

$$u' \longrightarrow 0 \quad when \quad u = \frac{u'}{u_0} \quad \implies \quad u' = uu_0$$

therefore,

$$0 = uu_0$$

$$\therefore u \longrightarrow 0 \qquad (3.53)$$

Again,

$$T \longrightarrow T_\infty \quad and \quad \vartheta = \frac{T - T_\infty}{T_w - T_\infty} \quad \implies \quad T = T_\infty + \vartheta(T_w - T_\infty)$$

$$T_\infty = \vartheta(T_w - T_\infty) + T_\infty$$

$$T_\infty - T_\infty = \vartheta(T_w - T_\infty)$$

simplifying the above further yields:

$$\therefore \vartheta \dashrightarrow 0 \qquad (3.54)$$

Finally,

$$C \longrightarrow C_\infty \quad and \quad \varphi = \frac{C - C_\infty}{C_w - C_\infty} \qquad \Longrightarrow \qquad C = C_\infty + \varphi(C_w - C_\infty)$$

$$C_\infty = \varphi(C_w - C_\infty) + C_\infty$$

$$C_\infty - C_\infty = \phi(C_w - C_\infty)$$

simplifying the above further yields:

$$\therefore \phi \dashrightarrow 0 \qquad (3.55)$$

From the above, transformation of governing equations momentum, concentration, energy and the boundary constraints are transformed to become:

$$\frac{\partial u}{\partial t} - (1 + \epsilon A e^{nt})\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + Gr\vartheta + Gm\varphi - M^2 u - A_1\left(\frac{\partial^3 u}{\partial t \partial y^2} - (1 + \epsilon e^{nt})\frac{\partial^3 u}{\partial y^3}\right) \tag{3.56}$$

$$\frac{\partial \vartheta}{\partial t} - (1 + \epsilon A e^{nt})\frac{\partial \vartheta}{\partial y} = \left(\frac{1 + Rr}{Pr}\right)\frac{\partial^2 \vartheta}{\partial y^2} + Ec\left(\frac{\partial u}{\partial y}\right)^2 + Du\frac{\partial^2 \varphi}{\partial y^2} + \delta\vartheta - \Delta u\vartheta \tag{3.57}$$

$$\frac{\partial \varphi}{\partial t} - (1 + \epsilon A e^{nt})\frac{\partial \varphi}{\partial y} = \frac{1}{S_c}\frac{\partial^2 \varphi}{\partial y^2} - k_r^2\varphi + S_r\frac{\partial^2 \vartheta}{\partial y^2} \tag{3.58}$$

here *Gr*, *Gm*, *Pr*, *Rr*, *Ec*, *Sc*, *kr*, *Du* ,*Sr*, α, Δ, and δ are thermal Grashof, mass Grashof, Prandtl number, radiation term, Eckert, Schmidt, chemical reaction term, Dufour, Soret, viscoelastic term, heat generation/absorption coefficient, and heat source/sink term respectively.

The boundary conditions are:

$$u = 1,\ \vartheta = \varphi = 1 + \epsilon e^{nt} \text{ at } y = 0 \qquad (3.59)$$

$$u \dashrightarrow 0,\ \vartheta \dashrightarrow 0,\ \phi \dashrightarrow 0, \text{ at } y \dashrightarrow \infty \qquad (3.60)$$

Engineering quantities interest are skin friction (C_f), local Nusselt (Nu) and Sherwood number (Sh). Skin friction coefficient is define as:

$$C_f = \frac{\tau_w}{\rho U_0 v_0} \quad where \;\; \tau_w = \mu \frac{\partial u'}{\partial y'} - \frac{K_0}{\rho}\left[\frac{\partial^3 u'}{\partial t' \partial y'^2} + \frac{\partial u'}{\partial y'^3}\right]$$

$$\therefore \tau_w = \mu \frac{U_0 v_0^2}{\nu}\frac{\partial u}{\partial y} - \frac{K_0}{\rho}\left[\frac{U_0 v_0^5}{\nu^3}\frac{\partial^3 u}{\partial t \partial y^2} + \frac{U_0 v_0^5}{\nu^3}\frac{\partial^3 u}{\partial y^3}\right]$$

$$\tau_w = \frac{\rho U_0 v - 0^2}{\nu}\frac{\partial u}{\partial y} - \frac{K_0}{\rho}\frac{U_0 v_0^5}{\nu^3}\left[\frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial y^3}\right]$$

Hence,

$$\tau_w = \mu \frac{\partial u}{\partial y}\,|_{y=0} - A_1\left[\frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial y^3}\right]$$

The Nusselt and Sherwood number are:

$$Nu = -\frac{K q_w}{T_w - T_\infty} \quad and \;\; Sh = \frac{D\left(\frac{\partial C}{\partial y}\right)_{y=0}}{C_w - C_\infty}$$

where

$$q_w = -K\left(\frac{\partial T}{\partial y}\right)_{y=0} - \frac{4\sigma_e}{3k_e}\left(\frac{\partial T^4}{\partial y}\right)_{y=0}$$

3.3 Qualitative analysis of problem one

In this section, the stability, existence and uniqueness of solution of the transformed dimensionless flow equations of unsteady free convective motion of a viscoelastic liquid past a half-infinite vertical plate at initial time t = 0. Nonlinear differential equations mostly are difficult to solve analytically. Thus, qualitative as well as numerical approach is important. This section aimed at examining if the dimensionless coupled flow equations is solvable, and if solvable, is the solution unique?. To perform the qualitative properties of the problem under investigation, this study is interested in the initial unsteady solution at t = 0 (that is, the steady state of the problem) for the transformed equations (3.56)-(3.58) subject to (3.59) and (3.60) is considered and$_{\epsilon} <<$1 that it could be neglected. Thus at t = 0, we have

$$\frac{d^2u}{dy^2} + \frac{du}{dy} + \alpha\frac{d^3u}{dy^3} + Gr\vartheta + Gm\varphi - M^2u = 0 \quad (3.61)$$

$$\left(\frac{1+Rr}{Pr}\right)\frac{d^2\vartheta}{dy^2} + \frac{d\vartheta}{dy} + Ec\left(\frac{du}{dy}\right)^2 + Du\frac{d^2\varphi}{dy^2} + \delta\vartheta - \Delta u\vartheta = 0 \quad (3.62)$$

$$\frac{1}{Sc}\frac{d^2\varphi}{dy^2} + \frac{d\varphi}{dy} - k_r^2\varphi + Sr\frac{d^2\vartheta}{dy^2} = 0 \quad (3.63)$$

subject to

$$u = 1,\ \vartheta = 1,\ \phi = 1\ ,at\ y = 0 \quad (3.64)$$

$$u \dashrightarrow 0,\quad \vartheta \dashrightarrow 0,\ \phi \dashrightarrow 0\ ,as\ y \dashrightarrow \infty \quad (3.65)$$

First reduce (3.61)-(3.63) subject to (3.64) and (3.65) to system of first order ordinary differential equations.

$$u = \beta_1,\ \frac{du}{dy} = \frac{d\beta_1}{dy} = \beta_2 \quad (3.66)$$

$$\frac{d^2u}{dy^2} = \frac{d}{dy}\left(\frac{du}{dy}\right) = \frac{d\beta_2}{dy} = \beta_3,\quad \frac{d^3u}{dy^3} = \frac{d}{dy}\left(\frac{d^2u}{dy^2}\right) = \frac{d\beta_3}{dy}$$

$$\vartheta = \beta_4,\quad \frac{d\vartheta}{dy} = \frac{d\beta_4}{dy} = \beta_5,\quad \frac{d^2\vartheta}{dy^2} = \frac{d}{dy}\left(\frac{d\vartheta}{dy}\right) = \frac{d\beta_5}{dy}$$

$$\varphi = \beta_6,\quad \frac{d\varphi}{dy} = \frac{d\beta_6}{dy} = \beta_7,\quad \frac{d^2\varphi}{dy^2} = \frac{d}{dy}\left(\frac{d\varphi}{dy}\right) = \frac{d\beta_7}{dy}$$

(3.67)

(3.68)

(3.69)

putting equations (3.66)-(3.69) into (3.61)-(3.63) yields

$$\beta_3 + \beta_2 + A_1\frac{d\beta_3}{dy} + G\beta_4 + Gm\beta_6 - M\beta_1 = 0 \quad (3.70)$$

$$\left(\frac{1+Rr}{Pr}\right)\frac{d\beta_5}{dy} + \beta_5 + E(\beta_2)^2 + Du\frac{d\beta_7}{dy} + \beta_4 - \Delta\beta_4 = 0 \quad (3.71)$$

$$\frac{1}{Sc}\frac{d\beta_7}{dy} + \beta_7 - k_r^2\beta_6 + Sr\frac{d\beta_5}{dy} = 0 \quad (3.72)$$

simplifying equation (3.70) gives

$$\frac{d\beta_3}{dy} = \frac{M\beta_1 - G\beta_4 - Gm\beta_6 - \beta_2 - \beta_3}{A_1} \quad (3.73)$$

Also, simplifying equation (3.71)

$$\left(\frac{1+Rr}{Pr}\right)\frac{d\beta_5}{dy} = \Delta\beta_4 - \beta_4 - E(\beta_2)^2 - \beta_5 - Du\frac{d\beta_7}{dy}$$

$$\frac{d\beta_5}{dy} = \frac{\Delta\beta_4 - \beta_4 - E(\beta_2)^2 - \beta_5}{\frac{1+Rr}{Pr})} - \frac{Du}{\frac{1+Rr}{Pr})}\frac{d\beta_7}{dy} \quad (3.74)$$

putting equations (3.74) into (3.72), we have

$$\frac{1}{Sc}\frac{d\beta_7}{dy} + \beta_7 - k_r^2\beta_6 + Sr\left(\frac{\Delta\beta_4 - \beta_4 - E(\beta_2)^2 - \beta_5}{\frac{1+Rr}{Pr})} - \frac{Du}{\frac{1+Rr}{Pr})}\frac{d\beta_7}{dy}\right) = 0$$

$$\frac{1}{Sc}\frac{d\beta_7}{dy} + \beta_7 - k_r^2\beta_6 + \frac{Sr\Delta\beta_4 - Sr\beta_4 - SrE(\beta_2)^2 - s\beta_5}{\frac{1+Rr}{Pr})} - \frac{SrDu}{\frac{1+Rr}{Pr})}\frac{d\beta_7}{dy} = 0$$

$$\frac{d\beta_7}{dy}\left(\frac{1}{Sc} - \frac{SrDu}{\frac{1+Rr}{Pr})}\right) = k_r^2\beta_6\beta_7 - \frac{(Sr\Delta\beta_4 - Sr\beta_4 - Sr\beta_4 - SrE(\beta_2)^2 - S\beta_5)}{\frac{1+Rr}{Pr})}$$

Simplifying further we have

$$\frac{d\beta_7}{dy} = \frac{k_r^2\beta_6 Sc\ \frac{1+Rr}{Pr})}{\frac{1+Rr}{Pr}) - ScSrDu} - \frac{\beta_7 Sc\ \frac{1+Rr}{Pr})}{\frac{1+Rr}{Pr}) - ScSrDu}$$

$$-\frac{(Sr\Delta\beta_4 - Sr\beta_4 - Sr\beta_4 - SrE(\beta_2)^2 - S\beta_5)}{\frac{1+Rr}{Pr}) - ScSrDu} \quad (3.75)$$

Putting $\left[\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\right]$ equation (3.75) into (3.72) yields

$$\frac{1}{Sc}\quad \frac{k_r^2\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}-\frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}-\frac{(Sr\Delta\beta_1\beta_4-Sr\delta\beta_4-Sr\beta_4-SrEc(\beta_2)^2-S\beta_5)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}$$

$$\beta_7-k_r^2\beta_6+Sr\frac{d\beta_5}{dy}=0$$

$$\frac{d\beta_5}{dy}=-\frac{1}{ScSr}\left[\frac{k_r^2\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}-\frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}-\frac{(Sr\Delta\beta_1\beta_4-Sr\delta\beta_4-Sr\beta_4-SrEc(\beta_2)^2-S\beta_5)}{\left(\frac{1+Rr}{Pr}\right)-ScSrDu}\right]$$

$$+\frac{k_r^2\beta_6}{Sr}-\frac{\beta_7}{Sr}$$

$$\frac{d\beta_5}{dy}=\frac{k_r^2\beta_6}{Sr}-\frac{\beta_7}{Sr}-\frac{k_r^2\beta_6 Sc\ \frac{1+Rr}{Pr})}{ScSr\left[\left(\frac{1+Rr}{Pr}\right)-ScSrDu\right]}+\frac{\beta_7 Sc\ \frac{1+Rr}{Pr})}{ScSr\left[\left(\frac{1+Rr}{Pr}\right)-ScSrDu\right]}$$
$$+\frac{(ScSr\Delta\beta_1\beta_4-ScSr\delta\beta_4-ScS\beta_5-ScSrEc(\beta_2)^2)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right)-ScSrDu\right]}$$

(3.76)

Theorem 3.1: Let u,ϑ and ϕ be continuous function at all points in some neighborhood and $Pr>0, Rr>0, Sr>0, Du>0, A_1>0, \delta>0, \Delta>0, Sc>0, kr>0, Ec>0, M>0, Gr>0$ and $Gm>0$, then there exist a unique solution for

coupled nonlinear boundary value problem.

$$\frac{d^2u}{dy^2}+\frac{du}{dy}+A_1\frac{d^3u}{dy^3}+Gr\vartheta+Gm\varphi-M^2u=0$$

$$\left(\frac{1+Rr}{Pr}\right)\frac{d^2\vartheta}{dy^2}+\frac{d\vartheta}{dy}+Ec\left(\frac{du}{dy}\right)^2+Du\frac{d^2\varphi}{dy^2}+\delta\vartheta-\Delta u\vartheta=0$$

$$\frac{1}{Sc}\frac{d^2\varphi}{dy^2}+\frac{d\varphi}{dy}-k_r^2\varphi+Sr\frac{d^2\vartheta}{dy^2}=0$$

subject to

$u=1,\ \vartheta=1,\ \phi=1$, at $y=0$ $u\longrightarrow 0,\ \vartheta \longrightarrow 0,\ \phi\longrightarrow 0$, as $y-\rightarrow\infty$

on some interval k $y-y_0$ k$\leq a$, k y_0-y k$\leq b$ provided 3 k such that $k=$

$max(0,1,P_2,...,P_{12})$ and $0<k<\infty$.

PROOF: Writing the equations (3.66)-(3.69) in a compact form as:

$$
\begin{bmatrix}
\frac{d\beta_1}{dy}\\
\frac{d\beta_2}{dy}\\
\frac{d\beta_3}{dy}\\
\frac{d\beta_4}{dy}\\
\frac{d\beta_5}{dy}\\
\frac{d\beta_6}{dy}\\
\frac{d\beta_7}{dy}
\end{bmatrix}
=
\begin{bmatrix}
\beta_2\\
\beta_3\\
\frac{M^2\beta_1 - Gr\beta_4 - Gm\beta_6 - \beta_2\beta_3}{A_1}\\
\beta_5\\
\frac{k_r^2\beta_6}{Sr} - \frac{\beta_7}{Sr} - \frac{k_r^2\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} + \frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} + \frac{(ScSr\Delta\beta_1\beta_4 - ScSr\beta_4 - ScS\beta_5 - ScSrEc(\beta_2)^2)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}\\
\beta_7\\
\frac{d\beta_7}{dy} = \frac{k_r^2\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{(Sr\Delta\beta_1\beta_4 - Sr\beta_4 - Sr\beta_4 - SrEc(\beta_2)^2 - S\beta_5)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}
\end{bmatrix}
\tag{3.77}
$$

satisfying the conditions

$$
\begin{bmatrix}
\beta_1(0)\\
\beta_2(0)\\
\beta_3(0)\\
\beta_4(0)\\
\beta_5(0)\\
\beta_6(0)\\
\beta_7(0)
\end{bmatrix}
=
\begin{bmatrix}
1\\
\alpha\\
\gamma\\
1\\
\beta\\
1\\
Q
\end{bmatrix}
$$

Consider $\frac{\partial u_i}{\partial\beta_j}$ such that *i,j* = 1(1)7 to denote the nonlinear functions on the RHS of equation (3.77), as *i* = 1 and *j* = *counts*. when *i* = 1 and *j* = *counts*, we have

$$u_1 = \beta_2$$

$$\left|\frac{\partial u_1}{\partial\beta_1}\right| = \left|\frac{\partial u_1}{\partial\beta_3}\right| = \left|\frac{\partial u_1}{\partial\beta_4}\right| = \left|\frac{\partial u_1}{\partial\beta_5}\right| = \left|\frac{\partial u_1}{\partial\beta_6}\right| = \left|\frac{\partial u_1}{\partial\beta_7}\right| = 0 < \infty$$

$$\left|\frac{\partial u_1}{\partial\beta_2}\right| = 1 < \infty$$

when i = 2 and j = *counts*, we have

$$u_2 = \beta_3$$

$$\left|\frac{\partial u_2}{\partial \beta_1}\right| = \left|\frac{\partial u_2}{\partial \beta_2}\right| = \left|\frac{\partial u_2}{\partial \beta_4}\right| = \left|\frac{\partial u_2}{\partial \beta_5}\right| = \left|\frac{\partial u_2}{\partial \beta_6}\right| = \left|\frac{\partial u_2}{\partial \beta_7}\right| = 0 < \infty$$

$$\left|\frac{\partial u_2}{\partial \beta_3}\right| = 1 < \infty$$

when i = 3 and j = *counts*, we have

$$U_3 = \frac{M^2\beta_1 - Gr\beta_4 - Gm\beta_6 - \beta_2 - \beta_3}{A_1}$$

$$\left|\frac{\partial u_3}{\partial \beta_5}\right| = \left|\frac{\partial u_3}{\partial \beta_7}\right| = 0,\ \left|\frac{\partial u_3}{\partial \beta_1}\right| = \left|\frac{M^2}{\alpha}\right| = P_1 < \infty$$

$$\left|\frac{\partial u_3}{\partial \beta_2}\right| = |-1| = 1 < \infty,\ \left|\frac{\partial u_3}{\partial \beta_3}\right| = |-1| = 1 < \infty$$

$$\left|\frac{\partial u_3}{\partial \beta_4}\right| = |-Gr| = P_2 < \infty,\ \left|\frac{\partial u_3}{\partial \beta_6}\right| = |-Gm| = P_3 < \infty$$

when i = 4, and j = *counts*, we have

$$u_4 = \beta_5$$

$$\left|\frac{\partial u_4}{\partial \beta_1}\right| = \left|\frac{\partial u_4}{\partial \beta_2}\right| = \left|\frac{\partial u_4}{\partial \beta_3}\right| = \left|\frac{\partial u_4}{\partial \beta_4}\right| = \left|\frac{\partial u_4}{\partial \beta_6}\right| = \left|\frac{\partial u_4}{\partial \beta_7}\right| = 0 < \infty$$

$$\left|\frac{\partial u_4}{\partial \beta_5}\right| = 1 < \infty$$

when i = 5 and j = *counts* we have

$$u_5 = \frac{k_r^2\beta_6}{Sr} - \frac{\beta_7}{Sr} - \frac{k_r^2\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} + \frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}$$
$$+ \frac{\left(ScSr\Delta\beta_1\beta_4 - ScSr\delta\beta_4 - ScSr\beta_5 - ScSrEc(\beta_2)^2\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}$$

Utilizing the properties of absolute value of real numbers as described by Robert and Murray (2002) defined by

$$|a+b| \leq |a| + |b|$$

$$|\frac{\partial u_5}{\partial \beta_1}| = |\frac{ScSr\Delta\beta_4}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}| \leq \frac{ScSr\Delta|\beta_4|}{|ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]|} = P_4 < \infty$$

$$|\frac{\partial u_5}{\partial \beta_2}| = |\frac{-2ScSrEc\beta_2}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}| \leq \frac{ScSrEc|-2||\beta_2|}{|ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]|} = P_5 < \infty$$

$$|\frac{\partial u_5}{\partial \beta_4}| = |\frac{ScSr\Delta\beta_1 - ScSr\delta}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}| \leq \frac{ScSr\Delta|\beta_1| - ScSr\delta}{|ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]|} = P_6 < \infty$$

$$|\frac{\partial u_5}{\partial \beta_5}| = |\frac{-ScSr}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}| \leq \frac{|-ScSr|}{|ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]|} = P_7 < \infty$$

$$|\frac{\partial u_5}{\partial \beta_6}| = |\frac{k_r^2}{Sr} - \frac{k_r^2Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}| \leq \frac{|k_r^2|}{|Sr|} - \frac{|k_r^2Sc\left(\frac{1+Rr}{Pr}\right)|}{|ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} = P_8 < \infty$$

$$|\frac{\partial u_5}{\partial \beta_7}| = |-\frac{1}{Sr} \leq \frac{|-1|}{|Sr|} = P_9 < \infty$$

And,

$$|\frac{du_5}{d\beta_7}| = 0 < \infty$$

when *i* = 6 and *j* = *counts*, we have

$$u_6 = \beta_7$$

$$|\frac{du_6}{d\beta_1}| = |\frac{du_6}{d\beta_2}| = |\frac{\partial u_6}{\partial \beta_3}| = |\frac{\partial u_6}{\partial \beta_4}| = |\frac{\partial u_6}{\partial \beta_5}| = |\frac{\partial u_6}{\partial \beta_6}| = 0 < \infty$$

$$|\frac{\partial u_6}{\partial \beta_7}| = 1 < \infty$$

When *i* = 7 and *j* = *counts* we have

$$u_7 = \frac{k_r^2\beta_6Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{\beta_7Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{(Sr\Delta\beta_1\beta_4 - Sr\delta\beta_4 - Sr\beta_4 - SrEc(\beta_2)^2 - S\beta_5)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}$$

$$|\frac{\partial u_7}{\partial \beta_1}| = |\frac{-ScSr\Delta\beta_4}{\frac{1+Rr}{Pr}) - ScSrDu}| \leq \frac{|2||ScSrEc\beta_2|}{|\frac{1+Rr}{Pr}) - ScSrDu|} = P_{11} < \infty$$

$$|\frac{\partial u_7}{\partial \beta_4}| = |-\frac{ScSr\Delta\beta_1}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} + \frac{ScSr\delta}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}| \leq \frac{|-ScSr\Delta\beta_1|}{|\left(\frac{1+Rr}{Pr}\right) - ScSrDu|} + \frac{|ScSr\delta|}{|\left(\frac{1+Rr}{Pr}\right) - ScSrDu|} = P_{12} < \infty$$

$$|\frac{\partial u_7}{\boldsymbol{\partial} \ _5}| = |\frac{ScSr}{\frac{1+Rr}{Pr}) - ScSrDu}| \leq \frac{|ScSr|}{| \ \frac{1+Rr}{Pr}) - ScSrDu|} = P_{13} < \infty$$

$$|\frac{\partial u_7}{\boldsymbol{\partial} \ _6}| = |\frac{k_r^2 Sc \ \ \frac{1+Rr}{Pr})}{\frac{1+Rr}{Pr}) - ScSrDu}| \leq \frac{|k_r^2 Sc \ \ \frac{1+Rr}{Pr}) \ |}{| \ \frac{1+Rr}{Pr}) - ScSrDu|} = P_{14} < \infty$$

$$|\frac{\partial u_7}{\boldsymbol{\partial} \ _7}| = |\frac{-Sc \ \ \frac{1+Rr}{Pr})}{\frac{1+Rr}{Pr}) - ScSrDu}| \leq \frac{| - Sc \ \ \frac{1+Rr}{Pr}) \ |}{| \ \frac{1+Rr}{Pr}) - ScSrDu|} = P_{15} < \infty$$

And,

$$|\frac{\partial u_7}{\boldsymbol{\partial} \ _3}| = 0 < \infty$$

Thus, we have shown that

$$\frac{\partial u_i}{\boldsymbol{\partial} \ _j} \leq K \ such \ that \ i,j = 1(1)7$$

Obviously,

$$|\frac{\partial u_i}{\boldsymbol{\partial} \ _j}|_{i,j=1(1)7} \ is \ bounded \ for \ i = 1,2,...,7$$

there exists K such that $K = max(0,1,P_1,P_2,P_3,P_4,P_5,P_6,P_7,P_8,P_9,P_{10},P_{11},P_{12},P_{13},P_{14},P_{15})$ and $0 < K < \infty$. Therefore $u_i(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7)$ are Lipschitz continuous.

Hence, there exists a unique solution for the system of coupled differential equation.

3.3.1Stability analysis

Stability analysis for the solutions of differential equations that describe dynamical systems are of various categories. The common one is the stability of solutions near to a point of equilibrium. The Lyapunov theory is used to discuss stability at equilibrium. Consider the solution near the equilibrium point x_e stay near x_e forever, thus x_e is Lyapunov stable. Hence, if x_e is Lyapunov stable and solutions near x_e still converge to x_e, thus x_e is asymtotically stable.

Theorem 3.2: Poincare-Lyapunov theorem states that if eigenvalues of Jacobian matrix evaluated at the fixed point are not equal zero or are not pure imaginary numbers, then the trajectories of the system around the critical point behave the same way as the trajectories of the associated linear autonomous system which are equivalent to that of its nonlinear system. The theorem can be further classified based on the following.

(i) Nature of roots $\lambda_1,...,\lambda_6,\lambda_7$ of characteristics equation.

(ii) Nature of the critical point ($\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7$) of the autonomous system of nonlinear differential equation and

(iii) Stability of critical point ($\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7$)

We consider the system of first order differential equations with critical point of β_i' then, we also have the following critical points (0,0,0,0,0,0,0) and (0,1,0,0,0,0,0) Table 3.1: Nature of root(s) of characteristic equation, critical point and stability of critical point

Nature of roots $\lambda_1,\ldots,\lambda_7$ of characteristic equation	Nature of the critical point, of the autonomous system of nonlinear differential equations.	Stability of critical point
Real unequal and all eigenvalues are positive signs. $\lambda_1 = 1$, $\lambda_2 = 2$	Node	Asymptotically unstable
Real, unequal and all eigenvalues are negative signs. $\lambda_1 = -1$, $\lambda_2 = -2$	Node	Asymptotically stable
Real, unequal and all eigenvalues are opposite signs. $\lambda_1 = 1$, $\lambda_2 = -2$	Saddle point	Unstable
Real, equal and all eigenvalues are opposite signs. $\lambda_1 = 1$, $\lambda_2 = 1$	Node	Asymptotically unstable
Real, equal and all eigenvalues are negative signs. $\lambda_1 = 1$, $\lambda_2 = -1$ or $\lambda_1 = 0$, $\lambda_2 = -2$ (Zero is also a real number)	Node	Asymptotically stable
Conjugate complex with positive real part signs. $\lambda_1 = 1+2i$, $\lambda_2 = 1 - 2i$. Real part of the roots are positive	Spiral point	Asymptotically unstable

Conjugate complex with negative real part signs. $\lambda_1 = -1 + 2i$, $\lambda_2 - 1 - 2i$. Real part of the roots are positive	Spiral point	Asymptotically stable
Conjugate complex with pure imaginary $\lambda_1 = 2i$, $\lambda_2 = -2i$	Center	Stable but not asymptotically stable

Table cited from Shepley (1984)

Let:

$$\beta_1' = u_1(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \beta_2$$

$$\beta_2' = u_2(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \beta_3$$

$$\beta_3' = u_3(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \frac{M\beta_1 - Gr\beta_4 - Gm\beta_6 - \beta_2 - \beta_3}{\alpha}$$

$$\beta_4' = u_4(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \beta_5$$

$$\beta_5' = u_5(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \frac{k_r\beta_6}{Sr} - \frac{\beta_7}{Sr} - \frac{k_r\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} + \frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]} + \frac{(ScSr\Delta\beta_4 - ScSr\beta_4 - ScS\beta_5 - ScSrEc(\beta_2)^2)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}$$

$$\beta_6' = u_6(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) = \beta_7$$

$$\beta_7' = \frac{k_r\beta_6 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{\beta_7 Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} - \frac{(Sr\Delta\beta_4 - Sr\beta_4 - Sr\beta_4 - SrEc(\beta_2)^2 - S\beta_5)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}$$

Hence, the necessary and sufficient condition of Jacobian matrix is satisfied and the Jacobian matrix takes the form

$$A = \begin{bmatrix} \frac{\partial u_1}{\partial\beta_1} & \frac{\partial u_1}{\partial\beta_2} & \frac{\partial u_1}{\partial\beta_3} & \frac{\partial u_1}{\partial\beta_4} & \frac{\partial u_1}{\partial\beta_5} & \frac{\partial u_1}{\partial\beta_6} & \frac{\partial u_1}{\partial\beta_7} \\ \frac{\partial u_2}{\partial\beta_1} & \frac{\partial u_2}{\partial\beta_2} & \frac{\partial u_2}{\partial\beta_3} & \frac{\partial u_2}{\partial\beta_4} & \frac{\partial u_2}{\partial\beta_5} & \frac{\partial u_2}{\partial\beta_6} & \frac{\partial u_2}{\partial\beta_7} \\ \frac{\partial u_3}{\partial\beta_1} & \frac{\partial u_3}{\partial\beta_2} & \frac{\partial u_3}{\partial\beta_3} & \frac{\partial u_3}{\partial\beta_4} & \frac{\partial u_3}{\partial\beta_5} & \frac{\partial u_3}{\partial\beta_6} & \frac{\partial u_3}{\partial\beta_7} \\ \frac{\partial u_4}{\partial\beta_1} & \frac{\partial u_4}{\partial\beta_2} & \frac{\partial u_4}{\partial\beta_3} & \frac{\partial u_4}{\partial\beta_4} & \frac{\partial u_4}{\partial\beta_5} & \frac{\partial u_4}{\partial\beta_6} & \frac{\partial u_4}{\partial\beta_7} \\ \frac{\partial u_5}{\partial\beta_1} & \frac{\partial u_5}{\partial\beta_2} & \frac{\partial u_5}{\partial\beta_3} & \frac{\partial u_5}{\partial\beta_4} & \frac{\partial u_5}{\partial\beta_5} & \frac{\partial u_5}{\partial\beta_6} & \frac{\partial u_5}{\partial\beta_7} \\ \frac{\partial u_6}{\partial\beta_1} & \frac{\partial u_6}{\partial\beta_2} & \frac{\partial u_6}{\partial\beta_3} & \frac{\partial u_6}{\partial\beta_4} & \frac{\partial u_6}{\partial\beta_5} & \frac{\partial u_6}{\partial\beta_6} & \frac{\partial u_6}{\partial\beta_7} \\ \frac{\partial u_7}{\partial\beta_1} & \frac{\partial u_7}{\partial\beta_2} & \frac{\partial u_7}{\partial\beta_3} & \frac{\partial u_7}{\partial\beta_4} & \frac{\partial u_7}{\partial\beta_5} & \frac{\partial u_7}{\partial\beta_6} & \frac{\partial u_7}{\partial\beta_7} \end{bmatrix}$$

This becomes;

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ P_1 & 1 & 1 & P_2 & 0 & P_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ P_4 & P_5 & 0 & P_6 & P_7 & P_8 & P_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ P_{10} & P_{11} & 0 & P_{12} & P_{13} & P_{14} & P_{15} \end{bmatrix}$$

where;

$$P_1 = \frac{M^2}{A_1},\ P_2 = -Gr,\ P_3 = -Gm,\ P_4 = \frac{ScSr\Delta_4}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}$$

$$P_5 = -\frac{2ScSrEc\Delta_2}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]},\ P_6 = \frac{ScSr\Delta_1 - ScSr\delta}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]}$$

$$P_7 = -\frac{ScSr}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]},\ P_8 = \frac{k_r^2}{Sr} - \frac{k_r^2Sc\left(\frac{1+Rr}{Pr}\right)}{ScSr\left[\left(\frac{1+Rr}{Pr}\right) - ScSrDu\right]},\ P_9 = -\frac{1}{Sr}$$

$$P_{10} = -\frac{ScSr\Delta_4}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu},\ P_{11} = \frac{2ScSrEc\Delta_2}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu},$$

$$P_{12} = -\frac{ScSr\Delta_1}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu} + \frac{ScSr\delta}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}$$

$$P_{13} = \frac{ScSr}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu},\ P_{14} = \frac{k_r^2Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu},\ P_{15} = -\frac{Sc\left(\frac{1+Rr}{Pr}\right)}{\left(\frac{1+Rr}{Pr}\right) - ScSrDu}$$

Using the extracted parameter from Alao et al. (2016), the Jacobian matrix for the parameter $Gr = 2.0, Pr = 0.71, Ec = 0.001, Sc = 0.6, Rr = kr = A = 0.5, Du =$

$0.2, Sr = 0.5, t = 1.0, M = 1.0, \Delta = 0.01, \delta = 0.001$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ A_1 & 1 & 1 & A_2 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ A_4 & A_5 & 0 & A_6 & A_7 & A_8 & A_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ A_{10} & A_{11} & 0 & A_{12} & A_{13} & A_{14} & A_{15} \end{bmatrix}$$

where;

$$A_1 = 0.25,\ A_2 = -2,\ A_3 = -2,\ A_4 = \frac{0.003}{0.61578}\beta_4,\ A_5 = \frac{0.0006}{0.61578}\beta_2,\ A_6 = \frac{0.003\beta_1 - 0.0003}{0.61578}$$

$$A_7 = -0.487186,\ A_8 = -0.0146,\ A_9 = -2,\ A_{10} = \frac{0.003}{2.0526}\beta_4,\ A_{11} = \frac{0.0006}{2.0526}\beta_2,$$

$$A_{12} = -\frac{0.003}{2.0526}\beta_1 + 0.00014,\ A_{13} = 0.1461,\ A_{14} = 0.1543,\ A_{15} = -0.6175$$

Evaluating the Jacobian matrix at the critical point

$$(\beta_1 = 0, \beta_2 = 0, \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 0, \beta_7 = 0)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 1 & & & & \\ 0 & 0 & 0 & 0.0004871 & -0.487186 & -0.0146 & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}$$

0 0 1 0 0 0 0

−2 0 −2 0

0 0 0 0 1 0 0

−2

0 0 0 0 0 0 1

00014 0.14610.1543 −0.6175

The eigenvalues of the matrix above can be calculated with $|A - \lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & & & & & & \\ 0 & -\lambda & & & & & \\ 0.25 & 1 & 1-\lambda & & & & \\ 0 & 0 & 0 & & & & \\ & & & & & & \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 0.00014 & 0.1461 & 0.1543 & -0.6175-\lambda \end{vmatrix}$$

$(0.4382502550\lambda^2 + 1.104686\lambda^3 - 0.07306052405\lambda + \lambda^4 + 0.00007720353)(\lambda^2 - \lambda^3 + \lambda + 0.25)$

$\lambda_1 = 0.001063509260,\ \lambda_2 = 0.1229392294,\ \lambda_3 = -0.6143443693{+}0.4615856068I,\ \lambda_4 = -0.6143443693{-}0.4615856068I,\ \lambda_5 = 1.682615007,\ \lambda_6 = -0.3413075034{+}0.1791296800I,\ \lambda_7 = -0.3413075034 - 0.1791296800I$

1 0 0 0 0 0 1 0 0 0 0

−2 0 −2 0

−λ 1 0 0

0 0 0 0.0004871 −0.487186 − λ −0.0146 −2

Evaluating the Jacobian matrix at the critical point

$$(\beta_1 = 0, \beta_2 = 1, \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 0, \beta_7 = 0)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.25 & 0.0009743 & 0 & 0.0004871 & -0.487186 & -0.0146 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.00014 & 0.1461 & 0.1543 & -0.6175 \end{bmatrix}$$

The eigenvalues is gotten using $|A - \lambda I| = 0$. Using the MAPLE software to solve the eigenvalues to obtain

$-0.017887528\lambda - 1.000000000\lambda^7 + 0.000019301 + 0.639335428\lambda^3 + 0.035091298\lambda^2 + 1.666435747\lambda^5 + 1.865996779\lambda^4 - 0.1046859994\lambda^6$ $\lambda_1 = 0.001081359289$, $\lambda_2 = 0.1226328147$, $\lambda_3 = 1.682396371$, $\lambda_4 = -0.6144017664 + 0.46348955581I$, $\lambda_5 = -0.3409965060 + 0.1725658636I$, $\lambda_6 = -0.3409965060 - 0.1725658636I$, $\lambda_7 = -0.6144017664 - 0.46348955981I$

Most of the eigenvalues are conjugate complex with negative real part signs and the remaining three are real, unequal positive sign. Based on the theorem (PoincareLyapunov theorem) earlier stated and the eigenvalues in table () the stability of critical point is Asymptotically Stable.

3.4 Solution technique to problem one

The dimensionless system of PDEs is solved by utilizing the SRM. This is a numerical techniques which follows the iterative steps of Gauss-siedel relaxation techniques to linearize and decoupled the coupled system of equations. The linearized equations are further discretized and solved by utilizing Chebyshev pseudo-spectral approach (Motsa, 2012). The linear functions are iterated at current level given by r+1 while non-linear functions are considered to be known at existing level of iteration given by r. The basic procedure of SRM are highlighted as follows:

(i) decouple and rearrange the nonlinear equations in Gauss-Seidel approach.

(ii) discretize resulting the linear equations.

(iii) the discretized linear equations are iteratively solved by utilizing Chebyshev pseudo-spectral approach.

First rearrange the transformed flow equations to apply SRM. This gives

$$\frac{\partial u}{\partial t}+A_1\frac{\partial^3 u}{\partial t\partial y^2}=(1+\epsilon Ae^{nt})\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}+Gr\vartheta+Gm\varphi-M^2u-A_1(1+\epsilon e^{nt})\frac{\partial^3 u}{\partial y^3} \quad (3.78)$$

$$\frac{\partial \vartheta}{\partial t}=(1+\epsilon Ae^{nt})\frac{\partial \vartheta}{\partial y}+\left(\frac{1+Rr}{Pr}\right)\frac{\partial^2\vartheta}{\partial y^2}+Ec\left(\frac{\partial u}{\partial y}\right)^2+Du\frac{\partial^2\varphi}{\partial y^2}+\delta\vartheta-\Delta u\vartheta \quad (3.79)$$

$$\frac{\partial \varphi}{\partial t}=(1+\epsilon Ae^{nt})\frac{\partial \varphi}{\partial y}+\frac{1}{Sc}\frac{\partial^2\varphi}{\partial y^2}-k_r^2\varphi+Sr\frac{\partial^2\vartheta}{\partial y^2} \quad (3.80)$$

subject to

$$u=1,\ \vartheta=\varphi=1+\epsilon e^{nt}\ \text{at}\ y=0 \quad (3.81)$$

$$u \dashrightarrow 0,\ \vartheta \dashrightarrow 0,\ \phi \dashrightarrow 0,\ \text{at}\ y \dashrightarrow \infty \quad (3.82)$$

Utilizing the SRM on the non-linear coupled PDEs (3.78)-(3.80) leads to:

$$\frac{\partial u_{r+1}}{\partial t}+A_1\frac{\partial^2 u_{r+1}}{\partial t^2}=\beta\ \frac{\partial^3 u_{r+1}}{\partial y^3}+\frac{\partial^2 u_{r+1}}{\partial y^2}+\beta\ \frac{\partial u_{r+1}}{\partial y}+Gr\vartheta_r+Gm\varphi_r-M^2u_{r+1} \quad (3.83)$$

$$Pr\frac{\partial \vartheta_{r+1}}{\partial t}=(1+Rr)\frac{\partial^2\vartheta_{r+1}}{\partial y^2}$$

$$+Pr\beta\ \frac{\partial \vartheta_{r+1}}{\partial y}+PrEc\left(\frac{\partial u_{r+1}}{\partial y}\right)^2+PrDu\frac{\partial^2\varphi_r}{\partial y^2}+Pr\delta\vartheta_{r+1}+Pr\Delta u_r\vartheta_{r+1} \quad (3.84)$$

$$Sc\frac{\partial \varphi_{r+1}}{\partial t}=\frac{\partial^2\varphi_{r+1}}{\partial y^2}+Sc\beta\ \frac{\partial \varphi_{r+1}}{\partial y}-Sck_r^2\varphi_{r+1}+ScSr\frac{\partial^2\vartheta_{r+1}}{\partial y^2} \quad (3.85)$$

subject to

$$u_{r+1}(0,t)=1,\ \ \vartheta_{r+1}(0,t)=1+\epsilon e^{nt},\ \ \varphi_{r+1}(0,t)=1+\epsilon e^{nt} \quad (3.86)$$

$$u_{r+1}(\infty,t)=0,\ \ \vartheta_{r+1}(\infty,t)=0,\ \ \varphi_{r+1}(\infty,t)=0 \quad (3.87)$$

where $\beta = 1+\epsilon Ae^{nt}$

setting

$$a_{0,r} = \beta = 1+\epsilon Ae^{nt},\;\; a_{1,r} = \beta = 1+\epsilon Ae^{nt},\;\; a_{2,r} = Gr\vartheta_r + Gm\varphi_r,\;\; b_{0,r} = (1+Rr),$$

$$b_{1,r} = P\beta = Pr(1+\epsilon Ae^{nt}),\; b_{2,r} = PrEc\left(\frac{\partial u_{r+1}}{\partial y^2}\right)^2,\;\; b_{3,r} = PrDu\frac{\partial^2\varphi_r}{\partial y^2},$$

$$b_{4,r} = Pr\Delta u_r,\; c_{0,r} = S\beta = Sc(1+\epsilon Ae^{nt}),\;\; c_{1,r} = ScSr\frac{\partial^2\vartheta_{r+1}}{\partial y^2} \tag{3.88}$$

substituting the above coefficient parameters into (3.83)-(3.85) to give

$$\frac{\partial u_{r+1}}{\partial t} + A_1\frac{\partial^3 u_{r+1}}{\partial t\partial y^2} = a_{1,r}\frac{\partial u_{r+1}}{\partial y} + \frac{\partial^2 u_{r+1}}{\partial y^2} + a_{2,r} - M^2u_{r+1} + a_{0,r}\frac{\partial^3 u_{r+1}}{\partial y^3} \tag{3.89}$$

$$\frac{\partial\vartheta_{r+1}}{\partial t} = b_{1,r}\frac{\partial\vartheta_{r+1}}{\partial y} + b_{0,r}\frac{\partial^2\vartheta_{r+1}}{\partial y^2} + b_{2,r} + b_{3,r} + Pr\delta\vartheta_{r+1} - b_{4,r}\vartheta_{r+1} \tag{3.90}$$

$$\frac{\partial\varphi_{r+1}}{\partial t} = c_{0,r}\frac{\partial\varphi_{r+1}}{\partial y} + \frac{\partial^2\varphi_{r+1}}{\partial y^2} - k_r^2\varphi_{r+1} + c_{1,r} \tag{3.91}$$

subject to

$$u_{r+1}(0,t) = 1,\;\; \vartheta_{r+1}(0,t) = 1+\epsilon e^{nt},\quad \varphi_{r+1}(0,t) = 1+\epsilon e^{nt} \text{ at } y=0 \tag{3.92}$$

$$u_{r+1}(\infty,t) = 0,\; \vartheta_{r+1}(\infty,t) = 0,\; \phi_{r+1}(\infty,t) = 0, \text{ at } y \longrightarrow \infty \tag{3.93}$$

The unknown functions in the resulting equations are defined using Gauss-Lobatto points defined as:

$$\xi_j = cos\frac{\pi j}{N},\;\; j = 0,1,2,...,N;\;\; 1 \le \xi \le -1 \tag{3.94}$$

where N =number of collocation points. To solve the linearized equations above, we first transform the domain of the physical problem from $[0,\infty)$ to $[-1,1]$. The following transformation is used to map the interval together:

$$\frac{\eta}{L} = \frac{\xi+1}{2},\;\; -1 \le \xi \le 1 \tag{3.95}$$

Here L means scaling term utilized in implementing the boundary constraints at infinity. The initial approximation for solving equations (3.89)-(3.91) are gotten at y = 0 which is considered to satisfy the boundary constraints (3.92) and (3.93). Therefore $U_0(y,t)$,$\theta_0(y,t)$ and $\varphi_0(y,t)$ are defined as:

$$u_0(y,t) = e^{-y},\;\; \vartheta_0(y,t) = \varphi_0(y,t) = e^{-y} + \epsilon e^{nt} \tag{3.96}$$

Equations (3.89)-(3.91) are iteratively tackled for all unknown terms commencing from the initial quess (3.96). The iterative schemes (3.89), (3.90) and (3.91) are iteratively solved for $\phi_{r+1}(y,t), U_{r+1}$ and $\vartheta_{y,t}$ as $r = 0,1,2$. To provide solution to equations (3.89)-(3.91). We first discretized using Chebyshev spectral collocation technique in y-direction while implicit finite difference approach in t-direction. The finite difference technique is further employed with centering about an average of t^{n+1} and t^n. The mid-point is expressed as:

$$t^{n+\frac{1}{2}} = \frac{t^{n+1} + t^n}{2} \qquad (3.97)$$

Thus, utilizing the centering in$t^{n+\frac{1}{2}}$ to functions $\phi(y,t),\vartheta(y,t)$ and $U(y,t)$ alongside their derivative leads to:

$$u(y_j, t^{n+\frac{1}{2}}) = u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2}, \left(\frac{\partial u}{\partial t}\right)^{n+\frac{1}{2}} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \qquad (3.98)$$

$$\vartheta(y_j, t^{n+\frac{1}{2}}) = \vartheta_j^{n+\frac{1}{2}} = \frac{\vartheta_j^{n+1} + \vartheta_j^n}{2}, \left(\frac{\partial \vartheta}{\partial t}\right)^{n+\frac{1}{2}} = \frac{\vartheta_j^{n+1} - \vartheta_j^n}{\Delta t}$$

$$\varphi(y_j, t^{n+\frac{1}{2}}) = \varphi_j^{n+\frac{1}{2}} = \frac{\varphi_j^{n+1} + \varphi_j^n}{2}, \left(\frac{\partial \varphi}{\partial t}\right)^{n+\frac{1}{2}} = \frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t}$$

(3.99)

(3.100)

The idea of spectral collocation technique is the use of matrix differentiation D to evaluate the unknown variables derivatives defined as:

$$\frac{d^r u}{dy^r} = \sum_{k=0}^{N} D_{ik}^r u(\xi_k) = D^r u, \quad i = 0, 1, \ldots N \qquad (3.101)$$

$$\frac{d^r \vartheta}{dy^r} = \sum_{k=0}^{N} D_{ik}^r \vartheta(\xi_k) = D^r \vartheta, \quad i = 0, 1, \ldots N \qquad (3.102)$$

$$\frac{d^r \varphi}{dy^r} = \sum_{k=0}^{N} D_{ik}^r \varphi(\xi_k) = D^r \varphi, \quad i = 0, 1, \ldots N \qquad (3.103)$$

First apply Chebyshev spectral collocation method on (3.89)-(3.91) before applying the finite differences.

$$\frac{du_{r+1}}{dt} + A_1 D^2 \frac{du_{r+1}}{dt} = a_{0,r} D^3 u_{r+1} + D^2 u_{r+1} + a_{1,r} D u_{r+1} + a_{2,r} - M^2 u_{r+1} \quad (3.104)$$

$$Pr \frac{d\vartheta_{r+1}}{dt} = b_{0,r} D^2 \vartheta_{r+1} + b_{1,r} D\vartheta_{r+1} + b_{2,r} + b_{3,r} + Pr\delta\vartheta_{r+1} + b_{4,r}\vartheta_{r+1} \qquad (3.105)$$

$$Sc\frac{d\varphi_{r+1}}{dt} = D^2\varphi_{r+1} + c_{0,r}D\varphi_{r+1} - Sck_r^2\varphi_{r+1} + c_{1,r} \tag{3.106}$$

subject to

$$u_{r+1}(x_0,t) = 1, \quad u_{r+1}(xNx,t) = 0, \quad \vartheta_{r+1}(x_0,t) = 1 + \epsilon e^{nt} \tag{3.107}$$

$$\vartheta_{r+1}(xNx,t) = 0, \quad \varphi_{r+1}(x_0,t) = 1 + \epsilon e^{nt}, \quad \varphi_{r+1}(xNx,t) = 0 \tag{3.108}$$

Simplifying equations (3.104)-(3.106) further lead to:

$$\frac{du_{r+1}}{dt} + A_1 D^2\frac{du_{r+1}}{dt} = (a_{0,r}D^3 + D^2 + a_{1,r}D - M^2)u_{r+1} + a_{2,r} \tag{3.109}$$

$$Pr\frac{d\vartheta_{r+1}}{dt} = (b_{0,r} + b_{1,r} + Pr\delta + b_{4,r})\vartheta_{r+1} + b_{2,r} + b_{3,1} \tag{3.110}$$

$$Sc\frac{d\varphi_{r+1}}{dt} = (D^2 + c_{0,r}D - Sck_r^2)\varphi_{r+1} + c_{1,r}$$

(3.111)

subject to (3.107) and (3.108) where

$$u_{r+1} = \begin{bmatrix} u_{r+1}(x_0,t) \\ u_{r+1}(x_1,t) \\ \vdots \\ u_{r+1}(x_{N_x-1},t) \\ u_{r+1}(x_{N_x},t) \end{bmatrix}, \quad a_{0,r} = \begin{bmatrix} a_{0,r}(x_0,t) & & & \\ & a_{0,r}(x_1,t) & & \\ & & \ddots & \\ & & & a_{0,r}(x_{N_x},t) \end{bmatrix} \tag{3.112}$$

$$\vartheta_{r+1} = \begin{bmatrix} \vartheta_{r+1}(x_0,t) \\ \vartheta_{r+1}(x_1,t) \\ \vdots \\ \vartheta_{r+1}(x_{N_x-1},t) \\ \vartheta_{r+1}(x_{N_x},t) \end{bmatrix}, \quad b_{0,r} = \begin{bmatrix} b_{0,r}(x_1,t) & & & \\ & b_{0,r}(x_2,t) & & \\ & & \ddots & \\ & & & b_{0,r}(x_{N_x},t) \end{bmatrix} \tag{3.113}$$

$$\phi_{r+1} = \begin{bmatrix} \phi_{r+1}(x_0,t) \\ \phi_{r+1}(x_1,t) \\ \vdots \\ \phi_{r+1}(x_{N_x-1},t) \\ \phi_{r+1}(x_{N_x},t) \end{bmatrix}, \quad c_{0,r} = \begin{bmatrix} c_{0,r}(x_0,t) & & & \\ & c_{0,r}(x_1,t) & & \\ & & \ddots & \\ & & & c_{0,r}(x_{N_x},t) \end{bmatrix} \tag{3.114}$$

The diagonal matrix is applicable for $a_{1,r}$, $a_{2,r}$, $b_{1,r}$, $b_{2,r}$, $b_{3,r}$, $b_{4,r}$, $c_{0,r}$ and $c_{1,r}$.

Implementing the forward finite scheme as defined in (3.98)-(3.100) on equations (3.109)-(3.11) we have

$$(1 + A_1 D^2)\left(\frac{u_{r+1}^{n+1} - u_{r+1}^{n}}{\Delta t}\right) = (a_{0,r}D^3 + D^2 + a_{1,r}D - M^2)\left(\frac{u_{r+1}^{n+1} + u_{r+1}^{n}}{2}\right) \tag{3.115}$$

$$Pr\left(\frac{\vartheta_{r+1}^{n+1} - \vartheta_{r+1}^{n}}{\Delta t}\right) = (b_{0,r}D^2 + b_{1,r}D + b_{4,r} + Pr\delta)\left(\frac{\vartheta_{r+1}^{n+1} + \vartheta_{r+1}^{n}}{2}\right) + b_{2,r} + b_{3,r} \tag{3.116}$$

$$Sc\left(\frac{\varphi_{r+1}^{n+1} - \varphi_{r+1}^{n}}{\Delta t}\right) = (D^2 + c_{0,r}D - Sck_r^2)\left(\frac{\varphi_{r+1}^{n+1} + \varphi_{r+1}^{n}}{2}\right) + c_{1,r} \tag{3.117}$$

Simplifying equations (3.115)-(3.117) leads to:

$$\frac{(1 + A_1 D^2)}{\Delta t}u_{r+1}^{n+1} - \frac{(1 + A_1 D^2)}{\Delta t}u_{r+1}^{n} = \frac{(a_{0,r}D^3 + D^2 + a_{1,r}D - M^2)}{2}u_{r+1}^{n+1}$$
$$\frac{(a_{0,r}D^3 + D^2 + a_{1,r}D - M^2)}{2}u_{r+1}^{n} + a_{2,r} \tag{3.118}$$

$$\frac{Pr}{\Delta t}\vartheta_{r+1}^{n+1} - \frac{Pr}{\Delta t}\vartheta_{r+1}^{n} = \frac{(b_{0,r}D^2 + b_{1,r}D + b_{4,r} + Pr\delta)}{2}\vartheta_{r+1}^{n+1}$$
$$+\frac{(b_{0,r}D^2 + b_{1,r}D + b_{4,r} + Pr\delta)}{2}\vartheta_{r+1}^{n}$$
$$+(b_{2,r} + b_{3,r} + b_{4,r}) \tag{3.119}$$

$$\frac{Sc}{\Delta t}\varphi_{r+1}^{n+1} - \frac{Sc}{\Delta t}\varphi_{r+1}^{n} = \frac{(c_{0,r}D + D^2 - Sck_r^2}{2}\varphi_{r+1}^{n+1} + \frac{(c_{0,r}D + D^2 - Sck_r^2}{2}\varphi_{r+1}^{n} + c_{1,r} \tag{3.120}$$

Upon further simplification, we obtain

$$\left[\frac{(1+A_1D^2)}{\Delta t}-\frac{(a_{0,r}D^3+D^2+a_{1,r}D-M^2)}{2}\right]u_{r+1}^{n+1}=$$

$$\left[\frac{(1+A_1D^2)}{\Delta t}+\frac{(a_{0,r}D^3+D^2+a_{1,r}D-M^2)}{2}\right]u_{r+1}^{n}+a_{2,r} \quad (3.121)$$

$$\left[\frac{Pr}{\Delta t}-\frac{(b_{0,r}D^2+b_{1,r}D+b_{4,r}+Pr\delta)}{2}\right]\vartheta_{r+1}^{n+1}=$$

$$\left[\frac{Pr}{\Delta t}+\frac{(b_{0,r}D^2+b_{1,r}D+b_{4,r}+Pr\delta)}{2}\right]\vartheta_{r+1}^{n}+(b_{2,r}+b_{3,r}$$

$$\left[\frac{Sc}{\Delta t}-\frac{(D^2+c_{0,r}D-Srk_r^2)}{2}\right]\varphi_{r+1}^{n+1}=$$

) (3.122)

$$\left[\frac{Sc}{\Delta t}-\frac{(D^2+c_{0,r}D-Srk_r^2)}{2}\right]\varphi_{r+1}^{n}+c_{1,r} \quad (3.123)$$

Upon further simplification gives

$$N_1u_{r+1}^{n+1} = H_1u_{r+1}^{n}+G_1 \quad (3.124)$$

$$N_2\vartheta_{r+1}^{n+1} = H_2\vartheta_{r+1}^{n}+G_2 \quad (3.125)$$

$$N_3\varphi_{r+1}^{n+1} = H_3\varphi_{r+1}^{n}+G_3 \quad (3.126)$$

Subject to the boundary constraints (3.127)-(3.129)

$$u_{r+1}(xN_x,t^n)=\vartheta_{r+1}(xN_x,t^n)=\varphi_{r+1}(xN_x,t^n)=0 \quad (3.127)$$

$$u_{r+1}(x_0,t^n)=1,\ \vartheta_{r+1}(x_0,t^n)=\varphi_{r+1}(x_0,t^n)=1+ve^{nt},\ n=1,2,... \quad (3.128)$$

$$u_{r+1}(y_j,0)=e^{-y_j},\ \vartheta_{r+1}(y_j,0)=\varphi_{r+1}(y_j,0)=e^{-y_j}+\epsilon e^{nt} \quad (3.129)$$

Thus, the matrices above gives:

$$N_1=\frac{(1+A_1D^2)}{\Delta t}-\frac{(a_{0,r}^{n+\frac{1}{2}}D^3+D^2+a_{1,r}^{n+\frac{1}{2}}D-M^2I)}{2}$$

$$H_1=\frac{(1+A_1D^2)}{\Delta t}+\frac{(a_{0,r}^{n+\frac{1}{2}}D^3+D^2+a_{1,r}^{n+\frac{1}{2}}D-M^2I)}{2}$$

$$N_2=\frac{Pr}{\Delta t}-\frac{(b_{0,r}^{n+\frac{1}{2}}D^2+b_{1,r}^{n+\frac{1}{2}}D+b_{4,r}^{n+\frac{1}{2}}+Pr\delta)}{2}$$

$$H_2=\frac{Pr}{\Delta t}+\frac{(b_{0,r}^{n+\frac{1}{2}}D^2+b_{1,r}^{n+\frac{1}{2}}D+b_{4,r}^{n+\frac{1}{2}}+Pr\delta)}{2}$$

$$N_3=\frac{Sc}{\Delta t}-\frac{(D^2+c_{0,r}^{n+\frac{1}{2}}D-Sck_r^2)}{2},\ H_3=\frac{Sc}{\Delta t}+\frac{(D^2+c_{0,r}^{n+\frac{1}{2}}D-Sck_r^2)}{2}$$

$$G_1=a_{1,r}^{n+\frac{1}{2}},\ G_2=(b_{2,r}^{n+\frac{1}{2}}+b_{3,r}^{n+\frac{1}{2}}),\ G_3=c_{1,r}^{n+\frac{1}{2}}$$

3.5 Formulation of the research problem two

A steady, two-dimensional, laminar free convective motion of an incompressible Casson along with Walters-B non-Newtonian liquid and conducting liquids through a vertical penetrable plate. The plate coordinate is (x,y) while x-coordinate is studied along the plate in a vertical direction while y-coordinate is studied normal to the plate as depicted in figure (3.2). The vicinity far from the plate is considered to be hot. The assumptions made in this study are:

(i) The wall concentration (C_w) along with wall temperature (T_w) are assumed constant.

(ii) A situation of $T_w < T_\infty$ along with $C_w < C_\infty$ which means a cooled plate is considered.

(iii) The penetrable medium is considered to be homogeneous.

(iv) A magnetism of uniform strength (B_0) is imposed perpendicular towards the liquid motion direction.

(v) A large level of species concentration is considered so that significant of Dufour and Soret can not be ignored.

(vi) The liquid attributes along with the penetrable medium are constant. (vii) The approximation of Bouddineqs is valid while the approximation of boundary layer is utilized. Based on all the assumptions stated above, the flow model equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (3.130)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+g\beta_t(T-T_\infty)+g\beta_c(C-C_\infty)-\frac{\sigma B_0^2}{\rho}u-\frac{\mu_0}{K}u \quad (3.131)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{1}{\rho c_p}\frac{\partial T}{\partial y}\frac{\partial k(T)}{\partial y}-\frac{1}{\rho c_p}\frac{\partial q_r}{\partial y}+\frac{\mu_0}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$
$$+\frac{Q_0}{\rho c_p}(T-T_\infty)+\frac{D_m k_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \quad (3.132)$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}-k_l(C-C_\infty)-\frac{\partial (V_T C)}{\partial y}+\frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} \quad (3.133)$$

subject to:

$$u = Bx, v = -v(x), T = T_w, C = C_w, at\ y = 0 \quad (3.134)$$

$$u \longrightarrow 0, T \longrightarrow T_\infty, C \longrightarrow C_\infty\ as\ y \longrightarrow \infty \quad (3.135)$$

The model of Walters-B and Casson liquid are considered simultaneously in this research. Thus, it results into two non-Newtonian liquids terms. Following the work of Fredrickson (1964) along with viscosity defined as$\left(\tau = \mu \frac{\partial u}{\partial y}|_{y=0}\right)$, the constitutive mode of Casson liquid is explained as:

$$\tau_{ij} = \left(\mu_b + \frac{P_y}{\sqrt{2\pi}}\right) 2e_{ij} \quad when \quad \pi > \pi_c$$

$$\tau_{ij} = \left(\mu_b + \frac{P_y}{\sqrt{2\pi_c}}\right) 2e_{ij} \quad when \quad \pi < \pi_c \qquad (3.136)$$

Here P_y means yield stress of the liquid defined as

$$P_y = \frac{\mu_b \sqrt{(2\pi)}}{\beta} \qquad (3.137)$$

μ_b means dynamic plastic viscosity, $\pi = e_{ij}e_{ij}$ implies rate of deformation component multiplying itself, e_{ij} means rate of deformation while π_c means critical numeric value subject to Casson liquid model. The Casson liquid motion where $\pi > \pi_c$, μ_0 is simply expressed as:

$$\mu_0 = \mu_b + \frac{P_y}{\sqrt{2\pi}} \qquad (3.138)$$

Using equation (3.137) in equation (3.138), thus kinematic viscosity becomes subject to plastic dynamic viscosity (μ_b), ρ means density while β means Casson term which

gives

$$\mu_0 = \frac{\mu_b}{\rho}\left(1 + \frac{1}{\beta}\right) \qquad (3.139)$$

Mehmood et al. (2008) explained that Walters-B liquid Cauchy stress tensor (S) gives an equations of motion of the form

$$S = -pT + \tau \qquad (3.140)$$

$$\tau = 2\eta_0 e - 2k_0 \frac{\delta e}{\delta t}$$

(3.141)

p means pressure while I means identity tensor. Thus, strain tensor rate e is given by:

$$2e = 5(v) + 5(v)^T \qquad (3.142)$$

v means velocity vector, O means gradient operator while $\frac{\delta}{\delta t}$ means convected differentiation of quantity of tensor relating to motion material. Thus, the strain tensor rate convected differentiation is given as:

$$\frac{\delta e}{\delta t} = \frac{\partial e}{\partial t} + v.\bigtriangledown(e) - e.\bigtriangledown(v) - (\bigtriangledown(v))^T.e \qquad (3.143)$$

η_0 = limiting kinematic viscosity at small shear rate and k_0 = the short memory coefficient for the Walters-B fluid which is defined as

$$\eta_0 = \int_0^\infty \lambda(\xi)d\xi \qquad (3.144)$$

$$k_0 = \int_0^\infty \tau\lambda(\xi)d\xi \qquad (3.145)$$

While Walters (1962) explained $\lambda(\xi)$ as relaxation spectrum. The equations of motion explained above is the rheological model for Walters-B liquid when short

memory is considered and any terms having $\int_0^\infty \tau^n\lambda(\tau)d\tau, \quad n \geq 2$ are forgone.

Based on the relation in equations (3.14)-(3.145) and following Tonekaboni et al. (2012), the component of stresses are written as:

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{bmatrix} \qquad (3.146)$$

$$\tau_{xx} = 2\mu_0\frac{\partial u}{\partial x} - 2k_0\left[u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} - 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)\right] \quad (3.147)$$

$$\tau_{yx} = \tau_{xy} = \mu_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$-2k_0\left[\frac{1}{2}u\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \frac{1}{2}v\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right) - \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right] \quad (3.148)$$

$$\tau_{yy} = 2\mu_0\frac{\partial v}{\partial y} - 2k_0\left[\frac{\partial^2 v}{\partial t\partial x} + u\frac{\partial^2 v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} - 2\left(\frac{1}{2}\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)^2\right)\right] \quad (3.149)$$

Where $\tau_{xx}, \tau_{xy}, \tau_{yx}$ and τ_{yy} are components of stress matrix. Differentiating the stress tensor above leads to

$$\rho\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right) = \mu_0\left(\frac{\partial^2 u}{\partial y^2}\right) - k_0\left(v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x\partial y^2} - 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) \quad (3.150)$$

To simplify the heat flux in the energy flow equation (3.132) on the flow, Rosseland diffusion simplication is preferred as elucidated in Alao et al. (2016) and Fagbade et al. (2016) such that:

$$q_r = -\frac{4\sigma_s}{3k_e}\frac{\partial T^4}{\partial y} \quad (3.151)$$

From the above equation (3.151), the Stefan-Boltzman constant is σ_s while ke means coefficient of mean absorption. A distinct temperature existing within the flow are so small such that T^4 can be simplified as a linear function by evaluating T^4 about T_∞ by utilizing Taylor series by neglecting higher order terms to obtain:

$$T^4 \approxeq 4T_\infty^3 T - 3T_\infty^4 \quad (3.152)$$

substituting (3.152) into (3.151) and substituting the outcome to the third term of the energy equation leads to:

$$-\frac{\partial q_r}{\partial y} = \frac{16\sigma_s T_\infty^3}{3k_e}\frac{\partial^2 T}{\partial y^2} \qquad (3.153)$$

According to Alam et al. (2009), the thermophoretic velocity V_T in equation (3.133) can be written as

$$V_T = -k\nu\frac{\nabla T}{T_{ref}} = -\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y} \qquad (3.154)$$

where k = thermophoretic coefficient defined as

$$k = \frac{2C_s\left(\frac{\lambda_g}{\lambda_p} + C_t K_n\right)\left[1 + K_n\left(C_1 + C_2 e^{\frac{-C_3}{K_n}}\right)\right]}{(1+3C_m K_n)\left(1 + 2\frac{\lambda_g}{\lambda_p} + 2C_t K_n\right)} \qquad (3.155)$$

$C_1, C_2, C_3, C_m, C_s, C_t$ are constants, λ_g and λ_p = thermal conductivities of the liquid and the diffused particles respectively, K_n means Knudsen number. Base on the above evaluations on the double non-Newtonian Casson alongside Walters'-B viscoelastic fluids in this research and substituting equations (3.139), (3.150), (3.153) and (3.154) into the flow equations (3.130)-(3.133) lead to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3.156)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu_b(T)}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial u}{\partial y}\frac{\partial \mu_b(T)}{\partial T}\frac{\partial T}{\partial y}$$
$$-\frac{k_0}{\rho}\left(v\frac{\partial^3 u}{\partial y^2} + u\frac{\partial^3 u}{\partial x \partial y^2} - 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)$$
$$+g\beta_t(T - T_\infty) + g\beta_c(C - C_\infty) - \frac{\sigma B_0^2}{\rho}u - \frac{\mu_b}{k\rho}\left(1+\frac{1}{\beta}\right)u \qquad (3.157)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho c_p}\frac{\partial T}{\partial y}\frac{\partial k(T)}{\partial y} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{\mu_b(T)}{\rho c_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2$$
$$+\frac{Q_0}{\rho c_p}(T - T_\infty) + \frac{Dk_T}{c_s c_p}\frac{\partial^2 C}{\partial y^2} \qquad (3.158)$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - k_l(C - C_\infty) - \frac{\partial}{\partial y}(V_T C) + \frac{Dk_T}{T_m}\frac{\partial^2 T}{\partial y^2} \qquad (3.159)$$

subject to:

$$u = Bx, v = -v(x), T = T_w, C = C_w \quad , at\ y = 0 \qquad (3.160)$$

$$u \longrightarrow 0, T \longrightarrow T_\infty, C \longrightarrow C_\infty, as\ y \longrightarrow \infty \qquad (3.161)$$

u and v represents $u = \partial\psi_{\partial y}$ and $v = -\partial\psi_{\partial x}$. In the defined function of u and v, $\psi(x,y)$ is the stream function which automatically satisfies the continuity equation (3.156). Similarity variables are defined as

$$\eta = \left(\frac{B}{\nu}\right)^{\frac{1}{2}} y \quad , \psi = (\nu B)^{\frac{1}{2}} x f(\eta) \quad (3.162)$$

The dimensionless temperature, concentration alongside thermal conductivity subject to temperature model in [Animasaun et al. (2016); Salem and Fathy (2012)] and viscosity temperature-dependent model in [Animasaun et al. (2016); Layek, Mukhopadhyay and Samad (2005)] are given by:

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \ , \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}$$

,

$$\mu_b(T) = \mu_b^*[a + b(T_w - T)] \ , k(T) = k^*[1 + \xi(T - T_\infty)] \quad (3.163)$$

Note that $^0a^0$ is a constant which is assumed to be 1 in the present study

3.5.1 Validation of the stream function used in research problem two

The stream function is defined as

$$\psi = (\nu B)^{\frac{1}{2}} x f(\eta), \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial (\nu B)^{\frac{1}{2}} x f}{\partial y} = (\nu B)^{\frac{1}{2}} \frac{\partial (x f)}{\partial y} = (\nu B)^{\frac{1}{2}} \left[x \frac{\partial f}{\partial y} + f \frac{\partial x}{\partial y} \right]$$

$$u = (\nu B)^{\frac{1}{2}} x \frac{\partial f}{\partial \eta} \times \frac{\partial \eta}{\partial y} = (\nu B)^{\frac{1}{2}} x f' \left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$u = Bxf' \quad (3.164)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{(\nu B)^{\frac{1}{2}} x f}{\partial x} = -(\nu B)^{\frac{1}{2}} \frac{\partial (x f)}{\partial x} = -(\nu B)^{\frac{1}{2}} \left[x \frac{\partial f}{\partial x} + f \frac{\partial x}{\partial x} \right]$$

$$v = -(\nu B)^{\frac{1}{2}} f \quad (3.165)$$

$$\frac{\partial u}{\partial x} = \frac{\partial (Bxf')}{\partial x} = B \frac{\partial (x f')}{\partial x} = B \left[x \frac{\partial f'}{\partial x} + f' \frac{\partial x}{\partial x} \right] = B \left[x \frac{\partial f'}{\partial \eta} \times \frac{\partial \eta}{\partial x} + f' \right]$$

$$\frac{\partial u}{\partial x} = Bf' \quad (3.166)$$

$$\frac{\partial v}{\partial y} = -\frac{\partial (\nu B)^{\frac{1}{2}} f}{\partial y} = -(\nu B)^{\frac{1}{2}} \frac{\partial f}{\partial y} = -(\nu B)^{\frac{1}{2}} \frac{\partial f}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial v}{\partial y} = -(\nu B)^{\frac{1}{2}} f' \left(\frac{B}{\nu}\right)^{\frac{1}{2}} = -Bf'$$

$$\frac{\partial v}{\partial y} = -Bf' \quad (3.167)$$

Substituting equations (3.166) and (3.167) into the continuity equation (3.156)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This implies that

$$Bf^0 - Bf^0 = 0$$

This shows that the stream function satisfied the continuity equation.

3.5.2 Non-dimensionalization of momentum equation of the research problem two

Since,

$$\frac{\partial u}{\partial x} = Bf' \ \ and \ \ u = Bxf'$$

$$\implies u\frac{\partial u}{\partial x} = Bxf' \times Bf'$$

$$u\frac{\partial u}{\partial x} = B^2x(f')^2 \tag{3.168}$$

$$\frac{\partial u}{\partial y} = \frac{\partial(Bf'x)}{\partial y} = B\left[x\frac{\partial f'}{\partial y} + f'\frac{\partial x}{\partial y}\right] = Bx\frac{\partial f'}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = Bx\frac{\partial f'}{\partial \eta} \times \left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial u}{\partial y} = -(\nu B)^{\frac{1}{2}} f \times Bx\frac{\partial f'}{\partial \eta} \times \left(\frac{B}{\nu}\right)^{\frac{1}{2}} = -B^2xff''$$

$$v\frac{\partial u}{\partial y} = -B^2xff''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(Bxf''\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\right) = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial(xf'')}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\left[x\frac{\partial f''}{\partial y} + f''\frac{\partial x}{\partial y}\right] = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}} x\frac{\partial f''}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}} xf'''\left(\frac{B}{\nu}\right)^{\frac{1}{2}} = \frac{B^2x}{\nu}f'''$$

(3.169)

Therefore

$$\frac{\mu_b(T)}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} = \frac{\mu_b(T)}{\rho}\left(1+\frac{1}{\beta}\right)\frac{B^2 x}{\nu}f'''$$

$$\frac{\mu_b(T)}{\rho}\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} = \left(1+\frac{1}{\beta}\right)B^2 x f''' \qquad (3.170)$$

$$\frac{\partial \mu_b(T)}{\partial T} = \frac{\partial}{\partial T}(\mu_b^*(1+b(T-T_\infty)) = \frac{\partial}{\partial T}(\mu_b^*(1+b(\theta(T_w-T_\infty))))$$

$$\frac{\partial \mu_b(T)}{\partial T} = (1+b(T_w-T_\infty$$

$$)) \qquad (3.171)$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial y} = \frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial \eta}\times\frac{\partial \eta}{\partial y} = (T_w-T_\infty)\theta'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial T}{\partial y} = (T_w-T_\infty)\theta'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial^3 u}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(\frac{B^2 x}{\nu}f'''\right) = \frac{B^2}{\nu}\frac{\partial}{\partial y}(xf''')$$

$$\frac{\partial^3 u}{\partial y^3} = \frac{B^2}{\nu}\left[x\frac{\partial f'''}{\partial y} + f'''\frac{\partial x}{\partial y}\right] = \frac{B^2}{\nu}x\frac{\partial f'''}{\partial y}$$

(3.172)

$$\frac{\partial^3 u}{\partial y^3} = \frac{B^2 x}{\nu}\frac{\partial f'''}{\partial \eta}\times\frac{\partial \eta}{\partial y} = \frac{B^2 x}{\nu}f^{iv}\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial^3 u}{\partial y^3} = -(\nu B)^{\frac{1}{2}}f\times\frac{B^2 x}{\nu}f^{iv}\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial^3 u}{\partial y^3} = -\frac{B^3 x}{\nu}ff^{iv}$$

$$\frac{\partial^3 u}{\partial x\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial x}\left(\frac{B^2 x}{\nu}f'''\right) = \frac{B^2}{\nu}\frac{\partial}{\partial x}(xf''')$$

$$\frac{\partial^3 u}{\qquad} = \frac{B^2}{\quad}\left[x\frac{\partial f'''}{\qquad}\quad f'''\frac{\partial x}{\quad}\right] = \frac{B^2}{\quad}f'''$$

(3.173)

$$\partial x \partial y^2 \quad \nu \quad \partial x \quad + \quad \partial x \quad \nu$$

$$u\frac{\partial^3 u}{\partial x \partial y^2} = Bxf' \times \frac{B^2}{\nu} f'''$$

$$u\frac{\partial^3 u}{\partial x \partial y^2} = \frac{B^3 x}{\nu} f' f'''$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(Bxf''\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}} \frac{\partial}{\partial x}(xf'') = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\left[x\frac{\partial f''}{\partial x} + f''\frac{\partial x}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x \partial y} = B\left(\frac{B}{\nu}\right)^{\frac{1}{2}} f''$$

$$2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} = Bf''\left(\frac{B}{\nu}\right)^{\frac{1}{2}} x \times B\left(\frac{B}{\nu}\right)^{\frac{1}{2}} f'' = 2\frac{B^3 x}{\nu}(f'')^2$$

$$2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} = 2\frac{B^3 x}{\nu}(f'')^2$$

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} = Bf' \times \frac{B^2 x}{\nu} f'''$$

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} = \frac{B^3 x}{\nu} f' f'''$$

(3.174)

(3.175)

(3.176) $g\beta_t(T - T_\infty) = g\beta_t\theta(T_w - T_\infty)$ (3.177) $g\beta_c(C - C_\infty) = g\beta_c\varphi(C_w - C_\infty)$ (3.178)

$$\frac{\sigma B_0^2}{\rho} u = \frac{\sigma B_0^2}{\rho} Bxf' \quad (3.179)$$

$$\frac{\mu_b(T)}{k\rho}\left(1+\frac{1}{\beta}\right)u = \frac{\mu_b(T)}{k\rho}\left(1+\frac{1}{\beta}\right)Bxf' \quad (3.180)$$

Substituting equations (3.168) – (3.180) into the momentum equation (3.157) to obtain

$$B^2x(f')^2 - B^2xff'' = \frac{1}{\rho}\left(1+\frac{1}{\beta}\right)Bxf''\left(\frac{B}{\nu}\right)^{\frac{1}{2}}(1+b\vartheta(T_w-T_\infty))(T_w-T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\theta'$$
$$+\left(1+\frac{1}{\beta}\right)B^2xf''' - \frac{K_0}{\rho}\left(-\frac{B^3x}{\nu}ff^{iv} + \frac{B^3x}{\nu}f'f''' - 2\frac{B^3x}{\nu}(f'')^2 - 3\frac{B^3x}{\nu}f'f'''\right)$$
$$+g\beta_t\theta(T_w-T_\infty) + g\beta_c\phi(C_w-C_\infty) - \frac{\sigma B_0^2}{\rho}Bxf' - \frac{\mu_b(T)}{k\rho}\left(1+\frac{1}{\beta}\right)Bxf'$$

divide all through by B^2x

$$(f')^2 - ff'' = \left(1+\frac{1}{\beta}\right)f''' + \frac{1}{\rho}(1+\gamma\theta)\left(1+\frac{1}{\beta}\right)\frac{1}{\nu}\theta' f'' + \frac{g\beta_c\phi(C_w-C_\infty)}{B^2x}$$
$$+\frac{g\beta_t\theta(T_w-T_\infty)}{B^2x} - \frac{k_0}{\rho}\left(-\frac{B}{\nu}ff^{iv} + \frac{B}{\nu}f'f''' - 2\frac{B}{\nu}(f'')^2 - 3\frac{B}{\nu}f'f'''\right) - \frac{\sigma B_0^2}{\rho B}f'$$
$$-\frac{\mu_b(T)}{k\rho B}\left(1+\frac{1}{\beta}\right)f'$$

Simplifying further to get

$$(f')^2 - ff'' = \left(1+\frac{1}{\beta}\right)f''' + \left(1+\frac{1}{\beta}\right)(1+\gamma\theta)\theta' f'' + Gr\theta + Gm\phi - M^2 f'$$
$$-\frac{1}{P_s}\left(1+\frac{1}{\beta}\right)f' + A_2(f^{iv}f + 2f'f''' + 2(f'')^2)$$

Therefore, the transformed momentum equation leads to

$$\left(1+\frac{1}{\beta}\right)f''' + \left(1+\frac{1}{\beta}\right)(1+\gamma\theta)\theta' f'' + Gr\theta + Gm\phi - M^2 f' + ff'' - (f')^2$$
$$-\frac{1}{P_s}\left(1+\frac{1}{\beta}\right)f' + A_2(f^{iv}f + 2f'f''' + 2(f'')^2 \qquad) = 0$$

(3.181)

where;

$$\gamma = b(T_w-T_\infty),\ Gr = \frac{g\beta_t(T_w-T_\infty)}{B^2x},\ Gm = \frac{g\beta_c(C_w-C_\infty)}{B^2x},\ A_2 = \frac{Bk_0}{\rho\nu},$$
$$M = \frac{\sigma B_0^2}{\rho B},\ P_s = \frac{k\rho B}{\mu_b(T)}$$

3.5.3 Transformation of the energy equation of the research problem two

$$\frac{\partial T}{\partial x}=\frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial x}=\frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial \eta}\times\frac{\partial \eta}{\partial x}$$

$$\frac{\partial T}{\partial x}=(T_w-T_\infty)\theta'$$

$$u\frac{\partial T}{\partial x}=Bxf'\times(T_w-T_\infty)\theta'\times 0$$

$$u\frac{\partial T}{\partial x}=0 \tag{3.182}$$

$$\frac{\partial T}{\partial y}=\frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial y}=\frac{\partial T}{\partial \theta}\times\frac{\partial \theta}{\partial \eta}\times\frac{\partial \eta}{\partial y}$$

$$\frac{\partial T}{\partial y}=(T_w-T_\infty)\theta'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial T}{\partial y}=-(\nu B)^{\frac{1}{2}}f\times(T_w-T_\infty)\theta'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial T}{\partial y}=-B(T_w-T_\infty)f\theta' \tag{3.183}$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial y}\left((T_w-T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\theta'\right)=(T_w-T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \theta'}{\partial y}$$

$$\frac{\partial^2 T}{\partial y^2}=(T_w-T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \theta'}{\partial \eta}\times\frac{\partial \eta}{\partial y}=(T_w-T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\theta''$$

$$\frac{\partial^2 T}{\partial y^2}=(T_w-T_\infty)\frac{B}{\nu}\theta''$$

$$\frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2}=\frac{k^*}{\rho c_p}(1+\xi(T-T_\infty))(T_w-T_\infty)\frac{B}{\nu}\theta''$$

$$\frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2}=\frac{k^*}{\rho c_p}(1+\xi\theta(T_w-T_\infty))(T_w-T_\infty)\frac{B}{\nu}\theta''$$

$$\frac{k(T)}{\rho c_p}\frac{\partial^2 T}{\partial y^2}=\frac{k^*}{\rho c_p}(1+\xi\theta(T_w-T_\infty))\frac{B}{\nu}(T_w-T_\infty)\theta'' \tag{3.184}$$

$$\frac{\partial k(T)}{\partial y}=\frac{\partial}{\partial y}k^*(1+\xi(T-T_\infty))=\frac{\partial}{\partial y}k^*(1+\xi\theta(T_w-T_\infty))$$

$$\frac{\partial k(T)}{\partial y}=\frac{\partial k^*}{\partial y}+\frac{\partial}{\partial y}(\xi\theta(T_w-T_\infty))=\xi\frac{\partial}{\partial y}(\theta(T_w-T_\infty))$$

$$\frac{\partial k(T)}{\partial y} = \xi(T_w - T_\infty)\frac{\partial \theta}{\partial y} = \xi(T_w - T_\infty)\frac{\partial \theta}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial k(T)}{\partial y} = \xi(T_w - T_\infty)\theta' \left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

Since $\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta' \ \frac{B}{\nu})^{\frac{1}{2}}$

Therefore,

$$\frac{1}{\rho c_p}\frac{\partial T}{\partial y}\frac{\partial k(T)}{\partial y} = \frac{1}{\rho c_p}(T_w - T_\infty)\theta' \left(\frac{B}{\nu}\right)^{\frac{1}{2}} \times \xi(T_w - T_\infty)\theta' \left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho c_p}\frac{\partial T}{\partial y}\frac{\partial k(T)}{\partial y} = \frac{B}{\rho c_p}(T_w - T_\infty)\frac{1}{\nu}\xi(T_w - T_\infty)(\theta')^2 \tag{3.185}$$

Recall that, $-\frac{\partial q_r}{\partial y} = \frac{16\sigma_s T_\infty^3}{3k_e}\frac{\partial^2 T}{\partial y^2}$ and$\frac{\partial^2 T}{\partial y^2} = \frac{B}{\nu}(T_w - T_\infty)\theta''$

Therefore;

$$-\frac{\partial q_r}{\partial y} = \frac{16\sigma_s T_\infty^3}{3k_e}\frac{B}{\nu}(T_w - T_\infty)\theta'' \tag{3.186}$$

$$\frac{\mu_b(T)}{\rho c_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\mu_b^*(1 + b(T_w - T_\infty))}{\rho c_p}\left(1 + \frac{1}{\beta}\right)\left(Bx\left(\frac{B}{\nu}\right)^{\frac{1}{2}} f''\right)^2$$

$$\frac{\mu_b(T)}{\rho c_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\mu_b^*}{\rho c_p}(1 + b(T_w - T_\infty))\left(1 + \frac{1}{\beta}\right)B^2 x^2 \frac{B}{\nu}(f'')^2$$

$$\frac{\mu_b(T)}{\rho c_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 = \frac{(Bx)^2}{c_p}\left(1 + \frac{1}{\beta}\right)(1 + b\theta(T_w - T_\infty))B(f'')^2 \tag{3.187}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial \phi} \times \frac{\partial \phi}{\partial \eta} \times \frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y}\left((C_w - C_\infty)\phi'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\right) = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \phi'}{\partial y}$$

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \phi'}{\partial \eta} \times \frac{\partial \eta}{\partial y} = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\phi''\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{B(C_w - C_\infty)}{\nu}\phi'' \tag{3.188}$$

$$\frac{Q_0}{\rho c_p}(T - T_\infty) = \frac{Q_0}{\rho c_p}\theta(T_w - T_\infty) \tag{3.189}$$

Substituting equations (3.182) – (3.189) into the energy equation (3.158) to give:

$$0 - B(T_w - T_\infty)f\theta' = \frac{k^*}{\rho c_p}(1 + \xi\theta(T_w - T_\infty))\frac{B}{\nu}(T_w - T_\infty)\theta'' + \frac{Q_0}{\rho c_p}\theta(T_w - T_\infty)$$
$$+ \frac{B}{\rho c_p}(T_w - T_\infty)\frac{1}{\nu}\xi(T_w - T_\infty)(\theta')^2 + \frac{16\sigma_s T_\infty^3}{3k_e}\frac{B}{\nu}(T_w - T_\infty)\theta'' + \frac{Dk_T}{c_s c_p}\frac{B(C_w - C_\infty)}{\nu}\phi''$$
$$+ \frac{(Bx)^2}{c_p}\left(1 + \frac{1}{\beta}\right)(1 + b\theta(T_w - T_\infty))B(f'')^2$$

Simplifying to obtain

$$-B(T_w - T_\infty)f\theta' = \frac{k^*}{\rho c_p \nu}(1+\epsilon\theta)B(T_w - T_\infty)\theta'' + \frac{16\sigma_s T_\infty^3}{3k_e}\frac{B}{\nu}(T_w - T_\infty)\theta''$$

$$+\frac{B(T_w - T_\infty)}{\rho c_p \nu}\epsilon(\theta')^2 + \frac{Dk_T}{c_s c_p}\frac{B(C_w - C_\infty)}{\nu}\phi'' + \frac{(Bx)^2}{c_p}\left(1+\frac{1}{\beta}\right)(1+\gamma\theta)B(f'')^2$$

$$+\frac{Q_0}{\rho c_p}\theta(T_w - T_\infty)$$

divide all through by $B(T_w - T_\infty)$

$$-f\theta' = \frac{k^*}{\rho c_p \nu}(1+\epsilon\theta)\theta'' + \frac{16\sigma_s T_\infty^3}{3k_e \rho c_p \nu}\theta'' + \epsilon(\theta')^2 + \frac{Dk_T(C_w - C_\infty)}{c_s c_p \nu (T_w - T_\infty)}\phi''$$

$$+\frac{(Bx)^2}{c_p(T_w - T_\infty)}\left(1+\frac{1}{\beta}\right)(1+\gamma\theta)(f'')^2 + \frac{Q_0}{\rho c_p B}\theta$$

Simplifying further,

$$-f\theta' = \frac{k^*}{\rho c_p \nu}[(1+\epsilon\theta) + Ra]\theta'' + \epsilon(\theta')^2 + Du\phi'' + Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\theta)(f'')^2$$

Therefore, the transformed energy equation becomes

$$\left(\frac{(1+\epsilon\theta)+Ra}{Pr}\right)\theta'' + f\theta' + \epsilon(\theta')^2 + Du\phi''$$

$$+Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\theta)(f'')^2 + \delta_x\theta = 0 \qquad (3.190)$$

3.5.4 Transformation of the concentration equation of the research problem two

$$\frac{\partial C}{\partial x} = \frac{\partial C}{\partial \phi}\times\frac{\partial \phi}{\partial x} = \frac{\partial C}{\partial \phi}\times\frac{\partial \phi}{\partial \eta}\times\frac{\partial \eta}{\partial x}$$

$$\frac{\partial C}{\partial x} = (C_w - C_\infty)\phi'$$

$$u\frac{\partial C}{\partial x} = Bxf'\times(C_w - C_\infty)\phi'\times 0$$

$$u\frac{\partial C}{\partial x} = 0 \qquad (3.191)$$

$$\frac{\partial C}{\partial y} = \frac{\partial C}{\partial \phi}\times\frac{\partial \phi}{\partial y} = \frac{\partial C}{\partial \phi}\times\frac{\partial \phi}{\partial \eta}\times\frac{\partial \eta}{\partial y}$$

$$\frac{\partial C}{} = (C_w - C_\infty)\phi'\left(\frac{B}{}\right)^{\frac{1}{2}}$$

$$\partial y \qquad\qquad\qquad \nu$$

$$v\frac{\partial C}{\partial y} = -(\nu B)^{\frac{1}{2}} f \times (C_w - C_\infty)\phi'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}$$

$$v\frac{\partial C}{\partial y} = -B(C_w - C_\infty)f\phi' \tag{3.192}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial y}\right) = \frac{\partial}{\partial y}\left((C_w - C_\infty)\phi'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \phi'}{\partial y} = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial \phi'}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\phi''\left(\frac{B}{\nu}\right)^{\frac{1}{2}} = (C_w - C_\infty)\frac{B}{\nu}\phi''$$

$$D\frac{\partial^2 C}{\partial y^2} = \frac{D}{\nu}B(C_w - C_\infty)\phi''$$

$$k_l(C - C_\infty) = k_l\phi(C_w - C_\infty)\phi''$$

(3.193)

(3.194)

Since $V_T = -\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}$ and $C = \varphi(C_w - C_\infty) + C_\infty$

Therefore,

$$V_T C = \left(-\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}\right) \times (\phi(C_w - C_\infty) + C_\infty)$$

$$V_T C = -\frac{k\nu}{T_{ref}}\phi\frac{\partial T}{\partial y} - \frac{k\nu}{T_{ref}}C_\infty\frac{\partial T}{\partial y}$$

Setting the free stream concentration $C_\infty = 0$

$$V_T C = -\frac{k\nu}{T_{ref}}\phi\frac{\partial T}{\partial y}$$

$$\frac{\partial}{\partial y}(V_T C) = \frac{\partial}{\partial y}\left(-\frac{k\nu}{T_{ref}}\phi\frac{\partial T}{\partial y}\right) = -\frac{k\nu}{T_{ref}}\frac{\partial}{\partial y}\left(\phi\frac{\partial T}{\partial y}\right)$$

$$\frac{\partial}{\partial y}(V_T C) = -\frac{k\nu}{T_{ref}}\left[\phi\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}\right]$$

$$\frac{\partial}{\partial y}(V_T C) = -\frac{k\nu}{T_{ref}}\left[\phi\frac{\partial^2 T}{\partial y^2} + \frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}\right]$$

$$\frac{\partial}{\partial y}(V_T C) = -\frac{k\nu}{T_{ref}}\left[\phi(T_w - T_\infty)\frac{B}{\nu}\theta'' + (T_w - T_\infty)\theta'\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\frac{\partial\phi}{\partial\eta}\times\frac{\partial\eta}{\partial y}\right]$$

$$\frac{\partial}{\partial y}(V_T C) = -\frac{k\nu}{T_{ref}}\left[(T_w - T_\infty)\frac{B}{\nu}\phi\theta'' + (T_w - T_\infty)\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\left(\frac{B}{\nu}\right)^{\frac{1}{2}}\theta'\phi'\right]$$

$$\frac{\partial}{\partial y}(V_T C) = -\frac{k\nu}{T_{ref}}\left[(T_w - T_\infty)\frac{B}{\nu}\phi\theta'' + \frac{B}{\nu}(T_w - T_\infty)\theta'\phi'\right]$$

$$-\frac{\partial}{\partial y}(V_T C) = \frac{k\nu B(T_w - T_\infty)}{\nu T_{ref}}[\phi\theta'' + \theta'\phi']$$

$$-\frac{\partial}{\partial y}(V_T C) = \frac{kB(T_w - T_\infty)}{T_{ref}}[\phi\theta'' + \theta'\phi' \quad] \tag{3.195}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\frac{B}{\nu}\theta'' \tag{3.196}$$

Substituting equations (3.191)-(3.196) into the concentration equation (3.159) to obtain

$$0 - B(C_w - C_\infty)f\phi' = \frac{D}{\nu}B(C_w - C_\infty)\phi'' - k_l\phi(C_w - C_\infty)$$
$$+\frac{kB(T_w - T_\infty)}{T_{ref}}[\phi\theta'' + \theta'\phi'] + \frac{Dk_T}{T_m}(T_w - T_\infty)\frac{B}{\nu}\theta''$$

dividing all through by $B(C_w - C_\infty)$ to obtain

$$-f\phi' = \frac{D}{\nu}\phi'' - \frac{k_l}{B}\phi + \frac{k(T_w - T_\infty)}{T_{ref}(C_w - C_\infty)}[\phi\theta'' + \theta'\phi']$$
$$+\frac{Dk_T(T_w - T_\infty)}{\nu T_m(C_w - C_\infty)}\theta''$$
$$-f\phi' = \frac{1}{Sc}\phi'' - C_r\phi + \tau(\phi\theta'' + \theta'\phi') + S_r\theta''$$

Therefore, the transformed concentration equation leads to

$$\varphi^{00} - ScC_r\varphi + Scf\varphi^{0} + Sc\tau(\varphi\theta^{00} + \theta^{0}\varphi^{0}) + ScS_r\theta^{00} = 0 \tag{3.197}$$

3.5.5 Transformation of the boundary conditions of the research problem two

Since $u = Bxf^0$ and $u = Bx$

$$\Longrightarrow Bxf^0 = Bx$$

divide all through by Bx to obtain

$$f^0(\eta) = 1 \tag{3.198}$$

Also, $v = -(\nu B)^{\frac{1}{2}} f$ and $v = -\nu(x)$

$$-(\nu B)^{\frac{1}{2}} f = -\nu(x)$$
$$f(\eta) = \frac{\nu}{(\nu B)^{\frac{1}{2}}} = \sqrt{\frac{\nu}{B}}$$
$$f(\eta) = f_w \tag{3.199}$$

Now $T = \theta(T_w - T_\infty) + T_\infty$ and $T = T_w \therefore \theta(T_w - T_\infty) + T_\infty$

$= T_w$

$$\theta(T_w - T_\infty) = T_w - T_\infty$$

divide all through by $T_w - T_\infty$

$$\theta(\eta) = 1 \tag{3.200}$$

also, $C = \varphi(C_w - C_\infty) + C_\infty$ and $C = C_w \therefore \varphi(C_w - C_\infty) + C_\infty =$

C_w

$$\varphi(C_w - C_\infty) = C_w - C_\infty$$

divide all through by $C_w - C_\infty$

$$\varphi(\eta) = 1 \tag{3.201}$$

Also, $u = Bxf^0$ and $u \longrightarrow 0$

$$\therefore Bxf^0 = 0$$

$$f^0(\eta) \longrightarrow 0 \tag{3.202}$$

$T = \theta(T_w - T_\infty) + T_\infty$ *and* $T \longrightarrow T_\infty\, \theta(T_w - T_\infty) +$
$T_\infty = T_\infty$
$\theta(T_w - T_\infty) = T_\infty - T_\infty\, \theta(T_w -$
$T_\infty) = 0$

$$\theta(\eta) \dashrightarrow 0 \qquad (3.203)$$

$$C = \varphi(C_w - C_\infty) + C_\infty and C \dashrightarrow C_\infty \varphi(C_w - C_\infty) + C_\infty = C_\infty \varphi(C_w - C_\infty) = C_\infty - C_\infty \varphi(C_w - C_\infty) = 0$$

$$\varphi(\eta) \dashrightarrow 0 \qquad (3.204)$$

Therefore, the transformed momentum, energy and concentration equations with the boundary conditions of the research problem two are

$$\left(1 + \frac{1}{\beta}\right) f''' + \left(1 + \frac{1}{\beta}\right)(1 + \gamma\theta)\theta' f'' + Gr\theta + Gm\phi - M_p^2 f' + ff'' - (f')^2$$
$$-\frac{1}{P_s}\left(1 + \frac{1}{\beta}\right) f' + A_2(f^{iv} f + 2f' f''' + 2(f'')^2 \quad) = 0 \qquad (3.205)$$

$$\left(\frac{(1 + \epsilon\theta) + Ra}{Pr}\right)\theta'' + f\theta' + \epsilon(\theta')^2 + Du\phi'' + \delta_x\theta$$
$$+Ec\left(1 + \frac{1}{\beta}\right)(1 + \gamma\theta)(f'')^2$$
$$\phi'' - ScC_r\phi + Scf\phi' + Sc\tau(\phi\theta'' + \theta'\phi') + ScS_r\theta''$$
$$= 0 \qquad (3.206)$$

$$= 0 \qquad (3.207)$$

together with the boundary conditions

$$f^0 = 1, f = f_w, \theta = 1, \varphi = 1 \; at \; \eta = 0 \qquad (3.208)$$

$$f^0 \dashrightarrow 0, \theta \dashrightarrow 0, \varphi \dashrightarrow 0 \; as \; \eta \dashrightarrow 0 \qquad (3.209)$$

Note that$\gamma = b(T_w - T_\infty), Gr = \frac{\beta_t(T_w - T_\infty)}{B^2 x}, Gm = \frac{\beta_c(C_w - C_\infty)}{B^2 x}, A_2 = \frac{k_0 B}{\nu\rho}, P_s = \frac{\mu^*}{\rho B k}, Rd = \frac{4\sigma_0 T_\infty^3}{3k_e k^*}, Pr = \frac{\nu\rho c_p}{k^*}, \epsilon = \xi(T_w - T_\infty)\beta = \frac{\mu_b\sqrt{2\pi}}{P_y}, Ec = \frac{(Bx)^2}{c_p(T_w - T_\infty)}, \delta_x = \frac{Q_0}{B\rho c_p}, D_f = \frac{Dk_T(C_w - C_\infty)}{c_s c_p \nu(T - w - T_\infty)}, Sc = \frac{\nu}{D}, Cr = \frac{k_l}{B}, \tau = \frac{k^{th}(T_w - T_\infty)}{T_{ref}\nu}, So = \frac{Dk_T(T_w - T_\infty)}{T_m\nu(C_w - C_\infty)}$ are the controlling flow parameters. A_2 = Walters-B viscoelastic fluid parameter, P_s = permeability parameter, γ = temperature dependent viscosity parameter, Gr = thermal Grashof number, Gm = mass Grashof number, Rd = radiation parameter, Pr = Prandtl number, β = Casson parameter, Ec = Eckert number, δ_x = heat generation parameter, D_f = Dufour number, Sc = Schmidt number, Cr = chemical reaction parameter, τ = thermophoretic parameter and So = Soret number.

The physical quantities of engineering interest are the local skin friction coefficient, Nusselt number and Sherwood number. The first physical quantities of interest is the wall skin friction coefficient C_f, it is defined as

$$C_f = \frac{\tau_w}{\rho \ \frac{B}{\nu})^{\frac{1}{2}}} \ \ where \ \ \tau_w = \left(\mu_b + \frac{P_y}{\sqrt{2\pi}}\right)\frac{\partial u}{\partial y}\ |_{y=0} - \left(2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\ |_{y=0}$$

τ_w = shear stress or skin friction within hydrodynamics boundary layer. The other quantity of interest are the Nusselt and Sherwood number which are defined as:

$$Nu = \frac{Kq_w}{(T_w - T_\infty)} \ \ where \ \ q_w = -K\left(\frac{\partial T}{\partial y}\ |_{y=0} - \frac{4\sigma_0}{3k_e}\left(\frac{\partial T^4}{\partial y}\right)\ |_{y=0}\right)$$

$$Sh = \frac{Dh_w}{C_w - C_\infty} \ \ where \ \ h_w = D\left(\partial\partial C\partial y\ |_{y=0}\right)$$

3.6 Qualitative analysis of problem two

Reducing the system of equations (3.205) − (3.207) into first order ordinary differential equations to obtain

$$f = \omega_1\ ,\ \frac{df}{d\eta} = \frac{d\omega_1}{d\eta} = \omega_2,\ \frac{d^2 f}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d\omega_1}{d\eta}\right) = \frac{d\omega_2}{d\eta} = \omega_3 \quad (3.210)$$

$$\frac{d^3 f}{d\eta^3} = \frac{d}{d\eta}\left(\frac{d\omega_2}{d\eta}\right) = \frac{d\omega_3}{d\eta} = \omega_4,\ \frac{d^4 f}{d\eta^4} = \frac{d}{d\eta}\left(\frac{d\omega_3}{d\eta}\right) = \frac{d\omega_4}{d\eta} \quad (3.211)$$

$$\theta = \omega_5,\ \frac{d\theta}{d\eta} = \frac{d\omega_5}{d\eta} = \omega_6,\ \frac{d^2\theta}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d\omega_5}{d\eta}\right) = \frac{d\omega_6}{d\eta}$$

$$\phi = \omega_7,\ \frac{d\phi}{d\eta} = \frac{d\omega_7}{d\eta} = \omega_8,\ \frac{d^2\phi}{d\eta^2} = \frac{d}{d\eta}\left(\frac{d\omega_7}{d\eta}\right) = \frac{d\omega_8}{d\eta}$$

(3.212)

(3.213)

Substituting equations (3.210) − (3.213) into equations (3.205) − (3.207) to obtain

$$\left(1 + \frac{1}{\beta}\right)\omega_4 + \left(1 + \frac{1}{\beta}\right)\omega_6\omega_3 + \gamma\left(1 + \frac{1}{\beta}\right)\omega_5\omega_6\omega_3 + Gr\omega_5 + Gm\omega_7 - M^2\omega_2$$
$$+\omega_1\omega_3 - (\omega_2)^2 - \frac{1}{P_s}\left(1 + \frac{1}{\beta}\right)\omega_2 + A_2\left(\frac{d\omega_4}{d\eta}\omega_1 + 2\omega_2\omega_4 + 2\omega_3\omega_3\right) = 0 \quad (3.214)$$

$$\left(\frac{(1 + \epsilon\omega_5) + Ra}{Pr}\right)\frac{d\omega_6}{d\eta} + \omega_1\omega_6 + \epsilon\omega_6\omega_6 + Du\frac{d\omega_8}{d\eta}$$
$$+Ec\left(1 + \frac{1}{\beta}\right)(1 + \gamma\omega_5)\omega_3\omega_3 = 0 \quad (3.215)$$

$$\frac{d\omega_8}{d\eta} - ScCr\omega_7 + Sc\omega_1\omega_8 + Sc\tau\left(\omega_7\frac{d\omega_6}{d\eta} + \omega_6\omega_8\right) + ScSr\frac{d\omega_6}{d\eta} = 0 \quad (3.216)$$

Simplifying equation (3.214) to obtain

$$A_2\omega_1\frac{d\omega_4}{d\eta} = -\left(1+_{\beta}\frac{1}{}\right)\omega_4 - \left(1+_{\beta}\frac{1}{}\right)\omega_6\omega_3 - \gamma\left(1+_{\beta}\frac{1}{}\right)\omega_5\omega_6\omega_3 - Gr\omega_5$$

$$-Gm\omega_7 + M^2\omega_2 - \omega_1\omega_3 + \omega_2\omega_2 + \frac{1}{Ps}\left(1+_{\beta}\frac{1}{}\right)\omega_2 - 2\alpha\omega_2\omega_4 - 2\alpha\omega_3\omega_3$$

Therefore;

$$\frac{d\omega_4}{d\eta} = \frac{\left(1+_{\beta}\frac{1}{}\right)\omega_4}{A_2\omega_1} - \frac{\left(1+_{\beta}\frac{1}{}\right)\omega_6\omega_3}{A_2\omega_1} - \frac{\gamma\left(1+_{\beta}\frac{1}{}\right)\omega_5\omega_6\omega_3}{A_2\omega_1} - \frac{Gr\omega_5}{A_2\omega_1} - \frac{Gm\omega_7}{A_2\omega_1}$$

$$+\frac{M^2\omega_2}{\alpha\omega_1} - \frac{\omega_1\omega_3}{\alpha\omega_1} + \frac{\omega_2\omega_2}{\alpha\omega_1} + \frac{\frac{1}{Ps}\left(1+_{\beta}\frac{1}{}\right)\omega_2}{\alpha\omega_1} - \frac{2\alpha\omega_2\omega_4}{\alpha\omega_1} - \frac{2\alpha\omega_3\omega_3}{\alpha\omega_1} \tag{3.217}$$

Simplifying equation (3.215) to obtain

$$\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)\frac{d\omega_6}{d\eta} = -\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Du\frac{d\omega_8}{d\eta} - Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3$$

$$\frac{d\omega_6}{d\eta} = \frac{-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$-\frac{Du}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}\frac{d\omega_8}{d\eta} \tag{3.218}$$

Substituting equations (3.218) into (3.216) to obtain

$$\frac{d\omega_8}{d\eta} - ScCr\omega_7 + Sc\omega_1\omega_8 - \frac{Du(Sc\tau\omega_7 + ScSr)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}\frac{d\omega_8}{d\eta}$$

$$+\frac{(Sc\tau\omega_7 = ScSr)(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+_{\beta}\frac{1}{}\right))(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

Upon further simplification to obtain

$$\frac{d\omega_8}{d\eta} = \frac{ScCr\omega_7\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)} - \frac{Sc\omega_1\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)}$$

$$-\frac{(Sc\tau\omega_7 + ScSr)\left(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)} \tag{3.219}$$

Setting $Z_1 = \frac{ScCr\omega_7 \frac{(1+\ \omega_5)+\ Ra}{Pr}}{\frac{(1+\ \omega_5)+\ Ra}{Pr} - Du\ (Sc\tau\omega_7 + ScSr\ )} - \frac{Sc\omega_1\omega_8 \frac{(1+\ \omega_5)+\ Ra}{Pr}}{\frac{(1+\ \omega_5)+\ Ra}{Pr} - Du\ (Sc\tau\omega_7 + ScSr\ )}$

$$-\frac{(Sc\tau\omega_7+ScSr)\left(-\omega_1\omega_6-\epsilon\omega_6\omega_6-Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)-Du(Sc\tau\omega_7+ScSr)}$$

Substituting Z_1 into (3.219) and using the result in (3.218) to obtain

$$\frac{d\omega_6}{d\eta} = \frac{-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1 +_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$-\frac{Du}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)} Z_1 \tag{3.220}$$

Theorem 3.3: Let f,θ and φ be continuous function at all points in some neighborhood and considering $\beta > 0, \gamma > 0, Gr > 0, Gm > 0, M > 0, P_s > 0, \alpha > 0, \epsilon >$

$0, Ra > 0, Pr > 0, Du > 0, Ec > 0, Sc > 0, Cr > 0$ and $Sr > 0$, then there exist a

unique solution for the equations

$$\left(1 +_{\beta}\frac{1}{}\right)\frac{d^3 f}{d\eta} + \left(1 +_{\beta}\frac{1}{}\right)(1+\gamma\theta)\frac{d\theta}{d\eta}\frac{d^2 f}{d\eta^2} + Gr\theta + Gm\phi - M^2\frac{df}{d\eta} + f\frac{d^2 f}{d\eta^2}$$

$$-\frac{1}{P_s}\left(1 +_{\beta}\frac{1}{}\right)\frac{df}{d\eta} + A_2\left(f\frac{d^4 f}{d\eta^4} + 2\frac{df}{d\eta}\frac{d^3 f}{d\eta^3} + 2\left(\frac{d^2 f}{d\eta^2}\right)^2\right) = 0$$

$$\left(\frac{(1+\epsilon\theta)+Ra}{Pr}\right)\frac{d^2\theta}{d\eta^2} + f\frac{d\theta}{d\eta} + \epsilon\left(\frac{d\theta}{d\eta}\right) + Du\frac{d^2\phi}{d\eta^2} + Ec\left(1 +_{\beta}\frac{1}{}\right)(1+\gamma\theta)\left(\frac{d^2 f}{d\eta^2}\right)^2 = 0$$

$$\frac{d^2\phi}{d\eta^2} - ScCr\phi + Scf\frac{d\phi}{d\eta} + Sc\tau\left(\phi\frac{d^2\theta}{d\eta^2} + \frac{d\theta}{d\eta}\frac{d\phi}{d\eta}\right) + ScSr\frac{d^2\theta}{d\eta^2} = 0$$

subject to

$$\frac{df}{d\eta} = 1,\ f = f_w,\ \theta = 1,\ \phi = 1,\ \ at\ \ \eta = 0$$

$$\frac{df}{d\eta} \longrightarrow 0,\ \theta \longrightarrow 0\ \ ,\phi \longrightarrow 0,\ \ as\ \ \eta \longrightarrow \infty$$

on the interval $||y-y_0|| \leq a$, $||y_0-y|| \leq b$ provided $\exists k$ such that $k = max(0,1,P_1,...,P_n)$

and $0 < k < \infty$.

Proof Writing the systems of first order ordinary differential equations (3.210)(3.213) in compact form as

$$
\begin{bmatrix}
\frac{d\omega_1}{d\eta} \\
\frac{d\omega_2}{d\eta} \\
\frac{d\omega_3}{d\eta} \\
\frac{d\omega_4}{d\eta} \\
\frac{d\omega_5}{d\eta} \\
\frac{d\omega_6}{d\eta} \\
\frac{d\omega_7}{d\eta} \\
\frac{d\omega_8}{d\eta}
\end{bmatrix}
=
\begin{bmatrix}
\omega_2 \\
\omega_3 \\
\omega_4 \\
\frac{\left(1+\frac{1}{\beta}\right)\omega_4}{A_2\omega_1} - \frac{\left(1+\frac{1}{\beta}\right)\omega_6\omega_3}{A_2\omega_1} - \frac{\gamma\left(1+\frac{1}{\beta}\right)\omega_5\omega_6\omega_3}{A_2\omega_1} - \frac{Gr\omega_5}{A_2\omega_1} - \frac{Gm\omega_7}{A_2\omega_1} \\
+\frac{M^2\omega_2}{A_2\omega_1} - \frac{\omega_1\omega_3}{A_2\omega_1} + \frac{\omega_2\omega_2}{A_2\omega_1} + \frac{\frac{1}{Ps}\left(1+\frac{1}{\beta}\right)\omega_2}{A_2\omega_1} - \frac{2A_2\omega_2\omega_4}{A_2\omega_1} - \frac{2A_2\omega_3\omega_3}{A_2\omega_1}\omega_6 \\
\frac{-\omega_1\omega_6-\epsilon\omega_6\omega_6-Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)} - \frac{Du}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}Z_1 \\
\omega_8 \\
\frac{ScCr\omega_7\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)-Du(Sc\tau\omega_7+ScSr)} - \frac{Sc\omega_1\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)-Du(Sc\tau\omega_7+ScSr)} \\
-\frac{(Sc\tau\omega_7+ScSr)\left(-\omega_1\omega_6-\epsilon\omega_6\omega_6-Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\omega_5)\omega_3\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)-Du(Sc\tau\omega_7+ScSr)}
\end{bmatrix}
\tag{3.221}
$$

satisfying the conditions

$\omega_1(0)$ f_w

$\omega^2(0)$ 1

$\omega_3(0)$ α_1

$\omega_4(0)$ α_2

=

$\omega_5(0)$ 1

$\omega_6(0)$ α_3

$\omega_7(0)$ 1

$\omega_8(0)$ α_4

We shall consider $\frac{\partial f_i}{\partial \omega_j}$ such that *i,j* = 1(1)8 to denote the non-linear functions on the right hand side of equation (3.221) if and only if *i* = 1,...,8 and *j* = *counts*.

When *i* = 1 and *j* = *counts*, we obtain

$$f_1 = \omega_2$$

$$|\frac{\partial f_1}{\partial \omega_1}| = |\frac{\partial f_1}{\partial \omega_3}| = |\frac{\partial f_1}{\partial \omega_4}| = |\frac{\partial f_1}{\partial \omega_5}| = |\frac{\partial f_1}{\partial \omega_6}| = |\frac{\partial f_1}{\partial \omega_7}| = |\frac{\partial f_1}{\partial \omega_8}| = 0 < \infty$$

$$|\frac{\partial f_1}{\partial \omega_2}| = 1 < \infty$$

When *i* = 2 and *j* = *counts*, we obtain

$$f_2 = \omega_3$$

$$|\frac{\partial f_2}{\partial \omega_1}| = |\frac{\partial f_2}{\partial \omega_2}| = |\frac{\partial f_2}{\partial \omega_4}| = |\frac{\partial f_2}{\partial \omega_5}| = |\frac{\partial f_2}{\partial \omega_6}| = |\frac{\partial f_2}{\partial \omega_7}| = |\frac{\partial f_2}{\partial \omega_8}| = 0 < \infty$$

$$|\frac{\partial f_2}{\partial \omega_3}| = 1 < \infty$$

When *i* = 3 and *j* = *counts*, we obtain

$$f_3 = \omega_4$$

$$|\frac{\partial f_3}{\partial \omega_1}| = |\frac{\partial f_3}{\partial \omega_2}| = |\frac{\partial f_3}{\partial \omega_3}| = |\frac{\partial f_3}{\partial \omega_5}| = |\frac{\partial f_3}{\partial \omega_6}| = |\frac{\partial f_3}{\partial \omega_7}| = |\frac{\partial f_3}{\partial \omega_8}| = 0 < \infty$$

$$|\frac{\partial f_3}{\partial \omega_4}| = 1 < \infty$$

When *i* = 4 and *j* = *counts*, we obtain

$$f_4 = \frac{\left(1 +_{\beta}^{1}\right)\omega_4}{A_2\omega_1} - \frac{\left(1 +_{\beta}^{1}\right)\omega_6\omega_3}{A_2\omega_1} - \frac{\gamma\left(1 +_{\beta}^{1}\right)\omega_5\omega_6\omega_3}{A_2\omega_1} - \frac{Gr\omega_5}{A_2\omega_1} - \frac{Gm\omega_7}{A_2\omega_1}$$

$$+\frac{M^2\omega_2}{A_2\omega_1} - \frac{\omega_1\omega_3}{A_2\omega_1} + \frac{\omega_2\omega_2}{A_2\omega_1} + \frac{\frac{1}{Ps}\left(1 +_{\beta}^{1}\right)\omega_2}{A_2\omega_1} - \frac{2A_2\omega_2\omega_4}{A_2\omega_1} - \frac{2A_2\omega_3\omega_3}{A_2\omega_1}$$

$$|\frac{\partial f_4}{\partial \omega_1}| = |\frac{-\omega_3}{A_2}| \leq \frac{|-\omega_3|}{A-2} = Q_1 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_2}| = \frac{M^2 + 2|\omega_2| + \frac{1}{P_s}\left(1 +_{\beta}^{1}\right) - 2A_2\omega_4|}{\alpha|\omega_1}$$

$$\leq \frac{M^2 + 2|\omega_2| + \frac{1}{P_s}\left(1 +_{\beta}^{1}\right) - 2\alpha|\omega_4|}{\alpha|\omega_1} = Q_2 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_3}| = |\frac{-\left(1 +_{\beta}^{1}\right)\omega_6 - \gamma\left(1 +_{\beta}^{1}\right)\omega_5\omega_6 - \omega_1 - 4\alpha\omega_3}{A_2\omega_1}|$$

$$\leq \frac{-\left(1 +_{\beta}^{1}\right)|\omega_6| - \gamma\left(1 +_{\beta}^{1}\right)|\omega_5||\omega_6| - |\omega_1| - 4\alpha|\omega_3|}{A_2|\omega_1|} = Q_3 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_4}| = |\frac{-\left(1 +_{\beta}^{1}\right) - 2A_2\omega_2}{A_2\omega_1}| \leq \frac{-\left(1 +_{\beta}^{1}\right) - 2A_2|\omega_2|}{A_2|\omega_1|} = Q_4 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_5}| = |\frac{-\gamma\left(1+_{\beta}\frac{1}{}\right)\omega_6\omega_3}{A_2\omega_1}| \le \frac{-\gamma\left(1+_{\beta}\frac{1}{}\right)|\omega_6||\omega_3|}{A_2|\omega_1|} = Q_5 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_6}| = |\frac{-\left(1+_{\beta}\frac{1}{}\right)\omega_3 - \gamma\left(1+_{\beta}\frac{1}{}\right)\omega_5\omega_3}{A_2\omega_1}|$$

$$\le \frac{-\left(1+_{\beta}\frac{1}{}\right)|\omega_3| - \gamma\left(1+_{\beta}\frac{1}{}\right)|\omega_5||\omega_3|}{A_2|\omega_1|} = Q_6 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_7}| = |\frac{-Gm}{A_2\omega_1}| \le \frac{-Gm}{A_2|\omega_1|} = Q_7 < \infty$$

$$|\frac{\partial f_4}{\partial \omega_8}| = 0 < \infty$$

When i = 5 and j = *counts*, we obtain

$$f_5 = \omega_6$$

$$|\frac{\partial f_5}{\partial \omega_1}| = |\frac{\partial f_5}{\partial \omega_2}| = |\frac{\partial f_5}{\partial \omega_3}| = |\frac{\partial f_5}{\partial \omega_4}| = |\frac{\partial f_5}{\partial \omega_5}| = |\frac{\partial f_5}{\partial \omega_7}| = |\frac{\partial f_5}{\partial \omega_8}| = 0 < \infty$$

$$|\frac{\partial f_5}{\partial \omega_6}| = 1 < \infty$$

When i = 6 and j = *counts*, we obtain

$$f_6 = \frac{-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)} - \frac{Du}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}Z_1$$

$$|\frac{\partial f_6}{\partial \omega_1}| = |\frac{-Sc\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - (Sc\tau\omega_7 + ScSr)(-\omega_6)}{-(Sc\tau\omega_7 + ScSr)}|$$

$$\le \frac{-Sc|\omega_8|\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - (Sc\tau|\omega_7| + ScSr)(-|\omega_6|)}{-(Sc\tau|\omega_7| + ScSr)} = Q_8 < \infty$$

$$|\frac{\partial f_6}{\partial \omega_2}| = 0 < \infty$$

$$|\frac{\partial f_6}{\partial \omega_3}| = |\frac{-2Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3}{-(Sc\tau\omega_7 + ScSr)}| \le \frac{-2Ec\left(1+_{\beta}\frac{1}{}\right)(1+\gamma|\omega_5|)|\omega_3|}{-(Sc\tau|\omega_7| + ScSr)} = Q_9 < \infty$$

$$|\frac{\partial f_6}{\partial \omega_4}| = 0 < \infty$$

$$|\frac{\partial f_6}{\partial \omega_5}| = |\frac{\epsilon ScCrPr\omega_7 - Sc\epsilon Pr\omega_1\omega_8 - Ec\left(1+_{\beta}\frac{1}{}\right)\gamma\omega_3^2}{-(Sc\tau\omega_7 + ScSr)(Pr\epsilon)}|$$

$$\leq \frac{\epsilon ScCrPr|\omega_7| - Sc\epsilon Pr|\omega_1||\omega_8| - Ec\left(1 +_{\beta}\frac{1}{}\right)\gamma|\omega_3^2|}{-(Sc\tau|\omega_7| + ScSr)(Pr\epsilon)} = Q_{10} < \infty$$

$$|\frac{\partial f_6}{\partial \omega_6}| = |\frac{-(Sc\tau\omega_7 + ScSr)(-\omega_1 - 2\epsilon\omega_6)}{-(Sc\tau\omega_7 + ScSr)}|$$

$$\leq \frac{-(Sc\tau|\omega_7| + ScSr)(-|\omega_1| - 2\epsilon|\omega_6|)}{-(Sc\tau|\omega_7| + ScSr)} = Q_{11} < \infty$$

$$|\frac{\partial f_6}{\partial \omega_7}| = |\frac{ScCr\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Sc\tau}{-(Sc\tau\omega_7 + ScSr)(DuSc\tau)}|$$

$$\leq \frac{ScCr\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right) - Sc\tau}{-(Sc\tau|\omega_7| + ScSr)(DuSc\tau)} = Q_{12} < \infty$$

$$|\frac{\partial f_6}{\partial \omega_8}| = 0 < \infty$$

When *i* = 7 and *j* = *counts*, we obtain

$$f_7 = \omega_8$$

$$|\frac{\partial f_7}{\partial \omega_1}| = |\frac{\partial f_7}{\partial \omega_2}| = |\frac{\partial f_7}{\partial \omega_3}| = |\frac{\partial f_7}{\partial \omega_4}| = |\frac{\partial f_7}{\partial \omega_5}| = |\frac{\partial f_7}{\partial \omega_6}| = |\frac{\partial f_7}{\partial \omega_7}| = 0 < \infty$$

$$|\frac{\partial f_7}{\partial \omega_8}| = 1 < \infty$$

When *i* = 8 and *j* = *counts*, we obtain

$$f_8 = ScCr\omega_7 - Sc\omega_1\omega_8 -$$

$$\frac{(Sc\tau\omega_7 + ScSr)\left(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1 +_{\beta}\frac{1}{}\right)(1 + \gamma\omega_5)\omega_3\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$|\frac{\partial f_8}{\partial \omega_1}| = |\frac{-Sc\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - (Sc\tau\omega_7 + ScSr)(-\omega_6)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}|$$

$$\leq \frac{-Sc|\omega_8|\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right) - (Sc\tau|\omega_7| + ScSr)(-|\omega_6|)}{\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right)} = Q_{13} < \infty$$

$$|\frac{\partial f_8}{\partial \omega_2}| = 0 < \infty$$

$$|\frac{\partial f_8}{\partial \omega_3}| = |\frac{-(Sc\tau\omega_7 + ScSr)\left(-2Ec\left(1 +_{\beta}^{1}\right)(1+\gamma\omega_5)\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}|$$

$$\leq \frac{-(Sc\tau|\omega_7| + ScSr)\left(-2Ec\left(1 +_{\beta}^{1}\right)(1+\gamma|\omega_5|)|\omega_3|\right)}{\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right)} = Q_{14} < \infty$$

$$|\frac{\partial f_8}{\partial \omega_4}| = 0 < \infty$$

$$|\frac{\partial f_8}{\partial \omega_5}| = |\frac{Pr\epsilon ScCr\omega_7 - Pr\epsilon Sc\omega_1\omega_8 - (Sc\tau\omega_7 + ScSr)\left(-Ec\left(1 +_{\beta}^{1}\right)\gamma\omega_3\omega_3\right)}{Pr\epsilon}|$$

$$\leq \frac{Pr\epsilon ScCr|\omega_7| - Pr\epsilon Sc|\omega_1||\omega_8| - (Sc\tau|\omega_7| + ScSr)\left(-Ec\left(1 +_{\beta}^{1}\right)\gamma|\omega_3||\omega_3|\right)}{Pr\epsilon} = Q_{15} < \infty$$

$$|\frac{\partial f_8}{\partial \omega_6}| = |\frac{-(Sc\tau\omega_7 + ScSr)(-\omega_1 - 2\epsilon\omega_6)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}|$$

$$\leq \frac{-(Sc\tau|\omega_7| + ScSr)(-|\omega_1| - 2\epsilon|\omega_6|)}{\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right)} = Q_{16} < \infty$$

$$|\frac{\partial f_8}{\partial \omega_7}| = |ScCr - \frac{Sc\tau(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1 +_{\beta}^{1}\right)(1+\gamma\omega_5)\omega_3\omega_3)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}|$$

$$\leq ScCr - \frac{Sc\tau(-|\omega_1||\omega_6| - \epsilon|\omega_6||\omega_6| - Ec\left(1 +_{\beta}^{1}\right)(1+\gamma|\omega_5|)|\omega_3||\omega_3|)}{\left(\frac{(1+\epsilon|\omega_5|)+Ra}{Pr}\right)} = Q_{17} < \infty$$

$$|\frac{\partial f_8}{\partial \omega_8}| = |-Sc\omega_1| \leq -Sc|\omega_1| = Q_{18} < \infty$$

Hence,

$$|\frac{\partial f_i}{\partial \omega_j}| \leq K$$

such that

$$1, j = 1(1)8$$

. Obviously, $|\frac{\partial f_i}{\partial \omega_j}|_{i,j=1(1)8}$ is bounded for $1 = 1,2,...,8 \exists K$ such that $K = max[0,1,Q_1,Q_2,...,Q_{18}]$ and $0 < K < \infty$. It means that, $f_i(\omega_1,\omega_2\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8)$ are Lipschitz continuous, Hence, the solution for the system of coupled non-linear ordinary differential equations is unique. We proceed to proof the stated theorem 3.2 on the transformed equations (3.205)-(3.207) for the problem two by considering the system of first order differential equations with critical point of ω_i', then the following critical points

(0,0,0,0,0,0,0,0) and (0,1,0,0,0,0,0,0) are considered.

Consider:

$$\omega_1' = f_1(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \omega_2$$

$$\omega_2' = f_2(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \omega_3$$

$$\omega_3' = f_3(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \omega_4$$

$$\omega_4' = f_4(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \frac{\left(1+\frac{1}{\beta}\right)\omega_4}{A_2\omega_1}$$

$$-\frac{\left(1+\frac{1}{\beta}\right)\omega_6\omega_3}{A_2\omega_1} - \frac{\gamma\left(1+\frac{1}{\beta}\right)\omega_5\omega_6\omega_3}{A_2\omega_1} - \frac{Gr\omega_5}{A_2\omega_1} - \frac{Gm\omega_7}{A_2\omega_1}$$

$$\omega_5' = f_1(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \omega_6$$

$$\omega_6' = f_6(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \frac{-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\omega_5)\omega_3\omega_3}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$-\frac{Du}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}Z_1\omega_8$$

$$\omega_7' = f_7(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \omega_8$$

$$\omega_8' = f_8(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8) = \frac{ScCr\omega_7\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)}$$

$$-\frac{Sc\omega_1\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)}$$

$$-\frac{(Sc\tau\omega_7 + ScSr)\left(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1+\frac{1}{\beta}\right)(1+\gamma\omega_5)\omega_3\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Du(Sc\tau\omega_7 + ScSr)}$$

Thus, the necessary and sufficient condition of Jacobian matrix is satisfied and it takes the form

$$A = \begin{bmatrix}
\frac{\partial f_1}{\partial \omega_1} & \partial f_1 & \frac{\partial f_1}{\partial \omega_3} & \frac{\partial f_1}{\partial \omega_4} & \frac{\partial f_1}{\partial \omega_5} & \partial f_1 & \frac{\partial f_1}{\partial \omega_7} & \frac{\partial f_1}{\partial \omega_8} \\
\frac{\partial f_2}{\partial \omega_1} & & \frac{\partial f_2}{\partial \omega_3} & \frac{\partial f_2}{\partial \omega_4} & \frac{\partial f_2}{\partial \omega_5} & & \frac{\partial f_2}{\partial \omega_7} & \frac{\partial f_2}{\partial \omega_8} \\
\frac{\partial f_3}{\partial \omega_1} & & \frac{\partial f_3}{\partial \omega_3} & \frac{\partial f_3}{\partial \omega_4} & \frac{\partial f_3}{\partial \omega_5} & & \frac{\partial f_3}{\partial \omega_7} & \frac{\partial f_3}{\partial \omega_8} \\
\frac{\partial f_4}{\partial \omega_1} & & \frac{\partial f_4}{\partial \omega_3} & \frac{\partial f_4}{\partial \omega_4} & \frac{\partial f_4}{\partial \omega_5} & & \frac{\partial f_4}{\partial \omega_7} & \frac{\partial f_4}{\partial \omega_8} \\
\frac{\partial f_5}{\partial \omega_1} & & \frac{\partial f_5}{\partial \omega_3} & \frac{\partial f_5}{\partial \omega_4} & \frac{\partial f_5}{\partial \omega_5} & & \frac{\partial f_5}{\partial \omega_7} & \frac{\partial f_5}{\partial \omega_8} \\
\frac{\partial f_6}{\partial \omega_1} & & \frac{\partial f_6}{\partial \omega_3} & \frac{\partial f_6}{\partial \omega_4} & \frac{\partial f_6}{\partial \omega_5} & & \frac{\partial f_6}{\partial \omega_7} & \frac{\partial f_6}{\partial \omega_8} \\
\frac{\partial f_7}{\partial \omega_1} & & \frac{\partial f_7}{\partial \omega_3} & \frac{\partial f_7}{\partial \omega_4} & \frac{\partial f_7}{\partial \omega_5} & & \frac{\partial f_7}{\partial \omega_7} & \frac{\partial f_7}{\partial \omega_8} \\
\frac{\partial f_8}{\partial \omega_1} & & \frac{\partial f_8}{\partial \omega_3} & \frac{\partial f_8}{\partial \omega_4} & \frac{\partial f_8}{\partial \omega_5} & & \frac{\partial f_8}{\partial \omega_7} & \frac{\partial f_8}{\partial \omega_8}
\end{bmatrix}$$

$\partial\omega_2\,\partial\omega_6$

$\partial f_2\,\partial f_2$

$\partial\omega_2\,\partial\omega_6$

$$\frac{\partial f_3}{\partial \omega_2} \frac{\partial f_3}{\partial \omega_6}$$

$$\frac{\partial f_4}{\partial \omega_2} \frac{\partial f_4}{\partial \omega_6}$$

$$\frac{\partial f_5}{\partial \omega_2} \frac{\partial f_5}{\partial \omega_6}$$

$$\frac{\partial f_6}{\partial \omega_2} \frac{\partial f_6}{\partial \omega_6}$$

$$\frac{\partial f_7}{\partial \omega_2} \frac{\partial f_7}{\partial \omega_6}$$

$$\frac{\partial f_8}{\partial \omega_2} \frac{\partial f_8}{\partial \omega_6}$$

This becomes;

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ Q_8 & 0 & Q_9 & 0 & Q_{10} & Q_{11} & Q_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ Q_{13} & 0 & Q_{14} & 0 & Q_{15} & Q_{16} & Q_{17} & Q_{18} \end{bmatrix}$$

given that:

$$Q_1 = \frac{-\omega_3}{\alpha}$$

$$Q_2 = \frac{M^2 + 2|\omega_2| + \frac{1}{P_s}\left(1 + \frac{1}{\beta}\right) - 2A_2|\omega_4|}{\alpha|\omega_1|}$$

$$Q_3 \frac{-\left(1 + \frac{1}{\beta}\right)\omega_6 - \gamma\left(1 + \frac{1}{\beta}\right)\omega_5\omega_6 - \omega_1 - 4A_2\omega_3}{A_2\omega_1}$$

$$Q_4 = \frac{-\left(1 + \frac{1}{\beta}\right) - 2A_2\omega_2}{A_2\omega_1}$$

$$Q_5 = \frac{-\gamma\left(1 +_{\beta}\frac{1}{}\right)\omega_6\omega_3}{A_2\omega_1}$$

$$Q_6 = \frac{-\left(1 +_{\beta}\frac{1}{}\right)\omega_3 - \gamma\left(1 +_{\beta}\frac{1}{}\right)\omega_5\omega_3}{A_2\omega_1}$$

$$Q_7 = \frac{-Gm}{A_2\omega_1}$$

$$Q_8 = \frac{-Sc\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - (Sc\tau\omega_7 + ScSr)(-\omega_6)}{-(Sc\tau\omega_7 + ScSr)}$$

$$Q_9 = \frac{-2Ec\left(1 +_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3}{-(Sc\tau\omega_7 + ScSr)}$$

$$Q_{10} = \frac{\epsilon ScCrPr\omega_7 - Sc\epsilon Pr\omega_1\omega_8 - Ec\left(1 +_{\beta}\frac{1}{}\right)\gamma\omega_3^2}{-(Sc\tau\omega_7 + ScSr)(Pr\epsilon)}$$

$$Q_{11} = \frac{-(Sc\tau\omega_7 + ScSr)(-\omega_1 - 2\epsilon\omega_6)}{-(Sc\tau\omega_7 + ScSr)}$$

$$Q_{12} = \frac{ScCr\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - Sc\tau}{-(Sc\tau\omega_7 + ScSr)(DuSc\tau)}$$

$$Q_{13}\frac{-Sc\omega_8\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right) - (Sc\tau\omega_7 + ScSr)(-\omega_6)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$Q_{14} = \frac{-(Sc\tau\omega_7 + ScSr)\left(-2Ec\left(1 +_{\beta}\frac{1}{}\right)(1+\gamma\omega_5)\omega_3\right)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$Q_{15} = \frac{Pr\epsilon ScCr\omega_7 - Pr\epsilon Sc\omega_1\omega_8 - (Sc\tau\omega_7 + ScSr)\left(-Ec\left(1 +_{\beta}\frac{1}{}\right)\gamma\omega_3\omega_3\right)}{Pr\epsilon}$$

$$Q_{16} = \frac{-(Sc\tau\omega_7 + ScSr)(-\omega_1 - 2\epsilon\omega_6)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$Q_{17} = ScCr - \frac{Sc\tau(-\omega_1\omega_6 - \epsilon\omega_6\omega_6 - Ec\left(1 +\frac{1}{\beta}\right)(1+\gamma\omega_5)\omega_3\omega_3)}{\left(\frac{(1+\epsilon\omega_5)+Ra}{Pr}\right)}$$

$$Q_{18} = |-Sc\omega_1| \leq -Sc|\omega_1|$$

$Q_1 = 0,\ Q_2 = 0,\ Q_3 = 0,\ Q_4 = 0,\ Q_5 = 0,\ Q_6 = 0,\ Q_7 = 0,\ Q_8 = 0,\ Q_9 = 0,\ Q_{10} = 0,\ Q_{11} = 0,\ Q_{12} = -8.8114,\ Q_{13} = 0,\ Q_{14} = 0,\ Q_{15} = 0,\ Q_{16} = 0,\ Q_{17} = 0.6099$, $Q_{18} = 0$ Hence, the Jacobian matrix C becomes

0 1 0 0 0 0 0 0

$$B = \begin{bmatrix} \\ \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8.8104 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6099 & 1 \end{bmatrix}$$

0 0 1 0 0 0 0 0

We calculate the eigenvalues of the above matrix using $|B - \lambda I| = 0$, hence.

$$|B-\lambda I| = \begin{vmatrix} -\lambda & & 1 & 0 & 0 & & & & 0 & 0 & 0 & 0 \\ 0 & -\lambda & & & & & & & & & & \\ 0 & 0 & -\lambda & & & & & & & & & \\ & & & & & -\lambda & 1 & 0 & 0 & & & \\ & & & & & & -\lambda & 8.8104 & 0 & & & \\ & & & & & & & -\lambda & 0 & & & \\ & & & & & & & .6099 & -\lambda & & & \end{vmatrix}$$

1 0 0 0 0 0

1 0 0 0 0

0 0 0 $-\lambda$ 0 0 0 0

0 0 0 0

0 0 0 0 0

0 0 0 0 0 0

0 0 0 0 0 0 0

Using the MAPLE software, the eigenvalues gives:

$$|B-\lambda I| = \lambda^6(\lambda^2 - 0.6099)$$

Hence, the eigenvalues becomes

$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, λ_4, $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_7 = 0.7809609465$, $\lambda_8 = -0.7809609465$

Base on theorem 3.2, table 3.1 and all the cases considered in this study, we shall conclude that the system of coupled ordinary differential equation is Asymptotically

stable.

3.7 Solution technique to problem two

The fourth order coupled nonlinear total differential equations (3.205)-(3.207) subject to (3.208) and (3.209), the SHAM is utilized. SHAM as discussed by Motsa (2012) is based on the concept of Chebyshev pseudospectral method (CPM) and HAM as discussed by Liao (1992). To utilize SHAM, the immediate domain of the problem $[0,\infty)$ will be transformed by following the domain truncation approach and the numerical computational domain $[0,L]$. L is the length and it is considered to be more than the thickness of the boundary layer. Thus, the domain $[0,L]$ is first transformed to [−1,1], where the Chebyshev spectral techniques can be apply by using the algebraic function

$$\xi = \frac{2\eta}{L} - 1 \quad , \xi \in [-1,+1] \qquad (3.222)$$

In utilizing SHAM, the following transformation is used on the boundary conditions for it to be homogeneous

$$f(\eta) = f(\xi) + f_0(\eta) \, , f_0(\eta) = S_w + 1 - e^{-\eta}$$

$$\theta(\eta) = \theta(\xi) + \theta_0(\eta) \, , \theta_0(\eta) = e^{-\eta}$$

$$\varphi(\eta) = \theta(\xi) + \theta_0(\eta) \, , \theta_0(\eta) = e^{-\eta} \qquad (3.223)$$

where $f_0(\eta)$,$\theta_0(\eta)$ and $\varphi_0(\eta)$ are the initial guess considered in reference to the boundary conditions (3.208) − (3.209) respectively. Using equation (3.223) on the transformed flow equations (3.205)-(3.207) along the boundary conditions (3.208) and (3.209) leads to

$$\gamma\left(1+\frac{1}{\beta}\right)[\theta' f'' - \theta f''' + Psf'] + [1+\gamma]\left(1+\frac{1}{\beta}\right)[f''' - Psf'] + f''[a_2 + f + a_4 + a_{12} + 2A_2 f'']$$

$$+\theta[a_6 + Gr] + f'[a_{10} - M_p + a_5] + f[a_3 + a_8] + f''''[A_2 f + a_9]$$

$$+f^{000}[2A_2 f^0 + a_7 + a_{11}] + a_1\theta^0 - (f^0)^2 + G_m\varphi = \psi_1 \qquad (3.224)$$

$$\left(\left(1+\frac{4}{3}Rd\right) + \epsilon\theta + b_2\right)\theta'' - (Prf + b_4 + \theta' + b_5)\theta' + (b_1 + b_8 + Pr\delta_x)\theta + b_3 f$$

$$+\left(PrEc(1+\gamma)\left(1+\frac{1}{\beta}\right) - PrEc\gamma\theta\left(1+\frac{1}{\beta}\right) + b_9\right)(f'')^2$$

$$+(b_6 + b_7\theta + b_10)f'' + PrD_f\phi'' = \psi_2(\eta) \qquad (3.225)$$

$$\varphi^{00} + scf\varphi^0 + Scc_2 f + Sca_1\varphi^0 - ScCr\varphi + Sc\tau\theta^0\varphi^0 + Sc\tau c_2\theta^0 + Sc\tau b_2\varphi^0$$

$$+Sc\tau\varphi + Sc\tau b_3\varphi + Sc\tau c_1\theta^{00} + ScSo\theta^{00} = \psi_3(\eta) \qquad (3.226)$$

subject to

$$f(-1) = f(1) = f^0(-1) = f^0(1) = 0 \quad , \theta(-1) = \theta(1) = 0 \quad , \varphi(-1) = \varphi(1) = 0$$

(3.227)

where

$$a_1 = \gamma\left(1+\frac{1}{\beta}\right) f_0'' \ , a_2 = \gamma\left(1+\frac{1}{\beta}\right)\theta_0' \ , a_3 = f_0'' \ , a_4 = f_0 \ , a_5 = -2f_0',$$

$$a_6 = -\gamma\left(1+\frac{1}{\beta}\right) f_0''',\ a_7 = -\gamma\left(1+\frac{1}{\beta}\right)\theta_0,\ a_8 = A_2 f_0'''',\ a_9 = A_2 f_0,$$

$$a_{10} = 2A_2 f_0''',\ a_{11} = 2A_2 f_0',\ a_{12} = 4A_2 f_0''$$

$$\psi_1(\eta) = f_0'\left[M_p + [1+\gamma]\left(1+\frac{1}{\beta}\right)Ps - \gamma\left(1+\frac{1}{\beta}\right)Ps\right]$$

$$-f_0'''\left[(1+\gamma)\left(1+\frac{1}{\beta}\right) - \gamma\left(1+\frac{1}{\beta}\right)\theta_0 + 2A_2 f_0'\right] - f_0''\left[\gamma\left(1+\frac{1}{\beta}\right)\theta_0' + f_0\right]$$

$$+(f')^2 - Gr\theta_0 - Gm\phi_0 - A_2 f_0 f_0''''$$

$$b_1 = \epsilon\theta_0'' \ , b_2 = \epsilon\theta_0 \ , b_3 = -Pr\theta_0',\ b_4 = -Prf_0,\ b_5 = 2\theta_0',$$

$$b_6 = 2PrEc[1+\gamma]\left(1+\frac{1}{\beta}\right) f_0'',\ b_7 = -2PrEc\left(1+\frac{1}{\beta}\right)\gamma f_0'',$$

$$b_8 = -PrEc\gamma\left(1+\frac{1}{\beta}\right)(f_0'')^2,\ b_9 = -PrEc\gamma\left(1+\frac{1}{\beta}\right)\theta_0,$$

$$b_{10} = -2PrEc\gamma\left(1+\frac{1}{\beta}\right)\theta_0 f_0''$$

$$\psi_2(\eta) = -\left[1+\tfrac{4}{3}Rd\right]\theta_0'' - \epsilon\theta_0\theta_0'' + Prf_0\theta_0' - (\theta_0')^2 - PrEc[1+\gamma]\left(1+\tfrac{1}{\beta}\right)(f_0'')^2 + PrEc\left(1+\tfrac{1}{\beta}\right)\gamma\theta_0(f_0'')^2 - Pr\delta_x\theta_0 - PrD_f\phi_0''$$

$$c_1 = \phi_0 \ , c_2 = \phi_0' \ , \psi_3(\eta) = -\phi_0'' - Scf_0\phi_0' + ScCr\phi_0 - Sc\tau\theta_0'\phi_0' - Sc\tau\phi_0\theta_0'' - ScSo\theta_0''$$

The CPM is applied on the resulting equations (3.224)-(3.226) while the unknown functions $f(\xi)$, $\theta(\xi)$ and $\varphi(\xi)$.

$$f(\xi) \approx \sum_{k=0}^{N} f_k T_k(\xi_j),\ \theta(\xi) \approx \sum_{k=0}^{N} \theta_k T_k(\xi_j),$$

$$\varphi(\xi) \approx \sum_{k=0}^{N} \varphi_k T_k(\xi_j),\ j = 0,1,2,...,N \quad (3.228)$$

where $\xi_0, \xi_1, \chi_2, ..., \xi_N$ are Gauss-Lobatto collocation points (Canuto, Hussaini, Quarteroni and Zang 1998) defined by $\xi_j = \cos\frac{\pi j}{N}$, $j = 0,1,2,...,N$, T_k is the k^{th} Chebyshev polynomial and $N+1$ is the number of collocation points. The functions $f(\eta)$, $\theta(\eta)$ and $\varphi(\eta)$ derivatives at the collocation points are:

$$\frac{d^r f}{d\eta^r}=\sum_{k=0}^{N} D_{kj}^{r} f(\xi) \quad , \frac{d^r \theta}{d\eta^r}=\sum_{k=0}^{N} D_{kj}^{r} \theta(\xi) \quad , \frac{d^r \phi}{d\eta^r}=\sum_{k=0}^{N} D_{kj}^{r} \phi(\xi) \tag{3.229}$$

where r = differentiation and $\mathbf{D}=\frac{2}{L}D$ (see Canuto et al., 1988 and Trefethen, 2000) and $\mathbf{D}$ means Chebyshev spectral differential matrix. Using equations (3.228) and (3.229) together with Gauss-Lobatto collocation points into (3.224)-(3.226) to obtain

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} f_l \\ \theta_l \\ \varphi_l \end{bmatrix} = \begin{bmatrix} \psi_1(\eta) \\ \psi_2(\eta) \\ \psi_3(\eta) \end{bmatrix} \tag{3.230}$$

Equation (3.230) can be written as:

$$A\phi_l = G \tag{3.231}$$

subject to

$$f_l(\xi_N)=0 \,, \sum_{k=0}^{N} D_{Nk} f_l(\xi_k)=0 \,, \sum_{k=0}^{N} D_{0k} f_l(\xi_k)=0 \tag{3.232}$$

$$\theta_l(\xi_N)=0 \,, \theta_l(\xi_0)=0 \,, \varphi_l(\xi_N)=0 \,, \varphi_l(\xi_0)=0 \tag{3.233}$$

where

$$\phi_l = [f_l(\xi_0), f_l(\xi_1), f_l(\xi_2), \dots, f_l(\xi_N), \theta_l(\xi_0), \theta_l(\xi_1), \theta_l(\xi_2), \dots, \theta_l(\xi_N)]^T,$$

$$[\varphi_l(\xi_0), \varphi_l(\xi_1), \varphi_l(\xi_2), \dots, \varphi_l(\xi_N)]^T$$

$$G = [\psi_1(\eta_0), \psi_1(\eta_1), \psi_1(\eta_2), \dots, \psi_1(\eta_N), \psi_2(\eta_0), \psi_2(\eta_1), \psi_2(\eta_2), \dots, \psi_2(\eta_N),$$

$$\psi_3(\eta_0), \psi_3(\eta_1), \psi_3(\eta_2), \dots, \psi_3(\eta_N)]^T$$

A=

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{3.234}$$

And

$$A_{11} = a_9 D^4 + \left((1+\gamma)\left(1+\frac{1}{\beta}\right) + a_7 + a_{11}\right) D^3 + (a_2 + a_4 + a_{12}) D^2$$

$$+\left(a_5 - M_p - (1+\gamma)\left(1+\frac{1}{\beta}\right)Ps + \gamma\left(1+\frac{1}{\beta}\right)Ps + a_{10}\right)D + (a_3 + a_8)$$

$$A_{12} = a_1 D + a_6 + Gr,$$

,

$$A_{13} = GmI \quad, A_{21} = b_3 + (b_6 + b_{10})D^2,$$

$$A_{22} = \left(\left(1 + \frac{4}{3}Rd\right) + b_2\right)D^2 + (b_4 + b_5)D + (b_1 + b_8 + Pr\delta_x),$$

$$A_{23} = PrD_f D^2, \quad A_{31} = Scc_2 \quad, A_{32} = Sc\tau c_2 D + Sc\tau c_1 D^2 + ScSoD^2,$$

$$A_{33} = D^2 + Sca_1 D - ScC_r + Sc\tau b_2 D + Sc\tau D^2 + Sc\tau b_3$$

In other to get the approximate solution of SHAM, the linear operators are as follows:

$$L_f[f(\eta;q),\theta(\eta;q),\phi(\eta;q)] = a_1\theta_l' + a_2 f_l'' + a_3 f_l + a_4 f_l'' + a_5 f_l' + [1+\gamma]\left(1+\frac{1}{\beta}\right)f_l'''$$
$$+a_6\theta_l + a_7 f_l''' + Gr\theta_l + Gm\phi_l - M_p f_l' - [1+\gamma]\left(1+\frac{1}{\beta}\right)Psf_l'$$
$$+\gamma\left(1+\frac{1}{\beta}\right)Psf_l' + a_8 f_l + a_9 f_l'''' + a_{10} f_l' + a_{11} f_l''' + a_{12} f_l'' \qquad (3.235)$$

$$L_\theta[f(\eta;q),\theta(\eta;q),\phi(\eta;q)] = \left[1 + \frac{4}{3}Rd\right]\theta_l'' + b_1\theta_l + b_2\theta_l'' + b_3 f_l + b_4\theta_l' + b_5\theta_l' + b_6 f_l''$$
$$+b_8\theta_l + b_{10} f_l'' + Pr\delta_x\theta_l + PrD_f\phi_l'' \qquad (3.236)$$

$$L_\phi[f(\eta;q),\theta(\eta;q),\phi(\eta;q)] = \phi_l'' + Scc_2 f_l + Sca_1\phi_l' - ScSr\phi_l + Sc\tau c_2\theta_l'$$
$$+Sc\tau b_2\phi_l' + Sc\tau\phi_l'' + Sc\tau b_3\phi_l + Sc\tau c_1\theta_l'' + ScSo\theta_l'' \qquad (3.237)$$

3.7.1 Zero order of deformation

q 3 [0,1] = embedded parameter, $\bar{h}_f, \bar{h}_\theta$ and $\bar{h}_\varphi$ are non-zero auxiliary parameters and the unknown functions are $f(\eta;q)$,$\theta(\eta;q)$ and $\varphi(\eta;q)$. The following is the zeroth order of deformation

$$(1-q)L_f[f(\eta;q) - f_0(\eta)] = q\bar{h}_f\psi_f(\eta)N[f(\eta;q),\theta(\eta;q),\varphi(\eta;q)] \qquad (3.238)$$

$$(1-q)L_\theta[\theta(\eta;q) - \theta_0(\eta)] = q\bar{h}_\theta\psi_\theta(\eta)N[f(\eta;q),\theta(\eta;q),\varphi(\eta;q)] \qquad (3.239)$$

$$(1-q)L_{\varphi}[\varphi(\eta;q)-\varphi_0(\eta)] = q\bar{h}_{\varphi}\psi_{\varphi}(\eta)N[f(\eta;q),\theta(\eta;q),\varphi(\eta;q)] \tag{3.240}$$

subject to:

$$f(\eta=0;q)=S_w, f^0(\eta=0;q)=1, \theta(\eta=0;q)=\varphi(\eta=0;q)=1 \tag{3.241}$$

$$f^0(\eta \longrightarrow \infty;q)=1, \theta(\eta \longrightarrow \infty;q)=\varphi(\eta \longrightarrow \infty;q)=1 \tag{3.242}$$

The nonlinear operators are defined as follows:

$$N_f[f(\eta;q),\theta(\eta;q),\phi(\eta;q)] = \gamma\left(1+\frac{1}{\beta}\right)\theta' f'' + ff'' - (f')^2$$
$$-\gamma\left(1+\frac{1}{\beta}\right)\theta f''' + A_2 ff'''' + 2A_2 f'f''' + 2A_2(f'')^2 \tag{3.243}$$

$$N_\theta[f(\eta;q),\theta(\eta;q),\phi(\eta;q)] = \epsilon\theta\theta'' - Prf\theta' + (\theta')^2 + PrEc[1+\gamma]\left(1+\frac{1}{\beta}\right)(f'')^2$$
$$-Pr\left(1+\frac{1}{\beta}\right)Ec\gamma\theta(f'')^2 + b_7\theta f'' + b_9(f'')^2 \tag{3.244}$$

$$N_{\varphi}[f(\eta;q),\theta(\eta;q),\varphi(\eta;q)] = Scf\varphi^0 + Sc\tau\theta^0\varphi^0 \tag{3.245}$$

setting $q=0$, the zeroth order of deformation equations (3.238)-(3.240) leads to

$$L_f[f(\eta;0)-f_0(\eta)]=0,\ L_\theta[\theta(\eta;0)-\theta_0(\eta)]=0$$

$$L_{\varphi}[\varphi(\eta;0)-\varphi_0(\eta)]=0 \tag{3.246}$$

Due to the reason that $f(\eta;0)=f_0(\eta)$, $\theta(\eta;0)=\theta_0(\eta)$ and $\varphi(\eta;0)=\varphi_0(\eta)$ subject to (3.241) and (3.242).

Setting $q=1$, the zeroth order of deformation (3.238)-(3.240) leads to

$$0=\bar{h}_f\psi_f(\eta)N[f(\eta;1),\theta(\eta;1),\varphi(\eta;1)] \tag{3.247}$$

$$0=\bar{h}_\theta\psi_\theta(\eta)N[f(\eta;1),\theta(\eta;1),\varphi(\eta;1)] \tag{3.248}$$

$$0=\bar{h}_{\varphi}\psi_{\varphi}(\eta)N[f(\eta;1),\theta(\eta;1),\varphi(\eta;1)] \tag{3.249}$$

iff $\bar{h}_f\psi_f(\eta)$ 6= 0,$\bar{h}_\theta\psi_\theta(\eta)$ 6= 0 and $\bar{h}_{\varphi}\psi_{\varphi}(\eta)$ 6= 0.

Thus,

$$f(\eta;1)=f(\eta),\ \theta(\eta;1)=\theta(\eta),\ \varphi(\eta;1)=\varphi(\eta) \tag{3.250}$$

subject to

$f(\eta = 0;1) = S_w, f^0(\eta = 0;1) = 1, \theta(\eta = 0;1) = 1, \varphi(\eta = 0;1) = 1$

$f^0(\eta \longrightarrow 0;1) = 0, \theta(\eta \longrightarrow 0;1) = 0, \varphi(\eta \longrightarrow 0;1) = 0$ (3.251)

3.7.2 High order of deformation

Simplifying $f(\eta;q)$,$\theta(\eta;q)$ and $\varphi(\eta;q)$ using Taylor series with respect to the embedding parameter q leads to

$$f(\eta;q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)q^m, \quad \theta(\eta;q) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)q^m$$

$$\phi(\eta;q) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)q^m \qquad (3.252)$$

where

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta;q)}{\partial \eta^m}\ |_{q=0}, \quad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta;q)}{\partial \eta^m}\ |_{q=0}$$

$$\phi_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta;q)}{\partial \eta^m}\ |_{q=0}$$

The series in equation (3.252) converges at $q = 1$. To get the m^{th} order deformation, differentiate equations (3.238)-(3.240) m times with respect to q, divide by $m!$ and set $q = 0$ to obtain

$$L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \bar{h}_f \psi_f(\eta) R_m^f(\eta) \qquad (3.253)(3.253)$$

$$L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \bar{h}_\theta \psi_\theta(\eta) R_m^\theta(\eta) \qquad (3.254)(3.254)$$

$$L_\varphi[\varphi_m(\eta) - \chi_m \varphi_{m-1}(\eta)] = \bar{h}_\varphi \psi_\varphi(\eta) R_m^\varphi(\eta) \qquad (3.255)$$

subject to:

$$f_m(\eta = 0) = 0, \quad f'_m(\eta = 0) = 0, \quad \theta_m(\eta = 0) = 0, \quad \phi_m(\eta = 0) = 0 \quad (3.256)$$

$$f'_m(\eta \longrightarrow \infty) \longrightarrow 0, \quad \theta_m(\eta \longrightarrow \infty) \longrightarrow 0, \quad \phi_m(\eta \longrightarrow \infty) \longrightarrow 0 \quad (3.257)$$

where

$$R_m^f(\chi) = a_1\theta'_{m-1} + a_2 f''_{m-1} + a_3 f_{m-1} + a_4 f''_{m-1} + a_5 f'_{m-1} + [1+\gamma]\left(1 +_{\beta}\frac{1}{}\right) f'''_{m-1}$$

$$a_6\theta_{m-1} + a_7 f'''_{m-1} + Gr\theta_{m-1} + Gm\phi_{m-1} - M_p f'_{m-1} - [1+\gamma]\left(1 +_{\beta}\frac{1}{}\right) Psf'_{m-1}$$

$$+\gamma\left(1 +_{\beta}\frac{1}{}\right) Psf'_{m-1} + a_8 f_{m-1} + a_9 f^{iv}_{m-1} + a_{10} f'_{m-1} + a_{11} f'''_{m-1} + a_{12} f''_{m-1}$$

$$+\sum_{n=0}^{m-1} \gamma\left(1 +_{\beta}\frac{1}{}\right)\theta'_n f''_{m-1-n} + f_n f''_{m-1-n} - f'_n f'_{m-1-n} - \gamma\left(1 +_{\beta}\frac{1}{}\right)\theta_n f'''_{m-1-n}$$

$$+A_2 f_n f^{iv}_{m-1-n} + 2A_2 f'_n f'''_{m-1-n} + 2A_2 f''_n f''_{m-1-n} - \psi_1(\eta)(1-\chi_m \qquad)$$

$$R_m^\theta(\chi) = \left[1 + \frac{4}{3}Rd\right]\theta''_{m-1} + b_1\theta_{m-1} + b_2\theta''_{m-1} + b_3 f_{m-1} + b_4\theta'_{m-1} + b_5\theta'_{m-1}$$

$$+b_6 f''_{m-1} + b_8\theta_{m-1} + b_{10} f''_{m-1} + Pr\delta_x\theta_{m-1} + PrD_f\phi''_{m-1}$$

$$+\sum_{n=0}^{m-1} \epsilon\theta_n\theta''_{m-1-n} - Prf_n\theta'_{m-1-n} + \theta'_n\theta'_{m-1-n} + Pr\left(1 +_{\beta}\frac{1}{}\right) Ec[1+\gamma] f''_n f''_{m-1-n}$$

(3.258)

$$+b_7\theta_n f''_{m-1-n} + b_9 f''_n f''_{m-1-n}$$

$$+\sum_{i=0}^{n}\sum_{n=0}^{m-1} -PrEc\left(1 +_{\beta}\frac{1}{}\right)\gamma\theta_i f''_{n-i} f''_{m-1-n} - \psi_2(\eta)(1-\chi_m \qquad)$$

$$R_m^\phi(\chi) = \phi''_{m-1} + Scc_2 f_{m-1} + Sca_1\phi'_{m-1} - ScSr\phi_{m-1} + Sc\tau c_2\theta'_{m-1} + Sc\tau b_2\phi'_{m-1}$$

$$+Sc\tau\phi''_{m-1} + Sc\tau b_3\phi_{m-1} + Sc\tau c_1\theta''_{m-1} + ScSo\theta''_{m-1}$$

$$+\sum_{n=0}^{m-1} \; Scf_n\phi'_{m-1-n} + Sc\tau\theta'_n\phi'_{m-1-n}) - \psi_3(\eta)(1-\chi_m$$

(3.259)

) (3.260)

Thus,

$$Q_{1,m-1} = \gamma\left(1 +_{\beta}\frac{1}{}\right)(D\theta_n)(D^2 f_{m-1-n}) + f_n(D^2 f_{m-1-n}) - (Df_n)(Df_{m-1-n})$$

$$-\gamma\left(1 +_{\beta}\frac{1}{}\right)\theta_n(D^2 f_{m-1-n}) - A_2(Df_n)(D^3 f_{m-1-n})$$

$$+A_2 f_n(D^4 f_{m-1-n}) + A_2(Df_n)(D^2 f_{m-1-n}) - A_2(Df_n)(D^2 f_{m-1-n} \qquad) \quad (3.261)$$

$$Q_{2,m-1} = \sum_{n=0}^{m-1} \epsilon\theta_n(D^2\theta_{m-1-n}) - Prf_n(D\theta_{m-1-n}) + \epsilon(D\theta_n)(D\theta_{m-1-n})$$

$$+PrEc\left(1+\frac{1}{\beta}\right)(D^2 f_n)(D^2 f_{m-1-n}) + PrEc\epsilon\left(1+\frac{1}{\beta}\right)(D^2 f_n)(D^2 f_{m-1-n})$$

$$-2PrEc\epsilon\left(1+\frac{1}{\beta}\right)a_3\theta_n(D^2 f_{m-1-n}) - PrEc\epsilon\left(1+\frac{1}{\beta}\right)b_1(D^2 f_n)(D^2 f_{m-1-n})$$

$$+\sum_{i=0}^{n}\sum_{n=0}^{m-1} -PrEc\epsilon\left(1+\frac{1}{\beta}\right)\theta_i(D^2 f_{n-i})(D^2 f_{m-1-n}) \qquad (3.262)$$

$$Q_{3,m-1} = \sum_{n=0}^{m-1} Scf_n(D\varphi_{m-1-n}) + Sc\tau(D\theta_n)(D\varphi_{m-1-n}) \qquad (3.263)$$

3.8 Formulation of the research problem three

A steady, incompressible, laminar motion of Casson nanoparticles in a slanting plate is explored in this research. At an angle Φ ($0^o \leq \Phi \leq 90^o$), the plate is inclined horizontally. The concentration along with temperature of the nanoparticle volume fraction at the wall are φ_w and θ_w respectively. The ambient concentration along with ambient temperature of the nanoparticle volume fraction is given by φ_∞ and θ_∞ respectively. The following assumptions are made:

(i) The penetrable medium is assumed to be saturated and homogeneous with liquid in thermodynamic equilibrium;

(ii) At an angle Φ, the vertical plate is inclined;

(iii) The liquid viscosity along with thermal conductivity is assumed to vary on the hydrodynamics and thermal boundary layer;

(iv) Numeric magnetic Reynolds is taken to be small such that induced magnetism is avoided as compared with applied magnetic field; and

(v) All the attributes of liquid are kept constant except density added to the momentum equation.

(vi) All the attributes of fluid are kept uniform except density in the buoyancy term in the momentum flow equation.

(vii) Boundary layer approximation is considered valid. This study assumed that the rheological equation of a Casson fluid gives:

$$\tau_{ij} = \left(\mu_b + \frac{P_y}{\sqrt{2\pi}}\right)2e_{ij} \;\; when \;\; \pi > \pi_c$$

$$\tau_{ij} = \left(\mu_b + \frac{P_y}{\sqrt{2\pi_c}}\right)2e_{ij} \;\; when \;\; \pi < \pi_c$$

where P_y means liquid yield stress given as:

$$P_y = \frac{\mu_b\sqrt{(2\pi)}}{\beta}$$

μ_b means dynamic viscosity, $\pi = e_{ij}e_{ij}$ means rate of deformation component multiplying itself, e_{ij} means rate of deformation while π_c means Casson model critical value. In a situation whereby $\pi > \pi_c$ we have that:

$$\mu_0 = \mu_b + \frac{P_y}{\sqrt{2\pi}}$$

Based on the above, the kinematic viscosity is subject to the plastic dynamic viscosity (μ_b) while ρ means density and Casson term leads to:

$$\mu_0 = \frac{\mu_b}{\rho}\left(1 + \frac{1}{\beta}\right)$$

Based on the listed assumptions above and the rheological equation of Casson fluid, the flow equations becomes:

$$\frac{\partial w}{\partial x} + \frac{\partial n}{\partial y} = 0 \tag{3.264}$$

$$w\frac{\partial w}{\partial x} + n\frac{\partial w}{\partial y} = \left(1 + \frac{1}{\beta}\right)\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu_b(T)\frac{\partial w}{\partial y}\right) + g\beta_t\cos(\Phi)(\theta - \theta_\infty)$$

$$+ g\beta_c\cos(\Phi)(\phi - \phi_\infty) - \frac{\sigma B_0^2}{\rho}w - \frac{\mu_b(T)}{k\rho}\left(1 + \frac{1}{\beta}\right)w \tag{3.265}$$

$$w\frac{\partial \theta}{\partial x} + n\frac{\partial \theta}{\partial y} = \frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial \theta}{\partial y}\right) - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} + \frac{\mu_b(T)}{\rho c_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial w}{\partial y}\right)^2$$

$$+\tau\left[D_B\frac{\partial \phi}{\partial y}\frac{\partial \theta}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial \theta}{\partial y}\right)^2\right] + \frac{Q_0}{\rho c_p}(\theta - \theta_\infty) + \frac{Dk_T}{c_s c_p}\frac{\partial^2 \phi}{\partial y^2} \tag{3.266}$$

$$w\frac{\partial \phi}{\partial x} + n\frac{\partial \phi}{\partial y} = D\frac{\partial^2 \phi}{\partial y^2} - k_l(\phi - \phi_\infty) - \frac{\partial(V_T\phi)}{\partial y} + \frac{Dk_T}{T_m}\frac{\partial^2 \theta}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 \theta}{\partial y^2}\right) \tag{3.267}$$

subject to:

$w = ax$, $n = -\nu(x)$, $\theta = \theta_w$, $\varphi = \varphi_w$ *at* $\eta = 0$ (3.268) $w \longrightarrow 0$, $\theta \longrightarrow 0$, $\varphi \longrightarrow 0$

as $\eta \longrightarrow 0$ (3.269)

In the concentration flow equation (3.267), V_T is the thermophoretic velocity and following Alam et al. (2008) leads to

$$V_T = -k\nu\frac{\nabla T}{T_{ref}} = -\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y} \tag{3.270}$$

where k-thermophoretic coefficient is defined as

$$k = \frac{2c_s(\frac{\lambda_g}{\lambda_p} + C_t k_n)[1 + k_n(C_1 + C_2 \exp^{-} \frac{-C_3}{k_n})]}{(1 + 3C_m k_n)(1 + 2\frac{\lambda_g}{\lambda_p} + 2C_t k_n)} \quad (3.271)$$

where $C_1, C_2, C_3, C_m, c_s, C_t$ are constant, λ_g and λ_p are the thermal conductivities of diffused particles and fluid respectively with k_n = Knudsen number.

Consider that the flow regime is sufficiently small due to temperature difference and that θ^4 is expressed as a linear function of T_∞. Now,expanding θ^4 in Taylor's series about θ_∞ and forgone higher order terms as obtained in (Idowu and Falodun, 2018)

as:

$$\theta^4 = 4\theta_\infty^3 \theta - 3\theta_\infty^4 \quad (3.272)$$

Utilizing the Roseland approximation, we have the radiative heat flux is given as

$$q_r = -\frac{4\sigma_e}{3k_e}\frac{\partial \theta^4}{\partial y} \quad (3.273)$$

Here σ_e means Stefan-Boltmann constant while ke means coefficient of mean absorption. $\psi(\eta)$ is the stream function defined as $w = \frac{\partial \psi}{\partial y}$ and $n = -\frac{\partial \psi}{\partial x}$, With this stream function, the continuity equation (3.264) is valid. The transformation employed in this research are:

$$\Psi = (\nu\gamma)^{\frac{1}{2}} f(\eta), \quad \eta = \sqrt{\frac{\gamma}{\nu}} y, \quad T(\eta) = \frac{\theta - \theta_\infty}{\theta_w - \theta_\infty}, \quad C(\eta) = \frac{\phi - \phi_\infty}{\phi_w - \phi_\infty} \quad (3.274)$$

3.8.1 Validation of the stream function of the research problem three

Since the stream function is define as

$$\psi = (\nu\gamma)^{\frac{1}{2}} x f(\eta), \quad w = \frac{\partial \psi}{\partial y}, \quad n = -\frac{\partial \psi}{\partial x}, \quad \eta = \sqrt{\frac{\gamma}{\nu}} y$$

$$w = \frac{\partial \psi}{\partial y} = \frac{\partial (\nu\gamma)^{\frac{1}{2}} x f}{\partial y} = (\nu\gamma)^{\frac{1}{2}} \frac{\partial (xf)}{\partial y} = (\nu\gamma)^{\frac{1}{2}} \left[x\frac{\partial f}{\partial y} + f\frac{\partial x}{\partial y} \right]$$

$$w = (\nu\gamma)^{\frac{1}{2}} x \frac{\partial f}{\partial \eta} \times \frac{\partial \eta}{\partial y} = (\nu\gamma)^{\frac{1}{2}} x f' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$u = \gamma x f' \tag{3.275}$$

$$n = -\frac{\partial \psi}{\partial x} = -\frac{(\nu\gamma)^{\frac{1}{2}} x f}{\partial x} = -(\nu\gamma)^{\frac{1}{2}} \frac{\partial (xf)}{\partial x} = -(\nu\gamma)^{\frac{1}{2}} \left[x\frac{\partial f}{\partial x} + f\frac{\partial x}{\partial x} \right]$$

$$v = -(\nu\gamma)^{\frac{1}{2}} f \tag{3.276}$$

$$\frac{\partial w}{\partial x} = \frac{\partial (\gamma x f')}{\partial x} = \gamma \frac{\partial (x f')}{\partial x} = \gamma \left[x\frac{\partial f'}{\partial x} + f'\frac{\partial x}{\partial x} \right] = \gamma \left[x\frac{\partial f'}{\partial \eta} \times \frac{\partial \eta}{\partial x} + f' \right]$$

$$\frac{\partial w}{\partial x} = \gamma f' \tag{3.277}$$

$$\frac{\partial n}{\partial y} = -\frac{\partial (\nu\gamma)^{\frac{1}{2}} f}{\partial y} = -(\nu\gamma)^{\frac{1}{2}} \frac{\partial f}{\partial y} = -(\nu\gamma)^{\frac{1}{2}} \frac{\partial f}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial n}{\partial y} = -(\nu\gamma)^{\frac{1}{2}} f' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} = -\gamma f'$$

$$\frac{\partial n}{\partial y} = -\gamma f' \tag{3.278}$$

Substituting equations (3.277) and (3.278) into continuity equation

$$\frac{\partial w}{\partial x} + \frac{\partial n}{\partial y} = 0$$

This implies that

$$\gamma f^0 - \gamma f^0 = 0$$

Hence, the stream function satisfies the continuity equation.

3.8.2 Transformation of the momentum equation of the research problem three

Since $\frac{\partial w}{\partial x} = \gamma f_0$

$$w\frac{\partial w}{\partial x} = \gamma x f' \times \gamma f'$$

$$w\frac{\partial w}{\partial x} = \gamma^2 x (f')^2 \tag{3.279}$$

$$\frac{\partial w}{\partial y} = \frac{\partial(\gamma x f')}{\partial y} = \gamma\frac{\partial(x f')}{\partial y}$$

$$\frac{\partial w}{\partial y} = \gamma\left[x\frac{\partial f'}{\partial y} + f'\frac{\partial x}{\partial y}\right] = \gamma x\frac{\partial f'}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial w}{\partial y} = \gamma x f'' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$n\frac{\partial w}{\partial y} = -(\nu f)^{\frac{1}{2}} f \times \gamma x f'' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$n\frac{\partial w}{\partial y} = -\gamma^2 x f f''$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu_b(T)\frac{\partial w}{\partial y}\right) = \frac{1}{\rho}\left[\mu_b(T)\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) + \frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y}\right]$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu_b(T)\frac{\partial w}{\partial y}\right) = \frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} + \frac{1}{\rho}\frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y}$$

(3.280)

(3.281)

non-dimensionalize the first term at the RHS of equation (3.281) leads to

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b(T)}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{\mu_b(T)}{\rho}\frac{\partial}{\partial y}\left(\gamma x f'' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b(T)}{\rho}\gamma\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial}{\partial y}(x f'')$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b(T)}{\rho}\gamma\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\left[x\frac{\partial f''}{\partial y} + f''\frac{\partial x}{\partial y}\right]$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b(T)}{\rho}\gamma\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} x\frac{\partial f''}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b(T)}{\rho}\gamma\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} x f''' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = \frac{\mu_b^*}{\rho}(1 + b(\theta - \theta_\infty))\frac{\gamma^2}{\nu} x f''' = (1 + bT(\theta_w - \theta_\infty))\gamma^2 x f'''$$

$$\frac{\mu_b(T)}{\rho}\frac{\partial^2 w}{\partial y^2} = (1 + \nabla_a T)\gamma^2 x f''' \tag{3.282}$$

non-dimensionlize the second term at the RHS of equation (3.281)

$$\frac{\partial \mu_b(T)}{\partial y} = \frac{\partial}{\partial y}[\mu_b^*(1 + b(\theta - \theta_\infty))] = \mu_b^* \frac{\partial}{\partial y}(1 + bT(\theta_w - \theta_\infty))$$

$$\frac{\partial \mu_b(T)}{\partial y} = \mu_b^* b(\theta_w - \theta_\infty)\frac{\partial T}{\partial y} = \mu_b^*(b(\theta_w - \theta_\infty))\frac{\partial T}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \mu_b(T)}{\partial y} = \mu_b^*(b(\theta_w - \theta_\infty))T' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

Thus

$$\frac{1}{\rho}\frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y} = \frac{1}{\rho}\gamma x f'' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} \times \mu_b^*(b(\theta_w - \theta_\infty))T' \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho}\frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y} = \frac{\mu_b^*}{\rho}\frac{\gamma^2}{\nu}x(b(\theta_w - \theta_\infty))T' f''$$

$$\frac{1}{\rho}\frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y} = \gamma^2 x b(\theta_w - \theta_\infty)T' f''$$

$$\frac{1}{\rho}\frac{\partial w}{\partial y}\frac{\partial \mu_b(T)}{\partial y} = \gamma^2 x \nabla_a T' f'' \qquad (3.283)$$

where $O_a = b(\theta_w - \theta_\infty)$. Substituting equations (3.282) and (3.283) into equation (3.281) to obtain

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu_b(T)\frac{\partial w}{\partial y}\right) = \gamma^2 x f''' + \gamma^2 x \nabla_a T' f''$$

$$\left(1 + \frac{1}{\beta}\right)\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu_b(T)\frac{\partial w}{\partial y}\right) = \left(1 + \frac{1}{\beta}\right)\gamma^2 x[\nabla_a T' f'' + f'''] \qquad (3.284)$$

$$g\beta_t \cos(\alpha)(\theta - \theta_\infty) = g\beta_t \cos(\Phi)T(\theta_w - \theta_\infty) \qquad (3.285)$$

$$g\beta_c \cos(\alpha)(\phi - \phi_\infty) = g\beta_c \cos(\Phi)T(\phi_w - \phi_\infty) \qquad (3.286)$$

$$\frac{\sigma B_0^2}{\rho}w = \frac{\sigma B_0^2}{\rho}\gamma x f'$$

$$\frac{\mu_b(T)}{k\rho}\left(1 + \frac{1}{\beta}\right)w = \frac{\mu_b^*}{k\rho}(1 + b(\theta - \theta_\infty))\left(1 + \frac{1}{\beta}\right)\gamma x f'$$

$$\frac{\mu_b(T)}{k\rho}\left(1 + \frac{1}{\beta}\right)w = \frac{\mu_b^*}{k\rho}(1 + bT(\theta_w - \theta_\infty))\left(1 + \frac{1}{\beta}\right)\gamma x f'$$

$$\frac{\mu_b(T)}{k\rho}\left(1 + \frac{1}{\beta}\right)w = \frac{\mu_b^*}{k\rho}(1 + \nabla_a T)\left(1 + \frac{1}{\beta}\right)\gamma x f'$$

(3.287)

(3.288)

Substituting equations (3.279)-(3.288) into the momentum equation (3.265) to obtain

$$\gamma^2 x(f')^2 - \gamma^2 x f f'' = \left(1+\frac{1}{\beta}\right)\gamma^2 x[\nabla_s T' f'' + (1+\nabla_a T) f'''] + \not{g}_{\;t}\cos(\Phi) T(\theta_w - \theta_\infty)$$

$$+\not{g}_{\;c}\cos(\Phi) C(\phi_w - \phi_\infty) - \frac{\sigma B_0^2}{\rho}\gamma x f' - \frac{\mu_b^*}{k\rho}\left(1+\frac{1}{\beta}\right)(1+\nabla_a T)\gamma x f'$$

divide all through by $\gamma^2 x$

$$(f')^2 - f f'' = \left(1+\frac{1}{\beta}\right)\nabla_a T' f'' + \left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f''' + \frac{\not{g}_{\;t}\cos(\Phi) T(\theta_w - \theta_\infty)}{\gamma^2 x}$$

$$+\frac{\not{g}_{\;c}\cos(\Phi) C(\phi_w - \phi_\infty)}{\gamma^2 x} - \frac{\sigma B_0^2}{\rho\gamma} f' - \frac{\mu_b^*}{k\rho\gamma}\left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f'$$

$$(f')^2 - f f'' = \left(1+\frac{1}{\beta}\right)\nabla_a T' f'' + \left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f''' + \triangle_a \cos(\Phi) T$$

$$+\triangle_b \cos(\Phi) C - M^2 f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f'$$

Therefore, the transformed momentum equation becomes

$$\left(1+\frac{1}{\beta}\right)\nabla_a T' f'' + \left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f''' + f f'' - (f')^2 + \triangle_a \cos(\Phi) T + \triangle_b \cos(\Phi) C$$

$$-M^2 f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)(1+\nabla_a T) f' = 0 \qquad (3.289)$$

3.8.3 Transformation of the energy equation of the research problem three

$$\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial T}\times\frac{\partial T}{\partial x}=\frac{\partial\theta}{\partial T}\times\frac{\partial T}{\partial\eta}\times\frac{\partial\eta}{\partial x}$$

$$\frac{\partial\theta}{\partial x}=(\theta_w-\theta_\infty)T'\times 0=0$$

$$w\frac{\partial\theta}{\partial x}=\gamma x f'\times 0$$

$$w\frac{\partial\theta}{\partial x}=0 \qquad (3.290)$$

$$\frac{\partial\theta}{\partial}=\frac{\partial\theta}{\partial T}\times\frac{\partial T}{\partial y}=\frac{\partial\theta}{\partial T}\times\frac{\partial T}{\partial\eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{\partial\theta}{\partial y}=(\theta_w-\theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$n\frac{\partial\theta}{\partial y}=-(\nu\gamma)^{\frac{1}{2}}f\times(\theta_w-\theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}=-\gamma(\theta_w-\theta_\infty)fT'$$

$$n\frac{\partial\theta}{\partial y}=-\gamma(\theta_w-\theta_\infty)fT'$$

$$\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial\theta}{\partial y}\right)=k(T)\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right)+\frac{\partial\theta}{\partial y}\frac{\partial k(T)}{\partial y}$$

$$\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial\theta}{\partial y}\right)=\frac{1}{\rho c_p}\left[k(T)\frac{\partial^2\theta}{\partial y^2}+\frac{\partial\theta}{\partial y}\frac{\partial k(T)}{\partial y}\right]$$

(3.291)

(3.292)

non-dimensionalize the first term of the RHS of equation (3.292)

$$\frac{\partial\theta}{\partial y} = (\theta_w - \theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho c_p}k(T)\frac{\partial^2\theta}{\partial y^2} = \frac{1}{\rho c_p}k^*(1+\epsilon(\theta-\theta_\infty))\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right) = k^*(1+\epsilon(\theta-\theta_\infty))\frac{\partial}{\partial y}\left((\theta_w - \theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{1}{\rho c_p}k(T)\frac{\partial^2\theta}{\partial y^2} = \frac{1}{\rho c_p}k^*(1+\epsilon T(\theta_w-\theta_\infty))(\theta_w-\theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial T'}{\partial y}$$

$$\frac{1}{\rho c_p}k(T)\frac{\partial^2\theta}{\partial y^2} = \frac{1}{\rho c_p}k^*(1+\epsilon T(\theta_w-\theta_\infty))(\theta_w-\theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial T'}{\partial \eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{1}{\rho c_p}k(T)\frac{\partial^2\theta}{\partial y^2} = \frac{k^*}{\rho c_p}(1+\delta_y T)(\theta_w-\theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}T''\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho c_p}k(T)\frac{\partial^2\theta}{\partial y^2} = \frac{k^*}{\rho c_p\nu}(1+\delta_y T)\gamma(\theta_w-\theta_\infty)T'' \qquad (3.293)$$

Now,

$$\frac{\partial k(T)}{\partial y} = \frac{\partial}{\partial y}k^*(1+\epsilon(\theta-\theta_\infty)) = k^*\frac{\partial}{\partial y}(1+\epsilon T(\theta_w-\theta_\infty))$$

$$\frac{\partial k(T)}{\partial y} = k^*\epsilon(\theta_w-\theta_\infty)\frac{\partial T}{\partial y} = k^*\epsilon(\theta_w-\theta_\infty)\frac{\partial T}{\partial \eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{\partial k(T)}{\partial y} = k^*\epsilon(\theta_w-\theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho c_p}\frac{\partial\theta}{\partial y}\frac{\partial k(T)}{\partial y} = \frac{1}{\rho c_p}(\theta_w-\theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\times k^*\epsilon(\theta_w-\theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{1}{\rho c_p}\frac{\partial\theta}{\partial y}\frac{\partial k(T)}{\partial y} = \frac{k^*}{\rho c_p}\delta_y(\theta_w-\theta_\infty)\frac{\gamma}{\nu}(T')^2 \qquad (3.294)$$

Substituting equations (3.293) and (3.294) into equation (3.292)

$$\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(k(T)\frac{\partial\theta}{\partial y}\right) = \frac{k^*}{\rho c_p\nu}(1+\delta_y T)\gamma(\theta_w-\theta_\infty)T'' + \frac{k^*}{\rho c_p\nu}\gamma(\theta_w-\theta_\infty)\delta_y(T')^2 \quad (3.295)$$

Since $q_r = -\frac{4\sigma_e}{3k_e}\frac{\partial\theta^4}{\partial y}$, from equation (3.272)

$$\frac{\partial\theta^4}{\partial y} = 4\theta_\infty^3\frac{\partial\theta}{\partial y}$$

Thus,

$$q_r = -\frac{16\sigma_e\theta_\infty^3}{3k_e}\frac{\partial\theta}{\partial y} \implies \frac{\partial q_r}{\partial y} = -\frac{16\sigma_e\theta_\infty^3}{3k_e}\frac{\partial^2\theta}{\partial y}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right) = \frac{\partial}{\partial y}\left((\theta_w - \theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\partial^2\theta}{\partial y^2} = (\theta_w - \theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial T'}{\partial y} = (\theta_w - \theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial T'}{\partial \eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{\partial^2\theta}{\partial y^2} = (\theta_w - \theta_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}T''\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\gamma}{\nu}(\theta_w - \theta_\infty)T''$$

Therefore,

$$\frac{\partial q_r}{\partial y} = -\frac{16\sigma_e\theta_\infty^3}{3k_e}\frac{\gamma}{\nu}(\theta_w - \theta_\infty)T'' \tag{3.296}$$

$$\frac{\mu_b(T)}{\rho c_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\mu_b^*}{\rho c_p}(1+b(\theta-\theta_\infty))\left(1+\frac{1}{\beta}\right)\left(\gamma x\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}f''\right)^2$$

$$\frac{\mu_b(T)}{\rho c_p}\left(1+\frac{1}{\beta}\right)\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\mu_b^*}{\rho c_p}(1+bT(\theta_w-\theta_\infty))\left(1+\frac{1}{\beta}\right)(f'')^2\frac{\gamma^2x^2}{\nu} \tag{3.297}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial C}\times\frac{\partial C}{\partial y} = \frac{\partial\phi}{\partial C}\times\frac{\partial C}{\partial\eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = (\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\therefore \frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y} = (\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\times(\theta_w - \theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y} = (\phi_w - \phi_\infty)(\theta_w - \theta_\infty)\frac{\gamma}{\nu}C'T' \tag{3.298}$$

$$\frac{\partial\theta}{\partial y} = (\theta_w - \theta_\infty)T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\left(\frac{\partial\theta}{\partial y}\right)^2 = (\theta_w - \theta_\infty)(T')^2\frac{\gamma}{\nu} \tag{3.299}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial y}\left((\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\partial^2\phi}{\partial y^2} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial C'}{\partial y} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial C'}{\partial\eta}\times\frac{\partial\eta}{\partial y}$$

$$\frac{\partial^2\phi}{\partial y^2} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}C''\times\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2\phi}{\partial y^2} = (\phi_w - \phi_\infty)\frac{\gamma}{\nu}C'' \tag{3.300}$$

Substituting equations (3.290)-(3.300) into the energy equation (3.266) to obtain

$$
\begin{aligned}
0-\gamma(\theta_w-\theta_\infty)fT' &= \frac{k^*}{\rho c_p\nu}(1+\delta_y T)\gamma(\theta_w-\theta_\infty)T''+\frac{k^*}{\rho c_p\nu}\gamma(\theta_w-\theta_\infty)\delta_y(T')^2\\
&+\frac{1}{\rho c_p}\frac{16\sigma_e\theta_\infty^3}{3k_e}\frac{\gamma}{\nu}(\theta_w-\theta_\infty)T''+\frac{\mu_b^*}{\rho c_p}(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)(f'')2\frac{\gamma^2x^2}{\nu}\gamma\\
&+\tau\left[D_B(\phi_w-\phi_\infty)(\theta_w-\theta_\infty)\frac{\gamma}{\nu}C'T'+\frac{D_T}{T_\infty}(\theta_w-\theta_\infty)^2(T')^2\frac{\gamma}{\nu}\right]\\
&+\frac{Q_0}{\rho c_p}T(\theta_w-\theta_\infty)+\frac{Dk_T}{c_s c_p}(\phi_w-\phi_\infty)\frac{\gamma}{\nu}C''
\end{aligned}
$$

divide all through by $-\gamma(\theta_w-\theta_\infty)$

$$
\begin{aligned}
-fT' &= \frac{k^*}{\rho c_p\nu}(1+\delta_y T)T''+\frac{k^*}{\rho c_p\nu}\frac{16\sigma_e\theta_\infty^3}{3k_e k^*}T''+\frac{k^*}{\rho c_p\nu}\delta_y(T')^2\\
&+\frac{\mu_b^*(\gamma x)^2}{\rho c_p(\theta_w-\theta_\infty)}(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)\frac{1}{\nu}(f'')^2+\frac{Q_0}{\rho c_p\gamma}T+\frac{Dk_T(\phi_w-\phi_\infty)}{c_s c_p(\theta_w-\theta_\infty)\nu}C''\\
&+\tau\left[D_B(\phi_w-\phi_\infty)\frac{1}{\nu}C'T'+\frac{D_T}{T_\infty}\frac{(\theta_w-\theta_\infty)}{\nu}(T')^2\right]
\end{aligned}
$$

Simplifying further,

$$
\begin{aligned}
-fT' &= \frac{1}{Pr}(1+\delta_y T)T''+\frac{1}{Pr}R_pT''+\frac{1}{Pr}\delta_y(T')^2+HT\\
&+E_n(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)(f'')^2+DC''+N_bC'T'+N_t(T')^2
\end{aligned}
$$

$$
\begin{aligned}
-fT' = \frac{(1+\delta_y T)+R_p}{Pr}T''+\frac{\delta_y}{Pr}(T')^2+HT+E_n(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)(f'')^2\\
+D_fC''+N_bC'T'+N_t(T')^2
\end{aligned}
$$

Upon rearrangement, the transformed energy equation becomes

$$
\frac{(1+\delta_y T)+R_p}{Pr}T''+fT'+\frac{\delta_y}{Pr}(T')^2+HT+E_n(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)(f'')^2
$$

$$
+D_fC^{00}+N_bC^0T^0+N_t(T^0)^2=0 \qquad (3.301)
$$

where

$$
R_p=\frac{16\sigma_e\theta_\infty^3}{3k_e k^*},\ Pr=\frac{\rho c_p\nu}{k^*},\ H=\frac{Q_0}{\rho c_p\gamma},\ E_n=\frac{(\gamma x)^2}{c_p(\theta_w-\theta_\infty)}
$$

$$
D_f=\frac{Dk_T(\phi_w-\phi_\infty)}{c_s c_p\nu(\theta_w-\theta_\infty)},\ N_b=\frac{\tau D_b(\phi_w-\phi_\infty)}{\nu},\ N_t=\frac{\tau D_T(\theta_w-\theta_\infty)}{T_\infty\nu}
$$

3.8.4 Transformation of the concentration equation of the research problem three

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial C} \times \frac{\partial C}{\partial x} = \frac{\partial \phi}{\partial C} \times \frac{\partial C}{\partial \eta} \times \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = (\phi_w - \phi_\infty)T' \times 0 = 0$$

$$w\frac{\partial \phi}{\partial x} = \gamma x f' \times 0$$

$$w\frac{\partial \phi}{\partial x} = 0 \qquad (3.302)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial C} \times \frac{\partial C}{\partial y} = \frac{\partial \phi}{\partial C} \times \frac{\partial C}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = (\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$n\frac{\partial \phi}{\partial y} = -(\nu\gamma)^{\frac{1}{2}} f \times (\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} = -\gamma(\phi_w - \phi_\infty) f C'$$

$$n\frac{\partial \phi}{\partial y} = -\gamma(\phi_w - \phi_\infty) f C' \qquad (3.303)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left((\phi_w - \phi_\infty)C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial C'}{\partial y} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial C'}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial y^2} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}} C'' \times \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2 \phi}{\partial y^2} = (\phi_w - \phi_\infty)\left(\frac{\gamma}{\nu}\right) C'' \qquad (3.304)$$

$$-k_l(\varphi - \varphi_\infty) = -k_l C(\varphi_w - \varphi_\infty) \qquad (3.305)$$

Since, $V_T = -\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}, \quad \phi = C(\phi_w - \phi_\infty) + \phi_\infty$

$$V_T\phi = \left(-\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}\right)(C(\phi_w - \phi_\infty) + \phi_\infty) = -\frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}C(\phi_w - \phi_\infty) - \frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}\phi_\infty$$

Setting $\varphi_\infty = 0$

$$V_T\phi = -\frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)C\frac{\partial T}{\partial y} - \frac{k\nu}{T_{ref}}\frac{\partial T}{\partial y}\times 0$$

$$V_T\phi = -\frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)C\frac{\partial T}{\partial y}$$

$$-\frac{\partial}{\partial y}(V_T\phi) = -\frac{\partial}{\partial y}\left(-\frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)C\frac{\partial T}{\partial y}\right) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\frac{\partial}{\partial y}\left(C\frac{\partial T}{\partial y}\right)$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)+\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right] = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\frac{\partial^2 T}{\partial y^2}+\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)+\frac{\partial T}{\partial \eta}\times\frac{\partial \eta}{\partial y}\times\frac{\partial C}{\partial \eta}\times\frac{\partial \eta}{\partial y}\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial \eta}\times\frac{\partial \eta}{\partial y}\right)+T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}C'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\frac{\partial}{\partial y}\left(T'\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\right)+\frac{\gamma}{\nu}T'C'\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}\frac{\partial T'}{\partial y}+\frac{\gamma}{\nu}T'C'\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[C\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}T''\left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}+\frac{\gamma}{\nu}T'C'\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}(\phi_w-\phi_\infty)\left[\frac{\gamma}{\nu}CT''+\frac{\gamma}{\nu}T'C'\right]$$

$$-\frac{\partial}{\partial y}(V_T\phi) = \frac{k\nu}{T_{ref}}\frac{\gamma(\phi_w-\phi_\infty)}{\nu}[CT''+T'C' \qquad] \tag{3.306}$$

$\frac{\partial^2\theta}{\partial y^2} = \frac{\gamma}{\nu}(\theta_w-\theta_\infty)T''$

Since

$$\frac{Dk_T}{T_m}\frac{\partial^2\theta}{\partial y^2} = \frac{Dk_T}{T_m}\frac{\gamma}{\nu}(\theta_w-\theta_\infty)T'' \tag{3.307}$$

$$\frac{D_T}{T_\infty}\frac{\partial^2\theta}{\partial y^2} = \frac{D_T}{T_\infty}\frac{\gamma}{\nu}(\theta_w-\theta_\infty)T'' \tag{3.308}$$

Substituting equations (3.302)-(3.308) into the concentration equation (3.267) to obtain

$$0-\gamma(\phi_w-\phi_\infty)fC' = D\frac{\gamma}{\nu}(\phi_w-\phi_\infty)C'' - k_lC(\phi_w-\phi_i nfty) + \frac{Dk_T}{T_m}\frac{\gamma}{\nu}(\theta_w-\theta_\infty)T''$$
$$+\frac{D_T}{T_\infty}\frac{\gamma}{\nu}(\theta_w-\theta_\infty)T'' + \frac{k\nu}{T_{ref}}\frac{\gamma(\phi_w-\phi_\infty)}{\nu}[CT''+T'C']$$

divide all through by $\gamma(\varphi_w - \varphi_\infty)$

$$-fC'\frac{D}{\nu}C'' - \frac{k_l}{\gamma}C + \frac{Dk_T(\theta_w - \theta_\infty)}{T_m\nu(\phi_w - \phi_\infty)}T'' + \frac{D_T(\theta_w - \theta_\infty)}{T_\infty\nu(\phi_w - \phi_\infty)}T'' + \frac{k\nu}{T_{ref}\nu}[CT'' + T'C']$$

$$-fC' = \frac{1}{Sc}C'' - C_pC + \left(S_o + \frac{N_t}{L_nN_b}\right)T'' + \tau[CT'' + T'C']$$

Therefore, the transformed concentration equation becomes

$$\frac{1}{Sc}C'' + fC' - C_pC + \left(S_o + \frac{N_t}{L_nN_b}\right)T'' + \tau[CT'' + T'C'] = 0 \quad (3.309)$$

3.8.5 Transformation of the boundary conditions of the research problem three

Since $w = \gamma x f^0$ and $w = \gamma x$

$$\Longrightarrow \gamma x f^0 = \gamma x$$

divide all through by γx to obtain

$$f^0(\eta) = 1 \quad (3.310)$$

Also, $n = -(\nu\gamma)^{\frac{1}{2}}f$ and $n = -\nu(x)$

$$-(\nu\gamma)^{\frac{1}{2}}f = -\nu(x)$$

$$f(\eta) = \frac{\nu}{(\nu\gamma)^{\frac{1}{2}}} = \sqrt{\frac{\nu}{\gamma}}$$

$$f(\eta) = f_w \quad (3.311)$$

Now, $\theta = T(\theta_w - \theta_\infty) + \theta_\infty$ and $\theta = \theta_w$

$$\therefore T(\theta_w - \theta_\infty) + \theta_\infty = \theta_w$$

$$T(\theta_w - \theta_\infty) = \theta_w - \theta_\infty$$

divide all through by $\theta_w - \theta_\infty$

$$T(\eta) = 1 \quad (3.312)$$

also, $\varphi = C(\varphi_w - \varphi_\infty) + \varphi_\infty$ and $\varphi = \varphi_w$

$$\therefore C(\varphi_w - \varphi_\infty) + \varphi_\infty = \varphi_w$$

$$C(\varphi_w - \varphi_\infty) = \varphi_w - \varphi_\infty$$

divide all through by $\varphi_w - \varphi_\infty$

$$C(\eta) = 1 \qquad (3.313)$$

Also, $w = \gamma x f^0$ and $w \longrightarrow 0$

$$\therefore \gamma x f^0 = 0$$

$$f^0(\eta) \longrightarrow 0 \qquad (3.314)$$

$$\theta = T(\theta_w - \theta_\infty) + \theta_\infty and \theta \longrightarrow \theta_\infty$$

$$T(\theta_w - \theta_\infty) + \theta_\infty = \theta_\infty$$
$$T(\theta_w - \theta_\infty) = \theta_\infty - \theta_\infty$$

$$T(\theta_w - \theta_\infty) = 0$$

$$T(\eta) \longrightarrow 0 \qquad (3.315)$$

$$\varphi = C(\varphi_w - \varphi_\infty) + \varphi_\infty and \varphi \longrightarrow \varphi_\infty$$

$$C(\varphi_w - \varphi_\infty) + \varphi_\infty = \varphi_\infty$$

$$C(\varphi_w - \varphi_\infty) = \varphi_\infty - \varphi_\infty$$

$$C(\varphi_w - \varphi_\infty) = 0$$

$$C(\eta) \longrightarrow 0 \qquad (3.316)$$

Therefore, the transformed momentum, energy and concentration equations with the boundary conditions are

$$\left(1 + \frac{1}{\beta}\right) \nabla_a T' f'' + \left(1 + \frac{1}{\beta}\right) (1 + \nabla_a T) f''' + f f'' - (f')^2 + \triangle_a \cos(\Phi) T + \triangle_b \cos(\Phi) C$$
$$-M^2 f' - \frac{1}{P_0}\left(1 + \frac{1}{\beta}\right) (1 + \nabla_a T) f' = 0 \qquad (3.317)$$

$$\frac{(1 + \delta_y T) + R_p}{Pr} T'' + f T' + \frac{\delta_y}{Pr}(T')^2 + HT + E_n (1 + \nabla_a T) \left(1 + \frac{1}{\beta}\right) (f'')^2$$
$$+ D_f C'' + N_b C' T' + N_t (T')^2$$

$$\frac{1}{Sc} C'' + f C' - C_p C + \left(S_o + \frac{N_t}{L_n N_b}\right) T'' + \tau [C T'' + T' C'$$

= 0 (3.318)

] = 0 (3.319)

together with the boundary conditions

$$f^0 = 1, f = S_w, T = 1, C = 1 \; at \; \eta = 0 \tag{3.320}$$

$$f^0 \longrightarrow 0, T \longrightarrow 0, C \longrightarrow 0 \; as \; \eta \longrightarrow 0 \tag{3.321}$$

Note that: $O_a = b(\theta_w - \theta_\infty)$, $\triangle_a = \frac{\beta_t(\theta_w-\theta_\infty)}{a^2x}$, $\triangle_b = \frac{\beta_c(\phi_w-\phi_\infty)}{a^2x}$, $M = \frac{\sigma B_0^2}{\rho a}$, $P_0 = \frac{\mu_b^*}{k\rho a}$, $\delta_y = \xi(\theta_w - \theta_\infty)$, $R_p = \frac{16\sigma_e\theta_\infty^3}{3k_ek^*}$, $Pr = \frac{\rho c_p\nu}{k^*}$, $H = \frac{Q_0}{\rho c_p a}$, $E_n = \frac{\mu_b^*(ax)^2}{\rho c_p(\theta_w-\theta_\infty)}$, $D_f = \frac{Dk_T(\phi_w-\phi_\infty)}{c_sc_p(\theta_w-\theta_\infty)}$, $Nb = \frac{\tau D_B(\phi_w-\phi_\infty)}{\nu}$, $N_t = \frac{\tau D_T(\theta_w-\theta_\infty)}{\theta_\infty\nu}$, $Sc = \frac{\nu}{D}$, $C_p = \frac{k_l}{a}$, $S_o = \frac{Dk_T(\theta_w-\theta_\infty)}{T_m\nu(\phi_w-\phi_\infty)}$, $Ln = \frac{\nu}{D_B}$, $\tau = \frac{k(\theta_w-\theta_\infty)}{T_{ref}}$ are the varied viscosity term, thermal Grashof,

mass Grashof, magnetic term, porosity parameter, varied thermal conductivity term, radiation term, Prandtl, heat generation term, Eckert, Dufour, Brownian motion term, thermophoresis term, Schmidt, chemical reaction term, Soret, Lewis number and thermophoretic term.

The quantities of engineering concern are the wall skin friction (C_f), Nusselt number (Nu) and the Sherwood. The wall skin friction coefficient C_f is defined as:

$$C_f = \frac{\tau_w}{\rho(xB)^2} \quad where \quad \tau_w = \left(1 + \frac{1}{\beta}\right)\frac{\partial u}{\partial y}_{y=0}$$

τ_w is the shear stress within the boundary layer and it is simplify using the similarity transformation to obtain

$$\sqrt{R_{ex}}C_f = \left(1 + \frac{1}{\beta}\right)f''(0)$$

The Nusselt number is defined as

$$Nu = \frac{xq_w}{K(T_w - T_\infty)} \quad where \quad q_w = -K\frac{\partial T}{\partial y}_{y=0} - \frac{4\sigma_0}{3k_e}\left(\frac{\partial T^4}{\partial y}\right)|_{y=0}$$

q_w is the heat flux within the boundary layer. Using the similarity variables to obtain

$$Nu = -\sqrt{R_{ex}}T^0(0)$$

The Sherwood number is defined as

$$Sh = \frac{xJ_w}{D(C_w - C_\infty)} \quad where \quad J_w = -D\frac{\partial C}{\partial y}|_{y=0}$$

J_w is the mass flux within the boundary layer. Using the similarity variables to obtain

$$Sh = -\sqrt{R_{ex}}C'(0)$$

3.9 Qualitative analysis of problem three

Reducing the system of equations (3.317)-(3.319) into first order ordinary differential equations

$$f = m_1,\ \frac{df}{d\eta} = \frac{dm_1}{d\eta} = m_2,\ \frac{d^2f}{d\eta^2} = \frac{d}{d\eta}\left(\frac{dm_1}{d\eta}\right) = \frac{dm_2}{d\eta} = m_3,$$
$$\frac{d^3f}{d\eta^3} = \frac{d}{d\eta}\left(\frac{dm_2}{d\eta}\right) = \frac{dm_3}{d\eta} \quad (3.322)$$

$$T = m_4,\ \frac{dT}{d\eta} = \frac{dm_4}{d\eta} = m_5,\ \frac{d^2T}{d\eta^2} = \frac{d}{d\eta}\left(\frac{dm_4}{d\eta}\right) = \frac{dm_5}{d\eta}$$
$$C = m_6,\ \frac{dC}{d\eta} = \frac{dm_6}{d\eta} = m_7,\ \frac{d^2C}{d\eta^2} = \frac{d}{d\eta}\left(\frac{dm_6}{d\eta}\right) = \frac{dm_7}{d\eta}$$

(3.323)

(3.324)

Substituting equations (3.322)-(3.324) into equations (3.317)-(3.319) to obtain

$$\left(1+\frac{1}{\beta}\right)\nabla_a m_5 m_3 + \left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)\frac{dm_3}{d\eta} + m_1 m_3 - (m_2)^2$$
$$+\triangle_a\cos(\Phi)m_4 + \triangle_b\cos(\Phi)m_6 - M^2 m_2 - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)m_2 = 0 \quad (3.325)$$
$$((1+\delta_y m_4)+R_p)\frac{dm_5}{d\eta} + Prm_1 m_5 + \delta_y(m_5)^2 + Hm_4$$
$$+Ec(1+\nabla_a m_4)\left(1+\frac{1}{\beta}\right)(m_3)^2 + D_f\frac{dm_7}{d\eta} + N_b m_7 m_5 + N_t(m_5)^2 = 0$$
$$\frac{dm_7}{d\eta} + Scm_1 m_7 - C_p m_6 + \left(So + \frac{N_t}{LnN_b}\right)\frac{dm_5}{d\eta} + \eta\left(m_6\frac{dm_5}{d\eta} + m_5 m_7\right) = 0 \quad (3.327)$$

(3.326)

Simplifying equation (3.325) to obtain

$$\frac{dm_3}{d\eta} = \frac{-\left(1+\frac{1}{\beta}\right)\nabla_a m_5 m_3 - m_1 m_3}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)} - \frac{m_1 m_3}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)} + \frac{(m_2)^2}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}$$

$$-\frac{\triangle_a\cos(\Phi)m_4}{\left(1+_\beta\frac{1}{}\right)(1+\nabla_a m_4)}$$

$$-\frac{\triangle_b\cos(\Phi)m_6}{\left(1+_\beta\frac{1}{}\right)(1+\nabla_a m_4)}+\frac{M^2m_2}{\left(1+_\beta\frac{1}{}\right)(1+\nabla_a m_4)}+\frac{\frac{1}{P_0}\left(1+_\beta\frac{1}{}\right)(1+\nabla_a m_4)m_2}{\left(1+_\beta\frac{1}{}\right)(1+\nabla_a m_4)} \quad (3.328)$$

Simplifying equation (3.327) to obtain

$$((1+\delta_y m_4)+R_p)\frac{dm_5}{d\eta}=-Prm_1m_5-\delta_y(m_5)^2-Hm_4-E_n(1+\nabla_a m_4)\left(1+_\beta\frac{1}{}\right)(m_3)^2$$
$$-N_bm_7m_5-N_t(m_5)^2-D_f\frac{dm_7}{d\eta} \quad (3.329)$$

Setting:

$$J_1=-Prm_1m_5-\delta_y(m_5)^2-Hm_4-En(1+\nabla_a m_4)\left(1+_\beta\frac{1}{}\right)(m_3)^2-N_bm_7m_5-N_t(m_5)^2$$

With the above, equation (3.329) becomes

$$((1+\delta_y m_4)+R_p)\frac{dm_5}{d\eta}=J_1-D_f\frac{dm_7}{d\eta}$$

$$\frac{dm_5}{d\eta}=\frac{J_1-D_f\frac{dm_7}{d\eta}}{((1+\delta_y m_4)+R_p)} \quad (3.330)$$

From equation (3.327):

$$\frac{dm_7}{d\eta}+Scm_1m_7-C_pm_6+\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\frac{dm_5}{d\eta}+\tau m_5m_7=0$$

Substituting equation (3.330) to obtain:

$$\frac{dm_7}{d\eta}=-Scm_1m_7+C_pm_6-\tau m_5m_7-\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{J_1-D_f\frac{dm_7}{d\eta}}{((1+\delta_y m_4)+R_p)}\right]$$

$$\frac{dm_7}{d\eta}=-Scm_1m_7+C_pm_6-\tau m_5m_7-\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]$$
$$\left[\frac{J_1}{(1+\delta_y m_4)+R_p}-\frac{D_f}{(1+\delta_y m_4)+R_p}\frac{dm_7}{d\eta}\right]$$

$$\frac{dm_7}{d\eta}=-Scm_1m_7+C_pm_6-\tau m_5m_7-\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]$$
$$\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{D_f}{(1+\delta_y m_4)+R_p}\frac{dm_7}{d\eta}\right]$$

Simplifying to obtain

$$\frac{dm_7}{d\eta} = \frac{-Scm_1m_7 + C_pm_6 - \tau m_5m_7 - \left[\left(So + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]}{\left[\left[\left(So + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{D_f}{(1+\delta_y m_4)+R_p}\frac{dm_7}{d\eta}\right]\right]} \tag{3.331}$$

Setting: $J_2 = \frac{-Scm_1m_7+C_pm_6-\tau m_5m_7-\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]}{\left[\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{D_f}{(1+\delta_y m_4)+R_p}\frac{dm_7}{d\eta}\right]\right]}$ Substituting the above into equation (3.331) to obtain

$$\frac{dm_7}{d\eta} = J_2 \tag{3.332}$$

Substituting equation (3.332) into equation (3.330) to obtain:

$$\frac{dm_5}{d\eta} = \frac{J_1 - D_f J_2}{(1+\delta_y m_4) + R_p} \tag{3.333}$$

Theorem 3.4: Let *f, T* and *C* to be continuous function at all points in some neighborhood and considering $\beta > 0$, O$_a > 0$, $4_a > 0$, $4_b > 0$, $\alpha > 0$, $M >$

0, $P_o > 0$, $\delta_y > 0$, $R_p > 0$, $Pr > 0$, $H > 0$, $En > 0$, $D_f > 0$, $N_b > 0$, $N_t >$

0, $Sc > 0$, $C_p > 0$, $S_o > 0$, $Ln > 0$ and $\tau > 0$, then there exist a unique solution

for the equations:

$$\left(1+\frac{1}{\beta}\right)\nabla_a \frac{dT}{d\eta}\frac{d^2f}{d\eta^2} + \left(1+\frac{1}{\beta}\right)(1+\nabla_a T)\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 + \triangle_a\cos(\Phi)T$$
$$+\triangle_b\cos(\Phi)C - M^2\left(\frac{df}{d\eta}\right) - \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)(1+\nabla_a T)\left(\frac{df}{d\eta}\right) = 0 \tag{3.334}$$

$$((1+\delta_y T)+R_p)\frac{d^2T}{d\eta^2} + Prf\frac{dT}{d\eta} + \delta_y\left(\frac{dT}{d\eta}\right)^2 + HT + En(1+\nabla_a T)\left(1+\frac{1}{\beta}\right)\left(\frac{d^2f}{d\eta^2}\right)^2$$
$$+D_f\frac{d^2C}{d\eta^2} + N_b\frac{dC}{d\eta}\frac{dT}{d\eta} + N_t\left(\frac{dT}{d\eta}\right)^2 = 0 \tag{3.335}$$

$$\frac{1}{Sc}\frac{d^2C}{d\eta^2} + f\frac{dC}{d\eta} - C_pC + \left(S_o + \frac{N_t}{LnN_b}\right)\frac{d^2T}{d\eta} + \tau\left(\frac{d^2T}{d\eta^2} + \frac{dT}{d\eta}\frac{dC}{d\eta}\right) = 0 \tag{3.336}$$

Subject to:

$$\frac{df}{d\eta} = 1,\ f = S_w,\ \theta = 1,\ \phi = 1,\ at\ \eta = 0 \tag{3.337}$$

$$\frac{df}{d\eta} \longrightarrow 0,\ \theta \longrightarrow 0,\ \phi \longrightarrow 0,\ as\ \eta \longrightarrow \infty \tag{3.338}$$

On the interval k $y - y_0$ k$\leq a$, k $y_0 - y$ k$\leq b$ provided $\exists K$ such that $K = max(0,1,U_1,...,U_n)$ and $0 < K < \infty$

Proof: The system of first order ordinary differential equations (3.322)-(3.324) in compact form is given as:

$$\begin{bmatrix} \frac{dm_1}{d\eta} \\ \frac{dm_2}{d\eta} \\ \frac{dm_3}{d\eta} \\ \frac{dm_4}{d\eta} \\ \frac{dm_5}{d\eta} \\ \frac{dm_6}{d\eta} \\ \frac{dm_7}{d\eta} \end{bmatrix} = \begin{bmatrix} m_2 \\ m_3 \\ \frac{-\left(1+\frac{1}{\beta}\right)\nabla_a m_5 m_3 - m_1 m_3 + (m_2)^2 - \triangle_a \cos(\Phi) m_4 - \triangle_b \cos(\Phi) m_6 + M^2 m_2 + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4) m_2}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)} \\ m_5 \\ \frac{J_1 - D_f J_2}{((1+\delta_y m_4)+R_p)} \\ m_7 \\ \frac{-Scm_1 m_7 + C_p m_6 - \tau m_5 m_7 - \left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]}{\left[\left[\left(So+\frac{N_t}{LnN_b}\right)+\tau m_6\right]\left[\frac{D_f}{(1+\delta_y m_4)+R_p}\frac{dm_7}{d\eta}\right]\right]} \end{bmatrix} \tag{3.339}$$

satisfying the conditions

S_w
$m_1(0)$
1
$m_2(0)$ γ_1
$m_3(0)$ 1
$m_4(0) =$ γ_2
$m_5(0)$ 1
$m_6(0)$ $\gamma_2 1$
$m_7(0)$
γ_3

Consider $\frac{\partial f_i}{\partial m_j}$ such that $i,j = 1(1)8$ to denote the non-linear functions on the right hand side of equation (11) if and only if $i = 1,...,7$ and $j = counts$. When $i = 1$ and $j = counts$, we obtain

$$f_1 = m_2$$

$$|\frac{\partial f_1}{\partial m_1}| = |\frac{\partial f_1}{\partial m_3}| = |\frac{\partial f_1}{\partial m_4}| = |\frac{\partial f_1}{\partial m_5}| = |\frac{\partial f_1}{\partial m_6}| = |\frac{\partial f_1}{\partial m_7}| = 0 < \infty$$

$$|\frac{\partial f_1}{\partial m_2}| = 1 < \infty$$

When i = 2 and j = *counts*, we have

$$|\frac{\partial f_2}{\partial m_1}| = |\frac{\partial f_2}{\partial m_2}| = |\frac{\partial f_2}{\partial m_4}| = |\frac{\partial f_2}{\partial m_5}| = |\frac{\partial f_2}{\partial m_6}| = |\frac{\partial f_2}{\partial m_7}| = 0 < \infty$$

$$|\frac{\partial f_2}{\partial m_3}| = 1 < \infty$$

When i = 3 and j = *counts*, we have

$$f_3 = \frac{-\left(1+\frac{1}{\beta}\right)\nabla_a m_5 m_3 - m_1 m_3 + (m_2)^2 - \triangle_a \cos(\Phi) m_4 - \triangle_b \cos(\Phi) m_6 + M^2 m_2 + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4) m_2}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}$$

$$|\frac{\partial f_3}{\partial m_1}| = |\frac{-m_3}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}|$$

$$\leq \frac{|-m_3|}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|)} = U_1 < \infty$$

$$|\frac{\partial f_3}{\partial m_2}| = |\frac{2m_2 + M^2 + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}| \leq \frac{2|m_2| + M^2 + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|)}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|)} = U_2 <$$

$$|\frac{\partial f_3}{\partial m_4}| = |\frac{-\triangle_a \cos(\Phi) + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)\nabla_a m_2}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4) + \left(1+\frac{1}{\beta}\right)(1+\nabla_a)}|$$

$$\leq \frac{-\triangle_a \cos(\Phi) + \frac{1}{P_o}\left(1+\frac{1}{\beta}\right)\nabla_a |m_2|}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|) + \left(1+\frac{1}{\beta}\right)(1+\nabla_a)} = U_4 < \infty$$

$$|\frac{\partial f_3}{\partial m_5}| = |\frac{-\left(1+\frac{1}{\beta}\right)\nabla_a m_3}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}| \leq \frac{-\left(1+\frac{1}{\beta}\right)\nabla_a |m_3|}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|)} = U_5 < \infty$$

$$|\frac{\partial f_3}{\partial m_6}| = |\frac{-\triangle_b \cos(\Phi)}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a m_4)}| \leq \frac{-\triangle_b \cos(\Phi)}{\left(1+\frac{1}{\beta}\right)(1+\nabla_a |m_4|)} = U_6 < \infty$$

$$|\frac{\partial f_3}{\partial m_7}| = 0 < \infty$$

When i = 4 and j = *counts*, we have

$$f_4 = m_5$$

$$|\frac{\partial f_4}{\partial m_1}| = |\frac{\partial f_4}{\partial m_2}| = |\frac{\partial f_4}{\partial m_3}| = |\frac{\partial f_4}{\partial m_4}| = |\frac{\partial f_4}{\partial m_6}| = |\frac{\partial f_4}{\partial m_7}| = 0 < \infty$$

$$|\frac{\partial f_4}{\partial m_5}| = 1 < \infty$$

When i = 5 and j = *counts*, we have

$$f_5 = \frac{J_1 - D_f J_2}{((1+\delta_y m_4) + R_p)}$$

$$|\frac{\partial f_5}{\partial m_1}| = |\frac{-Prm_5 + D_f Scm_7}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right][(1+\delta_y m_4) + R_p]}|$$

$$\leq \frac{-Pr|m_5| + D_f Sc|m_7|}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_5|)+R_p}\right)\right][(1+\delta_y|m_4|) + R_p]} = U_7 < \infty$$

$$|\frac{\partial f_5}{\partial m_2}| = 0 < \infty$$

$$|\frac{\partial f_5}{\partial m_3}| = |\frac{-2Ec(1+\nabla_a m_4)\left(1+\frac{1}{\beta}\right)m_3}{((1+\delta_y m_4) + R_p)}| \leq \frac{-2Ec(1+\nabla_a|m_4|)\left(1+\frac{1}{\beta}\right)|m_3|}{((1+\delta_y|m_4|) + R_p)} = U_8 < \infty$$

$$|\frac{\partial f_5}{\partial m_5}| = |\frac{-Prm_1 - 2\delta_y m_5 - N_b m_7 - 2N_t m_5 + \tau D_f m_7}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right][(1+\delta_y m_4) + R_p]}|$$

$$\leq \frac{-Pr|m_1| - 2\delta_y|m_5| - N_b|m_7| - 2N_t|m_5| + \tau D_f|m_7|}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_5|)+R_p}\right)\right][(1+\delta_y|m_4|) + R_p]} = U_{10} < \infty$$

$$|\frac{\partial f_5}{\partial m_4}| = |\frac{-H - E_n\nabla_a\left(1+\frac{1}{\beta}\right)(m_3)^2 - D_f(J_1\delta_y - (1+\delta_y m_4) + R_p)(-H - E_n\nabla_a\left(1+\frac{1}{\beta}\right)(m_3)^2)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right][(1+\delta_y m_4) + R_p]}|$$

$$\leq \frac{-H - E_n\nabla_a\left(1+\frac{1}{\beta}\right)|(m_3)^2| - D_f(J_1\delta_y - (1+\delta_y m_4) + R_p)(-H - E_n\nabla_a\left(1+\frac{1}{\beta}\right)|(m_3)^2)|}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_5|)+R_p}\right)\right][(1+\delta_y|m_4|) + R_p]} = U_9 < \infty$$

$$|\frac{\partial f_5}{\partial m_6}| = |\frac{-D_f(C_p - \tau)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right][(1+\delta_y m_4) + R_p]}|$$

$$\leq \frac{-D_f(C_p - \tau)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_5|)+R_p}\right)\right][(1+\delta_y|m_4|) + R_p]} = U_{11} < \infty$$

$$|\frac{\partial f_5}{\partial m_7}| = |\frac{-N_b m_5 + D_f(Scm_1 + \tau m_5)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right][(1+\delta_y m_4) + R_p]}|$$

$$\leq \frac{-N_b|m_5| + D_f(Sc|m_1| + \tau|m_5|)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_5|)+R_p}\right)\right][(1+\delta_y|m_4|) + R_p]} = U_{12} < \infty$$

When i = 6 and j = *counts*, we have

$$f_6 = m_7$$

$$|\frac{\partial f_6}{\partial m_1}| = |\frac{\partial f_6}{\partial m_2}| = |\frac{\partial f_6}{\partial m_3}| = |\frac{\partial f_6}{\partial m_4}| = |\frac{\partial f_6}{\partial m_5}| = |\frac{\partial f_6}{\partial m_6}| = 0 < \infty$$

$$|\frac{\partial f_6}{\partial m_7}| = 1 < \infty$$

When i = 7 and j = *counts*, we have

$$f_7 = \frac{-Scm_1m_7 + C_pm_6 - \tau m_5m_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$|\frac{\partial f_7}{\partial m_1}| = |\frac{-Scm_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-Prm_5}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{-Sc|m_7| - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau |m_6|\right]\left[\frac{-Pr|m_5|}{(1+\delta_y |m_4|)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau |m_6|\right)\left(\frac{D_f}{(1+\delta_y |m_4|)+R_p}\right)\right]} = U_{13} < \infty$$

$$|\frac{\partial f_7}{\partial m_2}| = 0 < \infty$$

$$|\frac{\partial f_7}{\partial m_3}| = |\frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-2E_n(1+\nabla_s m_4)\left(1+\frac{1}{\beta}\right)m_3}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right]\left[\frac{-2E_n(1+\nabla_a|m_4|)\left(1+\frac{1}{\beta}\right)|m_3|}{(1+\delta_y|m_4|)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_4|)+R_p}\right)\right]} = U_{14} < \infty$$

$$|\frac{\partial f_7}{\partial m_4}| = |\frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-H-En\nabla_a\left(1+\frac{1}{\beta}\right)(m_3)^2}{\delta_y(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{\delta_y(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right]\left[\frac{-H-En\nabla_a\left(1+\frac{1}{\beta}\right)|(m_3)^2|}{\delta_y(1+\delta_y|m_4|)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right)\left(\frac{D_f}{\delta_y(1+\delta_y|m_4|)+R_p}\right)\right]} = U_{15} < \infty$$

$$|\frac{\partial f_7}{\partial m_5}| = |\frac{\tau m_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-Prm_1-2\delta_y m_5-N_b m_7-2N_t m_5}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{\tau|m_7| - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau|m_6|\right]\left[\frac{-Pr|m_1|-2\delta_y|m_5|-N_b|m_7|-2N_t|m_5|}{(1+\delta_y|m_4|)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_4|)+R_p}\right)\right]} = U_{16} < \infty$$

$$|\frac{\partial f_7}{\partial m_6}| = |\frac{C_p}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{C_p}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau\right)\left(\frac{D_f}{(1+\delta_y|m_4|)+R_p}\right)\right]} = U_{17} < \infty$$

$$|\frac{\partial f_7}{\partial m_7}| = |\frac{-Scm_1 - \tau m_5}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}|$$

$$\leq \frac{-Sc|m_1| - \tau|m_5|}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau|m_6|\right)\left(\frac{D_f}{(1+\delta_y|m_4|)+R_p}\right)\right]} = U_{18} < \infty$$

From the above, we have shown that

$$|\frac{\partial f_i}{\partial m_j}| \leq K \ such\ that\ i,j = 1(1)7$$

It is obvious that

$$|\frac{\partial f_i}{\partial m_j}|_{i,j=1(1)7}\ is\ bounded\ for\ i = 1,2,...,7$$

hence K exist such that $K = max(0,1,U_1,...,U_{18})$ and $0 < K < \infty$. Thus,

$f_i(m_1,m_2,m_3,m_4,m_5,m_6,m_7)$ are Lipschitz continuous. Hence, the solution of the couple differential equations is unique. It is important to proof the theorem in 3.2

on the transformed equations for research problem three by considering the system of first order differential equations with critical point of $m^0{}_i$. Hence, the critical points (0,0,0,0,0,0,0) and (0,1,0,0,0,0,0) are considered. Consider:

$$m_1' = f_1(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = m_2$$

$$m_2' = f_2(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = m_3$$

$$m_3' = f_3(m_1, m_2, m_3, m_4, m_5, m_6, m_7)$$

$$= \quad \frac{\nabla}{(1+ \quad {}_a m_4)} \quad \frac{}{1+_{\beta}{}^{\frac{1}{}} \quad (1+\nabla_a m_4)} \quad \frac{}{1+_{\beta}{}^{\frac{1}{}} \quad (1+\nabla_a m_4)} \quad \frac{}{1+_{\beta}{}^{\frac{1}{}} \quad (1+\nabla_a m_4)}$$

$$-\frac{\triangle_b \cos(\Phi) m_6}{\left(1+_{\beta}{}^{\frac{1}{}}\right)(1+\nabla_a m_4)} + \frac{M^2 m_2}{\left(1+_{\beta}{}^{\frac{1}{}}\right)(1+\nabla_a m_4)} + \frac{1}{P_o} m_2$$

$$m_4' = f_4(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = m_5$$

$$m_5' = f_5(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = \frac{J_1 - D_f J_2}{((1+\delta_y m_4) + R_p)}$$

$$m_6' = f_6(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = m_7$$

$$m_7' = f_7(m_1, m_2, m_3, m_4, m_5, m_6, m_7)$$

$$= \frac{-Scm_1 m_7 + C_p m_6 - \tau m_5 m_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{J_1}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$- \quad \frac{{}_a m_5 m_3}{\nabla} - \frac{m_1 m_3}{(\quad)} + \frac{(m_2)^2}{(\quad)} - \frac{\triangle_a \cos(\Phi) m_4}{(\quad)}$$

Hence, the necessary and sufficient condition of Jacobian matrix is satisfied and the

Jacobian matrix takes the form

$$C = \begin{bmatrix}
\frac{\partial f_1}{\partial m_1} & \frac{\partial f_1}{\partial m_2} & \frac{\partial f_1}{\partial m_3} & \frac{\partial f_1}{\partial m_4} & \frac{\partial f_1}{\partial m_5} & \frac{\partial f_1}{\partial m_6} & \frac{\partial f_1}{\partial m_7} \\
\frac{\partial f_2}{\partial m_1} & \frac{\partial f_2}{\partial m_2} & \frac{\partial f_2}{\partial m_3} & \frac{\partial f_2}{\partial m_4} & \frac{\partial f_2}{\partial m_5} & \frac{\partial f_2}{\partial m_6} & \frac{\partial f_2}{\partial m_7} \\
\frac{\partial f_3}{\partial m_1} & \frac{\partial f_3}{\partial m_2} & \frac{\partial f_3}{\partial m_3} & \frac{\partial f_3}{\partial m_4} & \frac{\partial f_3}{\partial m_5} & \frac{\partial f_3}{\partial m_6} & \frac{\partial f_3}{\partial m_7} \\
\frac{\partial f_4}{\partial m_1} & \frac{\partial f_4}{\partial m_2} & \frac{\partial f_4}{\partial m_3} & \frac{\partial f_4}{\partial m_4} & \frac{\partial f_4}{\partial m_5} & \frac{\partial f_4}{\partial m_6} & \frac{\partial f_4}{\partial m_7} \\
\frac{\partial f_5}{\partial m_1} & \frac{\partial f_5}{\partial m_2} & \frac{\partial f_5}{\partial m_3} & \frac{\partial f_5}{\partial m_4} & \frac{\partial f_5}{\partial m_5} & \frac{\partial f_5}{\partial m_6} & \frac{\partial f_5}{\partial m_7} \\
\frac{\partial f_6}{\partial m_1} & \frac{\partial f_6}{\partial m_2} & \frac{\partial f_6}{\partial m_3} & \frac{\partial f_6}{\partial m_4} & \frac{\partial f_6}{\partial m_5} & \frac{\partial f_6}{\partial m_6} & \frac{\partial f_6}{\partial m_7} \\
\frac{\partial f_7}{\partial m_1} & \frac{\partial f_7}{\partial m_2} & \frac{\partial f_7}{\partial m_3} & \frac{\partial f_7}{\partial m_4} & \frac{\partial f_7}{\partial m_5} & \frac{\partial f_7}{\partial m_6} & \frac{\partial f_7}{\partial m_7}
\end{bmatrix}$$

This becomes;

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ U_7 & 0 & U_8 & U_9 & U_{10} & U_{11} & U_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ U_{13} & 0 & U_{14} & U_{15} & U_{16} & U_{17} & U_{18} \end{bmatrix}$$

where

$$U_1 = \frac{-m_3}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}$$

$$U_2 = \frac{2m_2 + M^2 + \frac{1}{P_o}\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}$$

$$U_3 = \frac{-\left(1 +_{\beta}^{1}\right)\nabla_a m_5 - m_1}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}$$

$$U_4 = \frac{-\triangle_a \cos(\Phi) + \frac{1}{P_o}\left(1 +_{\beta}^{1}\right)\nabla_a m_2}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4) + \left(1 +_{\beta}^{1}\right)(1+\nabla_a)}$$

$$U_5 = \frac{-\left(1 +_{\beta}^{1}\right)\nabla_a m_3}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}$$

$$U_6 = \frac{-\triangle_b \cos(\Phi)}{\left(1 +_{\beta}^{1}\right)(1+\nabla_a m_4)}$$

$$U_7 = \frac{\partial f_5}{\partial m_1}| = |\frac{-Prm_5 + D_f Scm_7}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right]\left[(1+\delta_y m_4)+R_p\right]}$$

$$U_8 = \frac{-2Ec(1+\nabla_a m_4)\left(1 +_{\beta}^{1}\right)m_3}{((1+\delta_y m_4)+R_p)}$$

$$U_9 = \frac{-H - E_n\nabla_a\left(1 +_{\beta}^{1}\right)(m_3)^2 - D_f(J_1\delta_y - (1+\delta_y m_4)+R_p)(-H - E_n\nabla_a\left(1 +_{\beta}^{1}\right)(m_3)^2)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right]\left[(1+\delta_y m_4)+R_p\right]}$$

$$U_{10} = \frac{\partial f_5}{\partial m_5}| = |\frac{-Prm_1 - 2\delta_y m_5 - N_b m_7 - 2N_t m_5 + \tau D_f m_7}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right]\left[(1+\delta_y m_4) + R_p\right]}$$

$$U_{11} = \frac{\partial f_5}{\partial m_6}| = |\frac{-D_f(C_p - \tau)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right]\left[(1+\delta_y m_4) + R_p\right]}$$

$$U_{12} = \frac{-N_b m_5 + D_f(Scm_1 + \tau m_5)}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_5)+R_p}\right)\right]\left[(1+\delta_y m_4) + R_p\right]}$$

$$U_{13} = \frac{-Scm_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-Prm_5}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$U_{14} = \frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-2E_n(1+\nabla_s m_4)\left(1+\frac{1}{\beta}\right)m_3}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$U_{15} = \frac{-\left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-H - En\nabla_a\left(1+\frac{1}{\beta}\right)(m_3)^2}{\delta_y(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right)\left(\frac{D_f}{\delta_y(1+\delta_y m_4)+R_p}\right)\right]}$$

$$U_{16} = \frac{\tau m_7 - \left[\left(S_o + \frac{N_t}{LnN_b}\right) + \tau m_6\right]\left[\frac{-Prm_1 - 2\delta_y m_5 - N_b m_7 - 2N_t m_5}{(1+\delta_y m_4)+R_p}\right]}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$U_{17} = \frac{C_p}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

$$U_{18} = \frac{-Scm_1 - \tau m_5}{\left[1 - \left(\left(S_o + \frac{N_t}{LnN_b}\right)\tau m_6\right)\left(\frac{D_f}{(1+\delta_y m_4)+R_p}\right)\right]}$$

Using the default values of parameters defined as β = 3.0,O$_a$ = 1.0,4_a = 4_b = 2.0,Φ = 30^o,M = 1.0,P_o = 2.0,δ_y = 1.0,R_p = 0.5,Pr = 0.71,H = 0.5,E_n = 0.01,D_f = S_o = 2.0,N_b = N_t = 1.0,Sc = 0.61,C_p = 1.0,Ln = 2.0 and τ = 2.0.

Evaluating the Jacobian matrix at the critical point ($m_1 = 0, m_2 = 0, m_3 = 0, m_4 = 0, m_5 = 0, m_6 = 0, m_7 = 0$).

$$U_1 = 0,\ U_2 = 1.2501,\ U_3 = 0,\ U_4 = 0.3248,\ U_5 = 0,\ U_6 = -0.6496,\ U_7 = 0,$$

$$U_8 = 0,\ U_9 = 0,\ U_{10} = 0,\ U_{11} = 0.44,\ U_{12} = 0,\ U_{13} = 0,\ U_{14} = 0,$$

$$U_{15} = -0.7518,\ U_{16} = 0,\ U_{17} = -0.0588,\ U_{18} = 0$$

Hence, the Jacobian matrix C becomes

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1.2501 & 0 & 0.3248 & 0 & -0.6496 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.44 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.7518 & 0 & -0.0588 & 0 \end{bmatrix}$$

The eigenvalues of the above matrix can be calculated using $|C - \lambda I| = 0$.

$$|C - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 1.2501 & -\lambda & 0.3248 & 0 & -0.6496 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0.44 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -0.7818 & 0 & -0.0588 & -\lambda \end{vmatrix}$$

Using the MAPLE software, the eigenvalues gives:

$$|C - \lambda I| - (\lambda^4 + 0.0588\lambda^2 + 0.343992)(\lambda 62 - 1.2501)\lambda$$

$\lambda_1 = 0$, $\lambda_2 = 1.118078709$, $\lambda_3 = -1.118078709$, $\lambda_4 = 0.5277822998 - 0.5549361729I$ $\lambda_5 = -0.5277822998 + 0.5549361729I$, $\lambda_6 = 0.5277822998 + 0.5549361729I$

$$\lambda_7 = -0.5277822998 - 0.5549361729I$$

Evaluating the Jacobian matrix at the critical point ($m_1 = 0, m_2 = 1, m_3 = 0, m_4 = 0, m_5 = 0, m_6 = 0, m_7 = 0$). Hence, the Jacobian matrix C becomes

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2.7505 & 0 & 0.5748 & 0 & -0.6496 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.44 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.7818 & 0 & -0.0588 & 0 \end{bmatrix}$$

The eigenvalues is gotten using the formula $|C - \lambda I| = 0$.

$$|C - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 2.7505 & -\lambda & 0.5748 & 0 & -0.6496 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0.44 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & -0.7818 & 0 & -0.0588 & -\lambda \end{vmatrix}$$

Using the MAPLE software, the eigenvalues gives:

$$|C - \lambda I| = -(\lambda^4 + 0.0588\lambda^2 + 0.343992)(\lambda^2 - 2.7505)\lambda$$

Hence, the eigenvalues gives

$\lambda_1 = 0$, $\lambda_2 = 1.658463144$, $\lambda_3 = -1.658463144$, $\lambda_4 = 0.5277822998 - 0.5549361729I$

$$\lambda_5 = -0.5277822998 + 0.5549361729I, \lambda_6 = 0.5277822998 + 0.5549361729I$$

$$\lambda_7 = -0.5277822998 - 0.5549361729I$$

The eigenvalues are conjugate complex with negative real part signs. Now, considering the theorem 3.2 and table 3.1, we shall conclude that the system of coupled ordinary differential equation is Asymptotically stable.

3.10 Solution technique to problem three

SHAM is utilized on the simplified equations (3.317)-(3.319) subject to (3.320) and (3.321). SHAM is the numerical techniques of HAM. HAM is significant in decomposing nonlinear systems of differential equation to linear differential equations. The decomposed linear ordinary differential equations is solved using Chebyshev spectral collocation method. The physical region of the problem is first transformed from $[0,\infty)$ to the $[-1,1]$ with the help of domain truncation. Hence, the solution is obtained within the interval $[0,\eta_\infty]$ and not $[0,\infty)$ again. This will lead us to the use of algebraic mapping

$$\zeta = \frac{2\eta}{L} - 1, \quad \xi \in [-1,1] \tag{3.340}$$

The boundary conditions is made homogeneous by applying the transformations

$$f(\eta) = f(\xi) + f_0(\eta),\ T(\eta) = T(\xi) + T_0(\eta),\ C(\eta) = C(\xi) + C_0(\eta) \tag{3.341}$$

substituting equation (3.341) into (3.334)-(3.336) to obtain

$$\begin{aligned}
&\left(1+\frac{1}{\beta}\right)\nabla_a T' f'' + \left(1+\frac{1}{\beta}\right)\nabla_a T' f_0'' + \left(1+\frac{1}{\beta}\right)\nabla_a T_0' f'' + \left(1+\frac{1}{\beta}\right)\nabla_a T_0' f_0'' \\
&+ \left(1+\frac{1}{\beta}\right) f''' + \left(1+\frac{1}{\beta}\right) f_0''' + \nabla_a \left(1+\frac{1}{\beta}\right) T f''' + \nabla_a \left(1+\frac{1}{\beta}\right) T f_0''' \\
&+\nabla_a \left(1+\frac{1}{\beta}\right) T_0 f''' + \nabla_a \left(1+\frac{1}{\beta}\right) T_0 f_0''' + f f'' + f f_0'' + f_0 f'' + f_0 f_0'' - (f')^2 \\
&-2f'f_0' - (f_0')^2 + \triangle_a \cos(\alpha) T + \triangle_a \cos(\alpha) T_0 + \triangle_b \cos(\alpha) C + \triangle_b \cos(\alpha) C_0 - M^2 f' \\
&-M^2 f_0' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right) f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right) f_0' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T f' \\
&-\frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a f f_0' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f_0' = 0
\end{aligned} \tag{3.342}$$

$$\begin{aligned}
&(1+R_p)T'' + (1+R_p)T_0'' + \delta_y T T'' + \delta_y T T_0 + \delta_y T_0 T'' + \delta_y T_0 T_0'' + Pr f T' + Pr f T_0' \\
&+Pr f_0 T' + Pr f_0 T_0' + \delta_y (T')^2 + 2\delta_y T' T_0' + \delta_y (T_0')^2 + Pr H T + Pr H T_0 \\
&+Pr E_n \left(1+\frac{1}{\beta}\right)(f'')^2 + 2 Pr E_n \left(1+\frac{1}{\beta}\right) f_0'' f'' + Pr E_n \left(1+\frac{1}{\beta}\right)(f_0'')^2 + D_f C''
\end{aligned}$$

$$+D_f C_0'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a T(f'')^2 + 2E_n\left(1+\frac{1}{\beta}\right)\nabla_a T f_0'' f'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a T(f_0'')^2$$
$$+E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0(f'')^2 + 2E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f_0'' f'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0(f_0'')^2$$
$$+N_b C'T' + N_b C'T_0' + N_b C_0'T' + N_b C_0'T_0' + N_t(T')^2 + 2N_t T'T_0' + N_t(T_0')^2 = 0 \quad (3.343)$$
$$C'' + C_0'' - S_c C_p C - S_c C_p C_0 + S_c f C' + S_c f C_0' + S_c f_0 C' + S_c f_0 C_0'$$
$$+\left(S_c S_o + \frac{S_c N_t}{L_n N_b}\right)T'' + \left(S_c S_o + \frac{S_c N_t}{L_n N_b}\right)T_0''$$
$$+\tau C T'' + \tau C T_0'' + \tau C_0 T'' + \tau C_0 T_0'' + \tau T'C' + \tau T'C_0' + \tau T_0'C' + \tau T_0'C_0' = 0 \quad (3.344)$$

Simplifying the above equations to obtain

$$\left(1+\frac{1}{\beta}\right)\nabla_a T'f'' + \left(1+\frac{1}{\beta}\right)\nabla_a T'f_0'' + \left(1+\frac{1}{\beta}\right)\nabla_a T_0'f'' + \left(1+\frac{1}{\beta}\right)f'''$$
$$+\nabla_a\left(1+\frac{1}{\beta}\right)Tf''' + \nabla_a\left(1+\frac{1}{\beta}\right)Tf_0''' + \nabla_a\left(1+\frac{1}{\beta}\right)T_0 f''' + ff'' + ff_0'' + f_0 f''$$
$$-(f')^2 - 2f'f_0' + \triangle_a\cos(\alpha)T + \triangle_b\cos(\alpha)C - M^2 f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)f' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T f'$$
$$-\frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T f_0' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f' = -\left(1+\frac{1}{\beta}\right)\nabla_a T_0' f_0'' - \left(1+\frac{1}{\beta}\right)f_0'''$$
$$-\nabla_a\left(1+\frac{1}{\beta}\right)T_0 f_0''' - f_0 f_0'' + (f_0')^2 - \triangle_a\cos(\alpha)T_0 - \triangle_b\cos(\alpha)C_0 + M^2 f_0'$$
$$+\frac{1}{P_0}\left(1+\frac{1}{\beta}\right)f_0' + \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f_0' \quad (3.345)$$
$$(1+R_p)T'' + \delta_y T T'' + \delta_y T T_0 + \delta_y T_0 T'' + PrfT' + PrfT_0' + Prf_0 T' + \delta_y(T')^2$$
$$+2\delta_y T'T_0' + PrHT + PrE_n\left(1+\frac{1}{\beta}\right)(f'')^2 + 2PrE_n\left(1+\frac{1}{\beta}\right)f_0''f'' + D_f C''$$
$$+E_n\left(1+\frac{1}{\beta}\right)\nabla_a T(f'')^2 + 2E_n\left(1+\frac{1}{\beta}\right)\nabla T f_0'' f'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a T(f_0'')^2$$
$$+E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0(f'')^2 + 2E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0 f_0'' f'' + N_a C'T' + N_b C'T_0' + N_b C_0'T'$$
$$+N_t(T')^2 + 2N_t T'T_0' = -(1+R_p)T_0'' - \delta_y T_0 T_0'' - \delta_y(T_0')^2 - PrHT_0$$
$$-PrE_n\left(1+\frac{1}{\beta}\right)(f_0'')^2 - D_f C_0'' - E_n\left(1+\frac{1}{\beta}\right)\nabla_a T_0(f'')^2$$
$$-Prf_0 T_0' - N_b C_0'T_0' - N_t(T')^2 \quad (3.346)$$

$$C'' - S_cC_pC + S_cfC' + S_cfC_0' + S_cf_0C' + \left(S_cS_o + \frac{S_cN_t}{L_nN_b}\right)T'' + \tau CT'' + \tau CT_0''$$

$$+\tau C_0T'' + \tau T'C' + \tau T'C_0' + \tau T_0'C' = C_0'' + S_nC_pC_0 - S_nf_0C_0'$$

$$-\left(S_cS_0 + \frac{S_cN_t}{L_nN_b}\right)T_0'' - \tau C_0T_0'' - \tau T_0'C_0' \quad (3.347)$$

Simplifying the above equations and setting

$$\alpha_1 = \left(1 + \frac{1}{\beta}\right)\nabla_a f_0'',\ \alpha_2 = \left(1 + \frac{1}{\beta}\right)\nabla_a T_0',\ \alpha_3 = \nabla_a\left(1 + \frac{1}{\beta}\right)f_0'',$$

$$\alpha_4 = \nabla_a\left(1 + \frac{1}{\beta}\right)T_0,\ \alpha_5 = f_0'',\ \alpha_6 = f_0,\ \alpha_7 = -2f_0',\ \alpha_8 = -\frac{1}{P_0}\left(1 + \frac{1}{\beta}\right)\nabla_a f_0',$$

$$\alpha_9 = -\frac{1}{P_0}\left(1 + \frac{1}{\beta}\right)\nabla_a T_0$$

$$G_1(\eta) = -\left(1 + \frac{1}{\beta}\right)\nabla_a T_0'f_0'' - \left(1 + \frac{1}{\beta}\right)f_0''' - \nabla_a\left(1 + \frac{1}{\beta}\right)T_0f_0''' - f_0f_0'' + (f_0')^2$$

$$-\triangle_a\cos(\alpha)T_0 - \triangle_b\cos(\alpha)C_0 + M^2f_0' + \frac{1}{P_0}\left(1 + \frac{1}{\beta}\right)f_0' + \frac{1}{P_0}\left(1 + \frac{1}{\beta}\right)\nabla_a T_0f_0'$$

$$\beta_1 = \delta_yT_0,\ \beta_2 = PrT_0',\ \beta_3 = Prf_0,\ \beta_4 = 2\delta_yT_0',\ \beta_5 = 2PrE_n\left(1 + \frac{1}{\beta}\right)f_0''$$

$$\beta_6 = 2E_n\left(1 + \frac{1}{\beta}\right)\nabla_a f_0'',\ \beta_7 = En\left(1 + \frac{1}{\beta}\right)\nabla_a(f_0'')^2,\ \beta_8 = E_n\left(1 + \frac{1}{\beta}\right)\nabla_a T_0,$$

$$\beta_9 = 2E_n\left(1 + \frac{1}{\beta}\right)\nabla_a T_0f_0'',\ \beta_{10} = N_bT_0',\ \beta_{11} = N_bC_0',\ \beta_{12} = 2N_tT_0'$$

$$G_2(\eta) = -(1 + R_p)T_0'' - \delta_yT_0T_0'' - \delta_y(T_0')^2 - PrHT_0 - PrE_n\left(1 + \frac{1}{\beta}\right)(f_0'')^2$$

$$-Prf_0T_0' - D_fC_0'' - E_n\left(1 + \frac{1}{\beta}\right)\nabla_a T_0(f_0'')^2 - N_bC_0'T_0' - N_t(T_0')^2$$

$$\gamma_1 = S_cC_0',\ \gamma_2 = S_cf_0,\ \gamma_3 = \tau T_0'',\ \gamma_4 = \tau C_0,\ \gamma_5 = \tau C_0',\ \gamma_6 = \tau T_0'$$

$$G_3(\eta) = -C_0'' + S_cC_pC_0 - S_cf_0C_0' - \left(S_cS_o + \frac{S_cN_t}{L_nN_b}\right)T_0'' - \tau C_0T_0'' - \tau T_0'C_0'$$

Substituting the above coefficient parameters into equations (3.345)-(3.347) to obtain

$$\left(1 + \frac{1}{\beta}\right)\nabla_a T'f'' + \alpha_1T' + \alpha_2f'' + \left(1 + \frac{1}{\beta}\right)f''' + \nabla_a\left(1 + \frac{1}{\beta}\right)Tf''' + \alpha_3T + \alpha_4f'''$$

$$+ff^{00} + \alpha_5f + \alpha_6f^{00} - (f^0)^2 + \alpha_7f^0 + 4_a\cos(\alpha)T + 4_b\cos(\alpha)C - M^2f^0$$

$$-\frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a T f' + \alpha_8 T + \alpha_9 f' = G_1(\eta) \tag{3.348}$$

$$(1+R_p)T'' + \delta_y T T'' + \beta_1 T + \beta_1 T'' + Pr f T' + \beta_2 f + \beta_3 T' + \delta_y (T')^2$$
$$+\beta_4 T' + PrHT + PrE_n\left(1+\frac{1}{\beta}\right)(f'')^2 + \beta_5 f'' + D_f C'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a T (f'')^2$$
$$+\beta_6 f'' + \beta_7 T + \beta_8 (f'')^2 + \beta_9 f'' + N_a C' T' + \beta_{10} C' + \beta_{11} T'$$
$$+N_t (T')^2 + \beta_{12} T = G_2(\eta) \tag{3.349}$$

$$C'' - S_c C_p C + S_c f C' + \gamma_1 f + \gamma_2 C' + \left(S_c S_0 + \frac{S_c N_t}{L_n N_b}\right) T''$$
$$+\tau C T'' + \gamma_3 C + \gamma_4 T'' + \tau T' C' + \gamma_5 T' + \gamma_6 C' = G_3(\eta) \tag{3.350}$$

From now, the derivatives of f,θ and φ are in respect of ξ given

$$\frac{d}{d\xi} = \frac{2}{L}\frac{d}{d\eta} \tag{3.351}$$

The above equation (3.351) is true because all functions of η are known functions represented by coefficient parameters. However, the following initial guess is chosen to satisfies the boundary conditions (3.320) and (3.321) at $\eta = 0$

$$f_0(\eta) = S_w + e^{-\eta} + 1,\ T_0(\eta) = C_(\eta) = e^{-\eta} \tag{3.352}$$

The nonhomogeneous linear part of equations (3.348) – (3.350) is solved to obtain initial solution of SHAM

$$\alpha_1 T_l' + \alpha_2 f_l'' + \left(1+\frac{1}{\beta}\right) f_l''' + \alpha_3 T_l + \alpha_4 f_l''' + \alpha_5 f_l + \alpha_6 f_l'' + \alpha_7 f_l'$$
$$+\triangle_a \cos(\alpha) T_l + \triangle_b \cos(\alpha) C_l - M^2 f_l' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right) f_l' + \alpha_8 T_l$$
$$+\alpha_9 f_l' = G_1(\eta) \tag{3.353}$$

$$(1+R_p)T_l'' + \beta_1 T_l + \beta_1 T_l'' + \beta_2 f_l + \beta_3 T_l' + \beta_4 T_l' + PrHT_l + \beta_5 f_l''$$
$$+D_f C_l'' + \beta_6 f_l'' + \beta_7 T_l + \beta_9 f_l'' + \beta_{10} C_l' + \beta_{11} T_l' + \beta_{12} T_l = G_2(\eta) \tag{3.354}$$

$$C_l'' - S_c C_p C_l + \gamma_1 f_l + \gamma_2 C_l' + \left(S_c S_o + \frac{S_c N_t}{L_n N_b}\right) T_l'' + \gamma_3 C_l + \gamma_4 T_l''$$
$$+\gamma_5 T_l' + \gamma_6 C_l' = G_3(\eta) \tag{3.355}$$

subject to:

$$f_l(-1) = f_l'(-1) = f_l'(1) = 0,\ T(-1) = T_l(1) = 0,\ C_l(-1) = C_l(1) = 0 \tag{3.356}$$

The Chebyshev pseudospectral method is applied on (30353)-(3.355) and the unknown functions $f_l(\xi)$,$T_l(\xi)$ and T_l will be approximated as a truncated series of Chebyshev polynomials given as:

$$f_l(\xi) \approx f_l^N(\xi_j) + \sum_{k=0}^{N} \bar{f}_k T_{1k}(\xi_j) \; ,j = 0,...,N \tag{3.357}$$

$$T_l(\xi) \approx T_l^N(\xi_j) + \sum_{k=0}^{N} \bar{T}_k T_{2k}(\xi_j) \; ,j = 0,...,N \tag{3.358}$$

$$C_l(\xi) \approx C_l^N(\xi_j) + \sum_{k=0}^{N} \bar{C}_k T_{3k}(\xi_j) \; ,j = 0,...,N \tag{3.359}$$

where T_{1k},T_{2k} and T_{3k} are the kth chebyshev polynomial and $\xi_0,\xi_1,...,\xi_N$ are GaussLobatto collocation point given as:

$$\xi_J = cos\left(\frac{\pi j}{N}\right) \;\; , \; j = 0, 1, ..., N \tag{3.360}$$

where N = number of collocation points and the derivatives of functions $f_l(\xi)$,$T_l(\xi)$ and $T_l(\xi)$ at the collocation point is given by

$$\frac{d^r f_l}{d\xi^r} = \sum_{k=0}^{N} D_{kj} f_l(\xi_j) = DF, \;\; \frac{d^r T_l}{d\xi^r} = \sum_{k=0}^{N} D_{kj} T_l(\xi_j) = DT, \;\; \frac{d^r C_l}{d\xi^r} = \sum_{k=0}^{N} D_{kj} C_l(\xi_j) = DC \tag{3.361}$$

where r = order of differentiation and D = Chebyshev spectral differentiation matrix. Substituting Equations (3.357)-(3.339) into equations (3.353)-(3.355) yields

$$MF_L = G \tag{3.362}$$

subject to

$$f_l(\xi_N) = -S_w, \sum_{k=0}^{N} D_{0,m} f_l(\xi_m) = 1 \; ,T_l(\xi_N) = C_l(\xi_N) = 1 \; ,T_l(\xi_0) = T_l(\xi_0) = 0 \tag{3.363}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ & M_{32} & M_{33} \end{bmatrix} \tag{3.364}$$

M_{31}

and

$$M_{11} = \alpha_2 D^2 + \left(1 + \frac{1}{\beta}\right) D^3 + \alpha_4 D^3 + \alpha_5 + \alpha_6 D^2 + \alpha_7 D - M^2 D - \frac{1}{P_0}\left(1 + \frac{1}{\beta}\right) D + \alpha_9 D$$

$$M_{12} = \alpha_1 D + \triangle_a \cos(\alpha) + \alpha_8, \quad M_{13} = \triangle_b \cos(\alpha), \quad M_{21} = -\beta_2 - \beta_5 D^2 - \beta_6 D^2 - \beta_9 D^2$$

$$M_{22} = (1 + R_p) D^2 - \beta_1 - \beta_1 D^2 - \beta_3 D - \beta_4 D + PrH - \beta_7 - \beta_{11} D - \beta_{12}$$

$$M_{23} = D_f D^2 - \beta_{10} D, \quad M_{31} = \gamma_1, \quad M_{32} = \left(S_n S_0 + \frac{S_n N_t}{L_n N_b}\right) D^2 + \gamma_4 D^2 + \gamma_5 D$$

$$M_{33} = D^2 - S_n C_p + \gamma_2 D + \gamma_3 + \gamma_6 D$$

$F_L = [f_l(\xi_0),...,f_l(\xi_N),T_l(\xi_0),...,T_l(\xi_N),C_l(\xi_0),...,C_l(\xi_N)]^T$

$G = [G_1(\eta_0),G_1(\eta_1),...,G_1(\eta_N),G_2(\eta_0),G_2(\eta_1),...,G_2(\eta_N),G_3(\eta_0),G_3(\eta_1),...,G_3(\eta_N)]$ $\alpha_i = diag([\alpha_i(\eta_0),\alpha_i(\eta_1),...,\alpha_i(\eta_{N-1})])$ $\beta_i = diag([\beta_i(\eta_0),\beta_i(\eta_1),...,b_i(\eta_{N-1})])$ $\gamma_i = diag([\gamma_i(\eta_0),\gamma_i(\eta_1),...,\gamma_i(\eta_{N-1})])$, $i = 1,2,3,4,5,6,7,8$

The superscript T means transpose, "*diag*" = diagonal matrix and I = identity matrix of size $(N+1)\times(N+1)$. To implement the boundary conditions, the first, last rows and columns of A are deleted during the computational analysis in MATLAB. Also, the first and last rows of $f_l(\xi),T_l(\xi),C_l(\xi)$ and G are deleted. Furthermore, the boundary conditions are imposed on the first and last rows of the matrix M.

Therefore, the values of $f_l(\xi_0),...,f_l(\xi_N),T_l(\xi_0),...,T_l(\xi_N),C_l(\xi_0),...,C_l(\xi_N)$ can be

determined from

$$F_L = M^{-1} G \qquad (3.365)$$

Equation (3.365) is the required solution of SHAM which provides the initial approximation. Hence, the linear operator is defined as follows to find the SHAM

solutions to (3.353)-(3.355)

$$L_f[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q) = \alpha_1 T_l' + \alpha_2 f_l'' + \left(1 + \frac{1}{\beta}\right) f_l''' + \alpha_3 T_l + \alpha_4 f_l'''$$
$$+\alpha_5 f_l + \alpha_6 f_l'' + \alpha_7 f_l' + \triangle_a \cos(\alpha) T_l + \triangle_b \cos(\alpha) C_l - M^2 f_l'$$
$$-\frac{1}{P_0}\left(1 + \frac{1}{\beta}\right) f_l' + \alpha_8 T_l + \alpha_9 f_l' \qquad (3.366)$$

$$L_T[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q) = (1 + R_p) T_l'' - \beta_1 T_l - \beta_1 T_l'' - \beta_2 f_l - \beta_3 T_l'$$

$$+\beta_4 T_{l0} + PrHT_l + \beta_5 f_{l00} + D_f C_{l00} + \beta_6 f_{l00} + \beta_7 T_l + \beta_9 f_{l00} + \beta_{10} C_{l0}$$

$$+\beta_{11}T_{l0} + \beta_{12}T_l \quad (3.367)$$

$$L_C[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q) = C_l'' - S_cC_pC_l + \gamma_1 f_l + \gamma_2 C' + \gamma_3 C_l + \left(S_cS_o + \frac{S_cN_t}{L_nN_b}\right)T_l'' + \gamma_4 T_l'' + \gamma_5 T_l' + \gamma_6 C_l' \quad (3.368)$$

In the above equations, $q \in [0,1]$ = embedding parameter and $\bar{f}(\eta;q)$, $\bar{T}(\eta;q)$ and $\bar{C}(\eta;q)$ are unknown functions. The zeroth order deformation equation is given by:

$$(1-q)L_f[\bar{f}(\eta;q) - \bar{f}_0(\eta)] = qh_fH_f(\eta)N_{hf}[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q)] \quad (3.369)$$

$$(1-q)L_T[\bar{T}(\eta;q) - \bar{T}_0(\eta)] = qh_TH_T(\eta)N_{hT}[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q)] \quad (3.370)$$

$$(1-q)L_C[\bar{C}(\eta;q) - \bar{C}_0(\eta)] = qh_CH_C(\eta)N_{hC}[\bar{f}(\eta;q),\bar{C}(\eta;q),\bar{C}(\eta;q)] \quad (3.371)$$

In the above equations, $\tilde{}_f$, $\tilde{}_T$ and $\tilde{}_C$ are nonzero convergence controlling auxiliary parameters and N_{hf}, N_{hT} and N_{hC} are the nonlinear operators defined by

$$N_{hf}[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{T}(\eta;q)] = \left(1+\frac{1}{\beta}\right)\nabla_a\bar{T}'\bar{f}'' + \nabla_a\left(1+\frac{1}{\beta}\right)\bar{T}\bar{f}''' + \bar{f}\bar{f}'' - \bar{f}'\bar{f}' - \frac{1}{P_0}\left(1+\frac{1}{\beta}\right)\nabla_a\bar{T}\bar{f}' \quad (3.372)$$

$$N_{hT}[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q)] = \delta_y\bar{T}\bar{T}'' + Pr\bar{f}\bar{T}' + \delta_y\bar{T}'\bar{T}' + PrE_n\left(1+\frac{1}{\beta}\right)\bar{f}''\bar{f}'' + E_n\left(1+\frac{1}{\beta}\right)\nabla_a\bar{T}\bar{f}'' + \beta_8\bar{f}''\bar{f}'' + N_a\bar{C}'\bar{T}' + N_t\bar{T}'\bar{T}' \quad (3.373)$$

$$N_{hC}[\bar{f}(\eta;q),\bar{T}(\eta;q),\bar{C}(\eta;q)] = S_c\bar{f}\bar{C}^{0} + \tau\bar{C}\bar{T}^{00} + \tau\bar{T}^{0}\bar{C}^{0} \quad (3.374)$$

Differentiating (3.369)-(3.371) m times with respect to q and setting $q = 0$ and finally dividing the resulting equations by $m!$, we obtain the mth order deformation equations:

$$L_{\bar{f}}[\bar{f}(\xi) - \chi_m\bar{f}_{m-1}(\xi)] = h_{\bar{f}}(\xi)R_m^{\bar{f}}(\xi) \quad (3.375)(3.375)$$

$$L_{\bar{T}}[\bar{T}(\xi) - \chi_m\bar{T}_{m-1}(\xi)] = h_{\bar{T}}(\xi)R_m^{\bar{T}}(\xi) \quad (3.376)(3.376)$$

$$L_{\bar{C}}[\bar{C}(\xi) - \chi_m\bar{C}_{m-1}(\xi)] = h_{\bar{C}}(\xi)R_m^{\bar{C}}(\xi) \quad (3.377)(3.377)$$

subject to:

$$\bar{f}_m(-1) = \bar{T}_m(-1) = \bar{C}_m(-1) = 0, \bar{f}^{0}_m(1) = \bar{T}_m(1) = \bar{C}_m(1) = 0 \quad (3.378)$$

where

$$R_m^f(\xi) = \alpha_1 T_{m-1} + \alpha_2 f''_{m-1} + \left(1 + \frac{1}{\beta}\right) f'''_{m-1} + \alpha_3 T_{m-1} + \alpha_4 f'''_{m-1} + \alpha_5 f_{m-1}$$

$$+\alpha_6 f''_{m-1} + \alpha_7 f'_{m-1} + \triangle_a \cos(\alpha) T_{m-1} + \triangle_b \cos(\alpha) C_{m-1} - M^2 f'_{m-1}$$

$$-\frac{1}{P_0}\left(1 + \frac{1}{\beta}\right) f'_{m-1} + \alpha_8 T_{m-1} + \alpha_9 f'_{m-1} + \sum_{n=0}^{m-1}\left(1 + \frac{1}{\beta}\right) \nabla_a T'_n f''_{m-1-n}$$

$$+\nabla_a \left(1 + \frac{1}{\beta}\right) T_n f'''_{m-1-n} + f_n f''_{m-1-n} - f'_n f'_{m-1-n}$$

$$-\frac{1}{P_0}\left(1 + \frac{1}{\beta}\right) \nabla_a T_n f'_{m-1-n} - G_1(\eta)(1-\chi_m)$$

$$R_m^T(\xi) = (1+R_p) T''_{m-1} + \beta_1 T_{m-1} + \beta_2 f_{m-1} + \beta_3 T'_{m-1} + \beta_4 T'_{m-1} + PrHT_{m-1}$$

$$+\beta_5 f''_{m-1} + D_f C''_{m-1} + \beta_6 f''_{m-1} + \beta_7 T_{m-1} + \beta_9 f''_{m-1} + \beta_{10} C'_{m-1} + \beta_{11} T'_{m-1} + beta_{12} T_{m-1}$$

$$+\sum_{n=0}^{m-1} \delta_y T_n T''_{m-1-n} + Pr f_n T'_{m-1-n} + \delta_y T'_n T'_{m-1-n} + PrE_n \left(1 + \frac{1}{\beta}\right) f''_n f''_{m-1-n}$$

$$+E_n \left(1 + \frac{1}{\beta}\right) \nabla_a T_n f''_{m-1-n} + \beta_8 f''_n f''_{m-1-n} + N_a C'_n T'_{m-1-n}$$

$$+N_t T'_n T'_{m-1-n} - G_2(\eta)(1-\chi_m)$$

$$R_m^C(\xi) = C''_{m-1} - S_c C_p C_{m-1} + \gamma_1 f_{m-1} + \gamma_2 C'_{m-1} + \left(S_c S_o + \frac{S_c N_t}{L_n N_b}\right) T''_{m-1}$$

$$+\gamma_3 C_{m-1} + \gamma_4 T''_{m-1} + \gamma_5 T'_{m-1} + \gamma_6 C'_{m-1} + \sum_{n=0}^{m-1} S_n f_n C'_{m-1-n} + \tau C_n T''_{m-1-n} + \tau T'_n C'_{m-1-n}$$

Applying the Chebyshev pseudo-spectral transformation on (3.338)-(3.338) gives

$$AF_m = (\chi_m + \hbar)AF_{m-1} - \hbar(1-\chi_m)G + \hbar Q_{m-1} \tag{3.379}$$

subject to the boundary conditions

$$f_m \xi_N = 0, \sum_{k=0}^{N} D_{Nk} f_m(\xi_k) = 0, \sum_{k=0}^{N} D_{0k} f_m(\xi_k) = 0 \tag{3.380}$$

$$T_m(\xi_N) = 0, T_m(\xi_0) = 0 \tag{3.381}$$

$$C_m(\xi_N) = 0, C_m(\xi_0) = 0 \tag{3.382}$$

where M and G are as defined above

$F_m = [f_m(\xi_0), f_m(\xi_1), ..., f_m(\xi_N), T_m(\xi_0), T_m(\xi_1), ..., T_m(\xi_N), C_m(\xi_0), C_m(\xi_1), ..., C_m(\xi_N)]^T$ The boundary conditions (3.350)-(3.352) are implemented on A on the left hand side of equation (3.349) in rows $1, N, N+1, N+2, N+3, 2(N+1)$ and $3(N+1)$ respectively as before with initial solution above. The corresponding rows, all column of A on the right hand side of (3.349), G, $Q_{1,m-1}$,$Q_{2,m-1}$ and $Q_{3,m-1}$ are all set to

zero. This results in the following recursive formula

$$F_m = (\chi_m + \hbar)A^{-1}.A\bar{F}_{m-1} + \hbar A^{-1}[Q_{m-1} - (1 - \chi_m)G] \qquad (3.383)$$

3.10.1 Convergence of SHAM solution

The convergence of SHAM is subject on the value selection for the auxiliary term (~). The term ~ controls the convergence of the SHAM series solution. The numeric value of ~ is taken on the horizontal segment of the ~-curves. Sibanda et al. (2012) discussed that the peak value of ~ to used is equivalent to the turning point of the second order ~- curve.

CHAPTER FOUR: Results and discussion of findings

4.1 Results and discussion of research problem one

Equations (3.56)-(3.58) subject to (3.59) and (3.60) have been solved using the spectral relaxation method (SRM). SRM employs the idea of Gauss-Seidel relaxation approach to linearize a decoupled system of nonlinear differential equations (Motsa et al., 2012). Using the SRM, numerical computations were carried out for the velocity, temperature, concentration, local skin friction, local Nusselt number and Sherwood number. Results are presented in tabular and graphical forms. All programmes were coded in MATLAB R2012a. The results were generated using the scaling parameter L = 15 and it is observed that increase in the value of L does not change the result to a reasonable extent. The number of collocation point used in generating the results was N_x = 120. The value of Prandtl number (Pr) used in this work is (Pr = 0.71) which denotes the Prandtl number for air at 1 atm. In the same vein, the value chosen for Schmidt number is (Sc = 0.20) which connote the Schmidt number for hydrogen. We have also chosen the value for magnetic parameter between 0.1 and 1 and all other parameters were set to be

$Gr = Gm = 2.0, Rr = A = kr = 0.5, t = 1.0, n = 0.5, Du = 0.2, Sr = 0.5, Ec =$

$0.01, A_1 = 0.01, \delta = 0.03, \Delta = 0.02, \epsilon = 0.001$. Hence, all numerical computations

correspond to the above stated values unless or otherwise stated. Remarkably, our results were compared with existing literature and was found to be in good agreement. It worths mentioning that, in a moment we set A_1 = 0 in this study, our model is categorized as a Newtonian fluid model.

Figs. (4.1)-(4.13) illustrates the effects of all the controlling parameters such as

Prandtl number Pr, thermal radiation parameter Rr, Soret parameter Sr, Dufour parameter Du, viscoelastic parameter A_1, heat source or sink parameter δ, heat generation/absorption coefficient parameter Δ, Schmidt number Sc, chemical reaction parameter kr, Eckert number Ec, magnetic parameter M, thermal Grashof number Gr, mass Grashof number Gm. The effect of the Prandtl number Pr on the ve-

locity, temperature and concentration profiles is presented in fig 4.1. It is observed that the fluid velocity is reducing with increasing value of Pr. These results are in agreement with that of Alao et al. (2016) but it is observed that the presence of parameters such as A_1,δ,Δ tends to influence the profile the more. From fig. 4.1 , as the value of Pr is increasing, it causes reduction in the velocity because higher Pr tend to reduce the velocity and the local skin friction. Also, from fig. 4.1, it is observed that increase in Pr decreases the temperature profile. It is noted that when $Pr < 1$, the fluid in the hydrodynamics, thermal and concentration boundary later is highly conducive. Figure 4.2 presents the influence of thermal radiation parameter Rr on the velocity, temperature and concentration profiles is depicted. It is noted from the fig. 4.2 that increasing Rr increases the velocity and temperature profiles. As a matter of fact, increasing thermal radiation parameter enhances the thermal condition of the fluid environment. When Rr is raised the temperature of the fluid will increase resulting to an increase in the profile. It is noted from fig. 4.2 that Rr does not have any effect on the concentration profile. This result is in correlation with that of Idowu and Falodun (2018) and Raju et al. (2016) but it is observed that increase in radiation parameter in the present study shows more impact on the velocity and temperature profiles compared to the work of Raju et al. (2016). Hence, this study conclude that radiation effect is more significant when Rr −→ 0 provided Rr 6= 0 and insignificant as Rr − →∞

The effects of Soret parameter Sr and Dufour parameter Du is investigated separately in this study. Fig. 4.3 presents the effect of Sr on the velocity, temperature and concentration profiles. It is found out from fig. 4.3 that increasing the values of Sr increases the velocity profile. This is due to the fact that, when Sr is raised, there will be greater thermal diffusion and this results to increase in the velocity of the fluid. It is observed from fig. 4.3 that the effect of Sr is negligible on the temperature profile while the concentration profile rises when increasing the Soret parameter as shown in fig. 4.3. Fig. 4.4 depicts the effect of Dufour parameter Du on the velocity, temperature and concentration profiles. An increase in the fluid velocity by increasing Du is noticed. From fig. 4.4 it is observed that as Du increases it gives a rise in the temperature profile. In fig. 4.4, it is noted that effect of Du on the concentration profile is negligible. The results presented in figure 4.3 and 4.4 of the present study is in excellent agreement with that of Omowaye et al. (2015) , Raju et al. (2016), Idowu and Falodun (2018). This shows the correctness of the code used in this study. The Soret term is added to the energy equation. Both Soret and Dufour term influence the fluid velocity but Soret term alters the fluid concentration while Dufour term alters the fluid concentration while Dufour term alters the fluid temperature. Fig. 4.5 illustrates the effect of the viscoelastic parameter A_1 on the velocity, temperature and concentration profiles. The viscoelastic parameter A_1 illustrates the effect of normal stress coefficient on the flow. Interestingly, very close to the plate the fluid velocity decreases and increases far away from the plate. This result is depicted in fig. 4.5. Fig. 4.5 presents the effect of A_1 on the temperature and concentration profiles are negligible. This indicates that a hike in the viscoelastic parameter has tendency of decreasing the boundary layer thickness, while a hike in the normal stress coefficient parameter has an opposing effect on the temperature profile of the flowing fluid. The effect of the viscoelastic fluid parameter as shown in figure 4.5 is in good agreement with that of Manglesh and Gorla (2012). It is noticed that the present result converges than Manglesh and Gorla (2012). The present result compared to Ramzan et al. (2016) is also the same but the effect close to the plate and far away from the plate differs from the result presented by Ramzan et al. (2016).

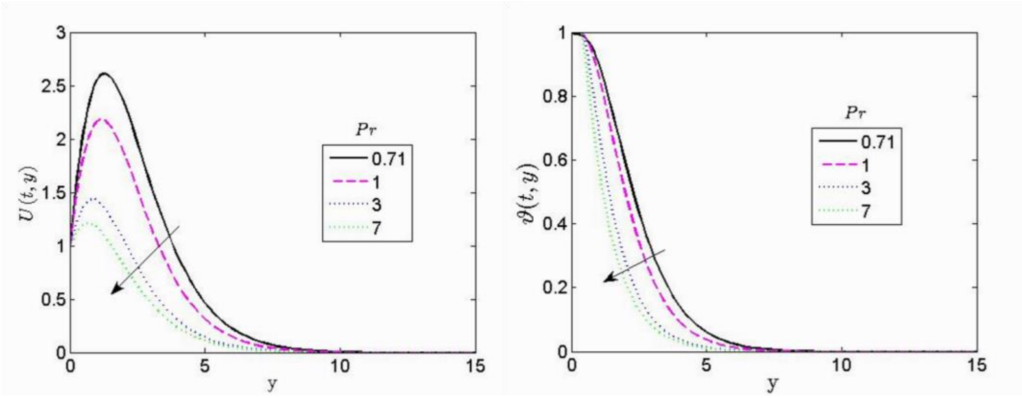
Fig. 4.6 illustrates the influence of heat source δ on the velocity, temperature and concentration profiles. Heat source add more heat energy into the boundary layer flow. It worths mentioning that the generation of heat enhances the velocity and temperature field. From fig. 4.6, an increase in the values of δ increases the velocity profile. In fig. 4.6, increase in the values of δ brings increase to the temperature profile. The effect of δ is negligible on the concentration profile as seen in fig. 4.6. The effect of heat generation coefficient parameter Δ on the velocity, temperature and concentration profiles is illustrated in fig. 4.7. Increasing the values of heat generation coefficient parameter Δ increases the velocity profile. When $\Delta > 0$, the behaviour of the fluid velocity changes and it drastically causes an increase. Δ increases the temperature profile as seen in fig. 4.7 because the thermal boundary layer gets thicker and resulted to the particles of the fluid getting warmer. From fig. 4.7 Δ does not have any effect on the concentration profile. Fig. 4.8 depicts the effect of the Schmidt number Sc on the velocity, temperature and concentration profiles. It is obvious from fig. 4.8 that increasing Sc retards the velocity profile. Clearly from fig. 4.8 Sc does not have any effect on the temperature profile. Fig. 4.8 shows that increase in the values of Sc drastically reduces the concentration profile. Fig. 4.9 shows the effect of the chemical reaction parameter kr on the velocity, temperature and concentration profiles. It is observed that there is a reduction in the velocity profile with increasing value of kr. From the fig. 4.9 effect of kr is negligible on the temperature profile while in fig. 4.9 the concentration profile decreases with increasing values of kr. The fluid motion is retarded on the account of chemical reaction destructive nature. When $kr > 0$ it brings a decrease in the concentration field which weakens the buoyancy effects due to concentration gradients. Thus, as the chemical reaction reduces the concentration thereby increasing its concentration gradient and concentration flux. The destructive nature of chemical reaction parameter on the velocity, temperature and concentration profiles in this study is noted to be in good agreement with that of Chamkha (2003) and Mahanthesh et al.

(2016).

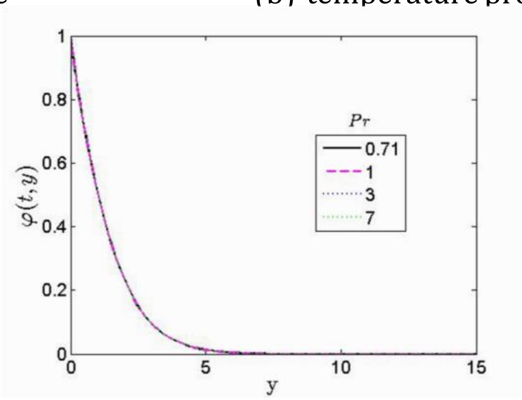
Fig. 4.10 depicts the influence of Eckert number Ec on the velocity, temperature and concentration profiles. Ec is the relationship between the kinetic energy in the flow and enthalpy. As shown in figure 4.10, the velocity profile increases with increase in the values of Eckert number. Also, the temperature profile increases with increase in the values of Ec. Scientifically, Ec add more energy to the hydrodynamics and thermal boundary layer. When the values of Ec increases, it accelerate both velocity and temperature profiles. The result in fig. 4.10 is true because at higher viscous dissipative energy, the velocity and temperature increases. It is seen from the fig. 4.10 that Ec is negligible or has no effect on the concentration profile. It is seen from the figure 4.10 that Ec is negligible or has no effect on the concentration profile. This result is in excellent agreement with the existing work of Mondal et al. (2018). The variation of different values of magnetic parameter M on the velocity, temperature and concentration profiles are shown in fig. 4.11. It is evident that, the applied magnetic field strength B_0 gives rise to a resistive force called Lorentz force. This force reduces the motion of an electrically conducting fluid. It is clearly seen in fig. 4.11 that, increasing the magnetic parameter causes a reduction in the velocity profile. Obviously, from fig. 4.11, the effect of M is negligible on both temperature and concentration profiles. The result as shown in figure 4.11 is in excellent agreement with the recent work of Liaquat et al. (2019) and Shah et al. (2019). Fig. 4.12 depicts the effect of thermal Grashof number Gr on the velocity, temperature and concentration profiles. Gr is the ratio of buoyancy to the viscous acting on the fluid. When the values of Gr increases, the velocity profile increases rapidly close to the plate and decreases to the free stream velocity. This graphical illustration is shown in fig. 4.12. It is obvious from the fig. 4.12 that the thermal Grashof number does not have any effect on the temperature and concentration. Fig. 4.13 illustrate the influence of mass Grashof number Gm on the velocity, temperature and concentration profiles. Obviously from the fig. 4.13, increasing the values of Gm intensifies the velocity profile. It is seen from the fig. 4.13 that Gm does not have any effect on both the temperature and concentration profiles respectively.

The comparison between the present study and existing literatures are presented in Table 4.1-4.3 while Table 4.4 shows the results of the present study. In table 4.1 , comparison of computational values for Sherwood number with the work of Chandra et al. (2015) and Mishra et al. (2013) with the present work in the absence of radiation parameter Rr, Eckert number Ec, Dufour parameter Du, heat generation/absorption parameter Δ, chemical reaction parameter kr, and Soret parameter (*i.e* $Du = Sr = R = Ec = \Delta = kr = 0$). From table 4.1 the present result is in

good agreement which shows the correctness of the code used in the present study. Table 4.2 and Table 4.3 shows the comparison between the present work for values of skin friction, Nusselt number and sherwood number with the work of Rao et al. (2013) by setting $A_1 = Du = \delta = \Delta = Sr = 0$ and that of Alao et al. (2016) by setting $A_1 = \delta = \Delta = 0$. Clearly, the results in table 4.2 and table 4.3 are in good agreement with that of Rao et al. (2013) and Alao et al. (2016). Table 4.4 presents the result obtained by varying different parameters for values of skin friction coefficient, Nusselt number, and Sherwood number. It is shown in table 4.4 that as Gr increases, there is a hike in the skin friction coefficient. A hike in the Prandtl number also gives a decrease to both the skin friction coefficient and Nusselt number. It is seen from the table 4.4 that a rise in the radiation parameter results to a hike in the skin friction and Nusselt number. Radiation parameter enhances convective flow because increase in Rr improves the thermal condition of the fluid environment. From table 4.4, increase in the values of Dufour number Du increases the Nusselt number (Nu) while increase in the values of Soret number increases the Sherwood number (Sh). Increase in the values of Soret number and Dufour number brought increase to the skin friction coefficient.It is noticed in table 4.4 that increase in the values of the viscous dissipative term (Eckert number) enhances the hydrodynamics and thermal boundary layer thickness by increasing the coefficient of skin friction and the Nusselt number.



(a) velocity profile (b) temperature profile

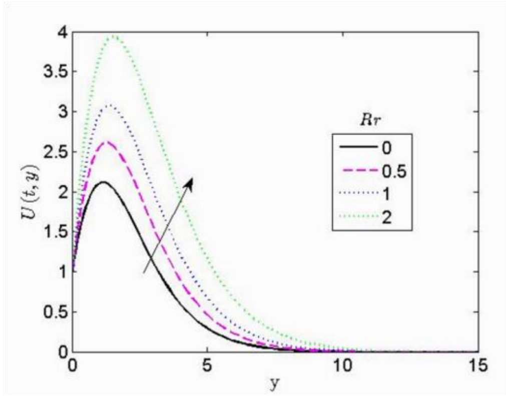


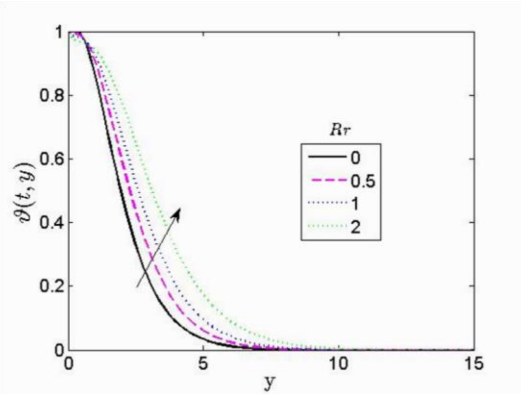
(c) concentration profile

Figure 4.1: Effect of Prandtl number Pr on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, Rr = A = kr =$

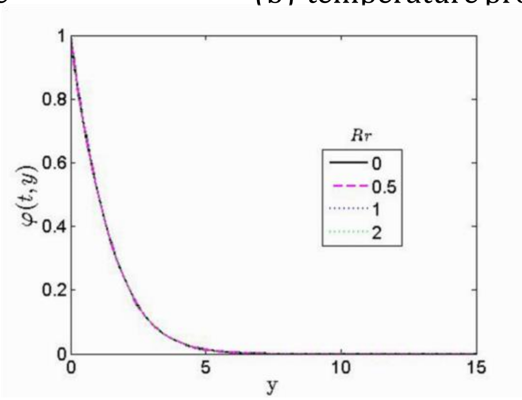
$0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02$ and

$\epsilon = 0.001$





(a) velocity profile (b) temperature profile

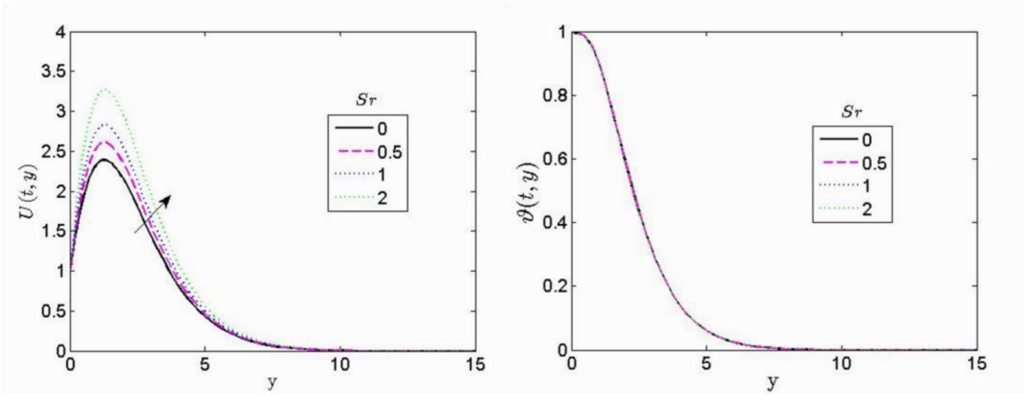


(c) concentration profile

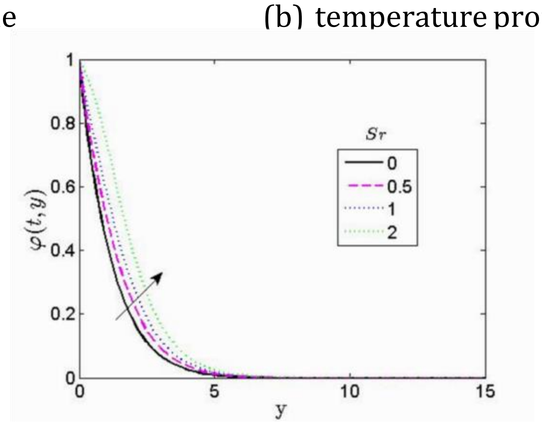
Figure 4.2: Effect of radiation parameter Rr on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr =$

$0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr =$

0.71 and $\epsilon = 0.001$

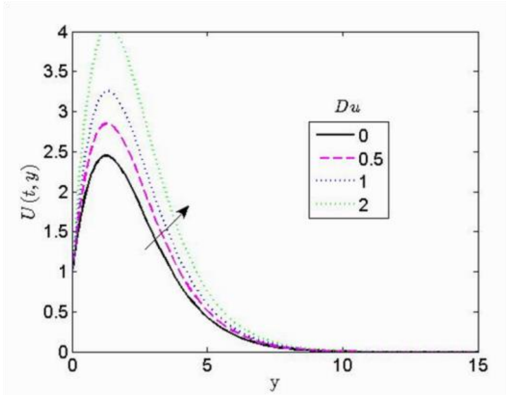


(a) velocity profile (b) temperature profile

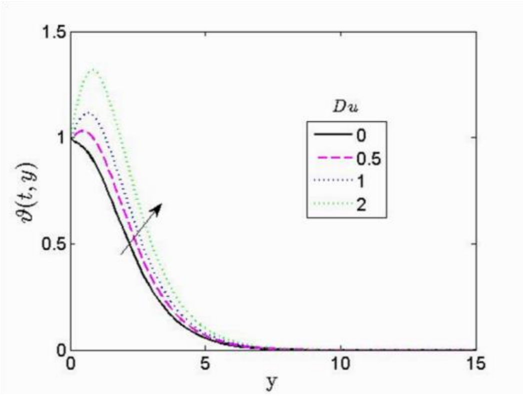


(c) concentration profile

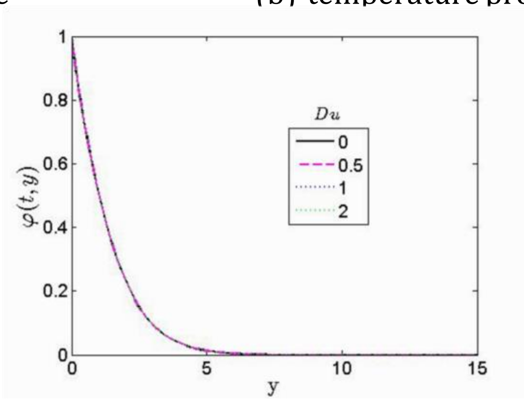
Figure 4.3: Effect of Soret number Sr on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$



(a) velocity profile

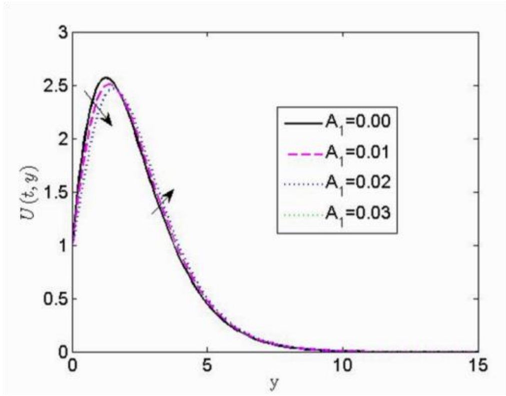


(b) temperature profile

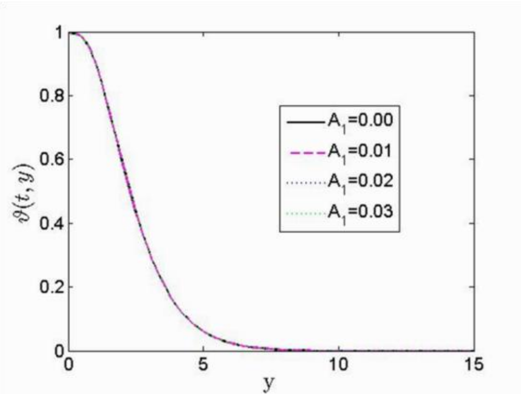


(c) concentration profile

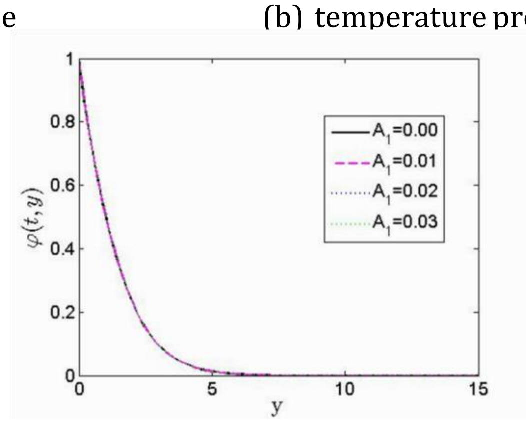
Figure 4.4: Effect of Dufour number Du on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0$, $Sc = 0.61$, $Gr = Gm = 2.0$, $A = kr = 0.5$, $t = 1$, $n = 0.5$, $Du = 0.2$, $Sr = 0.5$, $Ec = 0.01$, $A_1 = 1.0$, $\delta = 0.3$, $\Delta = 0.02$, $Pr = 0.71$ and $\epsilon = 0.001$



(a) velocity profile



(b) temperature profile

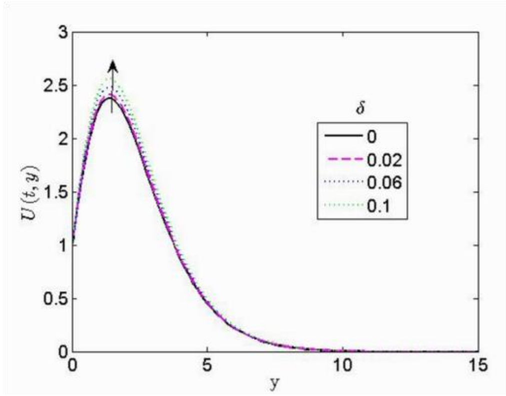


(c) concentration profile

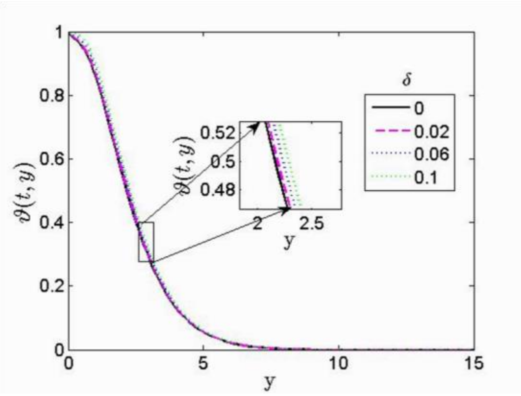
Figure 4.5: Effect of viscoelastic parameter α on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr =$

$0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr =$

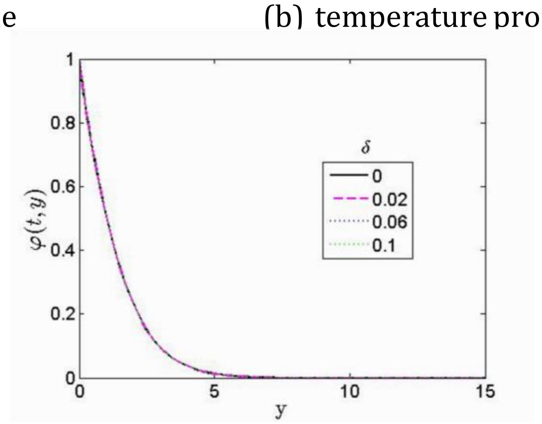
0.71 and $\epsilon = 0.001$



(a) velocity profile

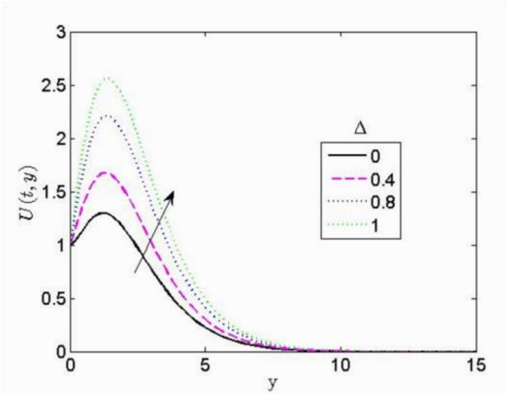


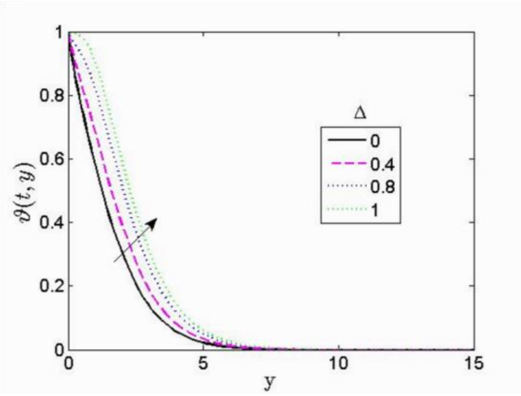
(b) temperature profile



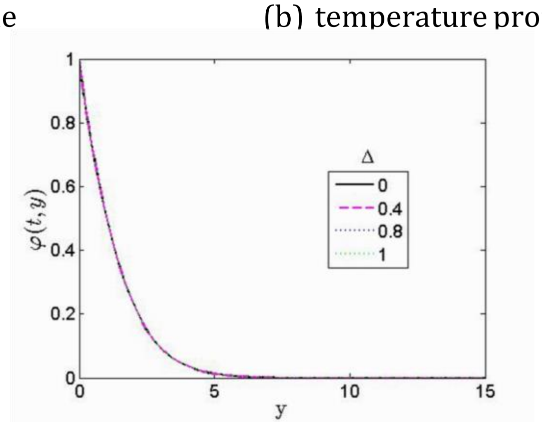
(c) concentration profile

Figure 4.6: Effect of heat source/sink parameter δ on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$





(a) velocity profile (b) temperature profile

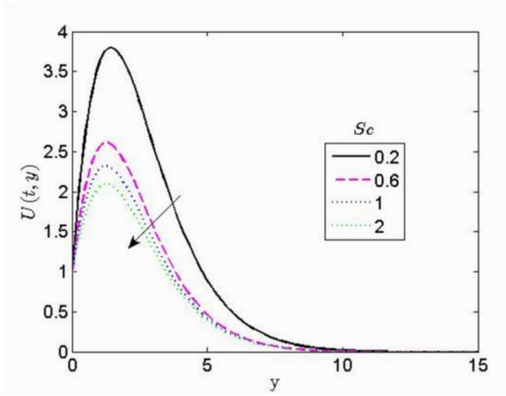


(c) concentration profile

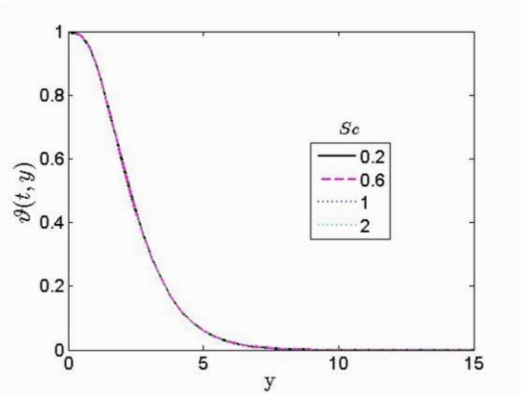
Figure 4.7: Effect of heat generation/absorption parameter Δ on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm =$

$2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta =$

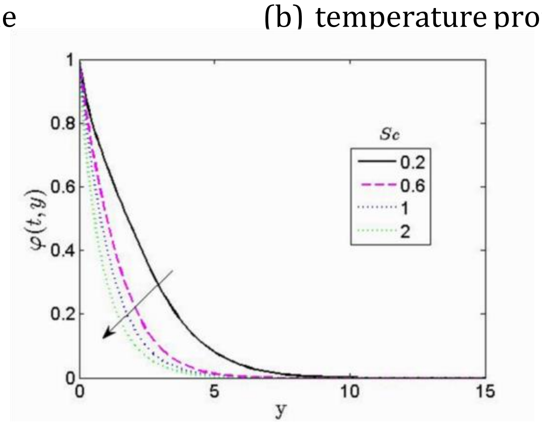
$0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$



(a) velocity profile

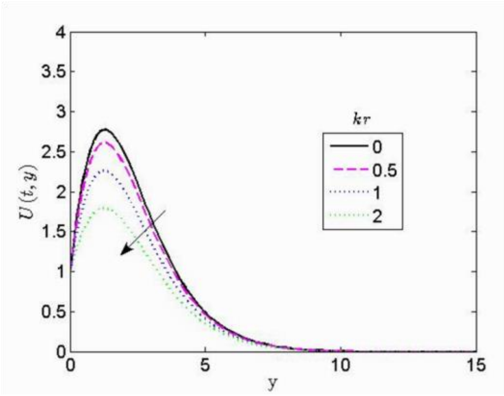


(b) temperature profile

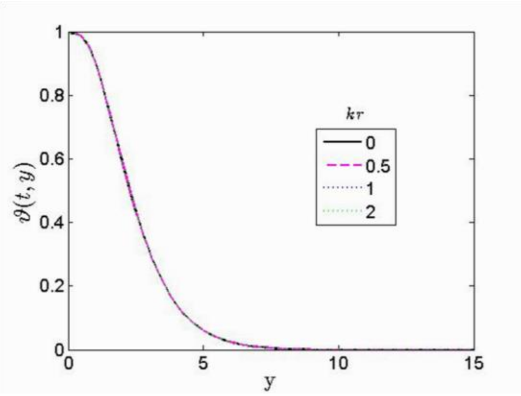


(c) concentration profile

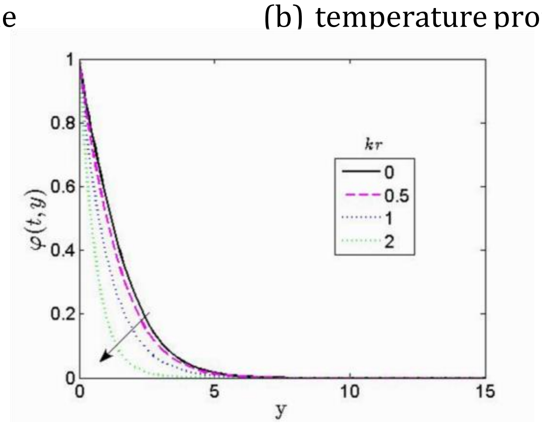
Figure 4.8: Effect of Schmidt number Sc on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$



(a) velocity profile

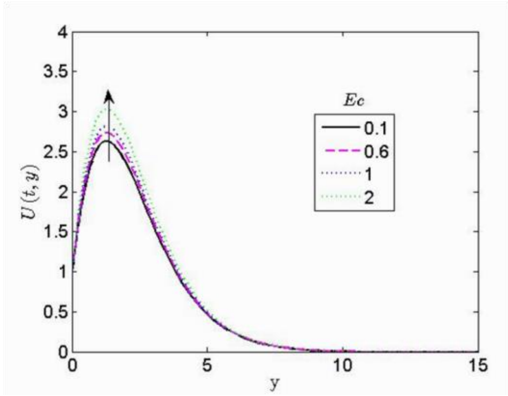


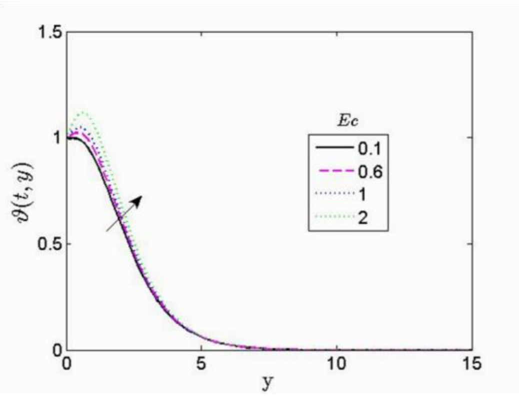
(b) temperature profile



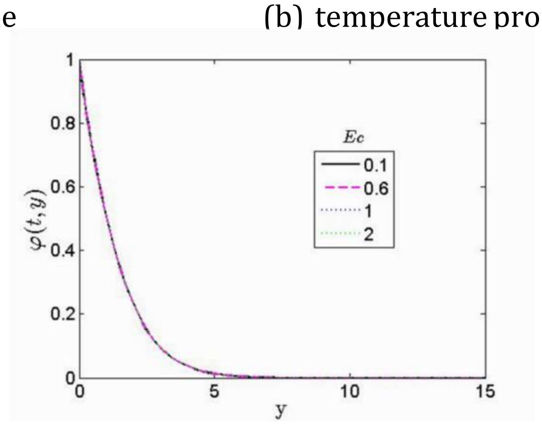
(c) concentration profile

Figure 4.9: Effect of chemical reaction parameter kr on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$



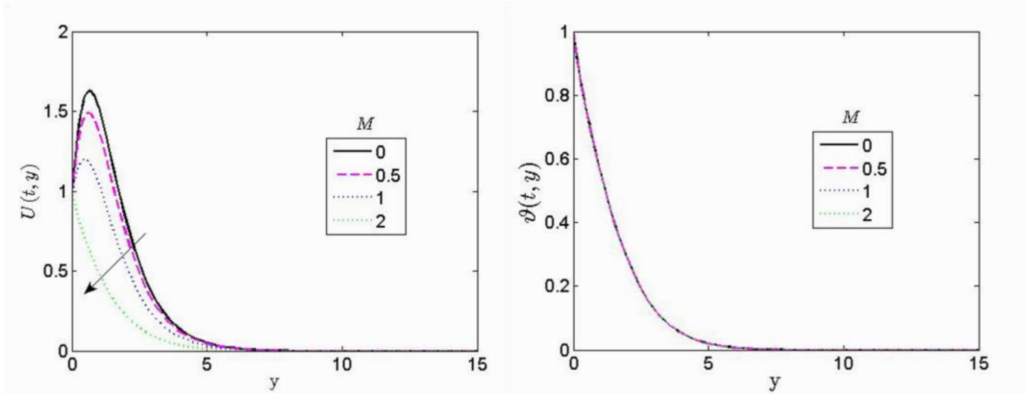


(a) velocity profile (b) temperature profile

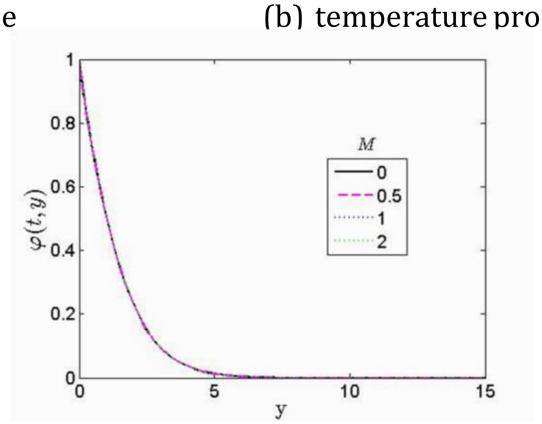


(c) concentration profile

Figure 4.10: Effect of Eckert number Ec on the velocity, temperature and concentration profiles when $M = 1.0, Sc = 0.61, Gr = Gm = 2.0, A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02, Pr = 0.71$ and $\epsilon = 0.001$

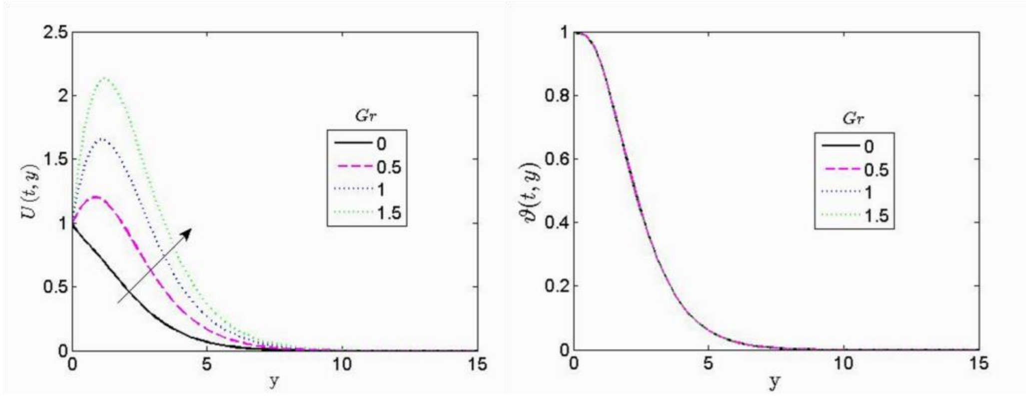


(a) velocity profile (b) temperature profile

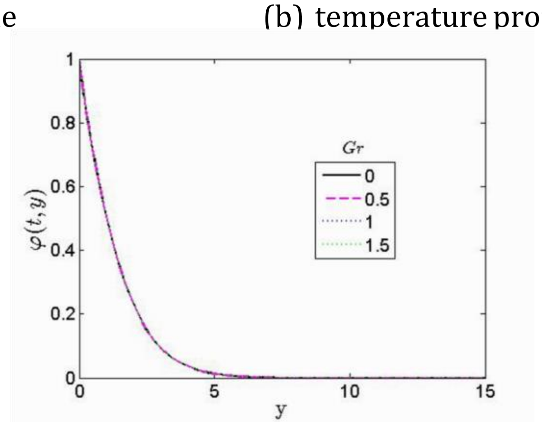


(c) concentration profile

Figure 4.11: Effect of Magnetic parameter M on the (a) velocity (b) temperature and (c) concentration profiles when $Pr = 0.71, Sc = 0.61, Gr = Gm = 2.0, Rr = A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02$ and $\epsilon = 0.001$

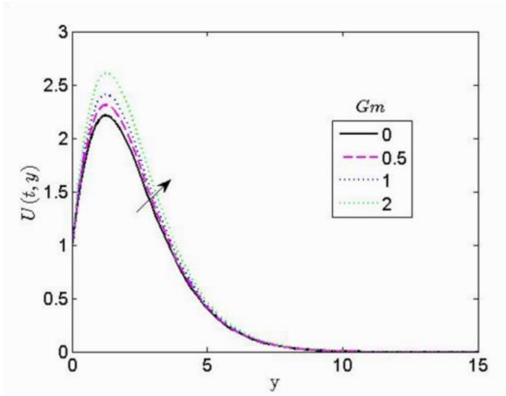


(a) velocity profile (b) temperature profile

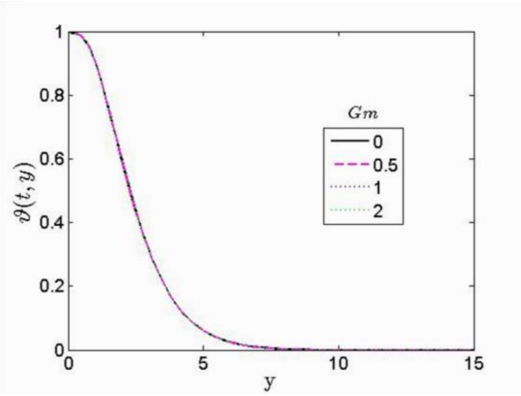


(c) concentration profile

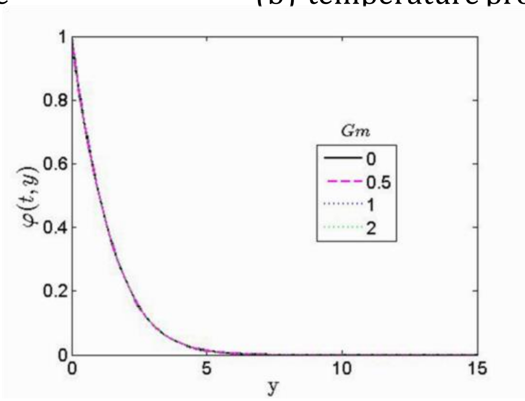
Figure 4.12: Effect of thermal Grashof number Gr on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Gm = 2.0, Pr = 0.71, Rr = A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta = 0.02$ and $\epsilon = 0.001$



(a) velocity profile



(b) temperature profile



(c) concentration profile

Figure 4.13: Effect of mass Grashof number Gm on the (a) velocity (b) temperature and (c) concentration profiles when $M = 1.0, Sc = 0.61, Pr = 0.71, Gr = 2.0, Rr =$

$A = kr = 0.5, t = 1, n = 0.5, Du = 0.2, Sr = 0.5, Ec = 0.01, A_1 = 1.0, \delta = 0.3, \Delta =$

0.02 and $\epsilon = 0.001$

Table 4.1: Comparison of computational values for Sherwood number (Sh) for different values of Sc when $R = Ec = Du = \Delta = kr = Sr = 0, Gr = 2.0, Gm = 2.0, M = 1.0, Pr = 0.71, \delta = 0.1, A_1 = 0.1.$

for validation of present work

	Present Study	Chandra et al. (2015)	Mishra et al. (2013)
Sc	Sh	Sh	Sh
0.22	0.2202	0.2201	0.219095
0.3	0.3002	0.3001	0.298966
0.66	0.6602	0.6601	0.658814
0.78	0.7804	0.7802	0.776574

Table 4.2: Computation values for skin-friction coefficient C_f and Sherwood number for various values of thermal radiation parameter compared with Rao et al. (2013)

when $A_1 = Du = \delta = \Delta = Sr = 0$

for validation of the present work.

	Present Study		Rao et al. (2013)	
Sc	C_f	Sh	Cf	Sh
0.22	3.1066	0.4512	3.1068	0.4515
0.60	2.4546	0.8429	2.4548	0.8431
0.78	2.2763	1.0211	2.2767	1.0214
0.94	2.1538	1.1742	2.1540	1.1745

Table 4.3: Computation values for skin-friction coefficient Cf and Nusselt number for various values of thermal radiation parameter compared with Alao et al. (2016)

when $A_1 = \delta = \Delta = 0$

	Present Study		Alao et al. (2016)	
R	C_f	$\theta^0(0)$	C_f	$\theta^0(0)$
0.0	2.1692	0.8291	2.1693	0.8291
0.5	2.4656	0.6154	2.4657	0.6154
1.0	2.6545	0.5087	2.6546	0.5087
2.0	2.9038	0.4019	2.9039	0.4019

Table 4.4: Computational values for skin friction coefficient (Cf), Nusselt number

(Nu), and sherwood number (Sh) for different values of Gr,Rr,Pr,Ec,Du and Sr

Parameters						Present Work		
Gr	Pr	Rr	Ec	Du	Sr	Cf	Nh	Sh
0.0	0.71	0.5	0.01	0.2	0.3	1.4626	0.6332	0.6993
0.5	0.71	0.5	0.01	0.2	0.3	1.4888	0.6332	0.6993
1.0	0.71	0.5	0.01	0.2	0.3	1.5151	0.6332	0.6993
2.0	0.71	0.5	0.01	0.2	0.3	2.9082	0.3527	0.6993
2.0	1.00	0.5	0.01	0.2	0.3	2.0129	0.7245	0.6993
2.0	3.00	0.5	0.01	0.2	0.3	1.5806	0.6377	0.6993
2.0	0.71	0.0	0.01	0.2	0.3	1.1436	0.5886	0.6993
2.0	0.71	0.5	0.01	0.2	0.3	1.5806	0.6377	0.6993

2.0	0.71	1.0	0.01	0.2	0.3	2.0831	0.7445	0.6993
2.0	0.71	0.5	0.01	0.2	0.3	1.3570	0.5092	0.6993
2.0	0.71	0.5	0.6	0.2	0.3	1.4465	0.5607	0.6993
2.0	0.71	0.5	1.0	0.2	0.3	1.5584	0.6250	0.6993
2.0	0.71	0.5	0.01	0.0	0.3	0.9572	0.4174	0.6993
2.0	0.71	0.5	0.01	0.5	0.3	1.3468	0.5551	0.6993
2.0	0.71	0.5	0.01	1.0	0.3	1.7364	0.6928	0.6993
2.0	0.71	0.5	0.01	0.2	0.0	1.2097	0.6377	0.5353
2.0	0.71	0.5	0.01	0.2	0.5	1.5806	0.6377	0.6993
2.0	0.71	0.5	0.01	0.2	1.0	1.9514	0.6377	0.8633

4.2 Results and discussion of research problem two

The set of coupled fourth order total differential equations (3.205)-(3.207) along with the constraints (3.208) and (3.209) have been profound solution numerically by utilizing SHAM. The contribution of changing the pertinent parameters on concentration ($\varphi(\eta)$), velocity $\left(\alpha\left(1+\frac{1}{\beta}\right)f(\eta)\right)$ and temperature $\theta(\eta)$ plots are depicted

using diagrams while Sherwood, Nusselt and coefficient of skin friction are tabulated. The impact of Casson liquid term on concentration, temperature and velocity are illustrated in figure 4.14. The physics of the problem for a case of constant thermal conductivity and viscosity, that is $\gamma = \xi = 0$, an increment in the fluid viscosity near to the plate owing to higher Casson liquid term (β) and lessens far from the plate. Too much of viscosity leads to degeneration in velocity. The outcome in figure 4.14 is for higher injection of thermal conductivity and viscosity to the motion of fluid, that is, $\gamma = \xi = 3.0$. However, raising β resist the fluid motion owing to the fact that, increase in β lead to reduction in the yield stress P_y of the Casson term degenerates and hereby leads to enhancement in the plastic dynamic viscosity. It is detected in figure 4.14 that the injected temperature owing to $\gamma = \xi = 3.0$ enhances the velocity near the plate alongside temperature of the thermal and hydrodynamic layer. Figure 4.14 illustrate the increment in the Casson term is negligible on the concentration plot. Figure 4.15 depicts the contribution of Walters-B term (Weissenberg number) on plots of concentration, velocity and temperature. The viscoelastic term A_2 connotes the contribution of normal stress on the flow. At a level in the flow domain, an increment in the velocity near the plate at the ambient vicinity for a case of varying thermal conductivity and viscosity. Physically, an increment in Walters-B term (A_2) contributes to the flow by enhancing the temperature plot but enhances the temperature, variable viscosity, thermal layer and variable thermal conductivity when the value of A_2 increases. Viscoelasticity portrays viscous and elastic features when there is occurrence of deformation. It worth noting that viscosity brings resistance to flow, hence properties of viscous in viscoelastic term tends to lessen the hydrodynamic layer for a constant thermal conductivity and viscosity. In otherwords, the varied viscosity γ = 3.0 assist the fluid flow to posses the loss energy and enhances the velocity. The thermal conductivity was injected ξ = 3.0 added more energy to the fluid temperature and thereby increase the thermal layer thickness by boasting the temperature plot as depicted in figure 4.15. The moment Casson and Walters-B liquid are mixed together and motion into the thermal and hydrodynamics layer over the penetrable wall, the Walters-B liquid will lessens the temperature and concentration. In addition, an enhancement in fluid velocity is noticeable as the two fluids mixed together. The proportion of Casson liquid term is more than Walters-B liquid.

The contribution of the two non-Newtonian liquid term explored in this research are plotted in figures 4.14 and 4.15. the variable conductivity and viscosity added to the energy and momentum equations (3.157 and (3.158) accelerate the velocities close to the plate and lessens the temperature plots and the whole thermal layer as illustrated in figures 4.14 and 4.15. Higher Casson liquid term in the hydrodynamic boundary layer becomes more than the Walters-B liquid term near the plate as illustrated in figure 4.14 and 4.15 due to γ = ξ = 3.0, the plastic dynamic viscosity lessens and lead to fluid motion enhancement. The thickness of thermal layers increases owing to elasticity stress term enhancement. In figures 4.14 and 4.15, the concentration plots are observed to slightly enhance far from the plate because of higher β and A_2. The concentration of fluid at the ambient vicinity where there is hotness as depicted in figure 3.1 enhances owing to the heat in this vicinity. The Casson and the Walters-B (tomato sauce and concentrated fruit juice) liquid lessens concentration at this hot vicinity. The contribution of Dufour-Soret are explored separately in this research. They portrays the diffusion-thermal and thermal-diffusion contribution in the research. The plots of concentration, temperature and velocity for distinct values of the Soret term (So) are illustrated in figure 4.16. The Soret phenomenon explains the temperature gradient contribution while varying concentration. Obviously, the entire hydrodynamic layer and velocity degenerates while raising So. This is owing to the variation of thermal conductivity which lowers the amount of thermal diffusion. In addition, the concentration as illustrated in figure 4.16 enhances near the plate but lessens far away from the plate. So is detected to be negligible on the entire thermal layer and temperature. The physics of the thermal layer posses high temperature owing to the hotness of the vicinity at the ambient. Hence, the concentrated fruit hereby mixed with Walters-B liquid at the layer and moves faster at the ambient boundary layer.

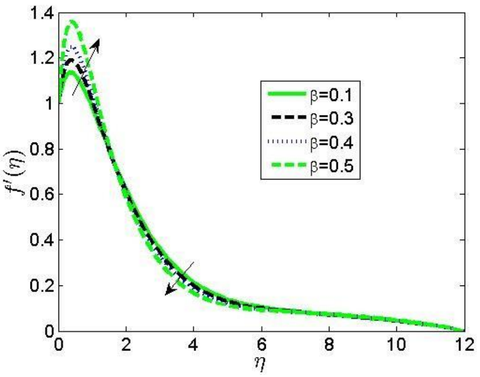
The variation of Dufour term (D_f) on the temperature, velocity and concentration distribution is depicted in figure 4.17. The Dufour phenomenon explains the impact of concentration gradients on the temperature as seen in equation (3.158). It acts as an assistance to the flow and capable of enhancing the thermal energy within the layers. This is illustrated in figure 4.17 for higher value of D_f, the temperature plot accelerates. It was detected that the contribution of Dufour (diffusion-thermo) greatly impacted the fluid temperature. Near the plate, higher D_f decreases the velocity. It was detected that higher temperature at the ambient vicinity raises the fluid velocity. Figure 4.18 illustrate the contribution of the permeability term (Ps) on the concentration, velocity and temperature plots. It is observed that the velocity plot degenerates as the porosity term increases. This

is correct because an additional resistance to the wall is owing to fluid motion and hereby retardate the hydrodynamic boundary layer. It hereby brings enhancement to the thermal layer as depicted in figure 4.18. this outcome is in excellent agreement with Salawu and Dada (2016). Experimentally, as the porosity term increases, it gives rooms for motion of the Walters-B and Casson liquids. The heat keeps rising at the free stream where there is hotness, the contribution of the varied thermal conductivity alters the temperature distribution. It hereby lowers the velocity and brings enhancement to the fluid temperature. After, the fluid moves over the penetrable vertical plate, the liquid becomes concentrated and the boundary layer becomes very thick. This is because the proportion of the Casson liquid term is more than the Walters-B liquid.
The concentration of the Casson liquids degenerates as it mixed together with the Walters-B liquid which is in the form of Water. Also, the thermal conductivity and viscosity varying on the boundary layer in correlation with the hot vicinity.

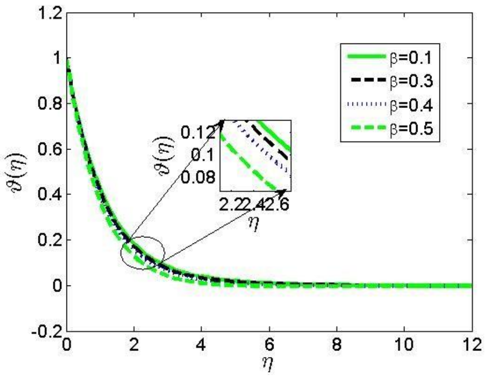
Figure 4.19 illustrate the concentration, velocity and temperature plots with distinct values of radiation term (Ra). Radiation term enhances convective motion (Idowu and Falodun; 2018). As the intensity of Ra rises, it is detected that velocity field and the who hydrodynamic boundary layer enhances. This outcome is in excellent agreement with the outcomes of Arifuzzaman et al. (2018). Because the thermal conductivity and viscosity varies coupled with the hot vicinity in this research, the velocity degenerates at the ambient environment. The temperature depicted in figure 4.19 enhances owing to the double non-Newtonian liquids explored in this research cooled down both the thermal layer and temperature at the wall but greatly affect the free stream. Practically, it implies the thermal energy has great impact on the flow regime. Owing to this fact, it is finalized that the contribution of radiation is importance as Ra 6= 0 and Ra −→∞. The contribution of Ra on

the concentration plot is observed to slightly degenerate as illustrated in figure 4.19. The contribution of (Pr) on the concentration, velocity and temperature is shown in figure 4.20. Pr explains the relationship existing with thermal conductivity and kinematic viscosity. It portrays the momentum diffusivity divided by the thermal diffusivity. The Pr manage the thickening of the momentum as well as thermal layers in the phenomena of heat transport. Practically, any liquid with higher Prandtl number resulted to greater viscosities. Hence, it serves to degenerate the velocity and the entire hydrodynamic layer. However, liquids with lower Pr posses greater thermal conductivities and hereby thickening the structures of the thermal layer. This give chance for heat to diffuse very fast compare to higher Pr. Furthermore, Pr can served as a term for enhancing the cooling rate in a flow. Higher Pr leads to degeneration in the velocity plot. In addition, higher values of Pr lowers the temperature plot because as $Pr < 1$, the liquid is highly conducive. In figure 4.20 , a high values of Pr is observed to slightly raise the concentration plot near the plate and lessens far away from the plate. Varying the magnetic field term (M_p) on concentration, velocity, temperature field are plotted in figure 4.21. The imposed magnetism to an electrically conducting liquids originates is a drag-like Lorentz force. The Lorentz force causes the fluid velocity to degenerate within the boundary immediately the magnetism opposes the phenomena of flow. The opposing force produced by the imposed magnetism, a higher values of M_p is observed to slow down the velocity plot and enhances the temperature plot owing to the generated friction by Lorentz force. This friction with variable thermal conductivity and viscosity lessens the heat energy and hereby enhances the temperature field in the flow. It is seen from figure 4.21 that M_p does not alter the concentration plot. Figure 4.22 illustrate the contribution of thermal buoyancy force term (Gr) on concentration, temperature and velocity field. The dimensionless Gr number which explains the division of buoyancy to the viscous force acting on fluid particles (Alao et al.; 2016). The hydrodynamic layer and fluid velocity accelerated upward because Gr behaves like a force exerted by the fluid owing to the placement of an object. Hence, the pressure exerted increases the depth. The pressure felt at the bottom is much than the force at the top. The temperature plot is detected to degenerate owing to increase in the values of thermal buoyancy term (Gr). Thus, an upward net force which accelerate the entire hydrodynamic layer and velocity but degenerates the thermal layer and temperature of fluid. The contribution of Schmidt term (Sc) on the concentration, velocity and temperature plots are illustrated in figure 4.23. Sc means the division of kinematic viscosity to the mass diffusivity of the liquid, meaning $\frac{\nu}{D}$. Practically, $\nu > D$ means that Sc is high and vice versa. On the other hand, concentration buoyancy contribution lessens the rate of mass transport and hereby decreases the concentration plot near the plate and increases far from the plate. The outcomes in figure 4.23 portrays that the fluid viscosity is higher than the mass diffusivity. This is owing to the variable viscosity resulting to an enhancement in the velocity plot far from the plate. With higher values of Sc, no contribution is observed on the temperature plot.

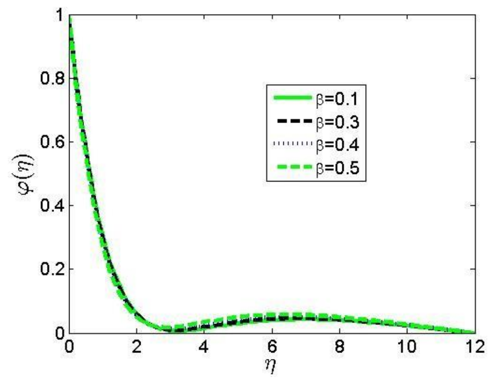
Table 4.5 illustrated the contribution of controlling terms Sc, M_p, A_2, Ra and Pr on the wall coefficient of skin friction (C_f), Nusselt (nu) and Sherwood (Sh). It is observed from table 4.5 that the skin friction enhances with higher Schmidt number. Also, a higher Schmidt term degenerates Sherwood and Nusselt number. In table 4.5, a higher magnetic term (M_p) shows degeneration in the wall skin friction coefficient and the Nusselt number while an enhancement in the Sherwood number is noticeable. Furthermore, a higher Walters-B term degenerates coefficient of skin friction but simultaneously enhances both the Sherwood and Nusselt number. In table 4.5, a higher thermal radiation term degenerates the Sherwood number and coefficient of skin friction. The radiation term is observed to accelerate the Nusselt number and hereby enhances the heat transport of the thermal layer shown in 4.5. Increase in the Prandtl term is detected to lessen the coefficient of skin friction and the Nusselt number shown in table 4.5. A higher values of Pr is observed to accelerate the Sherwood number in table 4.5. In table 4.6, the numerical calculations of numeric values of skin friction (C_f), Sherwood (Sh) and Nusselt number (Nu) for distinct values by varying thermal conductivity and viscosity. A higher values of varied conductivity ($\xi = 3.0$) shows degeneration in the skin friction and Nuseelt values. Also, fixing $\gamma = 3.0$ and raising the numeric values of ξ lead to skin friction degeneration alongside Sherwood number. A higher values of ξ with $\gamma = 3.0$ is detected to upsurge the Nusselt number in table 4.6. Table 4.7 indicates the presents outcomes in comparison to the work of Animasaun (2015) by setting $\delta_x = So = A_2 = 0$. The outcomes were in good correlation. This implies that the present outcomes is very correct. The contribution of Eckert number (Ec) on concentration, velocity, and temperature plots is illustrated in figure 4.24. The outcomes shows that a higher numeric value of Ec lessens velocity alongside temperature plots. Physically, Ec is derived from the simplification of viscous dissipation term added to the energy equation of motion. Ec symbolizes the relationship existing within kinetic energy as well as the enthalpy in the fluid motion (Idowu and Falodun, 2018). Higher values of Ec results to accelerated shear forces in the liquid. Owing to this increment in Ec alongside the varied thermal conductivity and viscosity, an elevation in velocity alongside temperature plots is noticed in figure 4.24. This is due to the fact that heat energy is gathered in the liquid owing to the frictional heating and resulted to elevation of the entire hydrodynamic and thickness of thermal layer. Furthermore, a higher values of Ec depicted in figure 4.24 degenerates the concentration plot near the plate and accelerates far from the plate. The contribution of thermophoretic term (τ) on the concentration, velocity and temperature plots is depicted in figure 4.25. In figure 4.25, the velocity plot degenerates close to the plate and accelerate far from the plate because of higher values of thermophoretic term has no contribution on the liquid temperature. Figure 4.25 implies that the concentration plot degenerates close to the plate and accelerate drastically at a far distance from the plate. Physically, an increment in thermophoresis, the solutal layer thickness elevates but degenerates the mass transport rate with the entire boundary layer thickness.



(a) velocity profile



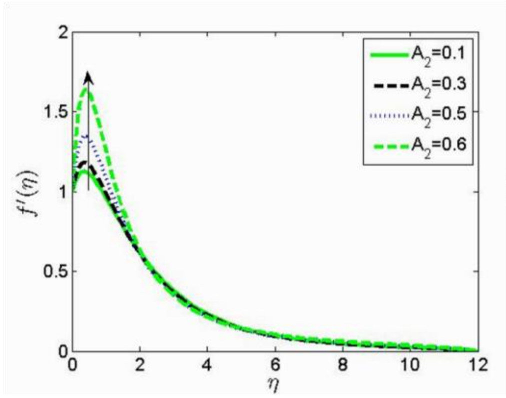
(b) temperature profile



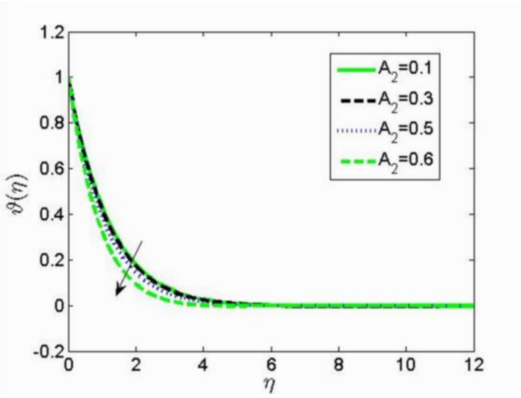
(c) concentration profile

Figure 4.14: Effect of β on the (a) velocity (b) temperature and (c) concentration

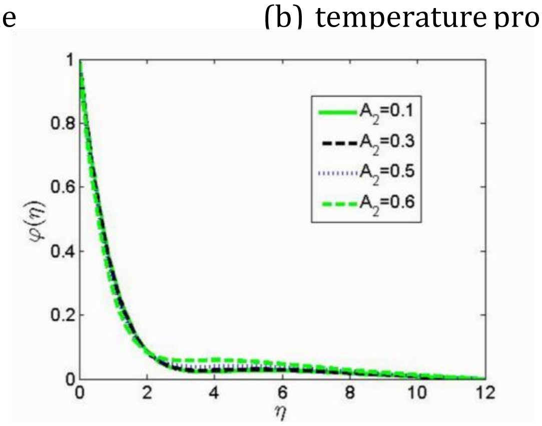
profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



(a) velocity profile



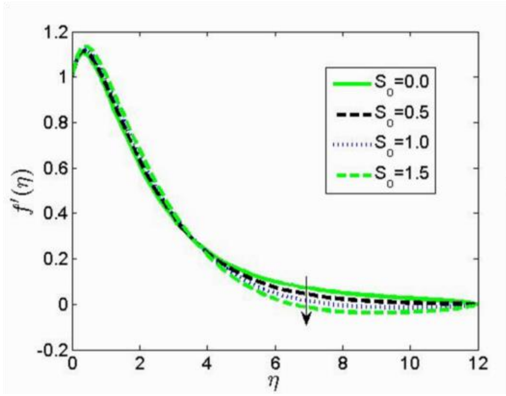
(b) temperature profile

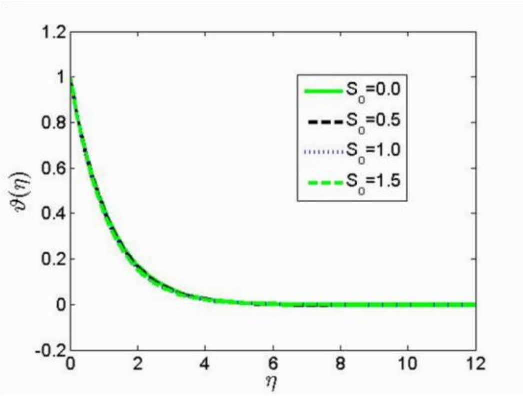


(c) concentration profile

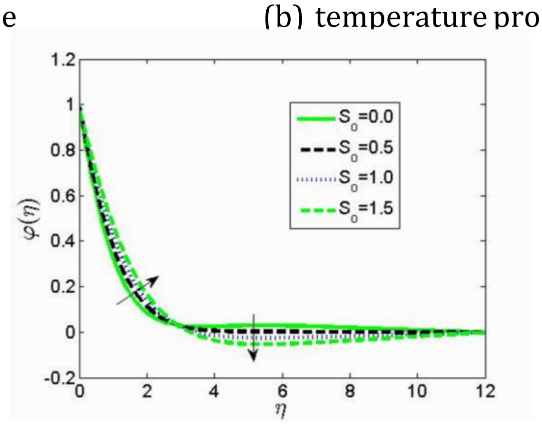
Figure 4.15: Effect of α on the (a) velocity (b) temperature and (c) concentration

profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



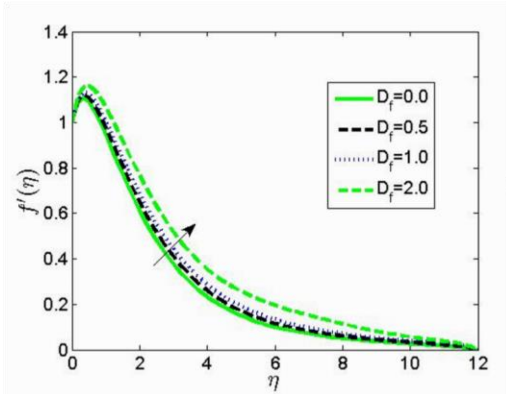


(a) velocity profile (b) temperature profile

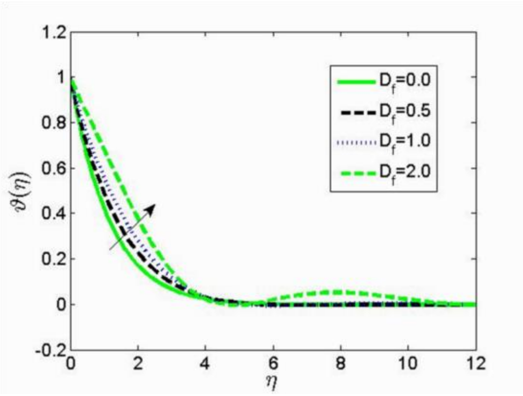


(c) concentration profile

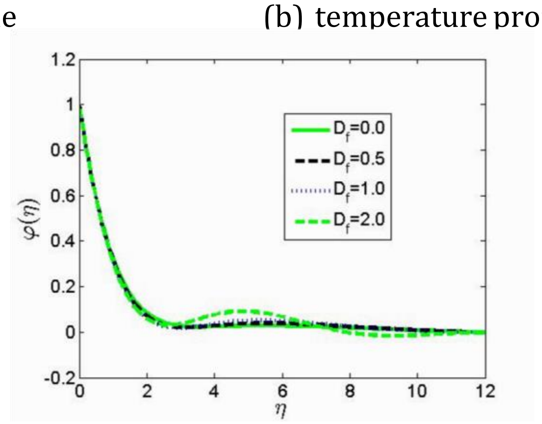
Figure 4.16: Effect of *So* on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



(a) velocity profile



(b) temperature profile



(c) concentration profile

Figure 4.17: Effect of Df on the (a) velocity (b) temperature and (c) concentration

profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$

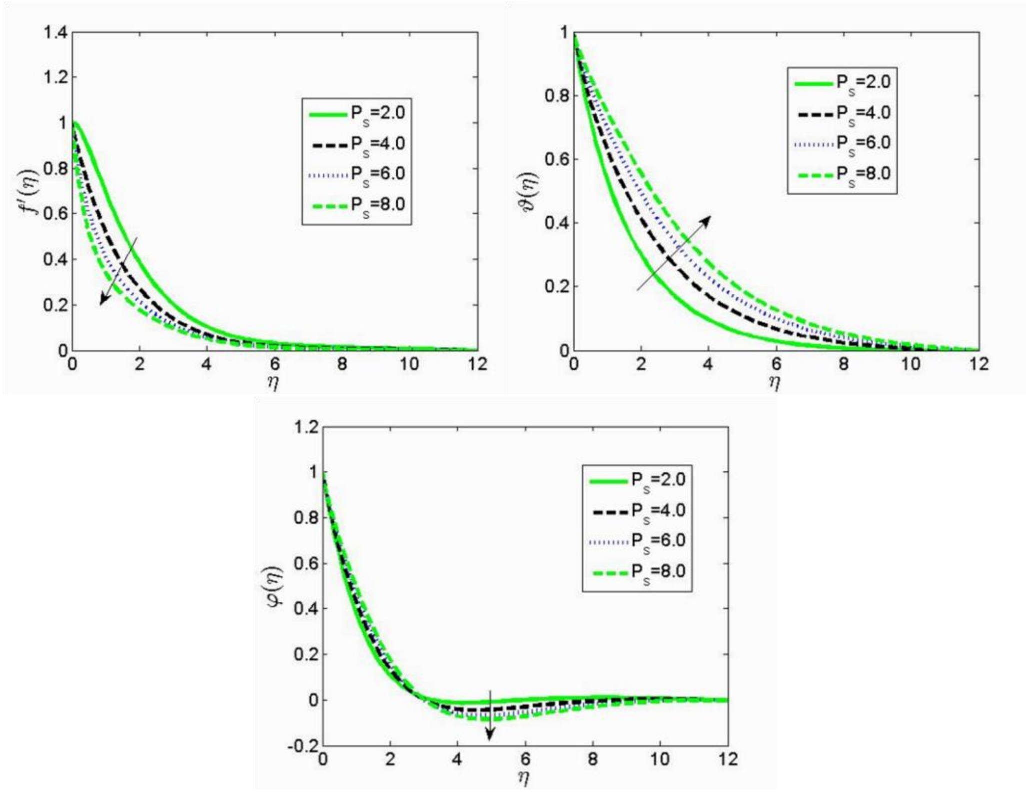
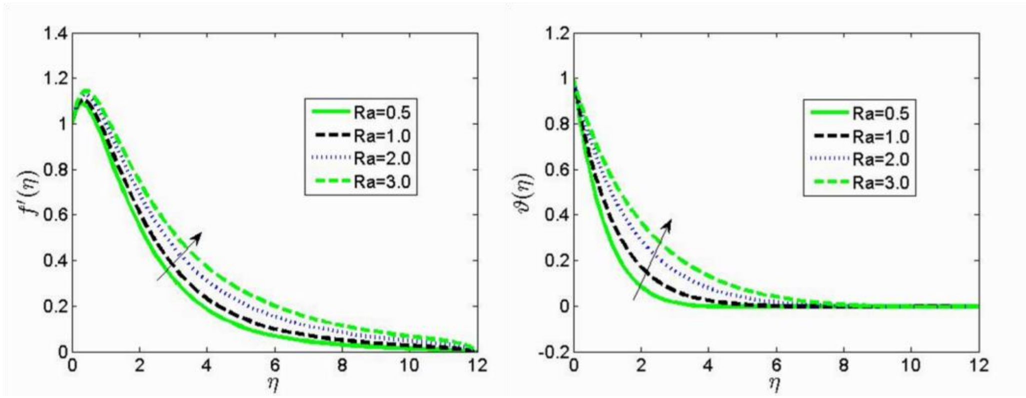
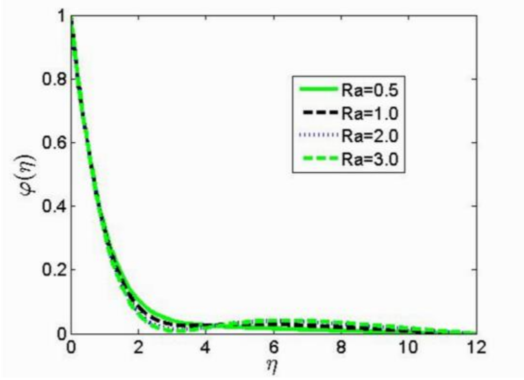


Figure 4.18: Effect of *Ps* on the (a) velocity (b) temperature and (c) concentration

profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



(a) velocity profile (b) temperature profile



(c) concentration profile

Figure 4.19: Effect of Ra on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$

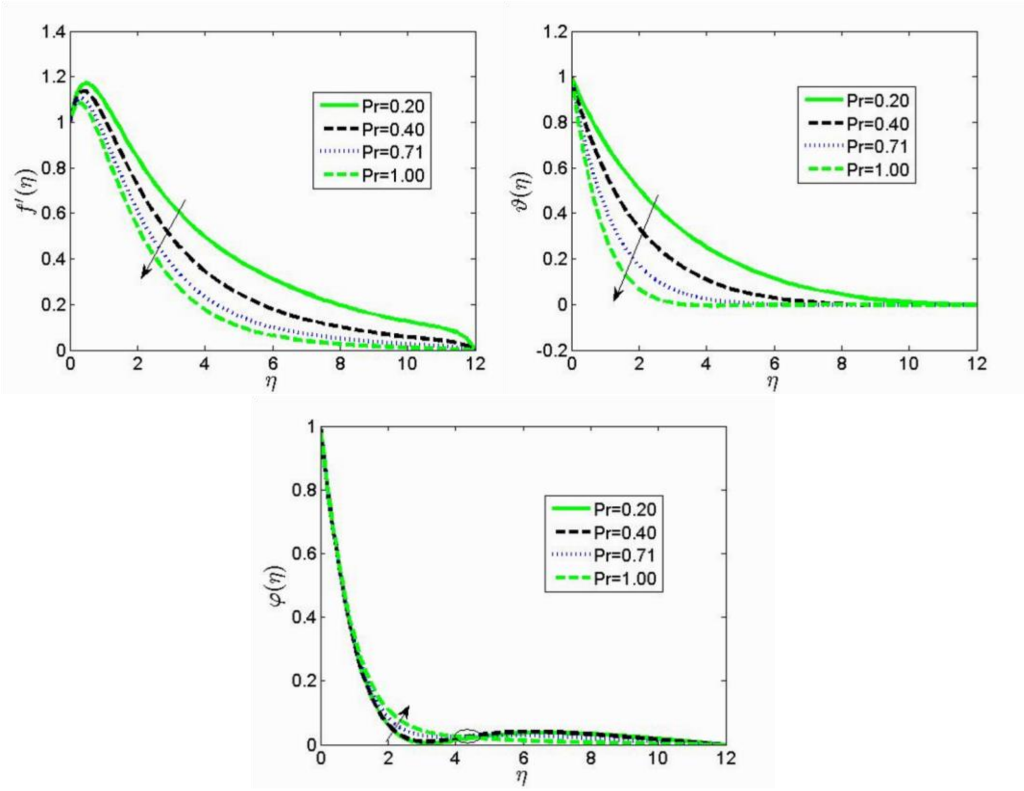
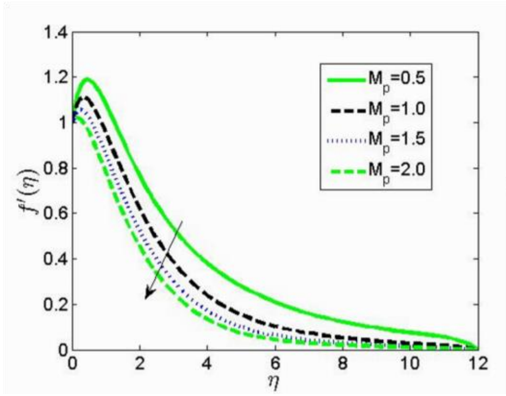
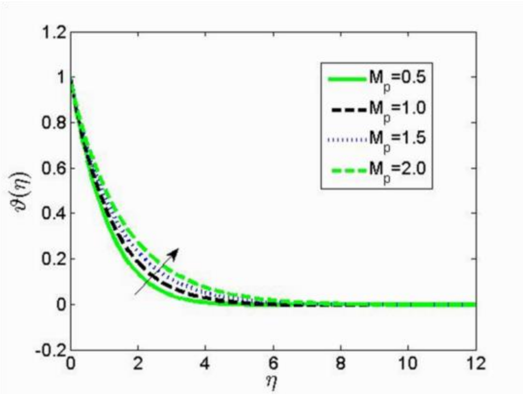


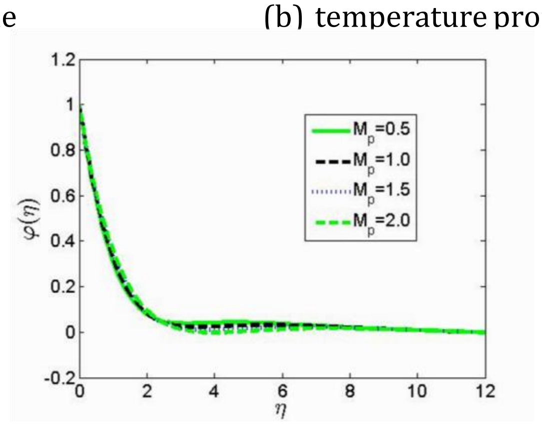
Figure 4.20: Effect of Pr on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



(a) velocity profile



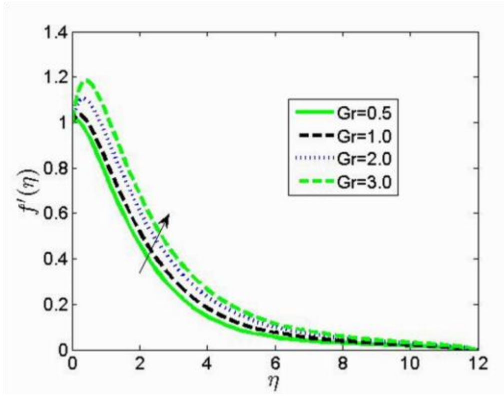
(b) temperature profile

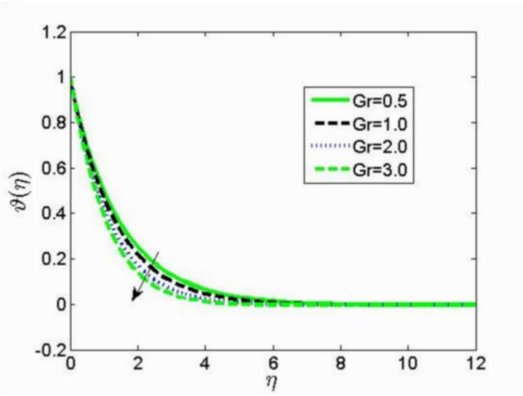


(c) concentration profile

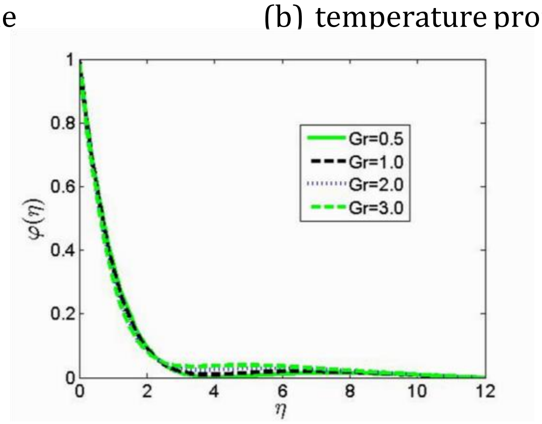
Figure 4.21: Effect of Mp on the (a) velocity (b) temperature and (c) concentration

profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



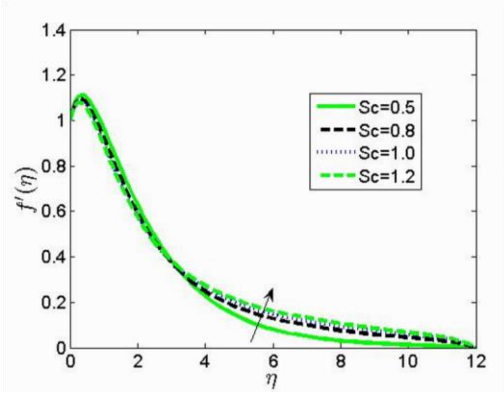


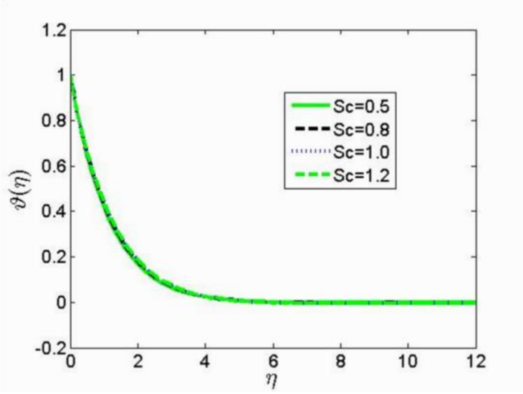
(a) velocity profile (b) temperature profile



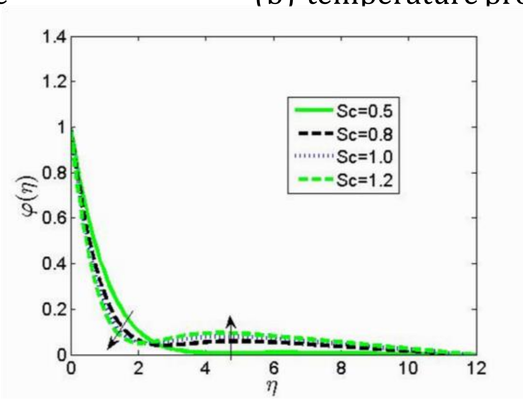
(c) concentration profile

Figure 4.22: Effect of Gr on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$





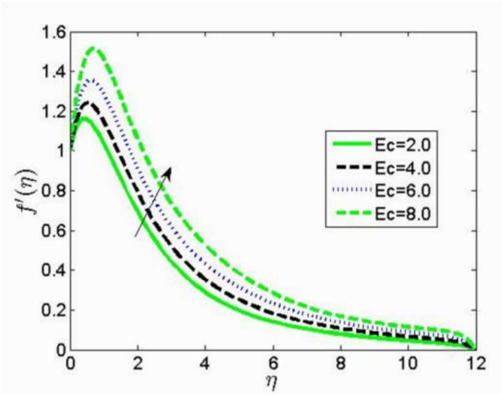
(a) velocity profile (b) temperature profile



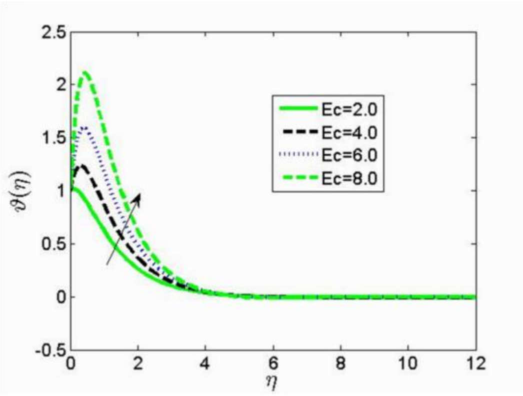
(c) concentration profile

Figure 4.23: Effect of Sc on the (a) velocity (b) temperature and (c) concentration

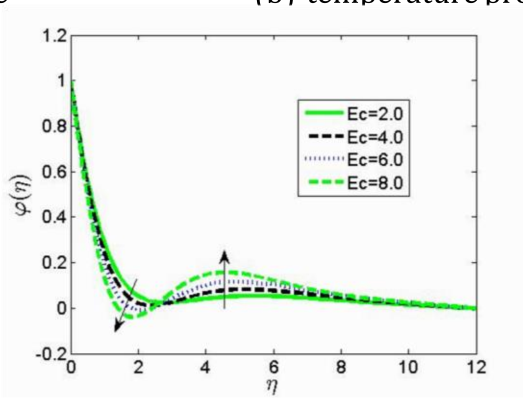
profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$



(a) velocity profile



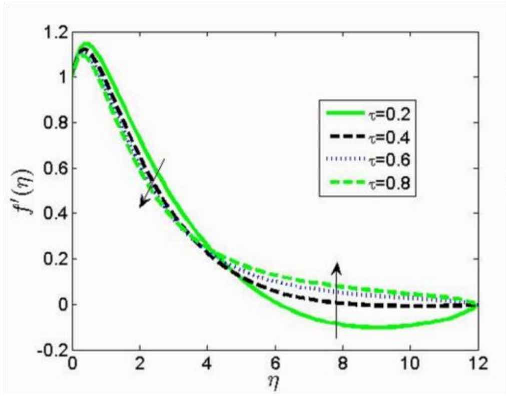
(b) temperature profile

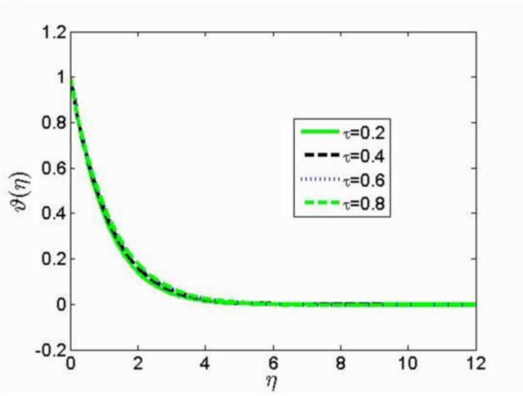


(c) concentration profile

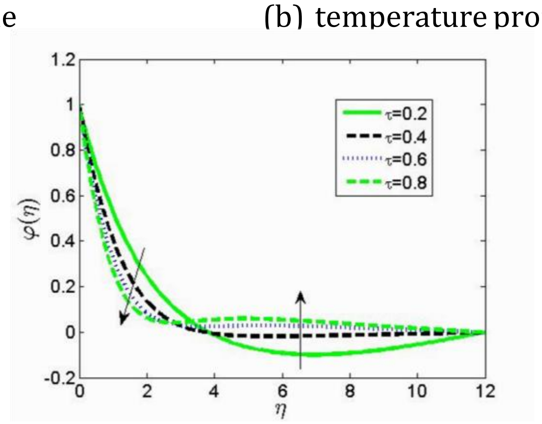
Figure 4.24: Effect of Ec on the (a) velocity (b) temperature and (c) concentration

profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$





(a) velocity profile (b) temperature profile



(c) concentration profiles

Figure 4.25: Effect of τ on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \gamma = \xi = 3.0, Gr = Gm = 2.0, M_p = \tau = f_w = 1.0, P_s = 0.6, A_2 = 0.1, Pr = 0.71, Ra = Cr = 0.5, Du = 0.3, Ec = 0.01, Sc = 0.61$ and $Sr = 0.2$

Table 4.5: Computational values for skin friction coefficient (C_f), Nusselt number (Nu), and sherwood number (Sh) for different values of Sc, M_p, A_2, Ra and Pr

	Parameters					Present Work	
Sc	M_p	A_2	Ra	Pr	Cf	Nh	Sh
0.61	1.00	0.10	0.50	0.71	1.82684215	1.25970112	0.04107226
1.00	1.00	0.10	0.50	0.71	1.83278163	1.22150067	0.04019208
2.00	1.00	0.10	0.50	0.71	1.83680754	1.19318509	0.03836566
0.61	0.00	0.10	0.50	0.71	1.82838919	1.184009703	0.03753980
0.61	0.51	0.10	0.50	0.71	1.72143962	1.18387211	0.03762533
0.61	1.00	0.10	0.50	0.71	1.63487194	1.18365490	0.03770776
0.61	1.00	0.10	0.50	0.71	1.98857346	1.18000334	0.03999742
0.61	1.00	0.30	0.50	0.71	1.57852676	1.17537798	0.04193522

0.61	1.00	0.50	0.50	0.71	1.41854012	1.16937701	0.04445257
0.61	1.00	0.10	0.50	0.71	1.83709818	1.14481507	0.09418621
0.61	1.00	0.10	1.00	0.71	1.83742769	1.15751245	0.07588727
0.61	1.00	0.10	2.00	0.71	1.68617411	1.18474433	0.03400623
0.61	1.00	0.10	0.50	0.20	1.71018187	0.12205103	1.52846420
0.61	1.00	0.10	0.50	0.40	1.68436341	0.02477757	1.69314037
0.61	1.00	0.10	0.50	0.71	1.63902616	0.01839299	1.90178334

Table 4.6: Computational values for skin friction coefficient (C_f), Nusselt number (Nu), and sherwood number (Sh) for different values of γ and ξ

γ	ξ	C_f	Nh	Sh
0.3	3.0	1.81056406	1.18653737	0.03808702
0.9	3.0	1.48607432	1.17522966	0.04233343
1.3	3.0	1.07528692	1.17458078	0.04245124
3.0	0.5	1.84218412	1.13768490	0.10847605
3.0	1.0	1.84093374	1.15266546	0.08561126
3.0	1.5	1.83955706	1.16811347	0.06198417

Table 4.7: Computation values for skin-friction coefficient Cf and Nusselt number for various values of Casson parameter, variable viscosity and thermal conductivity compared with Animasaun (2015) when $\delta_x = S_o = A_2 = 0$

	Animasaun (2015)			Present work		
	$-\left(1+\frac{1}{\beta}\right)f''(0)$	$-\theta'(0)$	$-\phi'(0)$	$-\left(1+\frac{1}{\beta}\right)f''(0)$	$-\theta'(0)$	$-\phi'(0)$
$\gamma=\xi=0.0$ $\beta=0.5$	0.9898295467	0.6049493121	0.2737555927	0.9898295462	0.6049493120	0.2737555925
$\gamma=\xi=0.0$ $\beta=1.0$	1.1585261426	0.5852503689	0.2601006893	1.1585261421	0.5852503686	0.2601006891
$\gamma=\xi=0.0$ $\beta=1.5$	1.2512959963	0.5741050139	0.2535407239	1.2512959959	0.0.5741050137	0.2535407237
$\gamma=\xi=0.0$ $\beta=2.0$	1.3104060264	0.5670520859	0.2496697911	1.3104060261	0.5670520857	0.2496697910
$\gamma=\xi=3.5$ $\beta=0.5$	0.2388807502	0.2644062864	0.3744355776	0.23888070501	0.2644062863	0.3744355775
$\gamma=\xi=3.5$ $\beta=1.0$	0.6054192627	0.2761129813	0.3891553685	0.6054192625	0.2761129812	0.3891553684
$\gamma=\xi=3.5$ $\beta=1.5$	0.7979598336	0.2809251967	0.3953339760	0.7979598332	0.2809251966	0.3953339758
$\gamma=\xi=3.5$ $\beta=2.0$	0.9176634348	0.2835728742	0.3987392025	0.9176634344	0.2835728743	0.3987392023

4.3 Results and discussion of the research problem three

The transformed governing equations (3.317)-(3.319) subject to the constraints (3.320) and (3.321) are coupled set of highly non-linear total differential equations. The coupled set of equations were numerically solved by utilizing SHAM. To explore the contribution of various key parameters such as radiation, Soret-Dufour, chemical reaction, heat generation, Casson parameter etc, a rigorous computation were carried out numerically.

Figure 4.26 depicts the contribution of Casson fluid term (β) on the concentration, temperature and velocity plots. With a high value of β, it is detected in figure 4.26 that rate of transportation is lessens within the thermal layer. A slight degeneration in the temperature plot is detected as the value of β is raised. It worth noting that an increment in Casson term close to infinity, the fluid acts like a Newtonian. Owing to increment in the elasticity stress term, a thick nature of the thermal layer is detected. Figure 4.27 explains the contribution of Dufour term (D_f) on the concentration, velocity and temperature plots. It is detected that a higher values of D_f elevate the thermal boundary layer, temperature, velocity and hydrodynamics layer. A higher values of D_f is discovered to be negligible on the concentration plot. The diffusion-thermal in the analysis contributes greatly to the temperature plot and thermal boundary layer.

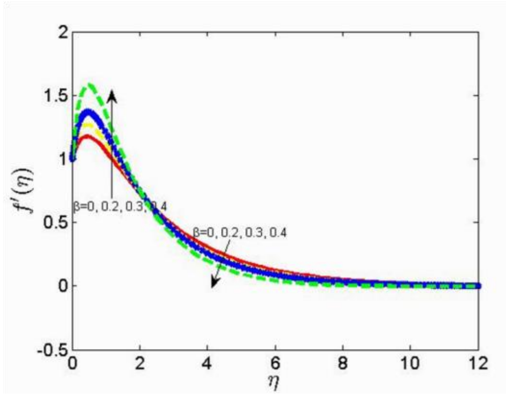
Figure 4.28 explains the contribution of Eckert number on the concentration, velocity and temperature plots. The flow term is derived from the viscous dissipation term added to the energy equation of motion. It contributes more heat energy to the flow due to frictional heating and hence enhanced the flow existing in the thermal and hydrodynamics boundary layer. E_n describe the relationship that occurs between the enthalpy and kinetic energy in the flow. A higher values of E_n is observed to be negligible on the concentration plot. The contribution of heat generation parameter (H) on the concentration, velocity and temperature is illustrated in figure 4.29. Figure 4.29 shows an increment in the fluid temperature and velocity because of increment in the values of heat generation term indicating that more heat is added to the temperature as the heat generation terms increases. An increment in the values of H has no contribution on the concentration plot. This is due to the heat added to the nanoliquid concentration in the solutal layer. Figure 4.30 explains the contribution of the Lewis number (Ln) on the concentration, velocity and temperature plots. The Lewis number connotes the division of thermal diffusivity to mass diffusivity. Hence, more Lewis numeric is as a result of large thermal diffusivity than mass diffusivity. From figure 4.30, the dimensionless wall velocity gradually decreases with a higher values of the Lewis number. The fluid temperature is observed to upsurge with a higher Lewis number. In addition, the concentration plot close to the plate is noticed to increase because of raising the values of Lewis number but decreases far from the plate because of the nanoparticles in the porous boundary layer region.

The contribution of magnetic term (M) is illustrated in figure 4.31. In figure 4.31, an increment in the values of magnetic term gives a damping impact on the velocity plot by originating a drag-like force refers to as Lorentz force. The force behaves in the reverse direction and leads to degeneration in the motion of an electrically conducting liquid. By raising the value of the magnetic term, the momentum layer thickness degenerates while the thermal layer elevate slightly far from the plate as depicted in figure 4.31. In addition, an increment in the values of M has no contribution on the concentration plot. Figure 4.32 explains the contribution of the

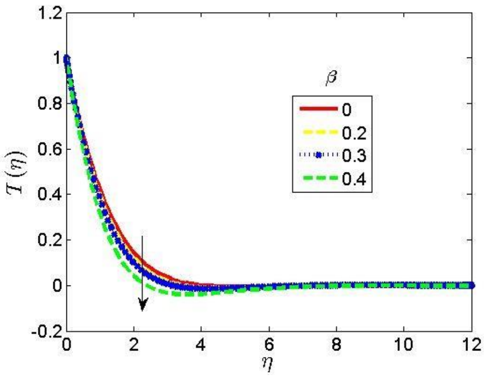
Brownian motion term (*Nb*) on the concentration, velocity and temperature plots. Incremental values of Brownian motion term existing in the hydrodynamic layer brings more fast movement of nanoparticles within the surrounding of the porous medium. An incremental values of *Nb* in figure 4.32 is detected to degenerate the concentration and velocity plots of the fluid nanoparticles. This is because of the random collision of fluid particles leading to fluid velocity degeneration. Obviously, figure 4.32 shows that higher values of *Nb* upsurge the thermal layer and temperature slightly owing to the nanofluid permeability. In addition, increment in *Nb* lessens the solutal concentration. Figure 4.33 shows the contribution of permeability term (*Ps*) on the concentration, velocity and temperature plots. An increment in the porosity add more holes and passage of nanoparticles within the thermal and hydrodynamics boundary layer. An increment in the value of porosity term *Ps* leads to degeneration on the momentum layer thickness and thereby lessens the fluid velocity as depicted in figure 4.33. owing to higher porosity term, an upsurge in the thickness of thermal boundary layer is noticeable. The contribution of *Ps* is noticed to be negligible on concentration plot. Figure 4.32 explains the contribution of radiation term (R_p) on the concentration, velocity and temperature plots. An incremental value of radiation term gives an increment to the fluid temperature plot. Owing to this fact, both thermal and hydrodynamics boundary layer elevates. Physically, a higher R_p added heat energy to the entire thermal layer. Hence, more temperature is added and the temperature plot hereby increases. The temperature plot as illustrated in figure 4.34 elevates at the whole thermal layer. An increment in R_p is noticed to lessen the nanoliquid concentration plot at the wall. Figure 4.35 shows the contribution of Soret term (*So*) on the temperature, concentration and velocity plots. An incremental values of *So* is detected to upsurge the velocity alongside the momentum boundary layer thickness in figure 4.35. In the same vein, incremental values of Soret term accelerates the concentration plots close to the plate and neglected at the ambient vicinity. The contribution of the Schmidt number (*Sc*) on the temperature, concentration and velocity plots is depicted in figure 4.36. The Schmidt number is a dimensionless number which connotes the division of the fluid viscosity to mass diffusivity. Therefore, if viscosity is more than mass diffusivity, Schmidt number becomes very large within the whole boundary layer. It is detected that higher *Sc* drastically lessens the velocity and concentration plot owing to the nanoliquid permeability within the solutal and hydrodynamic layer. A higher values of *Sc* results to an elevation in the solutal boundary layer. The contribution of the Grashof number (O_a) on the concentration, temperature and velocity plots is shown in figure 4.37. The force of buoyant behaves as favourable pressure gradient on the liquid motion and accelerates the nanoliquid existing in the boundary layers. This is depicted in figure 4.37 as increment in the Grashof number (O_a) elevates the momentum layer and lessens the mass and thermal layer slightly.

Figure 4.38 shows the contribution of mass Grashof (4_b) on the temperature, concentration and velocity plots. With increase in the values of 4_b, velocity and the entire thickness of hydrodynamic accelerates. An incremental values of 4_b decreases the temperature plot while contribution of 4_b was found to be negligible on the concentration plot. Figure 4.39 shows the contribution of Prandtl number (Pr) on the concentration, temperature and velocity plots. The velocity plot is detected to lessens with an incremental values of the Prandtl number (Pr). This is true because liquids with much Pr has too much viscosities which lessens the liquid velocities and degenerate the coefficient of wall skin friction. In addition, a higher numeric vale of Pr leads to degeneration in the liquid temperature and thickness of thermal layer. When the value of Pr is small, ($Pr < 1$) the liquid becomes very conducive. A higher value of Pr is detected to elevate liquid concentration at the wall.

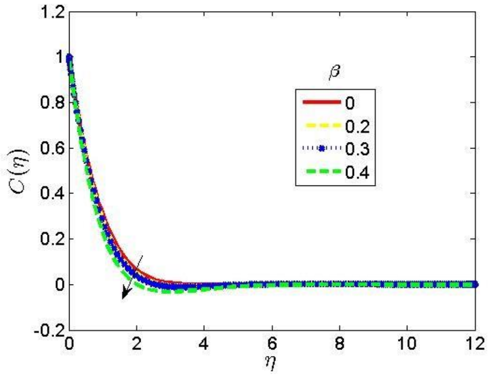
From table 4.8, a higher Lewis number elevates the coefficient of skin friction alongside Sherwood number whereas it degenerates the Nusselt number. In the table 4.8, a higher magnetic term (M) degenerates the three physical quantities of engineering interest (that is, Sherwood, Nusselt and coefficient of skin friction). In table 4.9, a higher Lewis number accelerates the transportation of heat by elevating the Nusselt number and lower the hydrodynamic and solutal layer by degenerating the skin friction and Sherwood number. In table 4.9, an incremental values of Brownian motion term (Nb) degenerates the coefficient of skin friction, Sherwood and Nusselt number. From table 4.9, an incremental values of Schmidt number lessens coefficient of wall skin friction alongside the Nusselt number while an elevation in Sherwood number is detected. In table 4.9, it is detected that the non-Newtonian liquid term (β) accelerates the Sherwood and Nusselt number but degenerates the skin friction. The implementation of all the results above are done in



(a) velocity profile

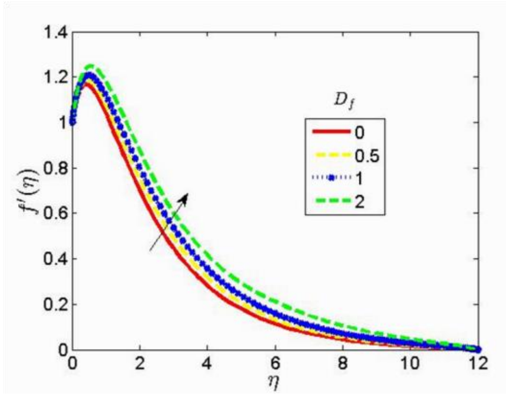


(b) temperature profile

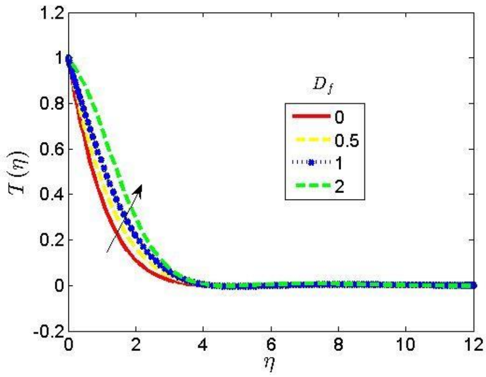


(c) concentration profile

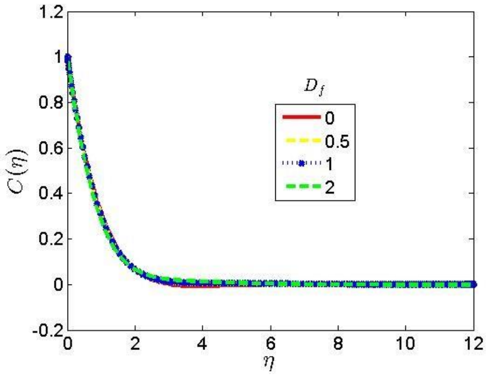
Figure 4.26: Effect of Casson parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, $4_a = 4_b = H$ = 2.0, Φ = 30^{deg}, $M = E_n = N_b = C_p = N_t = Ln = \tau$ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile

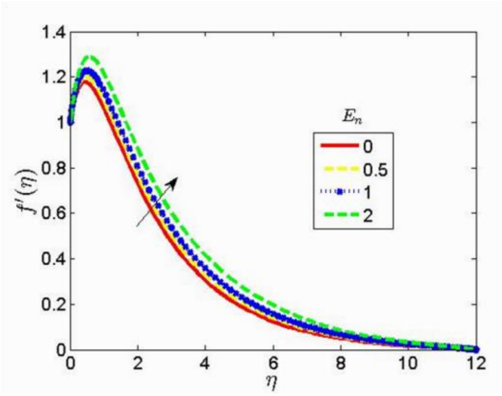


(b) temperature profile

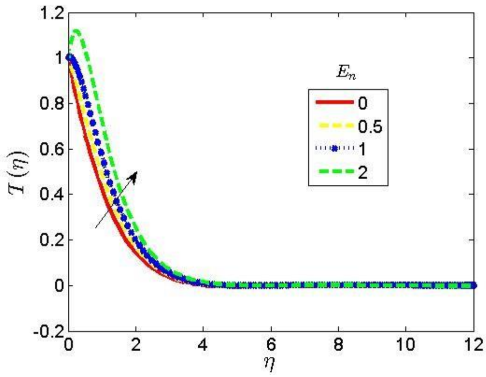


(c) concentration profile

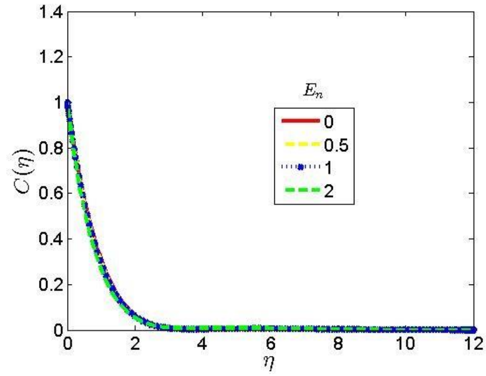
Figure 4.27: Effect of Dufour parameter on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = O_a = \delta_y = 3.0$, $4_a = 4_b = H = 2.0$, $\Phi = 30^{deg}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr = 0.71$, $R_p = 0.6$, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$



(a) velocity profile

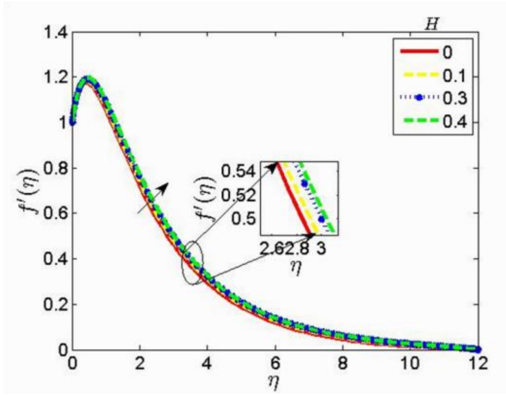


(b) temperature profile

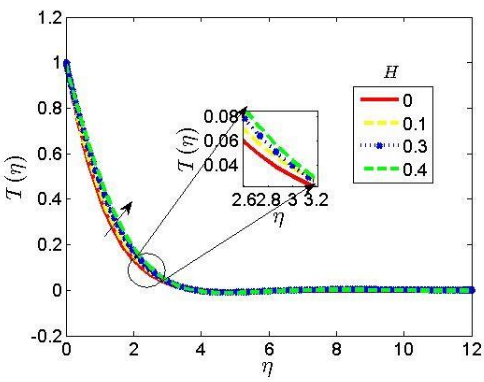


(c) concentration profile

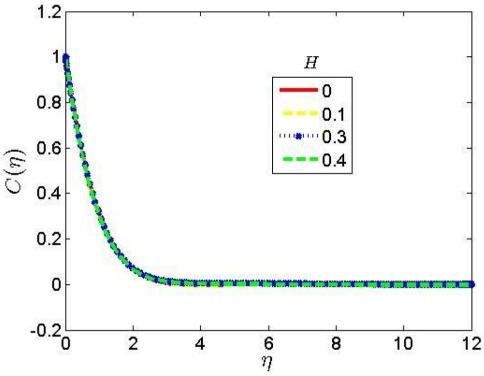
Figure 4.28: Effect of Eckert number on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30^{deg}, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile

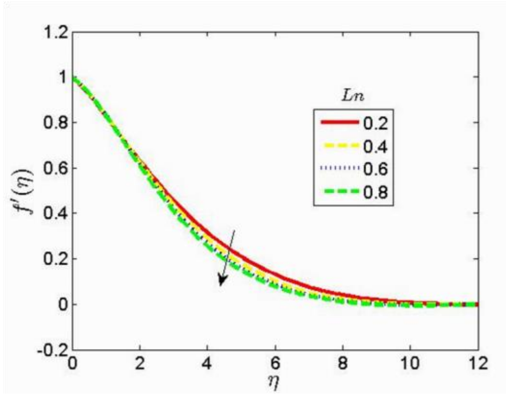


(b) temperature profile

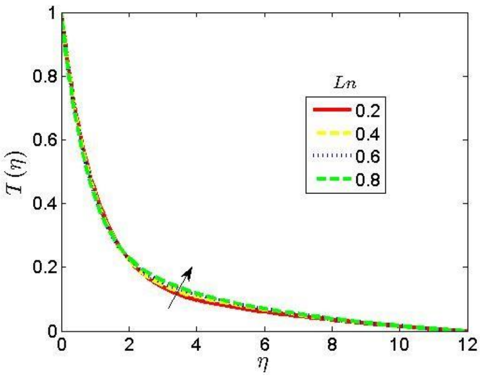


(c) concentration profile

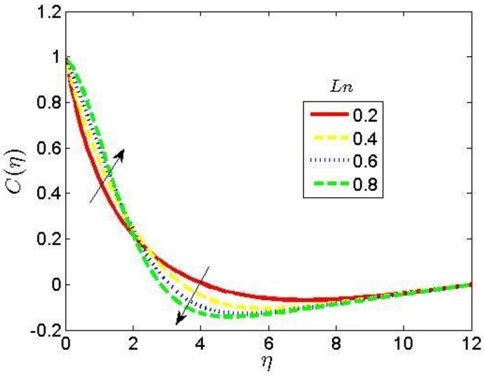
Figure 4.29: Effect of heat generation parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30^{deg}, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile



(b) temperature profile

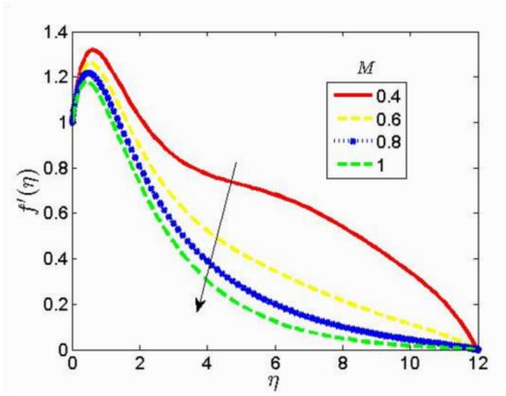


(c) concentration profile

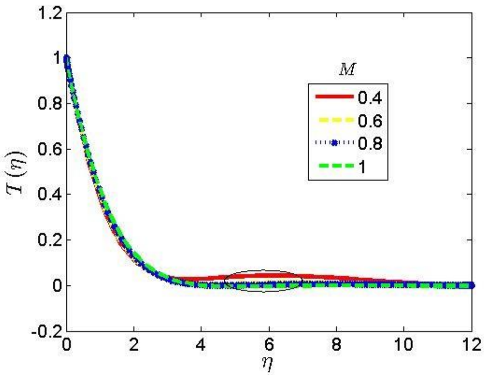
Figure 4.30: Effect of dimensionless Lewis number on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = O_a = \delta_y = 3.0$, $4_a = 4_b = H =$

2.0, $\Phi = 30^{\text{deg}}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr =$

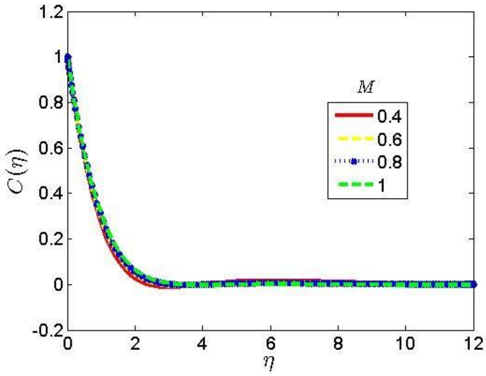
0.71, $R_p = 0.6$, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$



(a) velocity profile

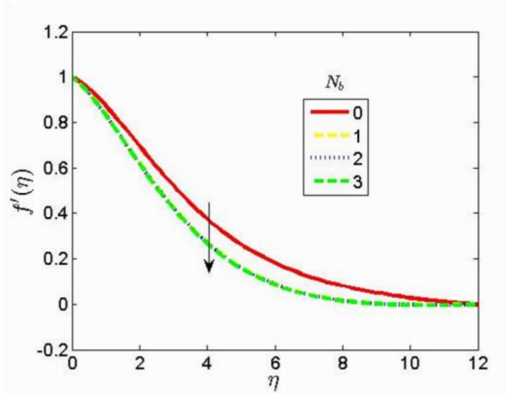


(b) temperature profile

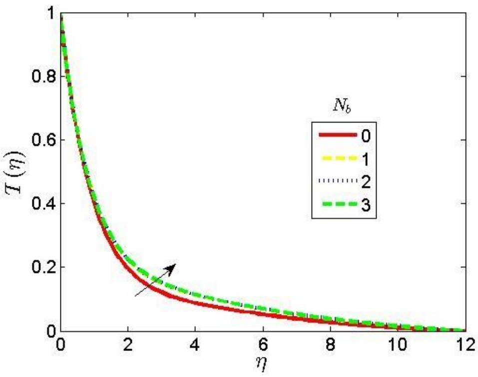


(c) concentration profile

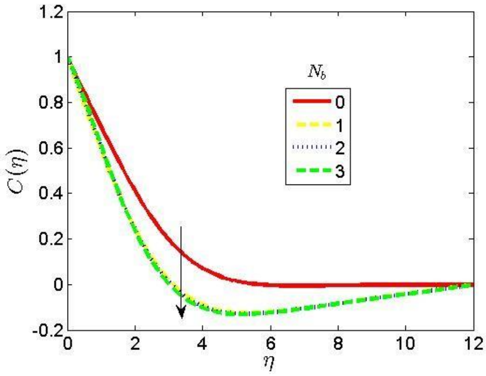
Figure 4.31: Effect of magnetic parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, $4_a = 4_b = H$ = 2.0, Φ = 30^{deg}, $M = E_n = N_b = C_p = N_t = Ln = \tau$ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile



(b) temperature profile

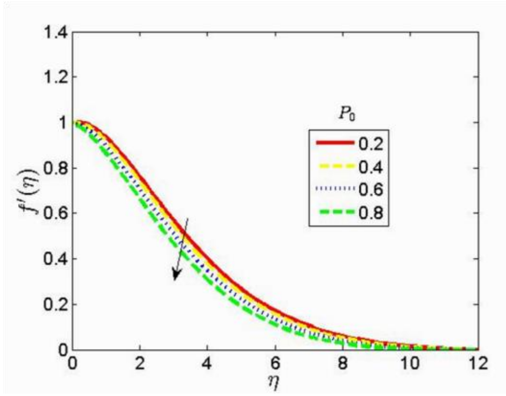


(c) concentration profile

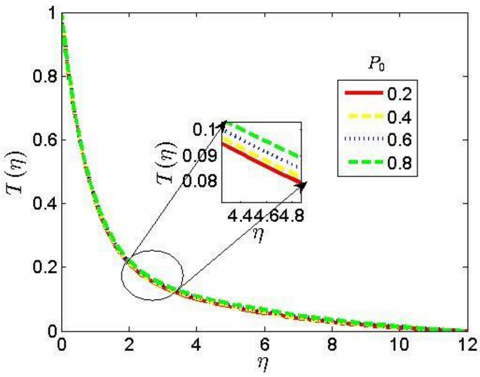
Figure 4.32: Effect of Brownian motion parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, $4_a = 4_b = H =$

2.0, $\Phi = 30^{\text{deg}}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr =$

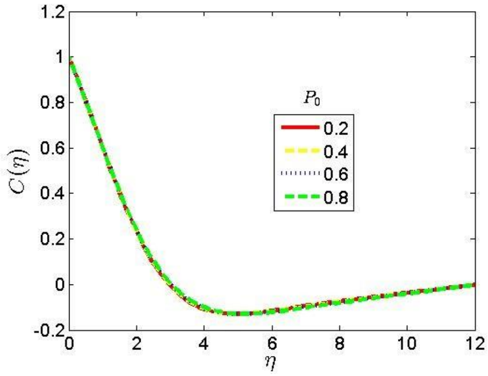
0.71, $R_p = 0.6$, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$



(a) velocity profile

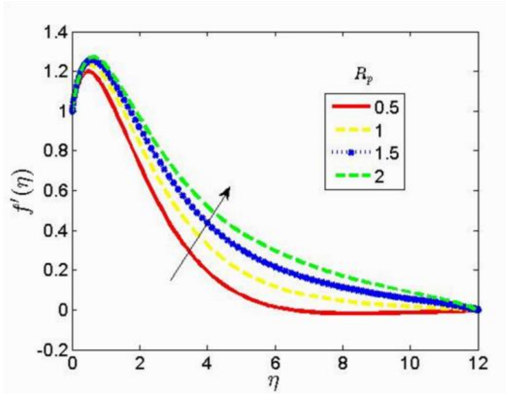


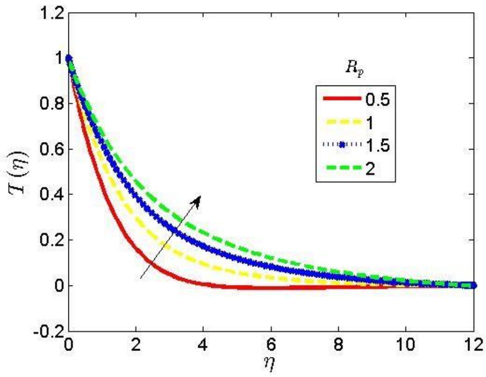
(b) temperature profile



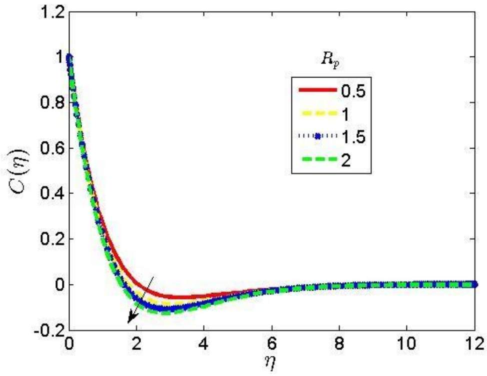
(c) concentration profile

Figure 4.33: Effect of porosity parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30$^{\text{deg}}$, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



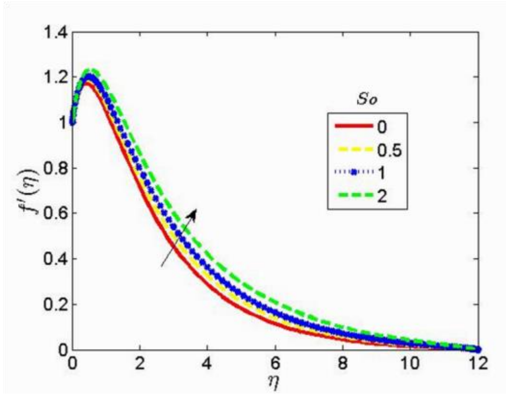


(a) velocity profile (b) temperature profile

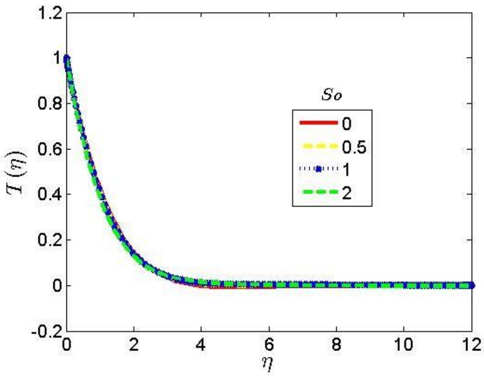


(c) concentration profile

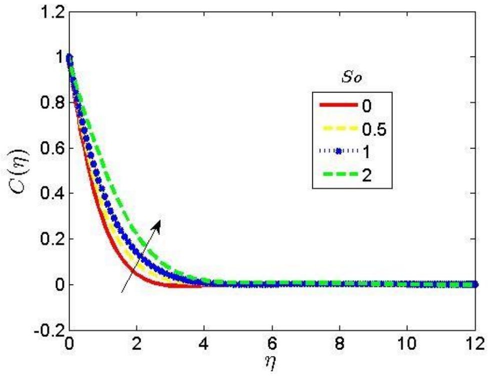
Figure 4.34: Effect of radiation parameter on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = O_a = \delta_y = 3.0$, $4_a = 4_b = H = 2.0$, $\Phi = 30^{deg}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr = 0.71$, $R_p = 0.6$, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$



(a) velocity profile

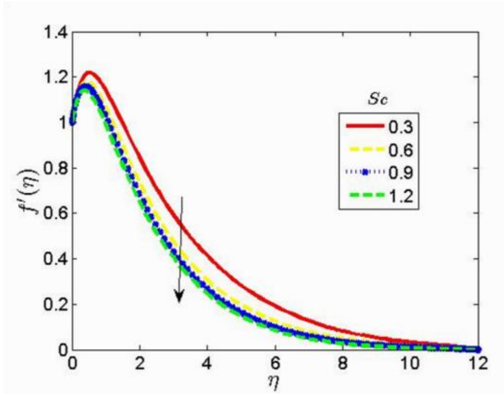


(b) temperature profile

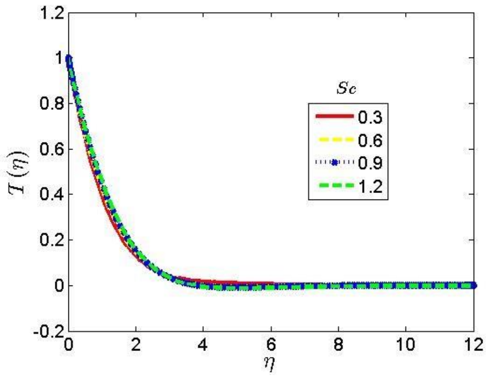


(c) concentration profile

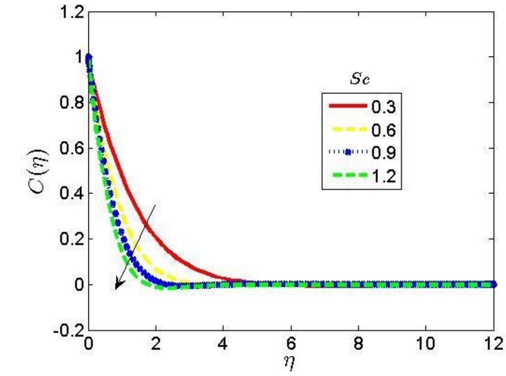
Figure 4.35: Effect of Soret parameter on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30$^{\text{deg}}$, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile

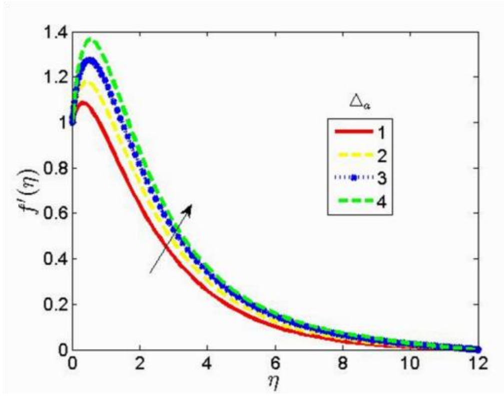


(b) temperature profile

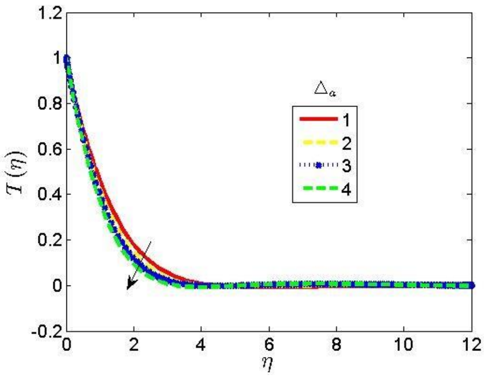


(c) concentration profile

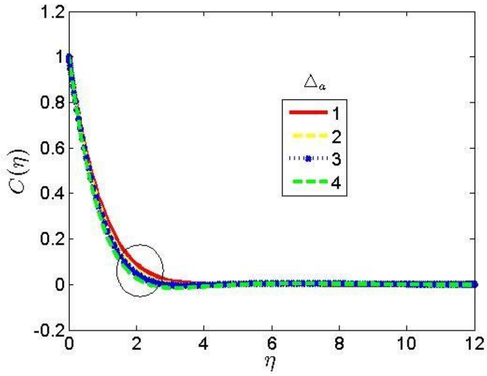
Figure 4.36: Effect of Schmidt number on the (a) velocity (b) temperature and (c) concentration profiles when $\beta = \mathrm{O}_a = \delta_y = 3.0$, $4_a = 4_b = H = 2.0$, $\Phi = 30^{\mathrm{deg}}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr = 0.71$, $R_p = 0.6$, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$



(a) velocity profile

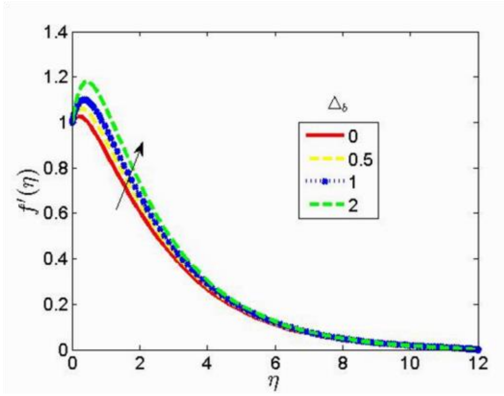


(b) temperature profile

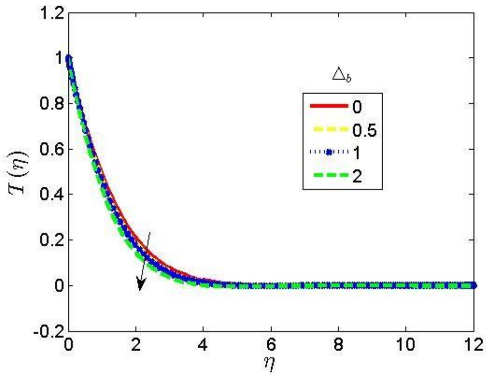


(c) concentration profile

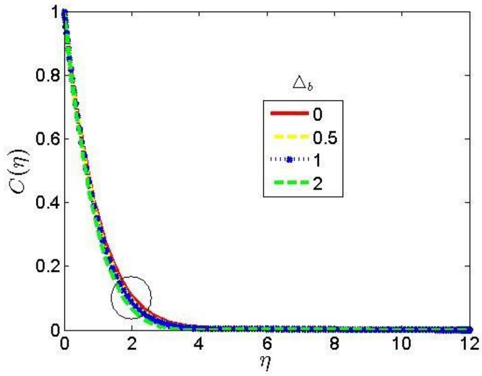
Figure 4.37: Effect of thermal Grashof number on the (a) velocity (b) temperature and (c) concentration profiles when β = O_a = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30^{deg}, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile

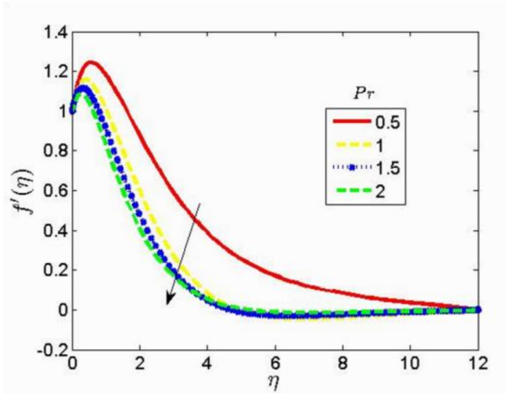


(b) temperature profile

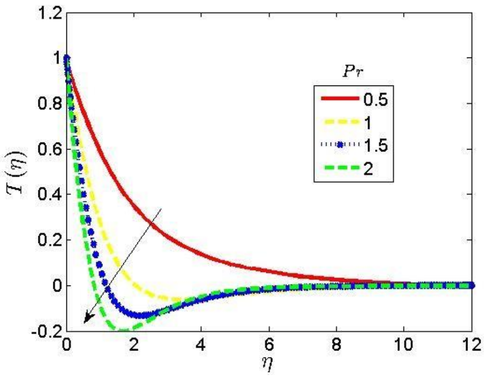


(c) concentration profile

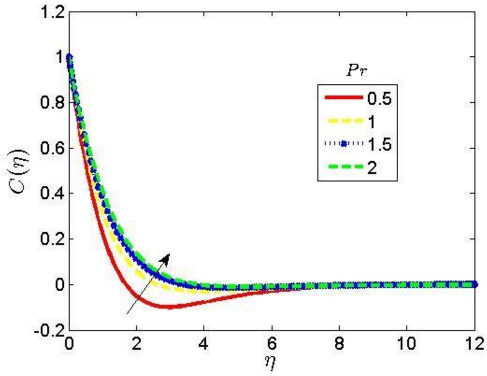
Figure 4.38: Effect of mass Grashof number on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, 4$_a$ = 4$_b$ = H = 2.0, Φ = 30$^{\text{deg}}$, $M = E_n = N_b = C_p = N_t = Ln = \tau$ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile

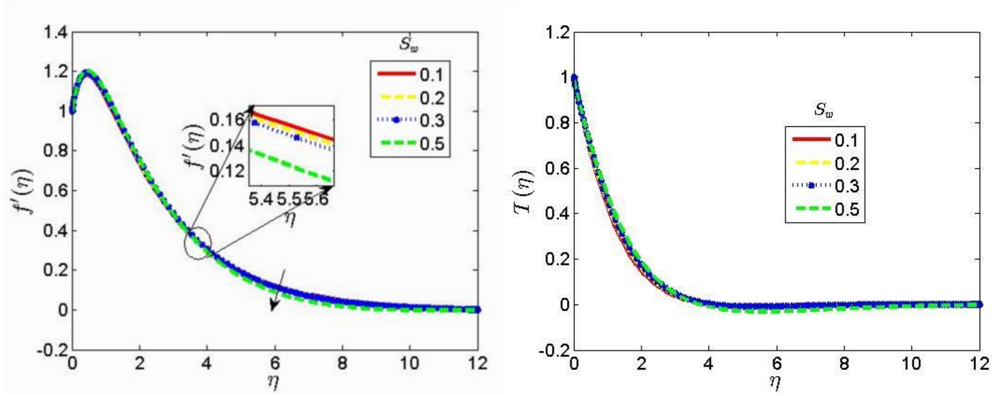


(b) temperature profile

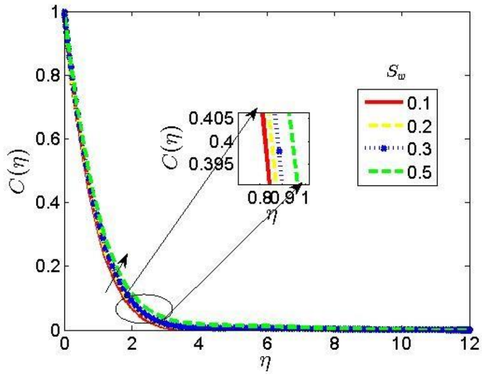


(c) concentration profile

Figure 4.39: Effect of Prandtl number on the (a) velocity (b) temperature and (c) concentration profiles when β = O_a = δ_y = 3.0, 4_a = 4_b = H = 2.0, Φ = 30^{deg}, M = E_n = N_b = C_p = N_t = Ln = τ = 1.0, P_o = 0.5, Pr = 0.71, R_p = 0.6, D_f = 2.0, Sc = 0.61, S_o = 3.0



(a) velocity profile (b) temperature profile



(c) concentration profile

Figure 4.40: Effect of suction velocity on the (a) velocity (b) temperature and (c) concentration profiles when β = O$_a$ = δ_y = 3.0, $4_a = 4_b = H = 2.0$, Φ = 30$^{\text{deg}}$, $M = E_n = N_b = C_p = N_t = Ln = \tau = 1.0$, $P_o = 0.5$, $Pr = 0.71$, R_p = 0.6, $D_f = 2.0$, $Sc = 0.61$, $S_o = 3.0$

Table 4.8: Computational values for skin friction coefficient (C_f), Nusselt number ($-T^0(0)$), and sherwood number ($-C^0(0)$) for different values of Ln and M

Ln	M	C_f	Nh	Sh
0.0	1.0	1.08233781	0.72334184	1.02490066
0.5	1.0	1.14285394	0.35872612	1.08228644
1.0	1.0	1.20739181	0.22940864	1.14323113
2.0	1.0	1.34923800	0.18349278	1.27658671
0.1	0.4	1.45926666	0.83996184	1.09810897
0.1	0.6	1.31269481	0.79692501	1.05550050
0.1	0.8	1.20198770	0.75406987	1.04802955

0.1	1.0	1.06855863	0.72694283	1.00977521

MATLAB language programming. During the implementation, a higher value of ~ is seen to give a spontaneous results. Hence, the maximum value of ~ used in this thesis is ~ = 0.1,L = 12 and N = 100.

Table 4.9: Computational values for skin friction coefficient (C_f), Nusselt number (Nu), and sherwood number ($-C^0(0)$) for different values of *Ln, Nb, Sc* and β

	Parameters			Present Work		
Ln	*Nb*	*Sc*	β	Cf	Nh	Sh
0.2	1.0	0.61	3.0	0.13265182	0.88477929	0.78652409
0.4	1.0	0.61	3.00	0.12774386	0.90996351	0.58398366
0.6	1.0	0.61	3.0	0.12488548	0.93522964	0.38152589
0.8	1.0	0.61	3.0	0.12289942	0.96147715	0.17200602
1.0	0.0	0.61	3.0	0.08967359	0.94746978	0.34074944
1.0	1.0	0.61	3.0	0.12519871	0.92975180	0.42525615
1.0	2.0	0.61	3.0	0.12488548	0.93522964	0.38152589
1.0	3.0	0.61	3.0	0.12448055	0.94174484	0.32969808
1.0	1.0	0.3	3.0	1.04462222	0.78476059	0.67090855
1.0	1.0	0.6	3.0	0.97607905	0.73329484	0.94697241
1.0	1.0	0.9	3.0	0.93182279	0.69528927	1.15619714
1.0	1.0	1.2	3.0	0.89901583	0.66435691	1.33199172
1.0	1.0	0.61	0.0	0.95154181	0.73022820	0.95324841
1.0	1.0	0.61	0.2	1.77694769	0.78778324	1.00573400
1.0	1.0	0.61	0.3	1.57443332	0.88549205	1.09642636
1.0	1.0	0.61	0.4	1.55314440	0.77899759	1.07697499

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The effort of this study is on mixed convective heat and mass transfer flow of nonNewtonian fluids through vertical plate. The study uses similarity variables to transform the system of partial differential equations into coupled nonlinear ordinary differential equations. All flow parameters such as radiation parameter, heat generation parameter, Soret-Dufour parameter, magnetic parameter, thermal and mass Grashof number were discussed using graphs. The effects of both constant and variable viscosity and thermal conductivity through a porous medium on the non-Newtonian fluids is extensively discussed. This study analyzed three problems and solved the three problems numerically.

The research problem one examined the effects of thermo-physical parameters on MHD heat and mass transfer of a viscoelastic fluid past a semi-infinite moving vertical plate using SRM. In the analysis, the plate moves towards the y^0- direction and the term, $\frac{\partial u'}{\partial x'}$ was neglected in the continuity equation. The magnetic field strength is applied opposite to the semi-infinite moving vertical plate (see figure 3.1). The problem assumed the magnetic Reynolds number to be small so that the induced magnetic field is neglected. The Roseland approximation were used because the fluid considered is optically thick.

The research problem two examined the effects of variable thermal conductivity and viscosity on non-Newtonian fluids flow through a vertical porous plate under Soret-Dufour influence. The two non-Newtonian fluids consider in the study are Casson and Walters'-B viscoelastic fluid. The vertical plate is porous and thereby allows the flow of both Casson and Walters'-B liquid (see figure 3.2). The magnetic field strength (B_0) is applied opposite to the flow of the fluid. At the boundary layer, Casson and Walters'-B liquid are mixed together and both have the same flow behaviour. It worths mentioning that as the Casson non-Newtonian fluid parameter approaches infinity, it behaves like a Newtonian fluid (that is, it obeys the Newton's law of viscosity). Furthermore, the fluid surroundings at the free stream is considered to be hot (see figure 3.2). At this environment, flow parameters such as radiation, heat generation, Eckert number, heat source/sink, Prandt number are very significant within the boundary layer

The research problem three examined the effects of thermophoresis, Soret-Dufour on mixed convective flow of MHD non-Newtonian nanofluid over an inclined plate embedded in a porous medium (see figure 3.3). A variable viscosity and thermal conductivity is considered in the problem. Similarity variables were used to reduce the governing partial differential equations into coupled ordinary differential equations.

All the problems solved in this study were found to be useful in many industrial applications such as geophysics, drying process and in the design of many advanced energy conversion system operating at higher temperature.

5.2 Conclusion

This study utilized SRM to solve the coupled third order PDEs that governs the thermo-physical contributions on MHD heat and mass transport of a viscoelastic liquid past a half-infinte vertical plate. The broad explanation on SRM was extensively explored in the previous section. Validation of the present outcomes was obtained with published works and was in good correlation. The present outcomes will be useful in understanding problems of complex nature of thermo-physical effects on MHD viscoelastic liquid past a half-infinite vertical plate. The SRM was found accurate and efficient as compared to other numerical techniques used in the discussed literature.

Dynamics of varying viscosity alongside thermal conductivity on a characterized Walters-B and Casson liquid (concentrated fruit juice and jelly). In figure 3.1, both fluids moves into the three layers from the vertical penetrable plate. In the boundary layer, the viscosity as well as thermal conductivity varies as the free stream vicinity is considered to be hot. The outcomes in the present study deduced that by varying viscosity and thermal conductivity accelerate the temperature and velocity plots for a higher Casson term and Walters-B viscoelastic term.

Owing to the hotness at the ambient environment, increment in radiation parameter degenerates the velocity existing in the free stream vicinity. As depicted in figure 3.1, the imposed magnetism strength (B_0) originate Lorentz force which causes opposition to the direction of flow and thereby degenerates the fluid velocity. A higher permeability term allows the entrance of more Casson and Walters-B liquid flow. The intensity of fluid flow increases the moment the permeability term is raised. Owing to this, more heat is generated at the boundary layer because of heat at the free stream which keeps increasing because of the hotness.

The outcomes of this research can be found significant in bioengineering, food processing and drilling operations. The practical usefulness are mainly in cooling system, oil-pipeline friction reduction and surfactant applications converted to large-scale heating. It is also found significant in the application of higher-polymer additives to enhance motion in petroleum pipe-lines that are useful for commercial purposes. Soret-Dufour contributions on the liquid flow is significant and hereby finds application in engineering such as separation of isotope. The present exploration acts a predominant role in the field of science and technology. The examples of Walters-B liquid considered in this research are industrial polymers namely ceramic processing liquid, chromatography liquid and polymethly methacrylate. These type of fluid are mainly used in bio-medical applications, communication, hardware appliances and agricultural activities. Also, jelly and concentrated fruit juice are the type of Casson fluid considered in this research. Hence, the present outcomes is of great interest in polymer engineering, manufacturing of ceramics, polymer production, particle deposition onto wafers in the microelectronics industry, metallurgy, magnetically controlled metal welding, magnetically monitored coating of metals.

From our numerical computations, we deduced the following:

- When Dufour parameter is increased, the velocity profile as well as the temperature profiles increases.
- Increasing the Soret parameter increases both the velocity and concentration

profiles.

- It is found out that the Soret term alters the concentration profile while Dufour term alters the temperature profile.

- It is found out that as the viscoelastic parameter increases, the velocity profile close to the plate decreases while far away from the plate, it increases slightly.

- The thermal Grashof number increases the hydrodynamic boundary layer thickness when it is increased.

5.3 Contribution to knowledge

This study explains the concept of non-Newtonian fluid flow with both constant and variable viscosity and thermal conductivity on mixed convective heat and mass transfer through a vertical plate which will guide scientists and experimentalists on the physics of pertinent flow parameters such as thermal Grashof number, permeability parameter, radiation parameter, heat generation parameter, Eckert number, Soret parameter, Dufour parameter, Casson and viscoelastic non-Newtonian parameter in food processing, drilling operations and bioengineering. The outcome of study will be useful in high-polymer additives to enhance flow in pipe-lines which is very useful for commercial purposes. The numerical methods used in this study is a useful tools for scientists and engineers in solving highly nonlinear differential equations.

5.4 Recommendations

Real life problems in sciences and engineering generally involve nonlinear differential equations. Also, problems on non-Newtonian fluids are generally complex due to their constitutive equations. The momentum boundary layer becomes coupled and highly nonlinear. SRM and SHAM are very good solution technique in solving governing equations of high complexity and nonlinearity. The methods of solution in this study is hereby recommended for the solutions of non-Newtonian fluids. Also, the results of this study is recommended for use in polymer industry for a perfect flow of fluid parameters such as heat generation parameter, radiation parameter, Eckert number, variable viscosity and thermal conductivity.

5.5 Suggestions for further studies

The study of MHD heat and mass transfer non-Newtonian fluids flow is of practical importance in real life situations. Thus, the following are suggestions for further

studies:

(i) Analyzing the problem of MHD heat and mass transfer non-Newtonian fluids through a stretching sheet.

(ii) Examining the problem of MHD heat and mass transfer non-Newtonian fluids through porous media flow in fuel cells.

(iii) Chaotic mixing in air filtration devices.

(iv) The use of spectral methods in solving more complex non-Newtonian fluids. (v) Examining a Newtonian model by setting the Casson and Walters'-B viscoelastic parameter to be zero.

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