

On the Stability Analysis of Infectious Model with Therapy

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Abstract

In this work on SEIR epidemic model was proposed to understand the impact of other efforts (such as sustainable environmental policy, quarantine, health education, etc) in addition with vaccination in generating speedy transition to recovery even at the 'exposed stage'. The possibility of transition from the exposed class to the recovery class was taken into consideration. The dynamics of the compartments were described by a system of ordinary differential equations and all the equations were solved to establish the disease free equilibrium state and the epidemic equilibrium state of the model. Stability analysis was also carried out on the disease free equilibrium state and the epidemic equilibrium state using the concepts of the characteristic equation of the Jacobian determinant and the Bellman and Cooke's theorem. Numerical simulations were carried out with the help of Mathematical Software (Maple) using parameter values from published data as the base. It was found out that the disease free equilibrium state and the epidemic equilibrium state of the model were stable. Stability of the equilibrium states of the model implied that the disease died out whenever there was an outbreak due to combination of vaccination and other factors.

Keywords: Epidemic model, sustainable policy, disease free equilibrium, epidemic equilibrium, stability, Numerical simulations

1.0. Introduction

Infectious diseases have tremendous influence on human life. Every year, millions of human beings suffer or die of various infectious diseases. In 1971, 22,931 cases of cholera and 2,945 deaths with a case fatality rate (CFR) of 12.8% were recorded in Nigeria [1].

The disease reoccurred in 1991. 59,478 cases and 7,654 deaths were reported. The CFR was 12.9% which remains the highest rate reported by the country to date [1].

In addition, Nigeria recorded the outbreak of cerebrospinal meningitis in 2009 with 38,586 cases and 2,172 deaths with a case fatality rate of 5.6%. In 2002, the country also experienced a serious outbreak of Lassa fever which subsided with more than 1700 cases with 112 deaths and a case fatality rate of 6.6% [2].

The outbreak and spread of disease has been questioned and studied for many years and controlling infectious diseases have been an increasingly complex issue in recent years. The ability to make predictions about diseases could enable scientists to evaluate inoculation or isolation plans and may have a significant effect on the mortality of a particular epidemic [3]. The modeling of infectious diseases is a tool which has been used to study the mechanism by which disease spread, to predict the future course of an outbreak and to evaluate strategies to control an epidemic [4]. Vaccination has been established as an indispensable instrument to fight against the propagation of epidemic diseases and a good number of wars against epidemic diseases has been won through Vaccination [5].

Frederick et al [6] investigated the control of measles by vaccination in a particular part of Kenya and claimed that where there is an outbreak of the disease and there is an introduction of a mass vaccination programme which can cover about 97% of the population, not everybody will become immune because vaccine efficacy is usually not 100% which means that part of the population will be immune and others will be vaccinated but not immune. They however placed emphasis on education and

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argued that more people should be educated in order to create awareness to the disease so that the community will of the deadly measles disease.

To support the impact of other efforts apart from vaccination in controlling infectious diseases, the application of vaccines has to take into consideration not only vaccine-related parameters such as coverage, efficacy and so forth, but also social factors (risk behavior) that may alter the expected theoretical predictions [7]. They maintained that vaccines that attempt to reduce susceptibility to infection should be characterized by at least three parameters which are coverage, efficacy and waning period. All these parameters, according to them, interact and determine to a certain degree the success or failure of a vaccination programme. Coverage and efficacy are related to cost either in terms of vaccination campaigns, accessibility of vaccines, propaganda, e.t.c., or in terms of intrinsic factors, e.g., quality of the vaccines so the inadequacies of vaccination are catered for and supplemented by other efforts apart from vaccination.

Most of the outbreaks especially in less developed countries have resulted from environmental factors and negative human behavior [8]. For instance, cholera is a very serious problem in many developing countries due to inadequate access to safe drinking water supply, improper treatment of reservoirs and improper sanitation. This disease is spread through human-to-human by unhygienic contact with cholera patient's faeces, vomit or corpse or through environment-to-human by ingesting the vibrio cholerae bacteria from contaminated food or water [9].

Since the outbreaks of some infectious diseases stem from the environmental factors and negative human behavior, this work is aimed at investigating the effect of the combination of vaccination and other factors like sustainable environmental policy, quarantine, isolation, personal hygiene, public enlightenment to mention but a few in fighting against the propagation of epidemic diseases.

2.0 Model Description

A deterministic compartmental model was adopted to illustrate the dynamism of the flow. In the model, πN is the recruitment rate into the population which was as a result of birth or loss of acquired immunity. γ is the rate of vaccination. ρ is the rate of death from disease related cases while μ is the rate of death from causes unrelated to the infection, β is the transmission constant (with the total number of infections per unit of time at time 't' being $\beta \frac{S(t)I(t)}{N(t)}$). α and σ are the rates of moving from exposed stage to recovered class and infectious class respectively though death due to the disease during the latent stage is neglected. γ is the recovery rate and ω is the rate of losing immunity. N is the total population and all parameters are non-negative.

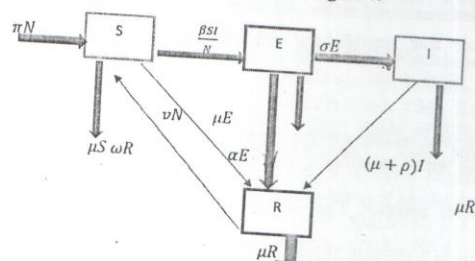


Fig. 1: Diagrammatic representation of the model

Fig. 1 showed that those who were successfully vaccinated would receive immunity and moved straight to the recovered class though some of them would die naturally. Besides, those who were not vaccinated or not successfully vaccinated would also die naturally. Those who were alive but not vaccinated or not successfully vaccinated would contact the disease and became exposed. As a result of the fact that certain proportion of the exposed individuals was not ignorant of their status through other efforts on ground apart from vaccination, a good number of them would be cured of the disease and moved to the recovered class at a rate α though some of them would die naturally. The remaining individuals in the exposed class who were totally ignorant of their status or not successfully cured of the disease would become infectious though some of them would die naturally before becoming infectious. Some of the infectious individuals would die either naturally or as a result of the infection while the remaining individuals would be cured of the disease and moved to the recovered class at a rate γ . Some of those who recovered from the disease would die naturally while the remaining people would become susceptible again at the expiration of immunity and the flow would continue like that.

The model equations are given as:

$$\frac{dS}{dt} = \pi N(t) - \frac{\beta S(t)I(t)}{N(t)} - \mu S + \omega R(t) - vN(t) \quad (1)$$

$$\frac{dE}{dt} = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \alpha + \sigma)E(t) \quad (2)$$

$$I'(t) = (\mu + \gamma + \rho)I(t)$$

$$N'(t) = vN(t) - (\omega + \mu)R(t) + \alpha E(t) + \gamma I(t)$$

(3)

(4)

3.0 Properties of the SEIR Model Equations

The basic properties of the model equations (1)-(4) are feasibility of solutions and positivity of solutions. The feasible solution describes the region in which the solution of the equations is biologically meaningful while the positivity of the solutions describes the non-negativity of the solutions of the equations (1) - (4) [6].

4.0 Feasible Solution

The feasible solution set which is positively invariant set of the model is given by $\{(S, E, I, R) \in \mathbb{R}^4 : S+E+I+R = N \leq N(\pi - \mu)\}$ $\mathbb{R}^4_+$.

5.0 Positivity of Solutions

It can be shown that all the variables in the model equations (1) - (4) are non-negative. The total population of (1) - (4) is $S+E+I+R$ which implies that $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$.

$$\begin{aligned} \frac{dN}{dt} &= \pi I(t) - \frac{\beta S(t)I(t)}{N(t)} - \mu S + \omega R(t) - vN(t) + \frac{\beta S(t)I(t)}{N(t)} - (\mu + \alpha + \sigma)E(t) + \sigma E(t) - (\mu + \gamma + \rho)I(t) + \\ &= \pi I(t) - (\omega + \mu)R(t) + \alpha E(t) + \gamma I(t) \\ &= \pi I(t) - \mu(S + E + I + R) - \rho I \end{aligned}$$

$$= I(\pi - \mu) - \rho I$$

$$= I(\pi - \mu)dt$$

Hence, the solution of the differential equation then becomes

$$I(t) = A e^{(\pi - \mu)t}, \text{ where } A = e^c$$

$$I(0) = 0, I(0) = A \text{ and}$$

$$N(t) = N(0) e^{(\pi - \mu)t}$$

Hence, ω is positively invariant.

6.0 Equilibrium States of the Model

6.1 The Disease Free Equilibrium state DFE

$$\pi I(t) - \frac{\beta S(t)I(t)}{N(t)} - \mu S + \omega R(t) - vN(t) = 0$$

(5)

$$(\mu + \alpha + \sigma)E(t) = 0$$

(6)

$$(\mu + \gamma + \rho)I(t) = 0$$

(7)

$$(\omega + \mu)R(t) + \alpha E(t) + \gamma I(t) = 0$$

(8)

$$\text{Using equation (6); } -(\mu + \alpha + \sigma)E = 0 \text{ since } I=0$$

Hence $E = 0$

Using equation (8);

$$vN(t) - (\omega + \mu)R(t) = 0 \text{ since } I=E=0$$

Then,

$$R = \frac{vN(t)}{\omega + \mu}$$

Using equation (5);

$$\begin{aligned} \pi N(t) - \mu S + \omega R(t) - vN(t) &= 0 \\ S &= \frac{1}{\mu} \left\{ \pi N(t) + vN(t) \left[\frac{\omega}{\omega + \mu} - 1 \right] \right\} \end{aligned}$$

Hence the Disease Free Equilibrium State is given as

$$(S, E, I, R) = \left(\frac{N}{\mu} \left\{ \pi - \frac{\mu v}{\omega + \mu} \right\}, 0, 0, \frac{vN(t)}{\omega + \mu} \right)$$

7.0 The Epidemic Equilibrium State

$$\text{Let } S(t) = k, E(t) = x, I(t) = y, R(t) = z$$

Therefore, the system of equations becomes:

$$\pi(t) - \frac{\beta k y}{N} - \mu k + \omega z - vN = 0$$

(9)

$$\frac{\beta ky}{N} - (\mu + \alpha + \sigma)x = 0 \quad (10)$$

$$\sigma x - (\mu + \gamma + \rho)y = 0 \quad (11)$$

$$vN - (\omega + \mu)z + \alpha x + \gamma y = 0 \quad (12)$$

$$\text{From equation (11); } x = \frac{(\mu + \gamma + \rho)}{\sigma} y \quad (13)$$

Putting equation (13) into (10) to obtain:

$$\frac{\beta ky}{N} - (\mu + \alpha + \sigma) \frac{(\mu + \gamma + \rho)}{\sigma} y = 0$$

$$y \left[\frac{\beta k}{N} - (\mu + \alpha + \sigma) \frac{(\mu + \gamma + \rho)}{\sigma} \right] = 0$$

$$y = 0 \text{ or } \frac{\beta k}{N} - (\mu + \alpha + \sigma) \frac{(\mu + \gamma + \rho)}{\sigma} = 0$$

$$\text{Hence; } k = \frac{N(\mu + \alpha + \sigma)(\mu + \gamma + \rho)}{\sigma \beta} \quad (14)$$

$$\text{Using equation (13) in equation (12), we have:}$$

$$vN - (\omega + \mu)z + \alpha \frac{(\mu + \gamma + \rho)}{\sigma} y + \gamma y = 0$$

$$z = \frac{1}{\omega + \mu} \left(vN + \frac{\alpha(\mu + \gamma + \rho) + \sigma \gamma}{\sigma} y \right) \quad (15)$$

$$\text{From equation (9); } \pi N(t) - \frac{\beta ky}{N} - \mu k + \omega z - vN = 0$$

$$z = \frac{1}{\omega} \left(\frac{\beta ky}{N} + \mu k - \pi N(t) + vN \right) \quad (16)$$

$$\text{Equating (15) and (16), we have:}$$

$$\frac{1}{\omega + \mu} \left(vN + \frac{\alpha(\mu + \gamma + \rho) + \sigma \gamma}{\sigma} y \right) = \frac{1}{\omega} \left(\frac{\beta ky}{N} + \mu k - \pi N(t) + vN \right)$$

$$\left(\frac{\alpha(\mu + \gamma + \rho) + \sigma \gamma}{(\omega + \mu)\sigma} - \frac{\beta k}{\omega N} \right) y = \frac{\mu}{\omega} k + \frac{N}{\omega} (v - \pi) - \frac{vN}{\omega + \mu}$$

$$\left(\frac{\alpha \omega N(\mu + \gamma + \rho) + \omega N \sigma \gamma - \sigma(\omega + \mu)\beta k}{\sigma \omega N(\omega + \mu)} \right) y = \frac{(\omega + \mu)\mu k - N(\omega + \mu)(\pi - v) - \omega v N}{\omega(\omega + \mu)}$$

$$y = \frac{\sigma \omega N(\omega + \mu)}{\alpha \omega N(\mu + \gamma + \rho) + \omega N \sigma \gamma - \sigma(\omega + \mu)\beta k} \left(\frac{(\omega + \mu)\mu k - N(\omega + \mu)(\pi - v) - \omega v N}{\omega(\omega + \mu)} \right)$$

$$\text{Using the value of } k \text{ as obtained in equation (14);}$$

$$y = \frac{N}{\beta} \left(\frac{\sigma \beta(\omega + \mu)(v - \pi) + \mu(\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho) - \omega v \beta \sigma}{\omega[\sigma \gamma + \alpha(\mu + \gamma + \rho)] - (\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho)} \right) \quad (17)$$

$$\text{From equation (13); } x = \frac{(\mu + \gamma + \rho)}{\sigma} y$$

$$x = \frac{N(\mu + \gamma + \rho)}{\sigma \beta} \left(\frac{\sigma \beta(\omega + \mu)(v - \pi) + \mu(\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho) - \omega v \beta \sigma}{\omega[\sigma \gamma + \alpha(\mu + \gamma + \rho)] - (\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho)} \right) \quad (18)$$

$$\text{From equation (15); } z = \frac{1}{\omega + \mu} \left(vN + \frac{\alpha(\mu + \gamma + \rho) + \sigma \gamma}{\sigma} y \right)$$

$$z = \frac{N}{\omega + \mu} \left(\frac{[\sigma \gamma + \alpha(\mu + \gamma + \rho)]}{\sigma \beta} \left(\frac{\sigma \beta(\omega + \mu)(v - \pi) + \mu(\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho) - \omega v \beta \sigma}{\omega[\sigma \gamma + \alpha(\mu + \gamma + \rho)] - (\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho)} \right) + v \right) \quad (19)$$

8.0 Dynamical Behavior of The Model

8.1 Stability Analysis of the Disease Free Equilibrium State

It has already been established that the system of equations (1) - (4) has disease free equilibrium state

$$E_1 = (S, E, I, R) = \left(\frac{N}{\mu} \left\{ \pi - \frac{\mu v}{\omega + \mu} \right\}, 0, 0, \frac{vN(t)}{\omega + \mu} \right)$$

Again, the general variational matrix corresponding to the system is given as

$$J = \begin{bmatrix} \frac{\beta I}{N} - \mu & 0 & \frac{\beta S}{N} & \omega \\ \frac{\beta I}{N} & -(\mu + \sigma + \alpha) & \frac{\beta S}{N} & 0 \\ 0 & \sigma & -(\mu + \gamma + \rho) & 0 \\ 0 & \alpha & \gamma & -(\mu + \omega) \end{bmatrix} \quad (20)$$

the disease free equilibrium state, using the expression E_1 , we obtain;

$$\begin{bmatrix} -\mu & 0 & -\frac{\beta}{\mu}(\pi - \frac{\mu v}{\omega + \mu}) & \omega \\ 0 & -(\mu + \sigma + \alpha) & \frac{\beta}{\mu}(\pi - \frac{\mu v}{\omega + \mu}) & 0 \\ 0 & \sigma & -(\mu + \gamma + \rho) & 0 \\ 0 & \alpha & \gamma & -(\mu + \omega) \end{bmatrix} \quad (21)$$

The characteristics equation is obtained from the Jacobian determinant with the values λ .

$$\begin{vmatrix} -\mu - \lambda & 0 & -\frac{\beta}{\mu}(\pi - \frac{\mu v}{\omega + \mu}) & \omega \\ 0 & -(\mu + \sigma + \alpha) - \lambda & \frac{\beta}{\mu}(\pi - \frac{\mu v}{\omega + \mu}) & 0 \\ 0 & \sigma & -(\mu + \gamma + \rho) - \lambda & 0 \\ 0 & \alpha & \gamma & -(\mu + \omega) - \lambda \end{vmatrix} = 0$$

The matrix reduces to

$$\begin{vmatrix} -(\mu + \omega - \lambda) & (\mu + \sigma + \alpha) - \lambda & \frac{\beta}{\mu}(\pi - \frac{\mu v}{\omega + \mu}) \\ 0 & \sigma & -(\mu + \gamma + \rho) - \lambda \end{vmatrix} = 0$$

The first two eigenvalues are:

$$\lambda_1 = -(\mu + \omega)$$

The remaining characteristic equation is given as:

$$(\mu + \sigma + \alpha + \rho + \gamma)\lambda + (\mu + \sigma + \alpha)(\mu + \rho + \gamma) - \frac{\sigma\beta(\pi(\omega + \mu) - \mu v)}{\mu(\omega + \mu)} = 0$$

Use criteria to check for the roots of the characteristics equation above.

Thus, The roots of the characteristic equation of a quadratic equation $\lambda^2 + p_1\lambda + a_0 = 0$ will all be negative, if and only if all the coefficients satisfy $a_n > 0$. [10].

For the quadratic equation:

$$(\mu + \sigma + \alpha + \rho + \gamma)\lambda + (\mu + \sigma + \alpha)(\mu + \rho + \gamma) - \frac{\sigma\beta(\pi(\omega + \mu) - \mu v)}{\mu(\omega + \mu)} = 0$$

We have all negative roots if

$$(\mu + \sigma + \alpha)(\mu + \rho + \gamma) > \frac{\sigma\beta(\pi(\omega + \mu) - \mu v)}{\mu(\omega + \mu)}$$

Since in the absence of the disease (DFE), β is zero, it follows that the above inequality holds, and therefore the system of equations is said to be locally asymptotically stable.

Stability Analysis of the Epidemic State

we shall apply the approach of [11] to establish the stability of the endemic equilibrium state

Theorem

Let $H(z) = P(z, e^z)$, $z \in \mathbb{C}$ where $P(z, e^z)$ is a polynomial with principal term, and suppose that $H(iy)$, $y \in \mathbb{R}$ can be separated into its real and imaginary parts such that

$$H(iy) = F(y) + iG(y)$$

If the zeros of $H(z)$ has negative real parts, then the zeros of $F(y)$ and $G(y)$ are real, simple and alternate and $G'(0) - G(0)F'(0) > 0$ for all $y \in \mathbb{R}$

(22)

Conversely, all zeros of $H(z)$ will be in the left half plane provided that either of the following conditions is satisfied:

- (i) All the zeros of $F(y)$ and $G(y)$ are real. Simple and alternate and the inequality (22) is satisfied for at least one y
- (ii) All the zeros of $F(y)$ are real and for each zero, the relation (22) is satisfied
- (iii) All zeros of $G(y)$ are real and for each zero, the relation (22) is satisfied.

Using the values of both S from equation (14) and I from equation (17) in the Jacobian matrix (20) and leaving both y and k in place of simplification, we have:

$$|J^* - \lambda I| = \begin{vmatrix} -y - \mu - \lambda & 0 & -\kappa & \omega \\ y & -(\mu + \sigma + \alpha) - \lambda & \kappa & 0 \\ 0 & \sigma & -(\mu + \gamma + \rho) - \lambda & 0 \\ 0 & \alpha & \gamma & -(\mu + \omega) - \lambda \end{vmatrix} = 0$$

$$(-y - \mu - \lambda) \left\{ -(\mu + \omega + \lambda) \left[-(\mu + \sigma + \alpha) - \lambda \right] - \frac{\sigma}{k} \left[-(\mu + \gamma + \rho) - \lambda \right] \right\} - y \left\{ 0 + k(-\sigma(\mu + \omega + \lambda)) + \omega(\sigma\gamma + \alpha(\mu + \gamma + \rho + \lambda)) \right\} = 0$$

$$(y + \mu + \lambda)(\mu + \omega + \lambda) \{ (\mu + \sigma + \alpha + \lambda)(\mu + \gamma + \rho + \lambda) - \sigma k \} - y \{ -\sigma k(\mu + \omega + \lambda) + \omega[\alpha\lambda + (\sigma\gamma + \alpha\mu + \alpha\rho + \alpha\gamma)] \} = 0$$

Hence,

$$p(\lambda) = \lambda^4 + \lambda^3[y + 4\mu + \omega + \sigma + \alpha + \gamma + \rho] + \lambda^2[y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k] + \lambda[(3\omega\mu^2 - k(2\sigma\mu + \omega\sigma)) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + 2\omega\gamma\alpha + 2\omega\gamma\sigma)] + [\mu^4 + \mu^3(\gamma + \rho + \alpha + \sigma + \omega) + \mu^2(\rho\alpha + \rho\sigma + \omega\alpha + \omega\gamma\sigma)] + k(\omega\mu\sigma + \sigma\mu^2) + \mu(\omega\rho\alpha + \omega\rho\sigma + \omega\gamma\alpha + \omega\gamma\sigma)$$

Now, we replace λ by $i\lambda$ in the characteristic equation above to obtain:

$$H(i\lambda) = \lambda^4 - i\lambda^3[y + 4\mu + \omega + \sigma + \alpha + \gamma + \rho] - \lambda^2[y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k] + i\lambda[(3\omega\mu^2 - k(2\sigma\mu + \omega\sigma)) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + 2\omega\gamma\alpha + 2\omega\gamma\sigma)] + [\mu^4 + \mu^3(\gamma + \rho + \alpha + \sigma + \omega) + \mu^2(\rho\alpha + \rho\sigma + \omega\alpha + \omega\gamma\sigma)] + k(\omega\mu\sigma + \sigma\mu^2) + \mu(\omega\rho\alpha + \omega\rho\sigma + \omega\gamma\alpha + \omega\gamma\sigma)$$

Separating the above into its real and imaginary part,
That is, $H(\lambda) = F(\lambda) + iG(\lambda)$

$$F(\lambda) = \lambda^4 - \lambda^2[(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k] + [y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k]$$

And

$$G(\lambda) = -\lambda^3[y + 4\mu + \omega + \sigma + \alpha + \gamma + \rho] + \lambda[(3\omega\mu^2 - k(2\sigma\mu + \omega\sigma)) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + 2\omega\gamma\alpha + 2\omega\gamma\sigma)] + [\mu^4 + \mu^3(\gamma + \rho + \alpha + \sigma + \omega) + \mu^2(\rho\alpha + \rho\sigma + \omega\alpha + \omega\gamma\sigma)] + k(\omega\mu\sigma + \sigma\mu^2) + \mu(\omega\rho\alpha + \omega\rho\sigma + \omega\gamma\alpha + \omega\gamma\sigma)$$

Obtaining the derivatives of both the real part and the imaginary part with respect to λ

$$F'(\lambda) = 4\lambda^3 - 2\lambda[y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k]$$

And

$$G'(\lambda) = -3\lambda^2[y + 4\mu + \omega + \sigma + \alpha + \gamma + \rho] + [(3\omega\mu^2 - k(2\sigma\mu + \omega\sigma)) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + 2\omega\gamma\alpha + 2\omega\gamma\sigma)] + [\mu^4 + \mu^3(\gamma + \rho + \alpha + \sigma + \omega) + \mu^2(\rho\alpha + \rho\sigma + \omega\alpha + \omega\gamma\sigma)] + k(\omega\mu\sigma + \sigma\mu^2) + \mu(\omega\rho\alpha + \omega\rho\sigma + \omega\gamma\alpha + \omega\gamma\sigma)$$

Setting $\lambda = 0$ in $F(0)$, $G(0)$, $F'(0)$ and $G'(0)$, the expressions reduced to:

$$F(0) = [y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma\gamma + \alpha(\gamma + \rho) + \rho\sigma - \sigma k]$$

$$G(0) = 0$$

$$F'(0) = 0$$

$$G'(0) = [(3\omega\mu^2 - k(2\sigma\mu + \omega\sigma)) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + 2\omega\gamma\alpha + 2\omega\gamma\sigma]$$

By using equation (22) of the statement of the theorem, we have:

$$F(0)G'(0) - F'(0)G(0) = [y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma(\gamma + \rho) + \rho\sigma - \sigma k][3\omega\mu^2 - k(2\sigma\mu + \omega\sigma) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + \omega\gamma\alpha + \omega\gamma\sigma)] > 0 \quad (23)$$

By substituting the values of y and k , $F(0)G'(0) - F'(0)G(0)$ becomes

$$F(0)G'(0) - F'(0)G(0) = \left[\frac{\sigma\beta(\omega + \mu)(\nu - \pi) + \mu(\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho) - \omega\nu\beta\sigma}{\omega[\sigma\gamma + \alpha(\mu + \gamma + \rho)] - (\omega + \mu)(\mu + \sigma + \alpha)(\mu + \gamma + \rho)} \right] (3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma(\gamma + \rho) + \rho\sigma - \sigma \frac{(\mu + \alpha + \sigma)(\mu + \gamma + \rho)}{\sigma} \left[\left(3\omega\mu^2 - \frac{(\mu + \alpha + \sigma)(\mu + \gamma + \rho)}{\sigma} (2\sigma\mu + \omega\sigma) + (2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + \omega\gamma\alpha + \omega\gamma\sigma) \right] > 0 \quad (24)$$

We then conclude that the endemic equilibrium is locally asymptotically stable.

10.0 Numerical Simulation of The Epidemic Equilibrium

From equation (23), let

$$F(0)G'(0) - F'(0)G(0) = [y(3\mu + \omega + \rho + \gamma + \sigma + \alpha) + \mu(3\omega + 6\mu + 3\alpha + 3\sigma + 3\gamma + 3\rho) + \omega(\gamma + \rho + \alpha - \sigma) + \sigma(\gamma + \rho) + \rho\sigma - \sigma k][3\omega\mu^2 - k(2\sigma\mu + \omega\sigma) + y(2\mu\alpha + 2\mu\sigma + 2\rho\mu + \rho\alpha + \rho\sigma + 2\gamma\mu + \gamma\alpha + \gamma\sigma + 3\mu^2 + \omega\sigma + \omega\rho + \omega\gamma) + 2\rho\alpha\mu + 2\rho\sigma\mu + 2\gamma\alpha\mu + 2\gamma\sigma\mu + 3\mu^2\alpha + 3\mu\sigma + 3\rho\mu^2 + 3\gamma\mu^2 + 4\mu^3 + 2\gamma\omega\mu + 2\omega\mu\alpha + 2\omega\mu\sigma + 2\omega\rho\mu + \omega\rho\alpha + \omega\rho\sigma + 2\omega\gamma\mu + \omega\gamma\alpha + \omega\gamma\sigma]$$

Table 1: Source of Initial Value of the Parameters

Parameter	Interpretation	Value	Source
β	Transmission constant	0.091	[6]
σ	Latent period	0.125	[6]
μ	Natural death rate	0.005	Hypothetical value
π	Recruitment rate	0.45	[7]
ω	Rate of losing immunity	0.36	[7]
ρ	Disease induced death	0.009	[6]
γ	Recovery rate	0.6	[7]
ν	Rate of vaccination	0.7	[7]
α	Transition rate from latent to recovery	0.075	Hypothetical value

Table 1 gives the source of initial value of each parameter except natural death rate and transition rate from latent period to recovery which cannot be quantified. The values of some parameters are then varied to evaluate J and the result is presented in Table 2.

Table 2: Numerical simulation of the Epidemic Equilibrium

β	σ	μ	π	ω	ρ	γ	ν	α	J	Remark
0.091	0.125	0.005	0.45	0.36	0.009	0.6	0.7	0.075	0.2238	Stable
0.065	0.125	0.005	0.45	0.36	0.005	0.6	0.55	0.045	0.3949	Stable
0.055	0.125	0.005	0.45	0.36	0.005	0.6	0.65	0.045	0.2490	Stable
0.045	0.125	0.005	0.45	0.36	0.005	0.6	0.75	0.045	0.1446	Stable
0.025	0.125	0.005	0.45	0.36	0.004	0.6	0.85	0.055	0.1117	Stable
0.015	0.125	0.005	0.45	0.36	0.003	0.6	0.95	0.060	0.1078	Stable
0.055	0.125	0.005	0.45	0.36	0.005	0.6	1.05	0.025	0.8588	Stable
0.056	0.125	0.005	0.45	0.36	0.006	0.6	1.05	0.010	1.0804	Stable
0.091	0.125	0.005	0.45	0.36	0.009	0.6	0.85	0.075	-0.0394	Unstable
0.091	0.125	0.005	0.45	0.36	0.009	0.6	0.90	0.075	-0.1272	Unstable
0.091	0.125	0.005	0.45	0.36	0.009	0.6	0.95	0.075	-0.2149	Unstable
0.091	0.125	0.005	0.45	0.36	0.009	0.6	1.00	0.075	-0.3027	Unstable

11.0 Results and Discussion

This work investigated stability analysis of infectious diseases with vaccination and other factors using an SEIR model as a frame. The positivity of solutions of the model equations was established and the disease free and epidemic equilibrium states were worked out. The Jacobian determinant was evaluated for the DFE state and the roots of the resulting equation were all

negative which established the local asymptotic stability of the model. We used Bellman and Cooke's theorem to obtain the parameter J and the numerical simulation was carried out by using published data as initial values. The results obtained showed that the epidemic equilibrium is locally asymptotically stable.

From the simulation table, we deduce that the parameter J tends to an unstable state for the epidemic equilibrium (stable state for the disease eradication) as a result of increase in vaccination rate with constant rate of applying other efforts apart from vaccination and decreasing rate of disease transmission due to immunity from vaccination. Besides, increase in vaccination rate together with increase in the rate of applying other efforts apart from vaccination tends to move the model to a more stable disease free equilibrium state because the values of the parameter J decreases which is as a result of decreasing effect on the transmission rate and disease induced death rate. Also, results from the table show that application of vaccination with poor emphasis on other social factors distorts the stability of the epidemic equilibrium state.

12.0 Conclusion

In this work, we have provided via mathematical modeling another understanding of managing infectious diseases. The results obtained in this work have serious implications to Africa and the entire less developed countries where hunger, disease, death and illiteracy are prominent. Regular education of the populace on diseases and their management through personal hygiene and regular medical check-up, formulation and implementation of sustainable environmental policies by the government and immediate response to disease outbreak through isolation and quarantine will do people better by reducing the infectivity of diseases. Through enlightenment, more exposed individuals would know their status, treat them and recovered from the illness. These individuals would not pass on to the group of infected individuals and this will be having a decreasing effect on the infected class in the population. So, the combination of vaccination and other therapy is an effective strategy that could lead to the eradication of infectious diseases.

13.0 References

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