

ON IDEMPOTENT OF SOME ORDERS-PRESERVING FULL CONTRACTION MAPPINGS IN METRIC SPACES

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Abstract

This paper discuss fixed point theorems on the idempotents of Order-Preserving Full Contraction Mappings $E(OCT_n)$ with property P and Q in metric spaces using Picard iterative scheme.

Keywords: Orders-Preserving, Full Contraction, Mappings & Order Preserving.

Introduction

The importance of mappings can be judged by the fact that every semigroup is isomorphic to a semigroup of mappings. The full transformation semigroup on $X_n = \{1, 2, 3, \dots, n\}$ under its natural order is denoted by T_n . The importance of the study of T_n as a naturally occurring semigroup is justified by its universal property in which every finite semigroup is embeddable in some T_n . In this regard, there is some interest to identify and study certain special sub-semigroups of T_n . The semigroup of order-preserving elements of T_n have been studied. In 1971, Howie showed that every element of order preserving is expressible as a product of idempotents and also obtained formulae for the number of elements of idempotents in order-preserving. The term contraction map in semigroup theory was first appeared in countable versus uncountable rank in finite semigroups of transformations and relations by Higgins et al. (2003) but algebraic and combinatorial studies of some semigroups of full contraction mappings of a finite chain were initiated by Dauda (2013). In addition, Zhao and Yang (2012) initiated the algebraic study of semigroups on order-preserving partial contraction of an n -chain where they referred to contractions as compressions and investigated the combinatorial properties of OCT_n and $ORCT_n$ with their idempotents. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. For a nice survey article concerning combinatorial problems in the full and partial transformation semigroups with their sub-semigroups (Garba, 1994; Fernandes, 1997; Laradji & Umar 2004; Mbah et al. 2012; Garba et al., 2017).

Fixed points of non-linear operators is an important research branch of non-linear analysis and has been applied in the study of semigroup theory. Fixed point theory is of interest because it provides a way of establishing the existence of a solution to a set of equations. For example, in theoretical economics such as general equilibrium theory, there comes a point where one needs to know whether the solution to a system of equations necessarily exist or under what conditions will a solution exist. The mathematical analysis of this question usually relies on fixed point theorems. It is well known that the Banach contraction principle is a fundamental result in the fixed point theory which has been used and extended in many different directions. The fixed point theorems are concerned with finding conditions on the structure that the set X

must be endowed as well as the properties of the operator $T : X \rightarrow X$ in order to obtain results on the existence and uniqueness of the fixed points.

The Banach fixed point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric space. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides constructive method to find those fixed points. The theorem was stated by Stefan Banach in (1922).

More results on fixed point may be seen in Choudhury and Dutta, (2008); Choudhury et al. (2011); Alata and Rauf (2016) and Wahab et al.(2016). The classical Banach contraction principle proved in complete metric spaces continues to be an indispensable and effective tool in theory as well as applications, which guarantees the existence and uniqueness of fixed points of contraction self mappings besides offering a contractive procedure to compute the fixed point of the underlying mapping.

In this article, the existence and uniqueness of common fixed points on the idempotent of Order Preserving Full Contraction Mapping $E(OCT_n)$ are investigated in metric spaces.

Basic Definitions of Relevant Terms

In this sub-section, we present some basic concepts which will be useful in the sequel.

Full Transformation Semigroup: A map $\alpha : \text{Dom}(\alpha) \subseteq X_n \rightarrow \text{Im}(\alpha) \subseteq X_n$ is called full (or total) transformation of X_n if $\text{Dom}(\alpha) = X_n$. The set of full transformations of X_n is denoted by T_n . The set T_n of a full transformations of X_n forms a semigroup under composition of mappings called the full transformation semigroup. A transformation α in T_n for which

$$|x\alpha - y\alpha| \leq |x - y|, \forall x, y \in X_n$$

is called a contraction.

Order-Preserving Transformation : A transformation $\alpha \in T_n$ is said to be order preserving if (for all $x, y \in \text{Dom} \alpha$) $x \leq y \Rightarrow x\alpha \leq y\alpha$ denoted by O_n .

Order-Preserving or Order-Reversing Transformation : A transformation $\alpha \in T_n$ for which (for all $x, y \in \text{Dom} \alpha$) $x \leq y \Rightarrow x\alpha \geq y\alpha$ is said to be order-reversing. The semigroup of order-preserving or reversing transformation of X_n is denoted by OR_n .

Definition (Jeong and Rhoades 2005): Let S be a self mapping of a metric space (M, d) with a non-empty fixed point $F(S)$, then S is said to satisfy the property P , if for each $n \in \mathbb{N}$

$$F(S) = F(S^n).$$

Definition (Jeong and Rhoades 2005): Let S and T be self mappings of a metric space (M, d) with fixed point set $F(S) \cap F(T) \neq \emptyset$, S and T are said to have property Q if

$$F(S^n) \cap F(T^n) = F(S) \cap F(T)$$

for each $n \in \mathbb{N}$.

Theorem (Banach 1922): Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an a -contraction, that is an operator satisfying

$$d(Tx, Ty) \leq kd(x, y),$$

for any $x, y \in X$ with $0 \leq a < 1$ fixed. Then,

- (i) T has a unique fixed point, that is $F(T) = \{x^*\}$;
- (ii) The Picard iteration associated to T , that is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \dots,$$

converges to x^* , for any initial guess $x_0 \in X$

- (iii) The a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{a^n}{1-a} \cdot d(x_0, x_1), n = 0, 1, 2, \dots$$

and

$$d(x_n, x^*) \leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n), n = 0, 1, 2, \dots$$

hold.

- (iv) The rate of convergence is given by

$$d(x_n, x^*) \leq a \cdot d(x_{n-1}, x^*) \leq a^n \cdot d(x_0, x^*), n = 1, 2, \dots$$

Main Results

We begin this section presenting results on Idempotent of Order Preserving Full Contraction Mappings $E(OCT_n)$ in metric spaces.

Let S be a semigroup, an element $e \in S$ is called an idempotent if $e^2 = e$ or if and only if $Ime = F(e)$.

Fixed point theorems on $E(OCT_n)$ in metric spaces

Theorem: Let S be a semigroup and (S, d) a complete metric space. Let $E(OCT_n)$ be the Idempotent of Order Preserving Full Contraction mapping and $A \in E(OCT_n)$ be a self map satisfying:

$$d(Ax, Ay) \leq c_1 \cdot d(x, y) + c_2 \cdot d(x, Ax) + c_3 \cdot d(y, Ay) \quad (3.1)$$

for all $x, y \in S$ and $c_1 + c_2 + c_3 < 1$. Then, A has a unique fixed point.

Proof: Let $x_0 \in S$ and $x_{n+1} = Ax_n, n = 0, 1, 2, \dots$ be the Picard iteration, then from (3.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Ax_{n-1}, Ax_n) \\ &\leq c_1 \cdot d(x_{n-1}, x_n) + c_2 \cdot d(x_{n-1}, Ax_{n-1}) + c_3 \cdot d(x_n, Ax_n) \\ &\leq c_1 \cdot d(x_{n-1}, x_n) + c_2 \cdot d(x_{n-1}, x_n) + c_3 \cdot d(x_n, x_{n+1}) \end{aligned}$$

This implies,

$$(1 - c_3)d(x_n, x_{n+1}) \leq (c_1 + c_2)d(x_{n-1}, x_n)$$

Now,

$$d(x_n, x_{n+1}) \leq \left(\frac{c_1 + c_2}{1 - c_3} \right) d(x_{n-1}, x_n)$$

Since, $c_1, c_2, c_3 \in (0, \frac{1}{2})$ then $0 < \left(\frac{c_1 + c_2}{1 - c_3} \right) < 1$

$$\text{Let } k = \left(\frac{c_1 + c_2}{1 - c_3} \right)$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) \\ &= kd(Ax_{n-2}, Ax_{n-1}) \\ &\leq k^2 d(x_{n-2}, x_{n-1}) \\ &= k^2 d(Ax_{n-3}, Ax_{n-2}) \\ &\leq k^3 d(x_{n-3}, x_{n-2}) \\ &\vdots \\ &\leq k^n d(x_0, x_1) \end{aligned}$$

Then, for $m, n \in \mathbb{N}; m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \dots + k^{m-1} d(x_0, x_1) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1}) d(x_0, x_1) \\ &= k^n (1 + k + k^2 + \dots + k^{m-n-1}) d(x_0, x_1) \\ &= \frac{k^n}{1 - k} d(x_0, x_1) \end{aligned}$$

This implies that, $d(x_n, x_m) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the sequence x_n is a Cauchy sequence in S and since S is complete, then the sequence converges to a point say u in S .

Moreso, there exists a point $v \in S$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$

Now substitute for $x = x_n$ and $y = v$ in (3.1) gives

$$\begin{aligned} d(x_n, Av) &= kd(Ax_{n-1}, Av) \\ &\leq c_1 \cdot d(x_{n-1}, v) + c_2 \cdot d(x_{n-1}, x_n) + c_3 \cdot d(v, Av) \end{aligned}$$

This implies that;

$$(1 - c_3)d(v, Av) \leq (c_1 + c_2)d(x_{n-1}, x_n)$$

Then,

$$d(v, Av) \leq \left(\frac{c_1 + c_2}{1 - c_3} \right) d(x_{n-1}, x_n)$$

$$\begin{aligned}
 &= k(d(x_{n-1}, x_n)) \\
 &\vdots \\
 &\leq k^n(d(x_0, x_1)) \rightarrow 0
 \end{aligned}$$

which implies, $d(v, Av) = 0$ if and only if $v = Av$. Let u be another fixed point, then

$$\begin{aligned}
 d(v, u) &= d(Av, Au) \\
 &\leq c_1.d(v, u) + c_2.d(v, Av) + c_3.d(u, Au) \\
 &= c_1.d(v, u) + c_2.d(v, v) + c_3.d(u, u) \\
 &\leq c_1.d(v, u) < d(v, u)
 \end{aligned}$$

That is, $(1 - c_1)d(v, u) \leq 0$.

But by the property of d , $d(v, u) = 0$ if and only if $v = u$ contradict the fact that, u is another fixed point. Hence, the proof.

Theorem: Let S be a semigroup and (S, d) a complete metric space. Let $E(OCT_n^*)$ be a generalized Idempotent of Order Preserving Full Contraction mapping and $A \in E(OCT_n)$ be a self map satisfying

$$d(Ax, Ay) \leq c_1.d(x, y) + c_2.d(x, Ax) + c_3.d(y, Ay) + c_4.d(x, Ay) + c_5.d(y, Ax) \quad (3.2)$$

for all $x, y \in S$ and $\sum_{i=1}^5 c_i < 1$.

Then, A has a unique fixed point.

Proof: Let $x_0 \in S$ and $x_n = Ax_{n-1}$ be an iterative sequence, then from (3.2) we have

$$\begin{aligned}
 d(x_n, x_{n+1}) &= d(Ax_{n-1}, Ax_n) \\
 &\leq c_1.d(x_{n-1}, x_n) + c_2.d(x_{n-1}, Ax_{n-1}) + c_3.d(x_n, Ax_n) + c_4.d(x_{n-1}, Ax_n) \\
 &\quad + c_5.d(x_n, Ax_{n-1}) \\
 &\leq c_1.d(x_{n-1}, x_n) + c_2.d(x_{n-1}, x_n) + c_3.d(x_n, x_{n+1}) \\
 &\quad + c_4.d(x_{n-1}, x_{n+1}) + c_5.d(x_n, x_n)
 \end{aligned} \quad (3.3)$$

Then, by triangular inequality;

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$$

Now (3.3) becomes

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq c_1.d(x_{n-1}, x_n) + c_2.d(x_{n-1}, x_n) + c_3.d(x_n, x_{n+1}) \\
 &\quad + c_4.[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + c_5.d(x_n, x_n)
 \end{aligned}$$

This implies,

$$(1 - c_3 - c_4)d(x_n, x_{n+1}) \leq (c_1 + c_2 + c_4)d(x_{n-1}, x_n)$$

Now,

$$d(x_n, x_{n+1}) \leq \left(\frac{c_1 + c_2 + c_4}{1 - c_3 - c_4} \right) d(x_{n-1}, x_n)$$

Since, $\sum_{i=1}^5 c_i < 1$ then, $0 < \left(\frac{c_1 + c_2 + c_4}{1 - c_3 - c_4} \right) < 1$

Let $k = \left(\frac{c_1 + c_2 + c_4}{1 - c_3 - c_4} \right)$

Therefore,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) \\ &= kd(Ax_{n-2}, Ax_{n-1}) \\ &\leq k^2 d(x_{n-2}, x_{n-1}) \\ &= k^2 d(Ax_{n-3}, Ax_{n-2}) \\ &\leq k^3 d(x_{n-3}, x_{n-2}) \\ &\vdots \\ &\leq k^n d(x_0, x_1) \end{aligned}$$

Then, for $m, n \in \mathbb{N}; m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \dots + k^{m-1} d(x_0, x_1) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \dots + k^{m-1}) d(x_0, x_1) \\ &= k^n (1 + k + k^2 + \dots + k^{m-n-1}) d(x_0, x_1) \\ &= \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

Then, there exists a point $v \in S$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$

Substitute for $x = x_n$ and $y = v$ in (3.2) gives

$$\begin{aligned} d(x_n, Av) &= kd(Ax_{n-1}, Av) \\ &\leq c_1 \cdot d(x_{n-1}, v) + c_2 \cdot d(x_{n-1}, Ax_{n-1}) + c_3 \cdot d(v, Av) \\ &\quad + c_4 \cdot d(x_{n-1}, Av) + c_5 \cdot d(v, Ax_{n-1}) \\ &\leq c_1 \cdot d(x_{n-1}, v) + c_2 \cdot d(x_{n-1}, x_n) + c_3 \cdot d(v, Av) \\ &\quad + c_4 \cdot d(x_{n-1}, Av) + c_5 \cdot d(v, x_n) \\ &\leq c_1 \cdot d(x_{n-1}, v) + c_2 \cdot d(x_{n-1}, v) + c_3 \cdot d(v, Av) \\ &\quad + c_4 \cdot d(x_{n-1}, Av) + c_5 \cdot d(v, v) \end{aligned} \tag{3.4}$$

But,

$$d(x_{n-1}, Av) \leq d(x_{n-1}, x_n) + d(x_n, Av)$$

and

$$d(x_{n-1}, v) \leq d(x_{n-1}, x_n) + d(x_n, v)$$

Now (3.4) becomes

$$\begin{aligned} d(v, Av) &\leq c_1 [d(x_{n-1}, x_n) + d(x_n, v)] + c_2 [d(x_{n-1}, x_n) + d(x_n, v)] + c_3 \cdot d(v, Av) \\ &\quad + c_4 [d(x_{n-1}, x_n) + d(x_n, Av)] \\ &\leq c_1 [d(x_{n-1}, x_n) + d(v, v)] + c_2 [d(x_{n-1}, x_n) + d(v, v)] + c_3 \cdot d(v, Av) \end{aligned}$$

$$\begin{aligned}
 &+ c_4[d(x_{n-1}, x_n) + d(v, Av)] \\
 &\leq c_1 d(x_{n-1}, x_n) + c_2 d(x_{n-1}, x_n) + c_3 d(v, Av) \\
 &+ c_4[d(x_{n-1}, x_n) + d(v, Av)]
 \end{aligned}$$

This implies that, $(1 - c_3 - c_4)d(v, Av) \leq (c_1 + c_2 + c_4)d(x_{n-1}, x_n)$

$$\text{That is, } d(v, Av) \leq \frac{(c_1 + c_2 + c_4)}{(1 - c_3 - c_4)} d(x_{n-1}, x_n)$$

$$\text{Let } k = \frac{(c_1 + c_2 + c_4)}{(1 - c_3 - c_4)}$$

Then,

$$\begin{aligned}
 d(v, Av) &\leq \left(\frac{c_1 + c_2 + c_4}{1 - c_3 - c_4} \right) d(x_{n-1}, x_n) \\
 &= k(d(x_{n-1}, x_n)) \\
 &\leq k(d(Ax_{n-2}, Ax_{n-1})) \\
 &= k^2(d(x_{n-2}, x_{n-1})) \\
 &\vdots \\
 &k^n(d(x_0, x_1)) \rightarrow 0
 \end{aligned}$$

Now by the property of d gives $d(v, Av) = 0$ if and only if $v = Av$.

Let u be another fixed point and assume that $u \neq v$, then

$$\begin{aligned}
 d(v, u) &= d(Av, Au) \\
 &\leq c_1 d(v, u) + c_2 d(v, Av) + c_3 d(u, Au) + c_4 d(v, Au) + c_5 d(u, Av) \\
 &= c_1 d(v, u) + c_2 d(v, v) + c_3 d(u, u) + c_4 d(v, u) + c_5 d(u, v) \\
 &\leq (c_1 + c_4 + c_5)d(v, u)
 \end{aligned}$$

This contradicts the fact that, u is another fixed point and $u \neq v$. Thus, $u = v$. Hence, the proof.

Theorem: Let (S, d) be a complete metric space and $A, B: S \rightarrow S$ be two self maps in $E(OCT_n)$ where B is the inverse of A . Then, if the operators satisfying

$$d(Ax, By) \leq c_1 d(x, y) + c_2 d(x, Ax) + c_3 d(y, By) + c_4 d(x, By) + c_5 d(y, Ax) \quad (3.5)$$

for any $x, y \in E(OCT_n) \in S$.

Then, for $c_1, c_2, c_3, c_4, c_5 \in \mathbb{C} < \frac{1}{2}$ and $\sum_{i=1}^5 c_i < 1$

(i) A and B have a unique common fixed point, that is $Av = Bv = v$

(ii) The Picard iteration associated to A and B , that is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = Ax_{n-1} = A^n x_0, n \geq 0$$

and

$$x_{n+1} = Bx_n = B^{n+1} x_0, n \geq 0$$

converges to v for any initial guess $x_0 \in S$.

Proof: Let $x_0 \in S$, assuming $u, v \in F(A)$ and $u \neq v$, then we get contradiction.

Now, we define $\{x_n\}_{n=0}^\infty \in A, B$ such that;

$x_{2n+1} = Ax_{2n}$ and $x_{2n} = Bx_{2n-1}$ when n is odd

and

$x_{2n} = Ax_{2n-1}$ and $x_{2n+1} = Bx_{2n}$ when n even

Then, from (3.5) we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(Ax_{2n}, Bx_{2n-1}) \\ &\leq c_1 \cdot d(x_{2n}, x_{2n-1}) + c_2 \cdot d(x_{2n}, Ax_{2n}) + c_3 \cdot d(x_{2n-1}, Bx_{2n-1}) \\ &\quad + c_4 \cdot d(x_{2n}, Bx_{2n-1}) + c_5 \cdot d(x_{2n-1}, Ax_{2n}) \\ &\leq c_1 \cdot d(x_{2n}, x_{2n-1}) + c_2 \cdot d(x_{2n}, x_{2n+1}) + c_3 \cdot d(x_{2n-1}, x_{2n}) \\ &\quad + c_4 \cdot d(x_{2n}, x_{2n}) + c_5 \cdot d(x_{2n-1}, x_{2n+1}) \end{aligned}$$

This implies,

$$(1 - c_2) d(x_{2n+1}, x_{2n}) \leq (c_1 + c_3) d(x_{2n}, x_{2n-1}) + c_5 \cdot d(x_{2n-1}, x_{2n+1}) \quad (3.6)$$

But by triangular inequality;

$$d(x_{2n-1}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})$$

Now, (3.6) becomes

$$(1 - c_2) d(x_{2n+1}, x_{2n}) \leq (c_1 + c_3) d(x_{2n}, x_{2n-1}) + c_5 \cdot [d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]$$

This implies,

$$(1 - c_2 - c_5) d(x_{2n+1}, x_{2n}) \leq (c_1 + c_3 + c_5) d(x_{2n-1}, x_{2n}) \quad (3.7)$$

When n is even, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Ax_{2n-1}, Bx_{2n}) \\ &\leq c_1 \cdot d(x_{2n-1}, x_{2n}) + c_2 \cdot d(x_{2n-1}, Ax_{2n-1}) + c_3 \cdot d(x_{2n}, Bx_{2n}) \\ &\quad + c_4 \cdot d(x_{2n-1}, Bx_{2n}) + c_5 \cdot d(x_{2n}, Ax_{2n-1}) \\ &\leq c_1 \cdot d(x_{2n-1}, x_{2n}) + c_2 \cdot d(x_{2n-1}, x_{2n}) + c_3 \cdot d(x_{2n}, x_{2n+1}) \\ &\quad + c_4 \cdot d(x_{2n-1}, x_{2n+1}) + c_5 \cdot d(x_{2n}, x_{2n}) \end{aligned}$$

This implies,

$$(1 - c_3) d(x_{2n}, x_{2n+1}) \leq (c_1 + c_2) d(x_{2n-1}, x_{2n}) + c_4 \cdot d(x_{2n-1}, x_{2n+1}) \text{ which is equivalent to}$$

$$(1 - c_3) d(x_{2n}, x_{2n+1}) \leq (c_1 + c_2) d(x_{2n-1}, x_{2n}) + c_4 \cdot [d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})] \text{ It gives,}$$

$$(1 - c_2 - c_4) d(x_{2n}, x_{2n+1}) \leq (c_1 + c_2 + c_4) d(x_{2n-1}, x_{2n}) \quad (3.8)$$

Now, add (3.7) and (3.8) gives

$$(2 - c_2 - c_3 - c_4 - c_5) d(x_{2n}, x_{2n+1}) \leq (2c_1 + c_2 + c_3 + c_4 + c_5) d(x_{2n-1}, x_{2n})$$

This implies that,

$$d(x_{2n}, x_{2n+1}) \leq \left(\frac{2c_1 + c_2 + c_3 + c_4 + c_5}{2 - c_2 - c_3 - c_4 - c_5} \right) d(x_{2n-1}, x_{2n})$$

Since $\sum_{i=1}^5 c_i < 1$, then

$$0 < \frac{2c_1 + c_2 + c_3 + c_4 + c_5}{2 - c_2 - c_3 - c_4 - c_5} < 1$$

$$\text{Let } k = \left(\frac{2c_1 + c_2 + c_3 + c_4 + c_5}{2 - c_2 - c_3 - c_4 - c_5} \right) < 1$$

Therefore,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq kd(x_{2n-1}, x_{2n}) \\ &= kd(Ax_{2n-2}, Ax_{2n-1}) \\ &\leq k^2 d(x_{2n-2}, x_{2n-1}) \\ &= k^2 d(Ax_{2n-3}, Ax_{2n-2}) \\ &\leq k^3 d(x_{2n-3}, x_{2n-2}) \\ &\vdots \\ &\leq k^{2n} d(x_0, x_1) \end{aligned}$$

Then, for $m, n \in \mathbb{N}; m > n$ we have

$$\begin{aligned} d(x_{2n}, x_{2m}) &\leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \dots + d(x_{2m-1}, x_{2m}) \\ &\leq k^{2n} d(x_0, x_1) + k^{2n+1} d(x_0, x_1) + \dots + k^{2m-1} d(x_0, x_1) \\ &\leq (k^{2n} + k^{2n+1} + k^{2n+2} + \dots + k^{2m-1}) d(x_0, x_1) \\ &= k^{2n} (1 + k + k^2 + \dots + k^{2m-2n-1}) d(x_0, x_1) \\ &= \frac{k^{2n}}{1-k} d(x_0, x_1) \end{aligned}$$

Now substitute for $x = x_{2n}$ and $y = v$ in (3.5) gives

$$\begin{aligned} d(x_{2n}, Bv) &= kd(Ax_{2n-1}, Bv) \\ &\leq c_1 d(x_{2n-1}, v) + c_2 d(x_{2n-1}, Ax_{2n-1}) + c_3 d(v, Bx_{2n}) \\ &\quad + c_4 d(x_{2n-1}, Bx_{2n}) + c_5 d(v, Ax_{2n-1}) \\ &\leq c_1 d(x_{2n-1}, x_{2n}) + c_2 d(x_{2n-1}, x_{2n}) + c_3 d(v, Bv) \\ &\quad + c_4 d(x_{2n-1}, Bv) + c_5 d(v, v) \end{aligned}$$

Also,

$$d(x_{2n-1}, Bv) \leq d(x_{2n-1}, x_n) + d(x_n, Bv)$$

This implies that;

$$(1 - c_3 - c_4) d(v, Bv) \leq (c_1 + c_2 + c_4) d(x_{2n-1}, x_{2n})$$

Then,

$$\begin{aligned} d(v, Bv) &\leq \left(\frac{1 - c_3 - c_4}{c_1 + c_2 + c_4} \right) d(x_{2n-1}, x_{2n}) \\ &= k(d(x_{2n-1}, x_{2n})) \\ &\leq k(d(Bx_{2n-2}, Bx_{2n-1})) \\ &= k^2(d(x_{2n-2}, x_{2n-1})) \\ &\vdots \\ &\leq k^{2n}(d(x_0, x_1)) \rightarrow 0 \end{aligned}$$

Implies that, $d(v, Bv) = 0$ if and only if $v = Bv$.

Therefore, by Theorem (3.1.2), $Av = Bv = v$ and $x_{2n} \rightarrow v$ as $n \rightarrow \infty$ for each $x_0 \in S$.

Let u be another fixed point, then

$$\begin{aligned} d(v, u) &= d(Av, Bu) \\ &\leq c_1.d(v, u) + c_2.d(v, Av) + c_3.d(u, Bu) + c_4.d(v, Bu) + c_5.d(u, Av) \\ &= c_1.d(v, u) + c_2.d(v, v) + c_3.d(u, u) + c_4.d(v, u) + c_5.d(u, v) \\ &\leq (c_1 + c_4 + c_5)d(v, u) < d(v, u) \end{aligned}$$

This contradict the fact that, u is another fixed point and $u \neq v$. Thus, $u = v$. Hence the proof.

Property P and Q on $E(OCT_n)$ in metric spaces

We want to prove that $E(OCT_n)$ satisfy property P and Q as earlier stated in Definitions (2.4) and (2.5).

Property P on $E(OCT_n)$

Theorem: Let (S, d) be a complete metric space and A a self map on $E(OCT_n)$ satisfying (3.1) with fixed point set $F(A) \neq \emptyset$. Then, for any given self map A and its inverse in $E(OCT_n)$ has property P .

Proof: From Theorem (3.1.1), A has a fixed point. This implies that $F(A) \neq \emptyset$ and $F(A^n)$ exist, since $x_n = Ax_{n-1} = A^n x_0$ and $x_{n+1} = Ax_n = A^{n+1} x_0$ for each $n \in \mathbb{N}$. Suppose that, A does not has a fixed point that is $v \neq Av$, then for positive integer $n \geq 1$ and from (3.1) we have

$$\begin{aligned} d(v, A^n v) &= d(A^n v, A^{n+1} v) \\ &\leq c_1.d(A^{n-1} v, A^n v) + c_2.d(A^{n-1} v, A^n v) + c_3.d(A^n v, A^{n+1} v) \\ &\leq c_1.d(A^{n-1} v, A^n v) + c_2.d(A^{n-1} v, A^n v) + c_3.d(v, A^n v) \end{aligned}$$

which gives,

$$d(v, A^n v) \leq \frac{c_1 + c_2}{1 - c_3} (d(A^{n-1} v, A^n v))$$

Since $k = \frac{c_1 + c_2}{1 - c_3} < 1$ then;

$$\begin{aligned} d(v, A^n v) &\leq k(d(A^{n-1} v, A^n v)) \\ &= k(d(AA^{n-2} v, AA^{n-1} v)) \\ &\leq k^2(d(A^{n-2} v, A^{n-1} v)) \\ &\vdots \\ &\leq k^n(d(v, Av)) \rightarrow 0 \end{aligned}$$

This shows that, $v = A^n v$ by the property of d and moreso by Theorem (3.1.1), $v = Av = A^n v$. Hence, $v \in F(A^n)$ implies that A satisfy property P .

Property Q on $E(OCT_n)$

Theorem: Let (S, d) be a complete metric space and $A, B: S \rightarrow S$ be two self maps in $E(OCT_n)$ where B is the inverse of A . Then, for any given self map A and its inverse in $E(OCT_n)$ satisfying (3.5) have property Q .

Proof: From Theorem (3.1.3), there exists a unique common fixed point of A and B . Moreover, any fixed of A is a fixed point of B and conversely. This implies that $F(A) \cap F(B) \neq \emptyset$ and so $F(A^n) \cap F(B^n) \neq \emptyset$ for each positive integer n . Then, for each $n \in \mathbb{N}$. Suppose that, B does not has a fixed point that is $v \neq Bv$, then for positive integer $n \geq 1$ and from (3.5) we have

$$\begin{aligned} d(v, Bv) &= d(A^n v, BB^n v) \\ &= d(AA^{n-1}v, B^{n+1}v) \\ &\leq c_1.d(A^{n-1}v, B^n v) + c_2.d(A^{n-1}v, AA^{n-1}v) + c_3.d(B^n v, B^{n+1}v) \\ &\quad + c_4.d(A^{n-1}v, B^{n+1}v) + c_5.d(B^n v, AA^{n-1}v) \\ &\leq c_1.d(A^{n-1}v, B^n v) + c_2.d(A^{n-1}v, v) + c_3.d(B^n v, Bv) \\ &\quad + c_4.d(A^{n-1}v, B^n v) + c_5.d(B^n v, v) \\ &\leq c_1.d(A^{n-1}v, B^n v) + c_2.d(v, v) + c_3.d(v, Bv) \\ &\quad + c_4.d(v, Bv) + c_5.d(v, v) \\ &\leq c_1.d(A^{n-1}v, B^n v) + c_3.d(v, Bv) + c_4.d(v, Bv) \end{aligned}$$

gives

$$d(v, Bv) \leq \left(\frac{c_1}{1 - c_3 - c_4} \right) d(A^{n-1}v, B^n v)$$

$$\text{Now, let } k = \left(\frac{c_1}{1 - c_3 - c_4} \right)$$

$$\begin{aligned} d(v, Bv) &\leq kd(A^{n-1}v, B^n v) \\ &= kd(AA^{n-2}v, BB^{n-1}v) \\ &\leq k^2 d(A^{n-2}v, B^{n-1}v) \\ &\vdots \\ &= k^n d(v, Bv) \rightarrow 0 \end{aligned}$$

This implies $d(v, Bv) = 0$ if and only if $v = Bv$, then by Theorem (3.1.3), $v = Av = Bv$ which contradict our assumption that $v \neq Bv$. Hence, $v \in F(A) \cap F(B)$ implies that A and B satisfy property Q .

Corollary: Let (S, d) be a complete metric space and A be a self map on $E(OCT_n)$ satisfying

$$d(A^n x, A^n y) \leq c_1.d(x, y) + c_2.d(x, A^n x) + c_3.d(y, A^n y) \quad (3.9)$$

for all $x, y \in S; c_1, c_2, c_3 \in (0, \frac{1}{2})$. Then, A^n has a unique fixed point.

Proof: Let $A = A^n$ in Theorem (3.1.1), then A^n has a unique point say $v \in S$. Since, $F(A) \neq \emptyset$, then by Theorem (3.2.1), $F(A^n)$ exists and $v \in F(A^n)$ which shows that $v = Av = A^n$.

Now, let v and u be two fixed points of A^n such that $v \leq u$ then letting $x = v$ and $y = u$ in (3.9) gives;

$$\begin{aligned} d(v, u) &= d(A^n v, A^n u) \\ &\leq c_1.d(v, u) + c_2.d(v, A^n v) + c_3.d(u, A^n u) \\ &\leq c_1.d(v, u) + c_2.d(v, v) + c_3.d(u, u) \\ &\leq c_1.d(v, u) \end{aligned}$$

This implies that, $d(v, u) = 0$ if and only if $v = u$. Hence, the proof.

Corollary: Let (S, d) be a complete metric space and A, B be two self maps on $E(OCT_n)$ satisfying:

$$d(A^n x, B^n y) \leq c_1.d(x, y) + c_2.d(x, A^n x) + c_3.d(y, B^n y) + c_4.d(x, B^n y) + c_5.d(y, A^n x) \quad (3.10)$$

for $\sum_{i=1}^5 c_i < 1$. Then, A^n, B^n have a unique common fixed point.

Proof: Let $A = A^n$ and $B = B^n$ in Theorem (3.1.3), then A^n and B^n have a unique fixed point say $v \in S$. Since, $F(A) \neq \emptyset$ and $F(B) \neq \emptyset$, then by Theorem (3.3.1), $F(A^n) \cap F(B^n)$ exists and $v \in F(A^n)$ which shows that $Av = Bv = v$ and $A^n = B^n = v$ is also a fixed point of A^n and B^n .

Now, let v and u be two fixed points of A^n and B^n such that $v \neq u$, then letting $x = v$ and $y = u$ in (3.10) gives

$$\begin{aligned} d(v, u) &= d(A^n v, B^n u) \\ &\leq c_1.d(v, u) + c_2.d(v, A^n v) + c_3.d(u, B^n u) + c_4.d(v, B^n u) + c_5.d(u, A^n v) \\ &\leq c_1.d(v, u) + c_2.d(v, v) + c_3.d(u, u) + c_4.d(v, u) + c_5.d(u, v) \\ &= (c_1 + c_4 + c_5).d(v, u) < d(v, u) \end{aligned}$$

By the property of d , $d(v, u) = 0$ if and only if $v = u$ which is a contradiction. Hence, the proof.

Example : Consider the subset $A, B \subset E(OCT_3)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

Also, let the function $f : [0, \infty) \rightarrow [0, \infty)$ be defined by

$$f_1(x) = \frac{x^2 + 1}{4} \text{ and } f_2(x) = \frac{x^2 - 1}{2x - 4}$$

satisfying A and B respectively with fixed point $x = 0.267949192$

Any problem of the form $f_1(x)$ and $f_2(x)$ above can be written equivalently as a fixed point problem $x = Ax = Bx$ where $A, B : S \rightarrow S$ are corresponding operators that allows us to use constructive fixed point tools in obtaining the solution.

Now, we want to show that for any self map and its inverse in $E(OCT_n)$ has a unique common fixed point. Let choose the initial guess $x_0 = 0.1$, then for $n \geq 0$ we have

$$\begin{aligned} Ax &= \frac{x^2 + 1}{4} \quad Bx = \frac{x^2 - 1}{2x - 4} \\ x_1 &= \frac{x_0^2 + 1}{4} \quad x_1 = \frac{x_0^2 - 1}{2x - 4} \\ &= 0.252500 = Ax_0 = 0.260526315 = Bx_0 \\ x_2 &= \frac{x_1^2 + 1}{4} \quad x_2 = \frac{x_1^2 - 1}{2x - 4} \\ &= 0.265939062 = Ax_1 = 0.267933354 = Bx_1 \\ x_3 &= \frac{x_2^2 + 1}{4} \quad x_3 = \frac{x_2^2 - 1}{2x - 4} \\ &= 0.267680896 = Ax_2 = 0.267949192 = Bx_2 \\ x_4 &= \frac{x_3^2 + 1}{4} \quad x_4 = \frac{x_3^2 - 1}{2x - 4} \\ &= 0.267913265 = Ax_3 = 0.267949192 = Bx_3 \\ x_5 &= \frac{x_4^2 + 1}{4} \quad x_5 = \frac{x_4^2 - 1}{2x - 4} \\ &= 0.267913265 = Ax_4 = 0.267949192 = Bx_4 \\ x_6 &= \frac{x_5^2 + 1}{4} \quad x_6 = \frac{x_5^2 - 1}{2x - 4} \\ &= 0.267948547 = Ax_5 = 0.267949192 = Bx_5 \\ x_7 &= \frac{x_6^2 + 1}{4} \quad x_7 = \frac{x_6^2 - 1}{2x - 4} \\ &= 0.267949106 = Ax_6 = 0.267949192 = Bx_6 \\ x_8 &= \frac{x_7^2 + 1}{4} \quad x_8 = \frac{x_7^2 - 1}{2x - 4} \end{aligned}$$

$$\begin{aligned} &= 0.26794918 = Ax_7 = Ax_{n-1} = 0.267949192 = Bx_7 = Bx_{n-1} \\ x_9 &= \frac{x_8^2 + 1}{4} x_9 = \frac{x_8^2 - 1}{2x - 4} \\ &= 0.26794919 = Ax_8 = Ax_n = 0.267949192 = Bx_8 = Bx_n \\ x_{10} &= \frac{x_9^2 + 1}{4} x_{10} = \frac{x_9^2 - 1}{2x - 4} \\ &= 0.267949192 = Ax_9 = Ax_{n+1} = 0.267949192 = Bx_9 = Bx_{n+1} \end{aligned}$$

This shows that, $Av = Bv = v$. Hence the result.

Conclusion

Fixed point theorems for the sub-semigroups of full contraction mapping in metric space is a new class in metric fixed point theory due to the fact that little work has been done in the area. In this manuscript, we considered a functional approach for the solution of sub-semigroups contraction mappings of T_n such as, the idempotents of order-preserving in metric fixed point theory.

In each case, a Picard iteration scheme was used to obtain a sequence that converges to a solution. Consequently, this study has facilitated the transition between linear algebra and functional analysis and also a valuable sources of new ideas and methods in the further development of the theory. Since semigroup of transformation are of three types which are: Full, Partial and Partial one-one transformation. We only considered one out of them which was full transformation. It is then recommended that, the remaining two should be considered in metric spaces. More so, there are more spaces we can as well extend this work to such as: Banach space, Cone metric space, Hilbert space, topological space, normed space, inner product spaces and so on. Again, extending this work to coupled and trippled fixed point in metric space would also be a promising area of study.

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