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Chapter 2

Radius Problems for a Certain Class of Analytic Functions Involving the Salagean Derivative

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Abstract

In this paper we determine the radii of starlikeness and convexity of functions of the class $T_n^\alpha(\beta)$ introduced in [3] by Opoola. It is well known that for $n=0$, this class consists largely of analytic functions, which are not necessarily univalent in the unit disk. For this case $n=0$, the radius of univalence is indicated by the sufficiency of starlikeness.

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1. Introduction

Let A denote the class of functions:

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Also let P be the class of functions:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.2}$$

which are analytic in E and have positive real part.

In [3] Opoola introduced the subclass $T_n^\alpha(\beta)$ consisting of functions $f \in T$ which satisfy the inequality:

$$\operatorname{Re} \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta \tag{1.3}$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, D^n $(n \in N_0 = \{0, 1, 2, \dots\})$ is the Salagean derivative operator defined as

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]' \tag{1.4}$$

with $D^0 f(z) = f(z)$ and powers in (1.3) meaning principal values only. The geometric condition (1.3) slightly modifies the one given originally in [3] (see [1]).

It is well-known that for $f \in A$ the condition that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \qquad |z| < r \tag{1.5}$$

is necessary and sufficient for starlikeness (and univalence) in the disk $|z| < r$, that is the function $f(z)$ maps the disk $|z| < r$ onto a starlike domain. Also a necessary and sufficient condition for $f \in A$ to be convex in the disk $|z| < r$ is that

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad |z| < r \tag{1.6}$$

If we know that a function maps the unit disk onto a domain having some nice geometric properties such as above, then we have the means of a more penetrating study of the function. These classes of functions were generalized by G.S. Salagean in [4]. Salagean stated that a function $f \in A$ belongs to the class $S_n(\lambda)$, $0 \leq \lambda < 1$ if and only if

$$\operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \lambda, \qquad |z| < r \tag{1.7}$$

where for $n = \lambda = 0$ we have (1.5) and $n = 1, \lambda = 0$ gives (1.6).

In the present paper we solve the conventional two radius problems for the class $T_n^\alpha(\beta)$. Our approach is to determine, for functions of this class, the radius of the largest disk $E_0 = \{z : |z| < r_0(\alpha, \beta)\}$ in which the conditions (1.7) holds for $\lambda = 0$. Thus the radii of starlikeness and convexity of functions in $T_n^\alpha(\beta)$ follows as simple corollaries. The class $T_n^\alpha(\beta)$ is a generalization of many known classes of function (see [3]). Thus the result of this work will naturally lead to those known about such classes. In particular, our main result generalizes those of Macgregor [2], Tuan and Anh [5] and Yamaguchi [6].

The main result of our paper is the following:

Theorem 1.1. *Let $f \in T_n^\alpha(\beta)$. Then $f \in S_n(0)$ for $|z| < r_0(\alpha, \beta)$ where*

$$r_0(\alpha, \beta) = \begin{cases} \frac{[(1+\alpha^2)(1-\beta)^2 - 2\alpha\beta(1-\beta)]^{\frac{1}{2}} + \beta(1+\alpha) - 1}{\alpha(1-2\beta)} & \text{if } 0 \leq \beta \leq \beta_0(\alpha) \\ \left(\frac{4[\alpha\beta(1-\beta)]^{\frac{1}{2}} - [(1-\alpha^2)(1-\beta) + 4\alpha\beta]}{(1+\alpha^2)(1-\beta) + 2\alpha(1-3\beta)}\right)^{\frac{1}{2}} & \text{if } \beta_0(\alpha) \leq \beta < 1,\ \beta \neq \frac{(1+\alpha)^2}{1+6\alpha+\alpha^2} \\ \left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}} & \text{if } \beta = \frac{(1+\alpha)^2}{1+6\alpha+\alpha^2}. \end{cases} \tag{1.8}$$

The results are the best possible.

Corollary 1.2. *Each function $f \in T_0^\alpha(\beta)$ maps the disk $|z| < r_0(\alpha,\beta)$ onto a starlike domain, where $r_0(\alpha,\beta)$ is defined by (1.8).*

The results are the best possible.

Corollary 1.3. *Each function $f \in T_1^\alpha(\beta)$ maps the disk $|z| < r_0(\alpha,\beta)$ onto a convex domain, where $r_0(\alpha,\beta)$ is defined by (1.8).*

The results are the best possible.

In Section 2., we state some preliminary lemmas, which we will depend on in the proof of our result in Section 3..

2. Preliminary Lemmas

We shall need the following lemmas:

Lemma 2.1 ([5]). *Let $p \in P$. Then for $0 \le \beta < 1$ and $|z| = r < 1$,*

$$\operatorname{Re}\left\{\frac{zp'(z)}{\frac{\beta}{1-\beta}+p(z)}\right\} \ge \begin{cases} \frac{-2(1-\beta)r}{(1+r)(1+(2\beta-1)r)}, & \text{for } R_1 \le R_2,\\ \frac{-\beta}{1-\beta}+\frac{1}{1-\beta}\left[2R_1-\frac{1-(2\beta-1)r^2}{1-r^2}\right] & \text{for } R_2 \le R_1,\end{cases} \tag{2.1}$$

where

$$R_1 = \left(\frac{\beta-\beta(2\beta-1)r^2}{1-r^2}\right)^{\frac{1}{2}} \quad \text{and} \quad R_2 = \frac{1+(2\beta-1)r}{1+r}. \tag{2.2}$$

The functions given by

$$p(z) = \begin{cases} \frac{1-z}{1+z} & \text{for } R_1 \le R_2,\\ \frac{1+z}{1-z} & \text{for } R_2 \le R_1,\end{cases} \tag{2.3}$$

show that the inequalities are sharp.

For a given $r < 1$, the transition from the first case $R_1 \le R_2$ to the second case $R_2 \le R_1$ takes place when $\beta = \beta_0 \in (0,1)$, where β_0 is determined by the equation $R_1 = R_2$.

Lemma 2.2. *Let $f \in A$, and $\alpha > 0$ be real. If $D^{n+1}f(z)^\alpha / D^n f(z)^\alpha$ takes a value which is independent of n, then*

$$\frac{D^{n+1}f(z)^\alpha}{D^n f(z)^\alpha} = \alpha\frac{D^{n+1}f(z)}{D^n f(z)}. \tag{2.4}$$

Proof. Let $D^{n+1}f(z)^\alpha / D^n f(z)^\alpha$ assume the same value for all $n \in N_0$. For $n = 0$, the assertion is easy to verify. Let $n = 1$. Then

$$\frac{D^2 f(z)^\alpha}{D^1 f(z)^\alpha} = 1 + \frac{zf''}{f'} + (\alpha-1)\frac{zf'}{f} = \frac{D^2 f(z)}{D^1 f(z)} + \left(1-\frac{1}{\alpha}\right)\frac{D^1 f(z)^\alpha}{D^0 f(z)^\alpha}.$$

But $D^1 f(z)^\alpha / D^0 f(z)^\alpha = D^2 f(z)^\alpha / D^1 f(z)^\alpha$ so that

$$\frac{D^2 f(z)^\alpha}{D^1 f(z)^\alpha} = \alpha \frac{D^2 f(z)}{D^1 f(z)}.$$

Next assume that (2.4) is true for some integer k. Then

$$\frac{D^{k+2} f(z)^\alpha}{D^{k+1} f(z)^\alpha} = (\alpha - 1)\frac{D^{k+1} f(z)}{D^k f(z)} + \frac{D^{k+2} f(z)}{D^{k+1} f(z)}$$

and

$$\frac{D^{k+2} f(z)^\alpha}{D^{k+1} f(z)^\alpha} = \left(1 - \frac{1}{\alpha}\right)\frac{D^{k+1} f(z)^\alpha}{D^k f(z)^\alpha} + \frac{D^{k+2} f(z)}{D^{k+1} f(z)}. \tag{2.5}$$

But $D^{k+1} f(z)^\alpha / D^k f(z)^\alpha$ has the same value for each k so that (2.5) can be written as

$$\frac{D^{k+2} f(z)^\alpha}{D^{k+1} f(z)^\alpha} = \left(1 - \frac{1}{\alpha}\right)\frac{D^{k+2} f(z)^\alpha}{D^{k+1} f(z)^\alpha} + \frac{D^{k+2} f(z)}{D^{k+1} f(z)},$$

that is,

$$\frac{D^{k+2} f(z)^\alpha}{D^{k+1} f(z)^\alpha} = \alpha \frac{D^{k+2} f(z)}{D^{k+1} f(z)}. \tag{2.6}$$

Hence the lemma follows by induction. □

We now turn to the proof of our main result.

3. Proof of the Main Result

Let $f \in T_n^\alpha(\beta)$, then there exists $p \in P$ such that

$$D^n f(z)^\alpha = \alpha^n z^\alpha [\beta + (1-\beta) p(z)]. \tag{3.1}$$

Logarithmic differentiation of (3.1) yields

$$\frac{D^{n+1} f(z)^\alpha}{D^n f(z)^\alpha} = \alpha + \frac{z p'(z)}{\frac{\beta}{1-\beta} + p(z)}. \tag{3.2}$$

Since the right hand of (3.2) is independent of n, it follows by Lemma 2.2 that

$$\alpha \frac{D^{n+1} f(z)}{D^n f(z)} = \alpha + \frac{z p'(z)}{\frac{\beta}{1-\beta} + p(z)}. \tag{3.3}$$

Now let R_1 and R_2 be as in Lemma 2.1 and suppose $R_1 \le R_2$. Then applying the lemma we have

$$\alpha \operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} \ge \alpha - \frac{2(1-\beta) r}{(1+r)(1+(2\beta-1)r)}. \tag{3.4}$$

It is easy to deduce from (3.4) that

$$\operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > 0, \tag{3.5}$$

provided $|z| = r < r_0(\alpha,\beta)$, where $r_0(\alpha,\beta)$ is the smallest positive root of the equation

$$\alpha(2\beta-1)r^2 + 2(\alpha\beta+\beta-1)r + \alpha = 0, \tag{3.6}$$

which is the first part of (1.8).

Evidently $r_0(\alpha,\beta)$, in this case, is not real except $\beta \le (1+\alpha^2)/(1+\alpha)^2$. Furthermore since $r_0(\alpha,\beta) < 1$, we must have $\beta < 1/(1+2\alpha)$.

Next we suppose $R_2 \le R_1$ then using Lemma 2.1 again, we have

$$\alpha \operatorname{Re} \frac{D^{n+1}f(z)}{D^n f(z)} \ge \alpha - \frac{\beta}{1-\beta} + \frac{1}{1-\beta}\left[2R_1 - \frac{1-(2\beta-1)r^2}{1-r^2}\right], \tag{3.7}$$

which implies that the condition (3.5) is satisfied if $|z| = r < r_0(\alpha,\beta)$, where $r_0(\alpha,\beta)$ is the smallest positive root of the equation

$$ar^4 + br^2 + c = 0, \tag{3.8}$$

where

$$\begin{aligned} a &= (1-\beta)^2(1+\alpha^2) + 2\alpha(1-\beta)(1-3\beta), \\ b &= 2(1-\beta)^2(1-\alpha^2) + 8\alpha\beta(1-\beta), \\ c &= \alpha\beta(\alpha\beta+2\beta-2\alpha) + (1-\beta)^2 + \alpha^2 - 2\alpha. \end{aligned}$$

If $\beta \ne (1+\alpha)^2/(1+6\alpha+\alpha^2)$, the solution of (3.8) is the second part of (1.8), otherwise we have the last segment of (1.8).

From equation (2.3) we deduce that the functions are given by

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \begin{cases} \frac{1-(1-2\beta)z}{1+z}, & \text{if} \quad 0 \le \beta \le \beta_0(\alpha), \\[2mm] \frac{1+(1-2\beta)z}{1-z}, & \text{if} \quad \beta_0(\alpha) \le \beta < 1, \end{cases} \tag{3.9}$$

show that the results are the best possible.

For any α, the number $\beta_0(\alpha) \in (0,1)$ is the solution of the equation:

$$\left(\frac{4[\alpha\beta(1-\beta)]^{\frac{1}{2}} - [(1-\alpha^2)(1-\beta)+4\alpha\beta]}{(1+\alpha^2)(1-\beta)+2\alpha(1-3\beta)}\right)^{\frac{1}{2}} = \frac{[(1+\alpha^2)(1-\beta)^2 - 2\alpha\beta(1-\beta)]^{\frac{1}{2}} + \beta(1+\alpha) - 1}{\alpha(1-2\beta)}. \tag{3.10}$$

For instance, if $\alpha = 1$, we have $\beta_0(1) = \frac{1}{10}$ as in [5, Theorem 3.1]. This completes the proof. □

Remark 3.1. *We note here that several cases of the parameters n, α and β lead to many known results. For instance, as indicated earlier, if we take $\alpha = 1$ in Corollaries 1.2 and 1.3, we have the results of Tuan and Anh [5, Theorems 3.1 and 4.1]. Also for $\beta = 0$, Corollary 1.2 gives a similar result to that of Yamaguchi [6, Theorem 1] while Corollary 1.3 similarly leads to a result of Macgregor [2, Theorem 2].*

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