

Comparative Analysis of the Numerical Effectiveness of the Four Kinds of Chebyshev Polynomials

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Abstract

In this paper, four kinds of Chebyshev polynomials are studied with the aim of identifying which of them gives more reliable results. It was eventually discovered, based on the numerical results, that the first and the fourth kinds perform better than the second and third kinds. While further comparison revealed that the second kind did better than the third kind.

Keyword: Chebyshev Polynomials, Canonical Polynomials, Recursive Formulation, Tau Approximant

1.0 Introduction

In the earlier work of Parker and Fox [1], the Chebyshev polynomials discussed were those of first and second kinds, with their derivation and application in the general interval $[a, b]$ were reported. The mini – max property of Chebyshev polynomials of the first kind in terms of error propagation was also discussed, where the superiority in performance of this polynomial over many others was x-rayed. Since then there have been some other attempts by some other scientist building on the work of Parker.

A rather more comprehensive work on Chebyshev polynomials (including exposition of its various kinds) was not reported until the famous work of Mason and Handscomb [2]. The reported in their masterpiece, the existence of two other kinds of Chebyshev polynomials in addition to the earlier two reported by Parker. The two were therefore tagged the third and fourth kinds Chebyshev polynomials respectively. The application of these various kinds of Chebyshev polynomials in the original tau method of Lanczos was carried out in the same work of theirs.

In this work, the four kinds of Chebyshev polynomials have been shifted from the traditional interval $[-1, 1]$ to the interval $[0, 1]$ (been the interval required for numerical problems of interest in this work). Thus, these shifted Chebyshev polynomials of the four kinds are discussed and applied in the Recursive Formulation of the tau method with the view of observing their individual performances. For the sake of completeness, a brief discussion on the various kinds of the Chebyshev polynomials will be stated here.

2.0 The Four Kinds of Chebyshev Polynomials

In this section, the four kinds of Chebyshev polynomials shall be discussed (although not in greater details) for the sake of completeness. Any reader that may be interested in further readings should do that in [2].

2.1 Chebyshev Polynomials of the First Kind

One of the solutions of the Chebyshev differential equation

$$(1 - x^2)y'' - 2xy' + n^2y = 0 \quad (2.1)$$

is

$$T_n(x) = \cos\{n\cos^{-1}x\} \quad (2.2)$$

with the substitution of $x = \cos \theta$, we obtain a trigonometric relation of the form

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta \quad (2.3)$$

From here the recurrence relation for the Chebyshev polynomials of the first kind is established as

$$T_{n+1}(x) = 2T_1(x)T_n(x) - T_{n-1}(x), \quad n \geq 1 \quad (2.4)$$

with $T_0(x) = 1$ and $T_1(x) = x$ (both obtained from (2.2)).

Since the interval $[-1, 1]$ is not most time the desired interval, then (2.2) is written in a more general interval $[a, b]$ to get

$$T_n^*(x) = \cos\left\{n\cos^{-1}\frac{(2x-a-b)}{b-a}\right\} \quad (2.5)$$

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which gives $T_0^*(x) = 1$ and $T_1^*(x) = \frac{2x-a-b}{b-a}$. And (2.4) becomes

$$T_{n+1}^*(x) = 2T_1^*(x)T_n^*(x) - T_{n-1}^*(x), \quad n \geq 1 \quad (2.6)$$

Chebyshev polynomials obtained from (2.6) are called shifted Chebyshev polynomials of the first kind. Let us restrict the interval to $[0, 1]$, the class to which most of the Ordinary Differential Equations (ODEs) do belong. Then, $T_0^*(x) = 1$, and $T_1^*(x) = 2x - 1$. Putting these in (2.6), we obtain the first few members of these polynomials as

$$n = 1: \quad T_2^*(x) = 8x^2 - 8x + 1$$

$$n = 2: \quad T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$

$$n = 3: \quad T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$$

and so on.

2.2 Chebyshev Polynomials of Second Kind

Another solution of (1) is

$$U_n(x) = \sin\{n\cos^{-1}x\}, \quad x \in [-1, 1] \quad (2.7)$$

The polynomials of this kind are derived from the trigonometric relation

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (2.8)$$

here x is again the same as $\cos\theta$. From (8), $U_0(x) = 1$ and $U_1(x) = 2\cos\theta$ (That is $U_0(x) = 1$ and $U_1(x) = 2x$).

Other members of these polynomials can as well be obtained from a recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geq 1 \quad (2.9)$$

The shifted Chebyshev polynomials of the second kind in the interval $[0, 1]$ are

$$U_{n+1}^*(x) = 2(2x-1)U_n^*(x) - U_{n-1}^*(x), \quad n \geq 1 \quad (2.10)$$

where $U_{n+1}^*(x) = 1$ and $U_1^*(x) = 4x - 2$. Some other few members are:

$$n = 1: \quad U_2^*(x) = 16x^2 - 16x + 3,$$

$$n = 2: \quad U_3^*(x) = 64x^3 - 96x^2 + 40x - 4$$

$$n = 3: \quad U_4^*(x) = 256x^4 - 512x^3 + 336x^2 - 80x + 5, \text{ etc.}$$

2.3 Chebyshev Polynomials of the Third Kind

These are obtained through the use of the trigonometric relation

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \quad n \geq 0 \quad (2.11)$$

From (11) $V_0(x)$ and $V_1(x)$ are obtained as $V_0(x) = 1$ and $V_1(x) = 2x - 1$.

Subsequent members of these polynomials are derived from the recurrence relation

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad n \geq 1 \quad (2.12)$$

The equivalent form of (12) when shifted to the interval $[0, 1]$ from the traditional interval $[-1, 1]$ is

$$V_{n+1}^*(x) = 2(2x-1)V_n^*(x) - V_{n-1}^*(x), \quad n \geq 1 \quad (2.13)$$

here $V_0^*(x) = 1$ and $V_1^*(x) = 4x - 3$. The other members include:

$$V_2^*(x) = 16x^2 - 16x + 3, \quad V_3^*(x) = 64x^3 - 112x^2 + 56x - 7,$$

$$V_4^*(x) = 256x^4 - 576x^3 + 432x^2 - 120x + 9, \text{ etc.}$$

2.4 Chebyshev Polynomials of the Fourth Kind

The Chebyshev polynomials of the fourth kind are obtained from the trigonometric relation

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \quad n \geq 0 \quad (2.14)$$

This gives $W_0(x) = 1$ and $W_1(x) = 2x + 1$.

The other members of these polynomials are equally derived from the recurrence relation

$$W_{n+1}(x) = 2xW_n(x) - W_{n-1}(x), \quad n \geq 1 \quad (2.15)$$

when the interval is now shifted from $[-1, 1]$ to the interval $[0, 1]$, we have the corresponding shifted Chebyshev polynomials as

$$W_{n+1}^*(x) = 2(2x-1)W_n^*(x) - W_{n-1}^*(x), \quad n \geq 1 \quad (2.15)$$

where $W_0^*(x) = 1$ and $W_1^*(x) = 4x - 1$. Few other members are:

$$W_2^*(x) = 16x^2 - 12x + 1$$

$$W_3^*(x) = 64x^3 - 80x^2 + 24x - 1$$

$$W_4^*(x) = 256x^4 - 448x^3 + 240x^2 - 40x + 1$$

3.0 Recursive Formulation of the TAU Method

Since the introduction of tau method by Lanczos [3], variants of the method have been developed. The choice of this variant (that is, the Recursive formulation) in this work was as a result of its advantages over the other two variants; the Differential formulation (otherwise referred to as the original formulation of the tau method) and the Integrated formulation. The said advantages are built around the existence of the so-called Canonical polynomials. The initial difficulty encountered by Lanczos was successfully resolved by Ortiz [4] when he generated them recursively. This therefore removed the initial restriction in application to the first order initial value problems (IVPs). The generalization of these canonical polynomials for the general class of both non-overdetermined and overdetermined m -th order (m not fixed) ODEs has been reported earlier [5,6], including the generalized forms of their derivatives. For the detail account of the tau method, interested readers should check [3-5]. Before now, it was shown in [10] that Legendre polynomials in some cases may also give numerically attractive results (the details of which can be found there in.)

4.0 Statement of the Problem and Methodology

In this work, the class of non-overdetermined m -th order IVP is considered

$$Ly(x) = \sum_{r=1}^m \left(\sum_{k=0}^{N_r} P_{r,k} x^k \right) y^{(r)}(x) = \sum_{r=0}^F f_r x^r \quad (4.1a)$$

with the supplementary conditions

$$L^* y(x_0) = y^{(k)}(x_0) = \alpha_k, \quad k = 0(1)m-1 \quad (4.1b)$$

where m is the order of the ODE, N_r and F are non-negative integers and $x_0, \alpha_k, f_r, P_{r,k}$ are given real numbers.

In solving problem (4.1) using the Recursive formulation, we seek an approximant

$$y_n = \sum_{r=0}^n a_r x^r, \quad n < +\infty \quad (4.2)$$

which now serves as the exact solution of the corresponding perturbed problem

$$Ly_n(x) = \sum_{r=0}^F f_r x^r + H_n(x) \quad (4.3)$$

where $H_n(x)$ is the perturbation term given as

$$H_n(x) = \sum_{k=1}^m \tau_k T_{n-k+1}^*(x) \quad (4.4)$$

where $\tau_r, r = 1(1)m$ are the tau parameters to be determined.

Unlike in some earlier [5, 7, 8], the Chebyshev polynomials terms (4.4) will assume each of the forms discussed in section 2.1, 2.2, 2.3 and 2.4 in succession. Integrating the canonical polynomials defined by the functional relation established by Lanczos [9] as

$$x^r = LQ_r(x) \quad (4.5)$$

into (4.3) yields

$$\alpha_\lambda = \sum_{r=0}^F f_r Q_r^{(\lambda)}(x) + \sum_{k=1}^m \tau_k \sum_{r=0}^{n-k+1} C_r^{(n-k+1)} Q_r^{(\lambda)}(x), \quad \lambda \geq 0(1)m-1 \quad (4.6)$$

where m is the order of the DE, λ is the order of the derivatives, r is the order of the canonical polynomial and α_λ is the set of initial values.

Also, the canonical polynomials $Q_r(x)$, and their derivatives $Q_r^{(\lambda)}(x)$ (with r and λ as defined above) as earlier reported [5] are:

$$Q_r(x) = \frac{x^r - \sum_{k=1}^m (\sum_{j=k}^m j! \binom{r}{j} P_{j,j-k} Q_{r-k}(x))}{\sum_{k=0}^m k! \binom{r}{k} P_{k,k}}, \quad r \geq 0, j \leq r \quad (4.7)$$

and

$$Q_r^{(\lambda)}(x) = \frac{\lambda! \binom{r}{\lambda} x^{r-\lambda} - \sum_{k=1}^m (\sum_{j=k}^m j! \binom{r}{j} P_{j,j-k}) Q_{r-k}^{(\lambda)}(x)}{\sum_{k=0}^m k! \binom{r}{k} P_{k,k}},$$

$$0 \leq \lambda \leq r, j \leq r \quad (4.8)$$

respectively.

(4.5) when expanded results into a tau system that is solved for constants, a 's and τ 's. The values thus obtained now substituted in the equation

$$y_n(x) = \sum_{r=0}^n Q_r(x)$$

to get the desired approximate solution.

5.0 Numerical Illustrations

The results obtained in the last section are applied to numerical problems of varying order constant coefficients, homogeneous and inhomogeneous initial value problems. The problems were handled by Mathematical - Tau - Program (MTP) [7].

Problem 5.1

Consider the variable coefficients homogeneous initial value problem

$$(1+x)y'(x) + y(x) = 0, \quad 0 \leq x \leq 1$$

with the initial value $y(0)=1$ and the analytic solution $y(x) = \frac{1}{x+1}$.

Table 1: Actual Error for the four kinds of Chebyshev polynomials for Problem 5.1

n	First Kind	Second Kind	Third Kind	Fourth Kind
6	2.61675E-5	1.66720E-5	2.93431E-5	1.27740E-5
7	1.81163E-7	2.88349E-6	5.09922E-6	2.13589E-6
8	1.44989E-7	5.00326E-7	8.76373E-7	3.58113E-7
9	1.07751E-9	8.47025E-8	1.50143E-7	6.06918E-8
10	7.67658E-10	1.48714E-8	1.59582E-8	1.02794E-8

Problem 5.2

Consider the variable coefficients inhomogeneous initial value problem

$$y'(x) - y(x) = 3(1+x), \quad 0 \leq x \leq 1$$

with the initial value $y(0)=2$ and the analytic solution $y(x) = 8e^x - 3x - 6$.

Table 2: Actual Error for the four kinds of Chebyshev polynomials for Problem 5.2

n	First Kind	Second Kind	Third Kind	Fourth Kind
6	9.94663E-7	1.20648E-6	2.04229E-6	7.78978E-7
7	2.92002E-8	3.64637E-8	6.30865E-8	2.45408E-8
8	7.51243E-10	1.04232E-9	1.72829E-9	6.46551E-10
9	1.80975E-11	2.53522E-11	4.22276E-11	1.63496E-11
10	4.05009E-13	5.96856E-13	9.73444E-13	3.55271E-13

Problem 5.3

Consider the third order constant coefficients homogeneous initial value problem

$$y'''(x) - 3y''(x) + 3y'(x) - y(x) = 0, \quad 0 \leq x \leq 1$$

with the initial conditions $y(0) = 2$, $y'(0) = 2$, $y''(0) = 10$ and the exact solution $y(x) = 2(1 + 2x^2)e^x$.

Table 3: Actual Error for the four kinds of Chebyshev polynomials for Problem 5.3

n	First Kind	Second Kind	Third Kind	Fourth Kind
6	8.23921E-3	8.68278E-3	4.14917E-1	2.49466E-3
7	2.81928E-4	4.50386E-4	1.08873E-3	9.66623E-5
8	9.03544E-6	1.97885E-5	4.59902E-5	3.37477E-6
9	2.70811E-7	7.48551E-7	1.69736E-6	1.08208E-7
10	7.55244E-9	2.48256E-8	5.52827E-8	3.11411E-9

Problem 5.4

Consider the second order constant coefficients homogeneous initial value problem

$$y''(x) - (1-x)y'(x) + y(x) = 0, \quad 0 \leq x \leq 1$$

with the initial conditions and exact $y(0) = 1$, $y'(0) = 1$, $y(x) = e^{x-\frac{x^2}{2}}$.

Table 4: Actual Error for the four kinds of Chebyshev polynomials for Problem 5.4

n	First Kind	Second Kind	Third Kind	Fourth Kind
6	2.61675E-5	4.65440E-5	1.00086E-4	1.34838E-5
7	1.81163E-7	1.83889E-7	1.40970E-6	7.20022E-8
8	1.44989E-7	4.12054E-7	8.81646E-7	8.12085E-8
9	1.07751E-9	4.11117E-9	1.98444E-9	9.67379E-10
10	7.67658E-10	2.79295E-9	5.97033E-9	4.15800E-10

Problem 5.5

Consider the second order constant coefficients inhomogeneous initial value problem

$$y''(x) + y(x) = x^2, \quad 0 \leq x \leq 1$$

with the initial values $y(0) = 1$, $y'(0) = 0$ and $y(x) = x^2 - 2 + 3\cos x$.

Table 5: Actual Error for the four kinds of Chebyshev polynomials for Problem 5.5

n	First Kind	Second Kind	Third Kind	Fourth Kind
6	4.53024E-7	7.80332E-7	1.81152E-6	2.65027E-7
7	2.84728E-8	5.37016E-8	1.15960E-7	1.57388E-8
8	3.92700E-10	8.74828E-10	1.9649E-9	1.98019E-10
9	1.62350E-11	4.67224E-11	9.86149E-11	1.02562E-11
10	1.99840E-13	6.11733E-13	1.34248E-12	1.14131E-13

The results presented above show that Chebyshev polynomials of the first and fourth kinds give results that are of the same order of actual errors. While the error propagation in the second kind is also evenly done, it nonetheless did not meet the level of accuracy of the aforementioned two. The third kind's numerical performance as depicted in the tables above justifies the earlier position of Mason et al. [2] on its performance.

6.0 References

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