

## COMBINATIONS OF GEOMETRIC EXPRESSIONS IMPLYING SCHLICHTNESS

K. O. BABALOLA

ABSTRACT. In this short paper we prove an interesting result that certain not-linear combinations of geometric expressions of the form

$$\psi(f, f', f'') + \lambda\phi(f, f', f'')$$

imply schlichtness of holomorphic functions, normalized by  $f(0) = 0$  and  $f'(0) = 1$ , in the unit disk  $E = \{z \in \mathbb{C} : |z| < 1\}$ .

### 1. INTRODUCTION

Let  $A$  be the class of functions of the form:

$$f(z) = z + a_2 z^2 + \dots$$

which are holomorphic in the unit disk  $E = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $f \in A$  is said to be of bounded turning, starlike and convex of order  $\alpha$ ,  $0 \leq \alpha < 1$  respectively if and only if, for  $z \in E$ ,  $\operatorname{Re} f'(z) > \alpha$ ,  $\operatorname{Re} z f'(z)/f(z) > \alpha$  and  $\operatorname{Re} (1 + z f''(z)/f'(z)) > \alpha$ . By usual notations we denote these classes of functions respectively by  $R(\alpha)$ ,  $S^*(\alpha)$  and  $C(\alpha)$ . These classes of functions all basically consist only of univalent functions in  $E$ . Similarly the condition that

$$\operatorname{Re} \frac{f^\lambda(z) f'(z)}{z^\lambda} > \alpha, \quad z \in E \tag{1.1}$$

for some real number  $\lambda \geq -1$  is also sufficient for schlichtness in the open unit disk  $E$ . Functions satisfying (1.1) are known as Bazilevic functions of order  $\alpha$ , type  $\lambda + 1$  and are generally schlicht in the open unit disk  $E$  [4]. Note that the classes of bounded turning and starlike functions respectively correspond to  $\lambda = 0$  and  $\lambda = -1$  in (1.1).

Investigations are well known about linear (convex) combinations of the form  $\gamma\psi(f, f', f'') + \lambda\phi(f, f', f'')$  where the real numbers  $\gamma$  and  $\lambda$  satisfy  $\gamma + \lambda = 1$ , and  $\psi$  and  $\phi$  represent some well known geometric expressions in geometric function theory. For example Miller et al [3] proved that for any real number  $\gamma$ , the positivity of the real part of the convex combination

$$(1 - \gamma) \frac{z f'(z)}{f(z)} + \gamma \left( 1 + \frac{z f''(z)}{f'(z)} \right)$$

implies univalence and starlikeness in the unit disk. However, very little is known about such combinations when the linearity conditions fail, except possibly for some indications which may be deduced from some results (see for example [1, 2]).

The object of the present paper is to prove that certain not-linear combinations of some geometric expressions imply schlichtness of normalized holomorphic functions  $f \in A$  in the

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open unit disk  $E$ . In particular, for some real numbers  $\lambda \geq -1$ , we shall prove that, under some mild restrictions, combinations of geometric expressions of the form

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) + \lambda \frac{zf'(z)}{f(z)} \quad (1.2)$$

imply schlichtness of holomorphic functions, normalized by  $f(0) = 0$  and  $f'(0) = 1$ , in  $E$ .

This is the idea we advance in this paper! In particular, we consider combinations (1.2) whose real parts are bounded below for the real number  $\lambda \geq -1$ . If we set  $\lambda = (1 - \gamma)/\gamma$ , ( $\gamma \geq 0$ ) in (1.2), we have the linear combination considered by Miller et al [3].

Our result shall depend on the following lemma of Frasin and Jahagiri [1].

**Lemma 1.1.** [1] *Let  $p$  be holomorphic in  $E$  with  $p(0) = 1$  and suppose that*

$$\operatorname{Re} \left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in E.$$

*Then  $\operatorname{Re} p(z) > \alpha$  in  $E$  and  $1/2 \leq \alpha < 1$ .*

Next we state and prove our result.

## 2. MAIN RESULT AND PROOF

**Theorem 2.1.** *If  $f \in A$  satisfies*

$$\operatorname{Re} \left\{ \left(1 + \frac{zf''(z)}{f'(z)}\right) + \lambda \frac{zf'(z)}{f(z)} \right\} > \lambda + \frac{3\alpha - 1}{2\alpha}, \quad z \in E.$$

*Then*

$$\operatorname{Re} \frac{f^\lambda(z)f'(z)}{z^\lambda} > \alpha, \quad 1/2 \leq \alpha < 1.$$

*Proof.* For  $z \in E$ , define a holomorphic function  $p(z)$  with  $p(0) = 1$  by

$$p(z) = \frac{f^\lambda(z)f'(z)}{z^\lambda}.$$

Then by simple differentiation we find that

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + \lambda \frac{zf'(z)}{f(z)} - \lambda$$

so that

$$\operatorname{Re} \left(1 + \frac{zp'(z)}{p(z)}\right) = \operatorname{Re} \left\{1 + \frac{zf''(z)}{f'(z)} + \lambda \frac{zf'(z)}{f(z)} - \lambda\right\} > \frac{3\alpha - 1}{2\alpha}$$

yields

$$\operatorname{Re} \left\{ \left(1 + \frac{zf''(z)}{f'(z)}\right) + \lambda \frac{zf'(z)}{f(z)} \right\} > \lambda + \frac{3\alpha - 1}{2\alpha},$$

which, by Lemma 1.1, implies

$$\operatorname{Re} \frac{f^\lambda(z)f'(z)}{z^\lambda} > \alpha, \quad 1/2 \leq \alpha < 1.$$

This concludes the proof. □

For specific real numbers  $\lambda$  and  $\alpha$  we have the following corollaries.

**Corollary 2.2.** *If  $f \in A$  satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > -\frac{1}{2}, \quad z \in E. \quad (1.3)$$

*Then*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}.$$

*That is  $f$  is starlike of order  $1/2$  in  $E$ .*

**Corollary 2.3.** *If  $f \in A$  satisfies*

$$\operatorname{Re} \left\{ 2 \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in E.$$

*Then*

$$\operatorname{Re} \frac{z^{1/2}f'(z)}{f^{1/2}(z)} > \frac{1}{2}.$$

*That is  $f$  is Bazilevic of order  $1/2$ , type  $1/2$  in  $E$ .*

**Corollary 2.4.** *If  $f \in A$  satisfies*

$$\operatorname{Re} \left\{ 2 \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \frac{zf'(z)}{f(z)} \right\} > 1, \quad z \in E.$$

*Then*

$$\operatorname{Re} \frac{f(z)^{1/2}f'(z)}{z^{1/2}} > \frac{1}{2}.$$

*That is  $f$  is Bazilevic of order  $1/2$ , type  $3/2$  in  $E$ .*

**Corollary 2.5.** *If  $f \in A$  satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in E.$$

*Then*

$$\operatorname{Re} f'(z) > \frac{1}{2}.$$

*That is  $f$  is bounded turning of order  $1/2$  in  $E$ .*

We note that Corollaries 2.2 and 2.5 agree respectively with Corollaries 2.4 and 2.5 in [1], and 2.7 and 2.6 in [2].

### 3. A NOTE ON EQUIVALENCE OF CLASSES OF FUNCTIONS

In the well written papers [1, 2], the authors erroneously proposed that the geometric conditions  $|\psi(f, f'f'', \dots) - 1| < 1 - \alpha$  and  $\operatorname{Re} \psi(f, f'f'', \dots) > \alpha$  are equivalent!

For instance Frasin and Jahangiri [1] introduced the class of functions  $\mathcal{B}(\mu, \alpha)$  as consisting of functions satisfying  $|f'(z)(z/f(z))^\mu - 1| < 1 - \alpha$  and remarked that  $\mathcal{B}(1, \alpha) \equiv \mathcal{S}_\alpha^*$  i.e. that  $|zf'(z)/f(z) - 1| < 1 - \alpha$  is equivalent to  $\operatorname{Re} zf'(z)/f(z) > \alpha$  as well as  $\mathcal{B}(0, \alpha) \equiv \mathcal{R}_\alpha$  i.e.  $|f'(z) - 1| < 1 - \alpha \equiv \operatorname{Re} f'(z) > \alpha$ .

In [2], Lupas and Catas followed similarly. In their Remark 2.2 they mentioned equivalences of the type indicated above. The notion is being used in the course of proofs and statements of results and corollaries. The present author has also come across the same notion being carried on in some other works in this field and thus finds it pertinent to call attention to the incorrectness of it.

This is since the first condition (i.e.  $|\psi(f, f'f'', \dots) - 1| < 1 - \alpha$ ) describes the interior of a circle centred at 1 of radius  $1 - \alpha$ , the diameter of which has endpoints  $\alpha$  and  $2 - \alpha$  on the real axis, while the later (i.e.  $\operatorname{Re} \psi(f, f'f'', \dots) > \alpha$ ) is the whole of the  $\alpha$ -half-plane from  $\alpha$  to infinity on the real axis minus the vertical  $\operatorname{Re} \psi = \alpha$ . Thus obviously, the former is contained in the later. Rather than equivalence, the later implies the former only and not conversely. This should be well known.

For further clarity, consider  $\psi(f, f'f'', \dots) = 1 + zf''(z)/f'(z)$ , then it is not difficult to see that the convex function  $f(z) = z/(1 - z)$  satisfies  $\operatorname{Re} 1 + zf''(z)/f'(z) > 0$  in the open unit disk while it does not satisfy  $|zf''(z)/f'(z)| < 1$  at  $z = 3/4$  or  $z = \frac{1}{2} + \frac{1}{2}i$  and in fact at many points, in the same disk. Similarly the well known Koebe function  $k(z) = z/(1 - z)^2$  satisfies  $\operatorname{Re} zf'(z)/f(z) > 0$  in the open unit disk but not  $|zf'(z)/f(z)| < 1$  at infinitely many points there.

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CURRENT: DEPARTMENT OF PHYSICAL SCIENCES, AL-HIKMAH UNIVERSITY, ILORIN, PERMANENT: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN  
*E-mail address:* kobabalola@gmail.com