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PATCHED SEGMENTED COLLOCATION TECHNIQUES FOR THE NUMERICAL SOLUTION OF SECOND ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT

This paper deals with the numerical solutions of boundary value problems by patched segmented collocation method. The method examined the numerical solutions of boundary value problems after the whole intervals of consideration have been partitioned into various sub-intervals and the solutions are sought in the various sub-intervals and then matched together using Chebyshev polynomials as the basis function . Numerical examples are given to illustrate the efficiency, accuracy and computational cost of the method. Results obtained are better than when the problems are solved within the whole intervals using the same approach.

Keywords: Patched Segmented Collocation Method, Boundary value problems, Chebyshev Polynomials, Accuracy and Approximation.

I.INTRODUCTION

The primary aim of this paper is to provide efficient and reliable methods for obtaining numerical solutions to problems that prove difficult in getting their solutions in closed form. For the purpose of our discussion, we have investigated the numerical solutions of second order boundary value problems of the form:

$$P(x)y^{//}(x)+Q(x)y^{/}(x)+R(x)y(x)=f(x), \quad a \leq x \leq b \tag{1}$$

together with the conditions

$$y(a) = A \tag{2}$$

and

$$y(b) = B \tag{3}$$

where y(a), y(b), A and B are known values which are valid in some intervals $a \leq x \leq b$ together with sufficient conditions imposed on the dependent variable at the two ends points x=a and x=b, where x is the independent variable , y(x) is an unknown function, P(x), Q(x), R(x) and f(x) are known smooth functions. Active research works have extensively being carried out in this area. Among the well known methods are Weighted Residual Galerkin, Collocation, Finite Element with power series and Canonical forms as basis functions, to mention a few (see[2,3,5,6]).

In this paper, we have investigated the same class of problems by partitioning the given intervals into segments and the sought results in these segments using Chebyshev Polynomials of degree N as our basis. We then patched the solutions together to give our required approximate solution. Here, we have used two numerical methods namely Standard and Perturbed methods without segmentation and Standard and Perturbed Patched Segmented collocation methods.

II. STANDARD COLLOCATION METHOD WITHOUT SEGMENTATION (SCMWS)

This method was discussed in [2]. In order to apply this method, we assume an approximate solution of the form:

$$y_N(x,a) = \sum_{i=0}^{N} a_i T_i(x) \tag{4}$$

where x represents the independent variables in the problem and the functions

$T_0(x), T_1(x),\ldots, T_N(x)$

are Chebyshev Polynomials defined in the interval [a,b] by

$$T_r(x) = Cos\left[r Cos^{-1}\left(\frac{2(x-a)}{b-a} - 1 \right) \right] \tag{5}$$

In order to solve equations (1)-(3) , we substitute equations (2) and (3) into equation (4), we obtain,

$$y_N(x,a) = \sum_{i=0}^{N} a_i T_i(x_a) = A \tag{6}$$

and,

$$y_N(x,b) = \sum_{i=0}^{N} a_i T_i(x_b) = B$$

(7)

Making a_0 the subject in equation (6), we obtain

$$a_0 = \frac{1}{T_0(x)}\left[A - \sum_{i=1}^{N} a_i T_i(x_a)\right]$$

(8)

Thus, substituting equation (8) into equation (7), after simplification, we obtain,

$$a_1 = \frac{1}{T_0(x_b)T_1(x_a) - T_0(x_a)T_1(x_b)}\left[(T_0(x_b)A - T_0(x_a)B) - \sum_{i=2}^{N} a_i (T_i(x_a)T_{i-2}(x_b) - T_{i-2}(x_a)T_i(x_b))\right] \quad (9)$$

Hence, substituting equation (9) into equation (8), after simplification, we obtain,

$$a_0 = \frac{1}{T_0(x_a)}\left[A - \frac{T_1(x_a)}{T_0(x_b)T_1(x_a) - T_0(x_a)T_1(x_b)}((T_0(x_b)A - T_0(x_a)B) - \sum_{i=2}^{N} a_i (T_i(x_a)T_{i-2}(x_b) - T_{i-2}(x_a)T_i(x_b))) - \sum_{i=2}^{N} a_i T_i(x_a)\right] \quad (10)$$

Finally, substituting equations (9) and (10) into equation (4) after simplification, we obtain,

$$a_0 = \frac{1}{T_0(x_a)}\left[A - \frac{T_1(x_a)}{T_0(x_b)T_1(x_a) - T_0(x_a)T_1(x_b)}((T_0(x_b)A - T_0(x_a)B) - \sum_{i=2}^{N} a_i (T_i(x_a)T_{i-2}(x_b) - T_{i-2}(x_a)T_i(x_b))) - \sum_{i=2}^{N} a_i T_i(x_a)\right] \quad (10)$$

Finally, substituting equation (9) into equation (8), after simplification, we obtain,

$$y_N(x,a) = \frac{T_0(x)}{T_0(x_a)}\left[A - \frac{T_1(x_a)}{T_0(x_b)T_1(x_a) - T_0(x_a)T_1(x_b)}((T_0(x_b)A - T_0(x_a)B) - \sum_{i=2}^{N} a_i (T_i(x_a)T_{i-2}(x_b) - T_{i-2}(x_a)T_i(x_b))) - \sum_{i=2}^{N} a_i T_i(x_a)\right]$$
$$+ \frac{T_1(x)}{T_0(x_b)T_1(x_a) - T_0(x_a)T_1(x_b)}\{(T_0(x_b)A - T_0(x_a)B) - \sum_{i=2}^{N} a_i (T_i(x_a)T_{i-2}(x_b) - T_{i-2}(x_a)T_i(x_b)$$
$$- T_i(x_a) - T_i(x))\} \quad (11)$$

We thus substitute equation (11) into equation (1), we obtain

$$P(x)y''_N(x,a) + Q(x)y'_N(x,a) + R(x)y_N(x,a) = f(x) \qquad (12)$$

Equation (12) is then collocated at points $x = x_k$, we obtain

$$P(x_k)y''_N(x_k,a) + Q(x_k)y'_N(x_k,a) + R(x_k)y_N(x_k,a) = f(x_k) \qquad (13)$$

where,

$$x_k = a + \frac{(b-a)k}{N}, \; j = 1,2,3.....,N-1 \qquad (14)$$

and a, b are respectively the lower and upper bounds of the interval . Hence , equation (13) gives rise to (N-1) linear systems of algebraic equations in (N-1) unknown constants a_i (i = 2, 3, . . . , N) which are then solved by Gaussian Elimination method to obtain the values of a_i (i = 2, 3, - - N) .The obtained values are then substituted back into our approximate solution given by equation (11).

III. PERTURBED COLLOCATION METHOD WITHOUT SEGMENTATION (PCMWS)

The whole idea of the perturbed collocation method is the addition of a small perturbed term , $H_N(x)$ as conceived by Lanczos [1] to equation (12). Thus , equation (12) now becomes

$$P(x)y''_N(x,a) + Q(x)y'_N(x,a) + R(x)y_N(x,a) = f(x) + H_N(x) \qquad (15)$$

where ,

$$H_N(x) = \tau_1 T_{N+2}(x) + \tau_2 T_{N+1}(x)$$

Here τ_1 *and* τ_2 are tau - parameters to be determined and $T_N(x)$ are Chebyshev polynomials defined by equation (5) .Thus equation (15) is then collocation at points x_j, we obtain

$$P(x_j)y''_N(x_j,a) + Q(x_j)y'_N(x_j,a) + R(x_j)y_N(x_j,a) = f(x_j) + \tau_1 T_{N+2}(x_j) + \tau_2 T_{N+1}(x_j). \quad . \qquad (16)$$

where,

$x_j = a + ((b-a)j)/(N+2)$,

$j = 1,2,3,\ldots,N+1$

Hence , equation (16) gives rises to (N + 1) linear systems of algebraic equations in (N+1) unknown constants a_i (i = 2, 3, - - - , N) τ_1 *and* τ_2. Also, these linear systems of algebraic equations are then solved using Gaussian elimination method to obtain the N + 1 unknown constants which are then substituted back into our approximate solution.

IV PATCHED STANDARD SEGMENTED COLLOCATION METHOD (PSSCM)

In this section , we consider the general second order differential equation of the form given in equation (1) together with the boundary conditions given in equations (2) and (3) . Also, our approximate solution given by equation (4) is ,modified as follows ;-

$$y_{mN}(x) = y_{1N}(x)\ U\ y_{2N}(x)\ U........U\ y_{mN}(x) \qquad (17)$$

where,

$$y_{1N}(x) = \sum_{i=0}^{N} a_{1i}T_{1N}(x), \quad x_a \le x \le x_1 \qquad (18)$$

$$y_{2N}(x) = \sum_{i=0}^{N} a_{2i}T_{2N}(x), \quad x_1 \le x \le x_2 \qquad (19)$$

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$$y_{mN}(x) = \sum_{i=0}^{N} a_{mi}T_{mN}(x), \quad x_m \le x \le x_N = x_b \qquad (20)$$

and m denotes the number of segments. To solve equation (1) in the first segment, a new condition is now added at the inter segment points to the initial condition given by equation (1), that is,

$$y(x_1) = A_1 \qquad (21)$$

where A_1 is the solution at $x = x_1$ in case the exact solution is known otherwise we forced the derivative to be equal to the initial condition. That is,

$$y(x_1) = y'(x_1)$$

Thus, putting equations (2) and (21) into equation (18) , after simplification , we obtain ,

$$y_{1N}(x_a, a) = \sum_{i=0}^{N} a_{1i}\, T_{1i}(x_a) = A \qquad (22)$$

and

$$y_{1N}(x_1, a) = \sum_{i=0}^{N} a_{1i}\, T_{1i}(x_1) = A_1 \qquad (23)$$

Making a_{10} the subject of the formula in equation (22) , we obtain,

$$a_{10} = \frac{1}{T_{10}(x)}\left[A - \sum_{i=0}^{N} a_{1i}T_{1I}(x_a)\right] \qquad (24)$$

Thus, substituting equation (24) into equation (23), after simplification and make a_{1i} the subject, we obtain,

$$a_{11} = \frac{1}{T_{10}(x_1)T_{11}(x_a) - T_{10}(x_a)T_{11}(x_1)}\left\{ (T_{10}(x_1)A - T_{10}(x_a)A_1) + \sum_{i=2}^{N} a_{1i}(T_{1i}(x_a)T_{1i-2}(x_1) - T_{1i-2}(x_a)T_{1i}(x_1)) - T_{1l}(x_a) \right\} \quad (25)$$

Hence, putting equation (25) into equation (24), after simplification, we obtain,

$$a_{10} = \frac{1}{T_{10}(x_a)}\left\{ \frac{T_{11}(x_a)}{T_{10}(x_1)T_{11}(x_a) - T_{10}(x_a)T_{11}(x_1)}((T_{10}(x_1)A - T_{10}(x_a)A_1) + \sum_{i=2}^{N} a_{1i}(T_{1i}(x_a)T_{1i-2}(x_1) - T_{1i-2}(x_a)T_{1i}(x_1))) - \sum a_{1i}T_{1i}(x_a) \right\} \quad (26)$$

Finally, substituting equations (25) and (26) into equation (18), thus reducing the number of unknown constants a_{1i} $(i = 2,3,...N)$ to a_{1j} $(j = 2,3,...,N-1)$.
That is,

$$\begin{aligned} y_N(x,a) = {} & \frac{T_{10}(x)}{T_{10}(x_a)}\left\{ A - \frac{T_{11}(x_a)}{T_{10}(x_1)T_{11}(x_a) - T_{10}(x_a)T_{11}(x_1)}[(T_{10}(x_1)A - T_{10}(x_a)A_1 + \sum_{i=2}^{N} a_{1i}(T_{1i}(x_a)T_{1i-2}(x_1) - T_{1i-2}(x_a)T_{1i}(x_1))) - \sum a_{1i}T_{1i}(x_a) \right\} \\ & + \frac{T_{11}(x)}{T_{10}(x_1)T_{11}(x_a) - T_{10}(x_a)T_1(x_1)}\{(T_{10}(x_1)A - T_{10}(x_a)A_1) + \sum_{i=2}^{N} a_{1i}(T_{1i}(x_a)T_{1i-2}(x_1) - T_{1i-2}(x_a)T_{1i}(x_1)) \\ & - T_{1i}(x))\} \end{aligned} \quad (27)$$

Thus equation (27) is our new approximate solution in the first segment with N -1 unknown constants. Thus equation (27) is then substituted into equation (1) to obtain,

$$P(x)y''_{1N}(x,a) + Q(x)y'_{1N}(x,a) + R(x)y_{1N}(x,a) = f(x) \quad (28)$$

We then collocate equation (28) at the point $x = x_j$, we obtain,

$$P(x_j)y''_{1N}(x_j,a) + Q(x_j)y'_{1N}(x_j,a) + R(x_j)y_{1N}(x_j,a) = f(x_j) \quad (29)$$

where

$$x_j = a + \frac{(x_1 - a)j}{N}, \; j = 1,2,3.....,N - 1 \qquad (30)$$

Thus, the collocation equation (29) gives rise to N -1 linear algebraic systems of equations in N -1 unknown constants. These equations are then solved by Gaussian Elimination to obtain the values of unknown constants which are then substituted back into equation (27) to obtain our approximate solution at the first segment. The process is repeated until the solutions are obtained throughout the other segments.

V. PATCHED PERTURBED SEGMENT COLLOCATION METHOD (PPSCM)

In this section, a perturbation parameter $H_{1N}(x)$ is added to equation (28), we obtain,

$$P(x)y_{1N}''(x,a) + Q(x)y_{1N}'(x,a) + R(x)y_{1N}(x,a) = f(x) + H_{1N}(x) \qquad (31)$$

where,

$$H_{1N}(x) = \tau_0 T_{1N-2}(x) + \tau_1 T_{1N-1}(x)$$

and τ_0 *and* τ_1 are free tau parameter to be determined. We then collocate equation (31) at the point $x=x_k$, we obtain,

$$P(x_k)y_{1N}''(x_k,a) + Q(x_k)y_{1N}'(x_k,a) + R(x_k)y_{1N}(x_k,a) = f(x_k) + \tau_0 T_{1N-2}(x_j) + \tau_1 T_{1N-1}(x_k) \,. \qquad (33)$$

where,

$$x_k = a + \frac{(x_1 - a)k}{N + 1}, \; k = 1,2,3.....,N + 1$$

Hence, the collocated equation (33) gives rise to N+1 linear algebraic systems of equations in N+1 unknown constants. These N+1 linear algebraic equations are then solved using Gaussian Elimination to obtain the values of the N+1 unknown constants which are then substituted back into our new approximate solution of equation (27).

VI. NUMERICAL EXAMPLES

Example 1: Solve the following problem:

$$12x^2\, y''(x) \square 24\, y'(x) \square\square 3x^4 \square 204x^3 \square 351x^2 \square 11x, \quad 0 \square x \square 1$$

with the boundary conditions,

$$y(0) \square 1 \ \textit{and}\ \ y(1) \square 2$$

$$y(x) \square \frac{1}{24}(\square 3x^4 \square 34x^3 \square 117x^2 \square 110x \square 24)$$

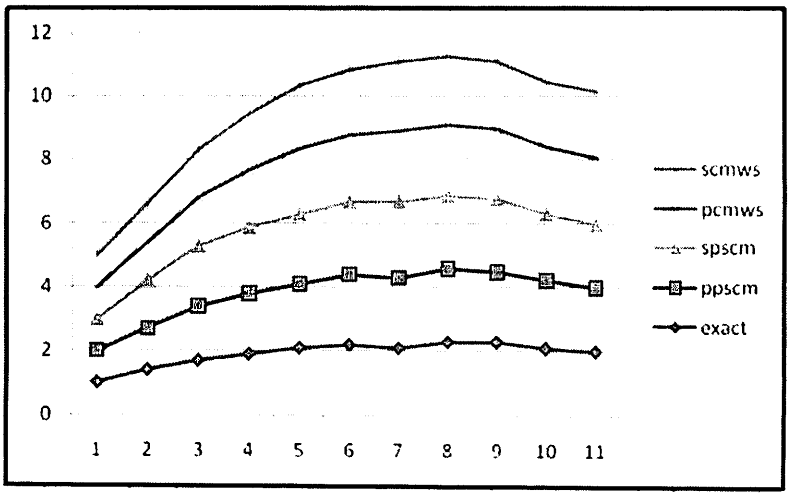

Fig .1 Graphical representation of the exact solution and the other approximate solutions.

Example 2:

$$x^3\, y''(x) \;\square x^2\, y'(x) \;\square 2 \;\square 0, \qquad 1 \,\square x \,\square 2$$

$$y(1) \,\square\, 2$$

$$\square x\, y'(x)|_{x\square 2} \;\square \frac{1}{2}$$

$$y(x) \,\square\, \frac{2}{x} \,\square \frac{\ln(x)}{2}$$

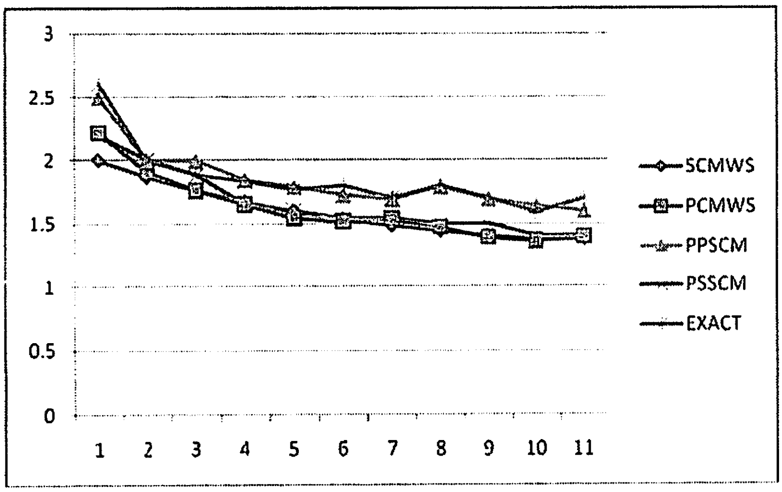

Fig. 2 Graphical representation of example 2 for the exact solution and other approximate solutions.

VII. CONCLUSION

In this paper, patched segmented collocation methods have been successfully applied to solve linear Boundary Value Problems. It was shown that these methods were very efficient and powerful to get closer to the solution as this can be seen from the graphs. We observed that as N increases, the numerical solution of the two methods tends towards the exact solution but the patched perturbed segmented collocation method moved closer to the exact solution than other method. Also, the two graphs show the accuracy of the methods as compared with the exact solution.

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